From geometry to high-energy scattering in N=4 Super Yang Mills at all orders

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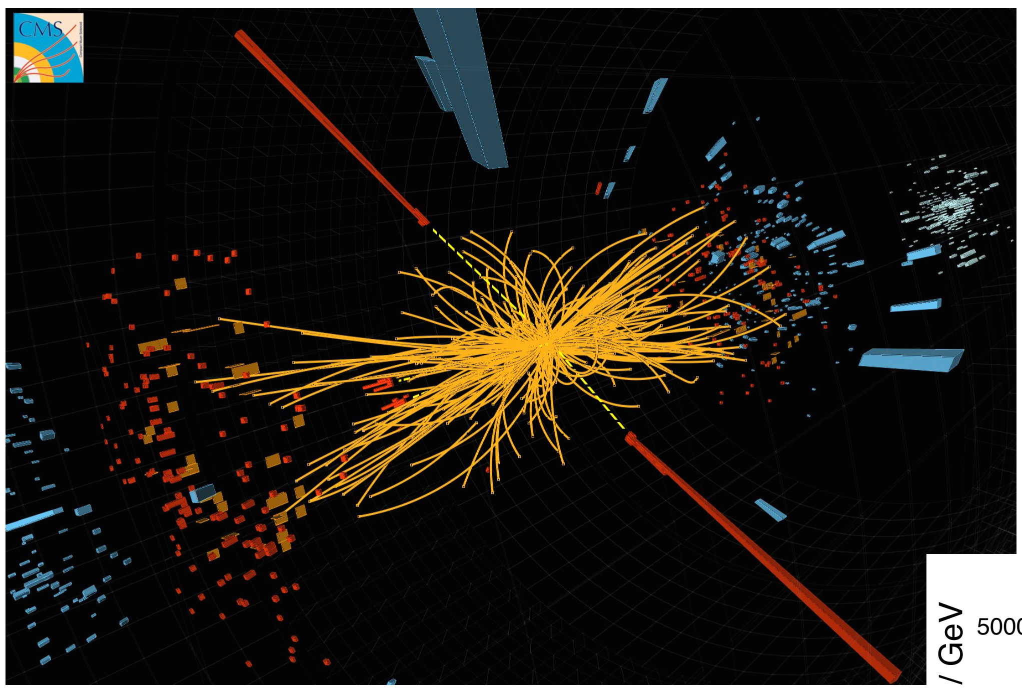
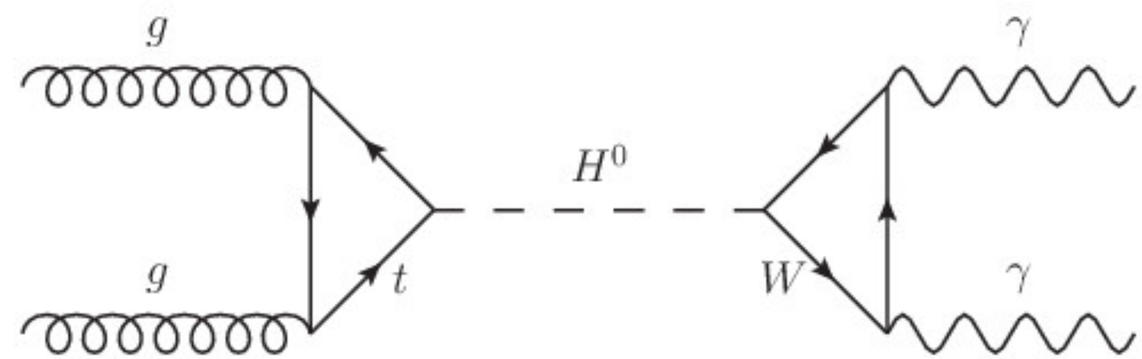
Scattering amplitudes

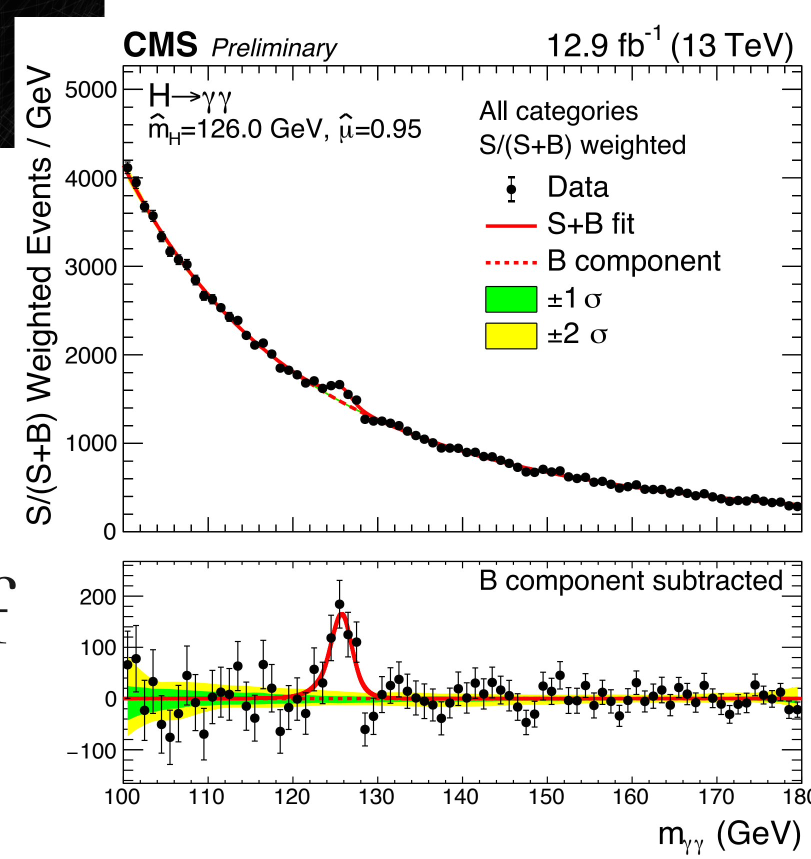
- Scattering amplitudes are among the most fundamental objects in particle physics.





➡ Proton = bound state of quarks and gluons.

➡ LHC = Collisions of quarks and gluons.



$$\text{Proba} \sim |\mathcal{A}_N|^2$$

➡ Need to compute amplitudes in gauge theory as efficiently as possible!

Scattering amplitudes

- In general we do not know how to compute amplitudes exactly.
  - ➡ Need to resort to perturbation theory.

$$\mathcal{A}_N = \mathcal{A}_N^{(0)} + \alpha_s\,\mathcal{A}_N^{(1)} + \alpha_s^2\,\mathcal{A}_N^{(2)} + \ldots \qquad \alpha_s = \text{ coupling constant}$$

  - ➡ Precision increases with the number of terms we compute.

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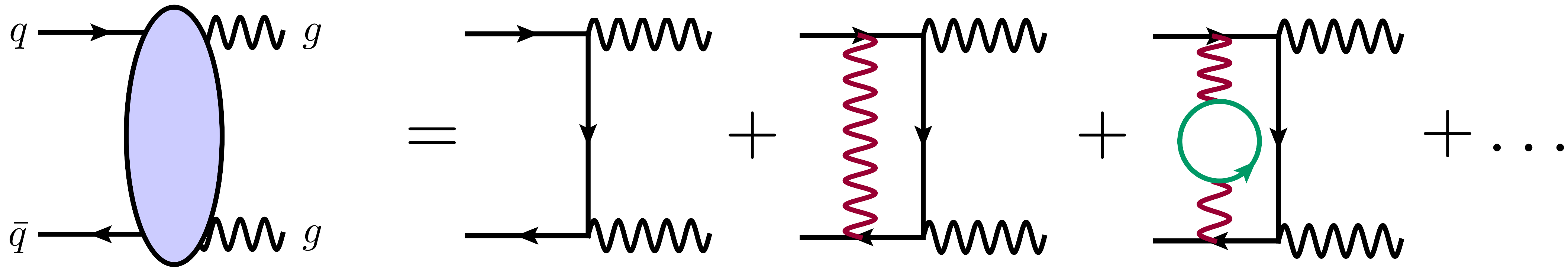
  - ➡ Precision increases with the number of terms we compute.
- $\mathcal{A}_N^{(L)}$ receives contributions from Feynman diagrams with L loops.



  - ➡ Each diagram translates into an analytic formula.
- In principle: can compute anything we like.

Scattering amplitudes

- In practise: Life is hard!
- The number of diagrams grows factorially with the number of external legs.
  - ➡ Example: # tree diagrams contributing to $g\,g \to (N-2)g$

$N-2$	2	3	4	5	6	...
# diagrams	4	45	510	5040	40320	...

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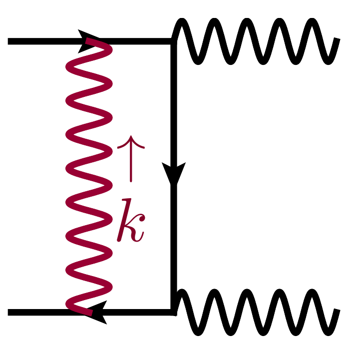
- Beyond tree level: very tough integrals over momentum of unresolved particle.

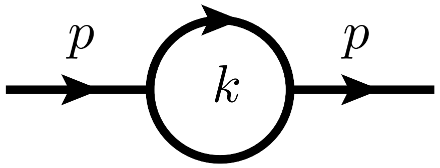
$\int d^4k$



- ➡ 1 loop: usually doable.
- ➡ 2 loop: some $2 \to 2$.
- ➡ 3 loop: some $2 \to 1$.

Scattering amplitudes

- What should we expect?



$$\sim \int \frac{d^4k}{k^2\,(k-p)^2} \sim \int_0^{\Lambda^2} \frac{dk^2}{(k-p)^2} \sim \log \frac{\Lambda^2}{p^2}$$

Scattering amplitudes

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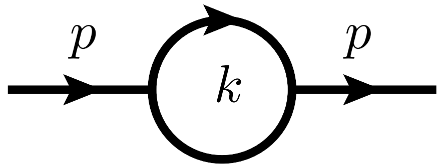

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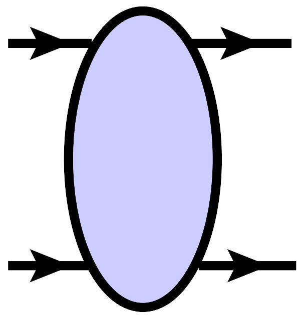
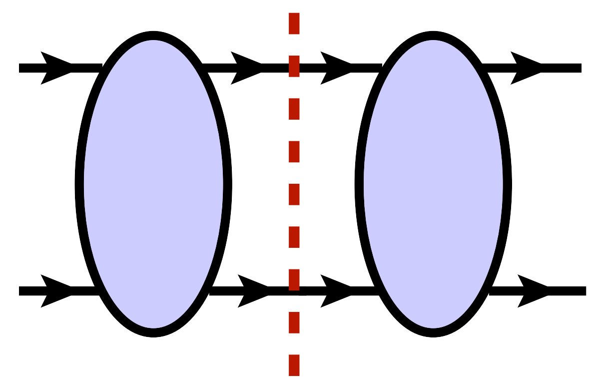
- Optical theorem: Branch cuts encode unitarity.

Im  $= \sum_i \int d\Phi$ 

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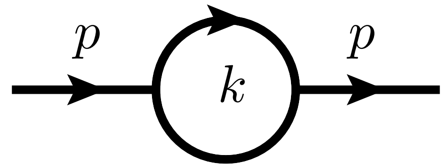

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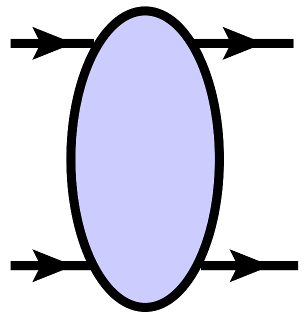
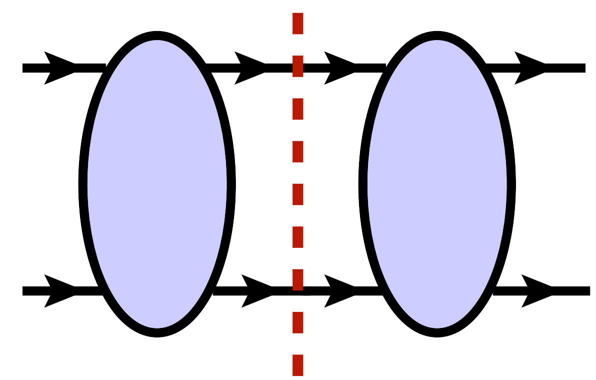
Im  $= \sum_i \int d\Phi$ 

- In general: multi-variable generalisations of logarithms.

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\,G(a_2,\ldots,a_n;t)$$

$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0,1;z) = -\mathrm{Li}_2(z)$$

- Beyond one loop: Also elliptic functions may appear.

N=4 Super Yang-Mills

N=4 Super Yang-Mills

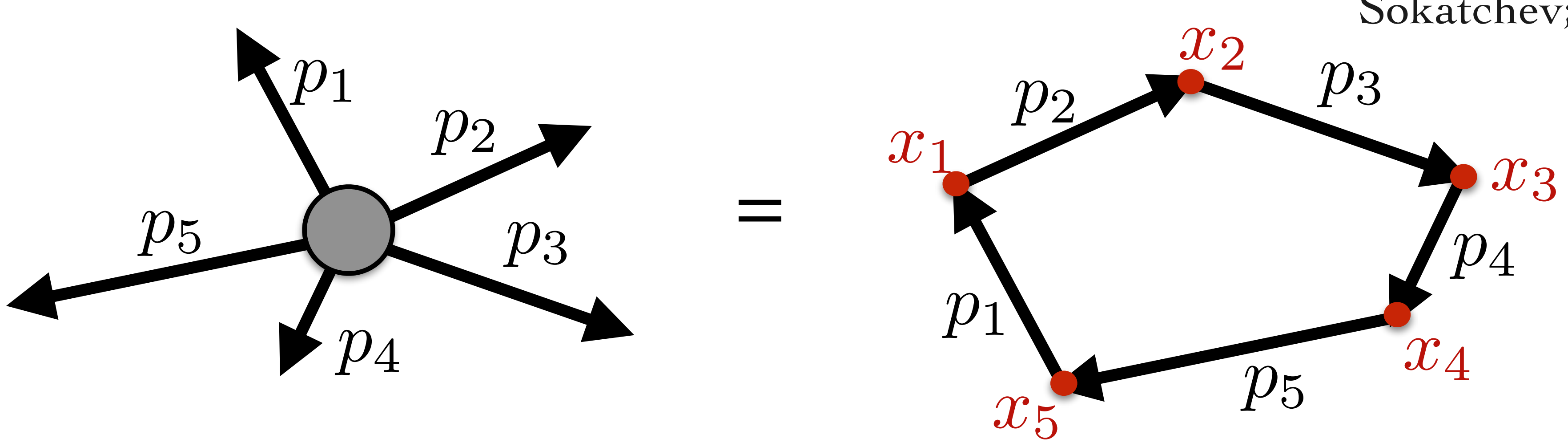
- Supersymmetric cousin of $SU(N_c)$ Yang-Mills theory.
- Spectrum:
  - ➡ Gluon (spin 1, 2 pol.)
  - ➡ Gluino (spin 1/2, 2 pol., 4 kinds)
  - ➡ Scalar (spin 0, 6 kinds)

  8 bosonic and
  8 fermionic d.o.f.
- Conformal at the quantum level.
- Expected to be dual to string theory on $AdS_5 \times S^5$ via AdS/CFT correspondence.
  - ➡ Allows to explore strongly coupled regime.
- Could be looking at the first exactly solvable gauge theory in 4D.
  - ➡ N=4 SYM is the 'hydrogen atom of the 21st centruy'.

A new way of doing QFT

- <u>Dual conformal symmetry:</u> [Drummond, Henn, Korchemsky, Sokatchev]
  - ➡ In the planar limit $N_c \to \infty$ scattering amplitudes in N=4 SYM have additional symmetries.
  - ➡ Closes with ordinary conformal symmetry into an infinite-dimensional Yangian symmetry. [Drummond, Henn, Plefka]

- <u>Amplitude/Wilson-loop duality:</u> [Alday, Maldacena; Brandhuber, Heslop, Spence, Travaglini; Drummond, Henn, Korchemsky, Sokatchev; Mason, Skinner; Caron-Huot]



$$p_i = x_i - x_{i-1}$$

  - ➡ Dual conformal symmetry fixes 4 & 5-point amplitudes completely!

A new way of doing QFT

- <u>'Maximal transcendentality':</u> [Kotikov, Lipatov]
  - ➡ An L loop amplitude only contains polylogarithms of 'transcendentality'/weight $2L$.

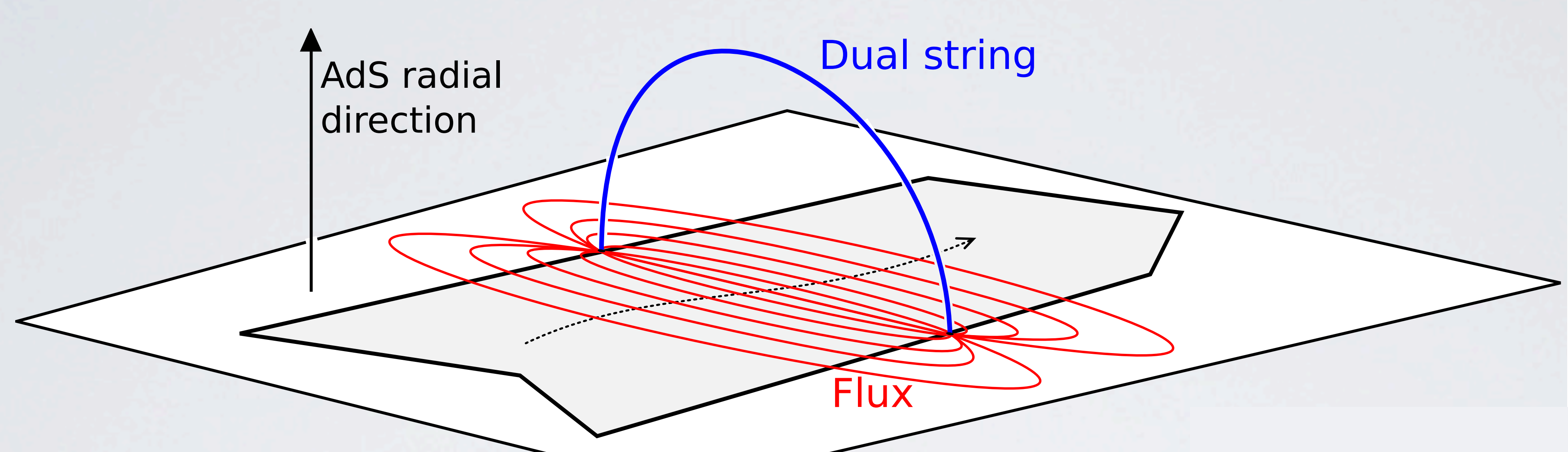
$$\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3} \qquad G(\underbrace{a_1,\ldots,a_n}_{\text{weight } n};z) = \int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)$$

  - ➡ MHV (--++...) amplitudes are 'pure': coefficients in front of polylogarithms are rational numbers (not functions!)

- <u>Geometric description of amplitudes:</u> [Golden, Goncharov, Spradlin, Volovich; Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka; Arkani-Hamed, Trnka]
  - ➡ Cluster algebras.
  - ➡ Positive Grassmannians.
  - ➡ Amplituhedron.
  - ➡ So far: only describes the loop integrand.

Flux tube picture

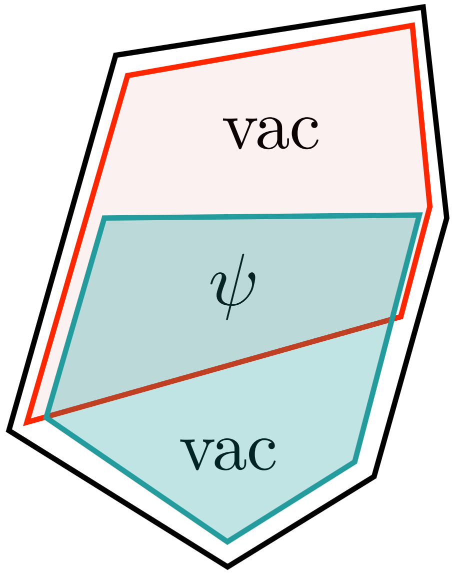
- The sides of the polygon source a flux tube.



[Basso@Amplitudes 16]

- Can describe the Wilson loop/amplitude via the excitations of the flux tube.

[Alday, Gaiotto, Maldacena, Sever, Vieira]

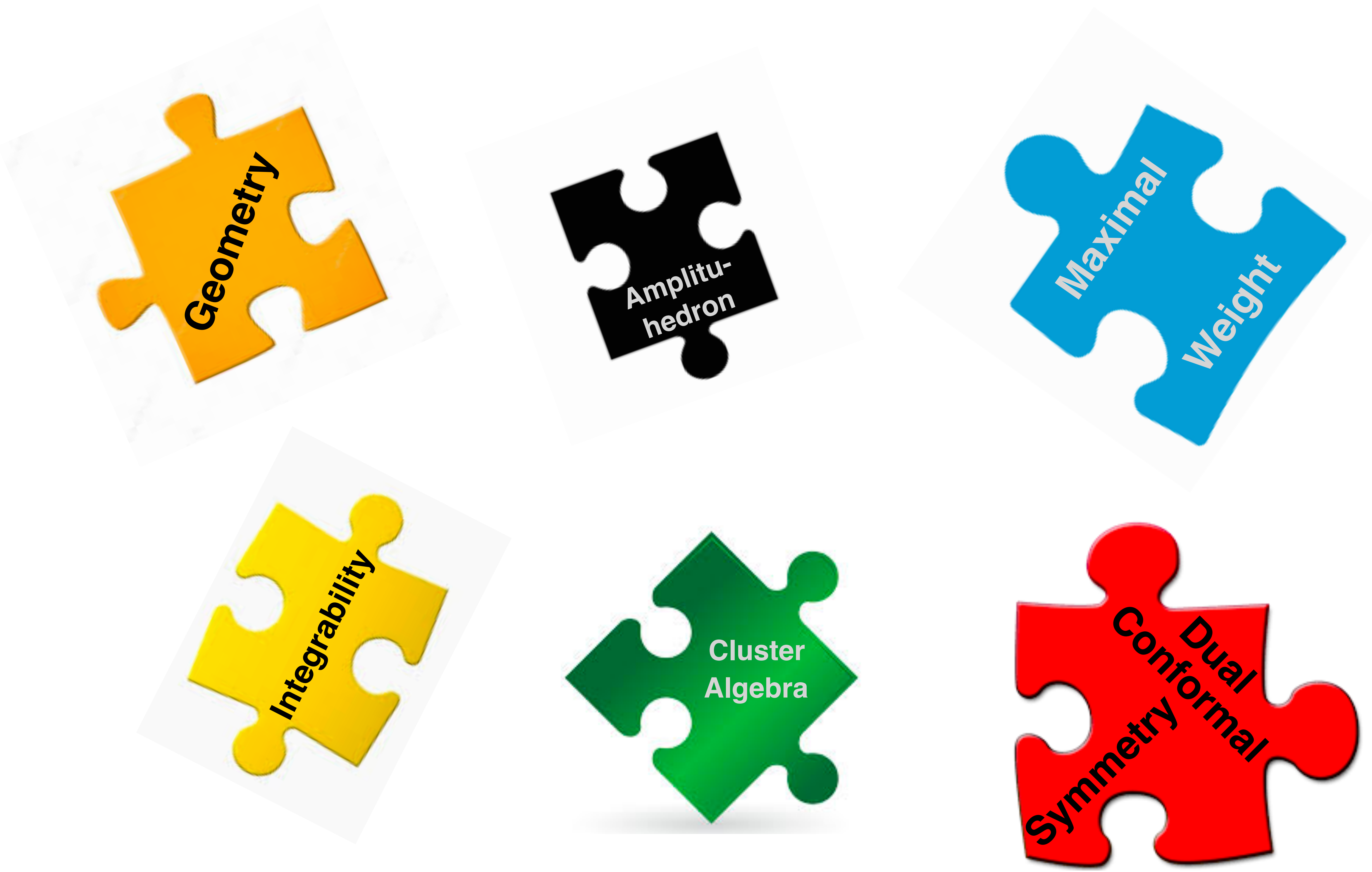


$$= \sum_{\psi} \int d\mu \, P(0|\psi) \, e^{-E\tau + ip + im\phi_1} \, P(\psi|0)$$

➡ Energy spectrum and S-matrix of excitations from integrability.

[Basso, Sever, Vieira]

The big puzzle

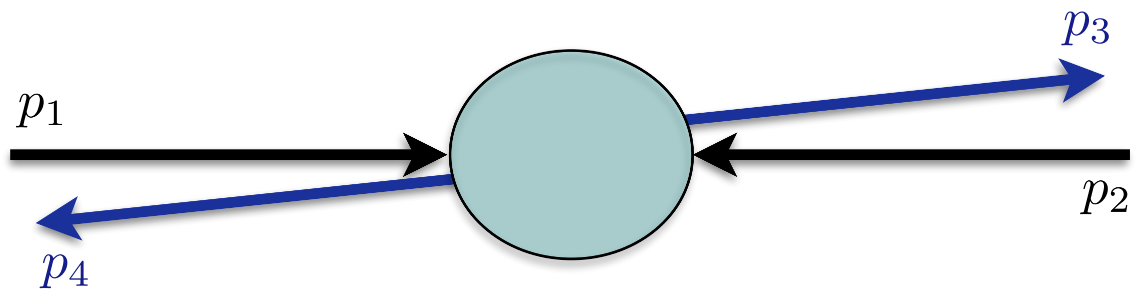


The high-energy limit

The integrability side of the story

The high-energy limit

- High-energy limit = Forward scattering.



$$s = (p_1 + p_2)^2 = E_{CM}^2 \gg |t| = -(p_1 - p_3)^2 = E_1 E_3 (1 - \cos\theta)$$

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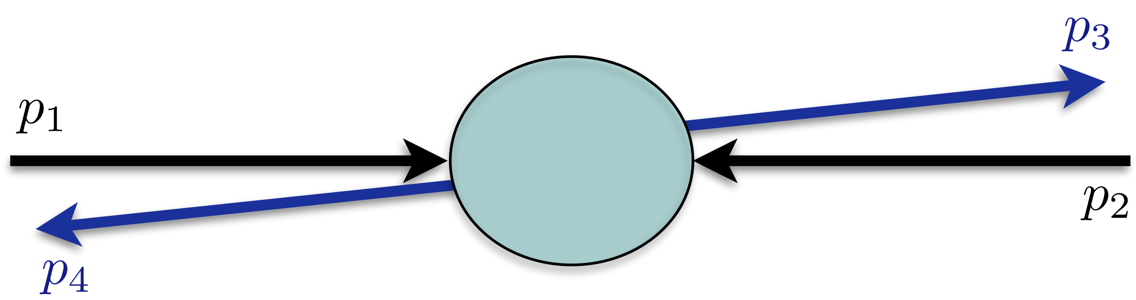

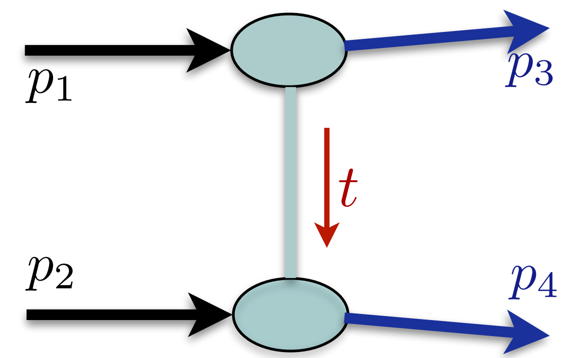

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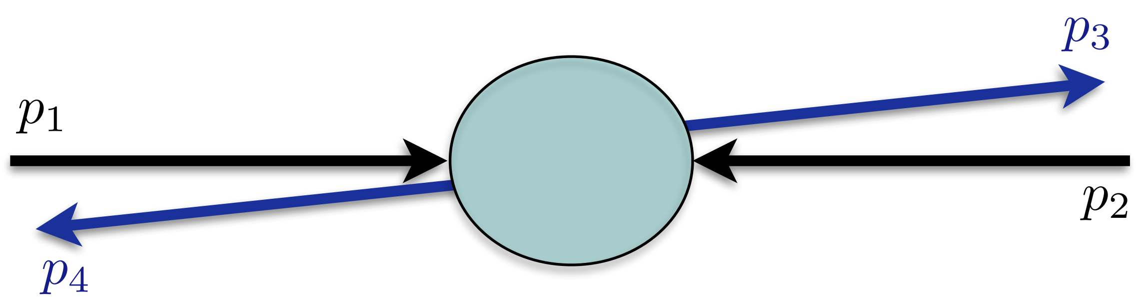
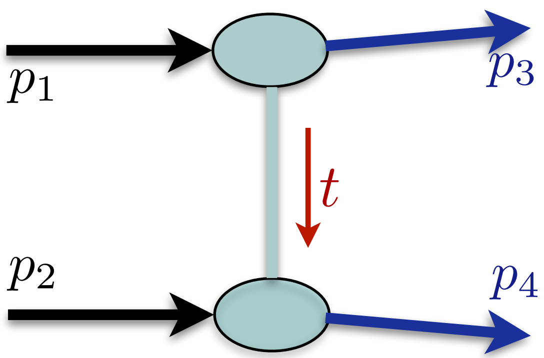
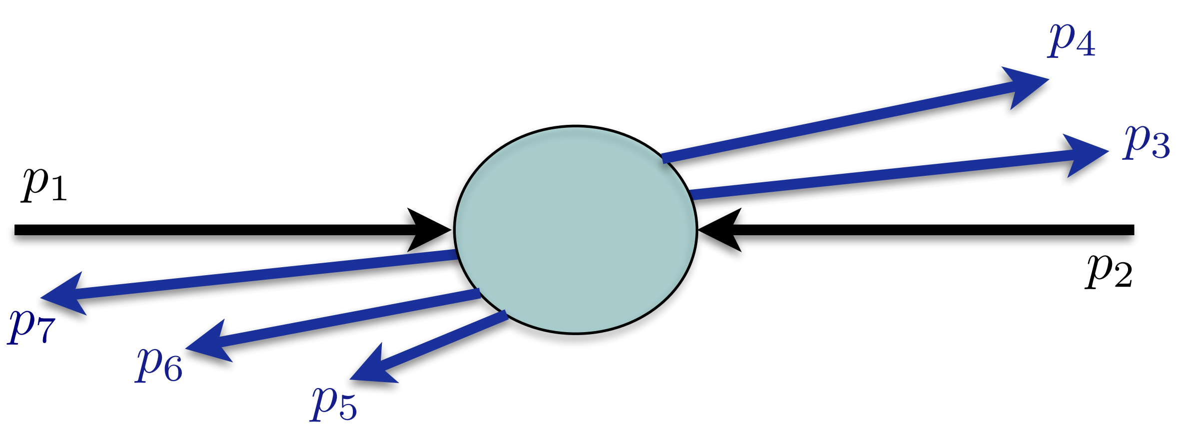
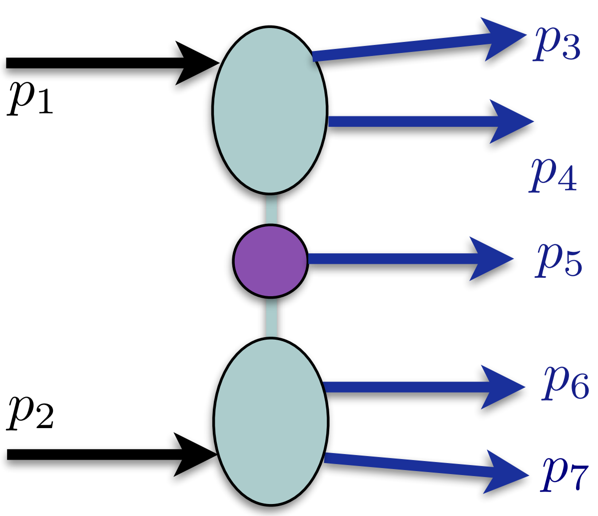




$$s = (p_1 + p_2)^2 = E_{CM}^2 \gg |t| = -(p_1 - p_3)^2 = E_1 E_3 (1 - \cos\theta)$$

- Generalises to more external legs (multi-Regge kinematics).





➡ Hierarchy in 'angles' with respect to the beam axis.

➡ No hierarchy in transverse plane.

The high-energy limit

- Amplitudes factorises into a set of building blocks:
  - ➡ Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation.

$$\sum_{n_1} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \int \frac{d\nu_1}{2\pi} |z_1|^{2i\nu_1} \chi_1^{h_3} \tau_1^{\omega_1} \chi_1^{-h_4}$$

$$\tau_i \sim \log \frac{s_i}{t_i}$$

$$\omega_i = \text{ BFKL eigenvalue}$$

$$\chi_i^h = \text{ impact factor}$$

$$C_{ij}^h = \text{ central emission vertex}$$

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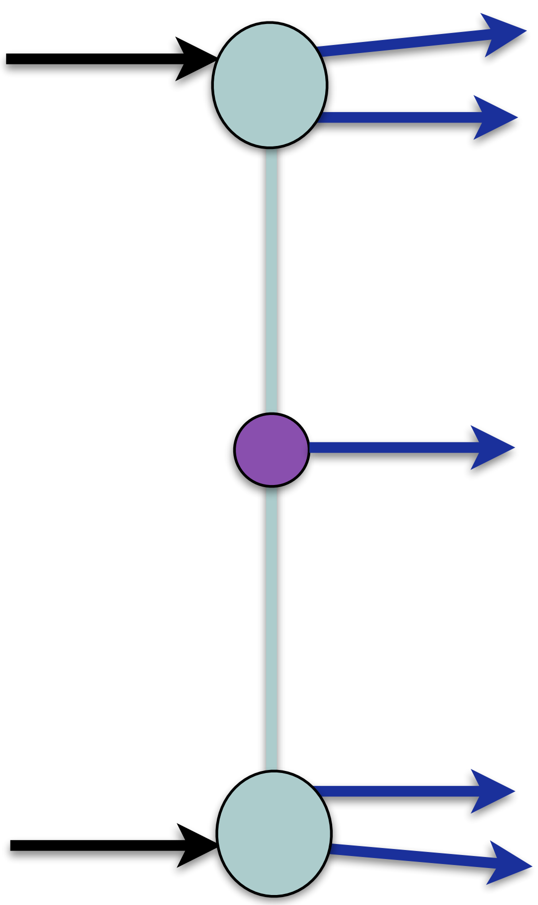

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$$\sum_{n_1}\left(\frac{z_1}{\bar z_1}\right)^{\frac{n_1}{2}}\int\frac{d\nu_1}{2\pi}|z_1|^{2i\nu_1}\sum_{n_2}\left(\frac{z_2}{\bar z_2}\right)^{\frac{n_2}{2}}\int\frac{d\nu_2}{2\pi}|z_2|^{2i\nu_2}$$

$$\chi_1^{h_3}\tau_1^{\omega_1}C_{12}^{h_4}\tau_2^{\omega_2}\chi_2^{-h_5}$$

$\omega_i =$ BFKL eigenvalue

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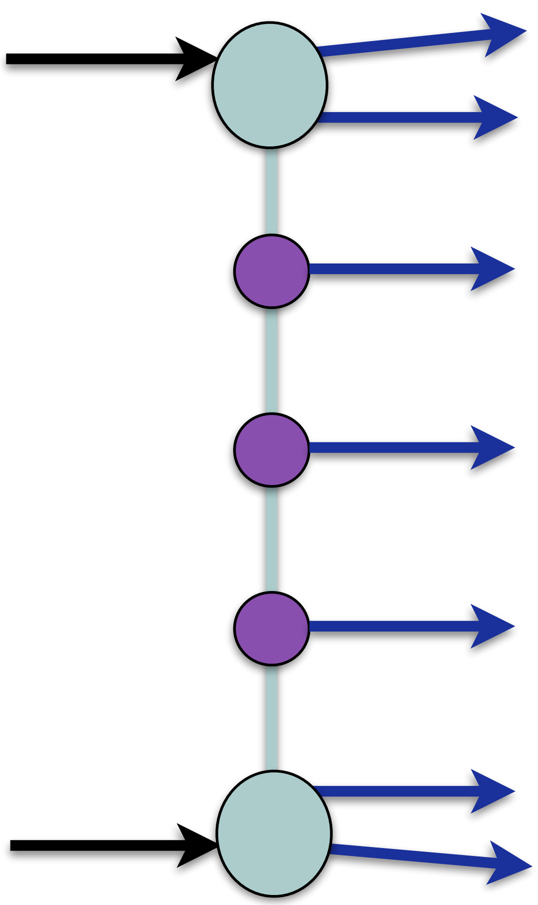

$$\sum_{n_1}\left(\frac{z_1}{\bar z_1}\right)^{\frac{n_1}{2}}\int\frac{d\nu_1}{2\pi}|z_1|^{2i\nu_1}\cdots\sum_{n_k}\left(\frac{z_k}{\bar z_k}\right)^{\frac{n_k}{2}}\int\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}$$

$$\chi_1^{h_3}\,\tau_1^{\omega_1}\,C_{12}^{h_4}\,\tau_2^{\omega_2}\cdots\tau_{k-1}^{\omega_{k-1}}\,C_{k-1,k}^{h_{k+2}}\,\tau_k^{\omega_k}\,\chi_k^{-h_{k+3}}$$

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$\omega_i =$ BFKL eigenvalue

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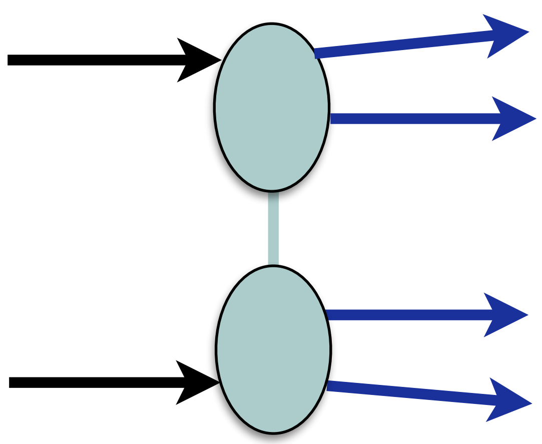
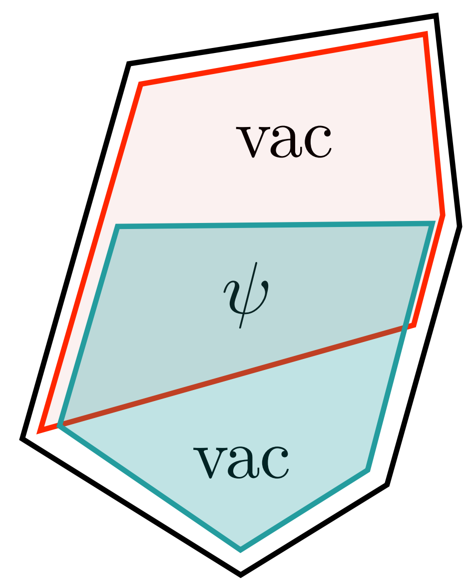
MRK vs Flux Tube

- BFKL-type equation very reminiscent of flux tube formula!

BFKL eigenvalue ⟷ Spectrum of excitations

Impact factor & central emission block ⟷ Transition probability $P(\psi_1|\psi_2)$





$$\sum_n \left(\frac{z}{\bar{z}}\right)^{n/2} \int \frac{d\nu}{2\pi} |z|^{2i\nu} \chi_1^{h_3} \tau_1^{\omega_1} \chi_1^{-h_4} \qquad \sum_\psi \int d\mu \, P(0|\psi)\, e^{-E\tau + ip + im\phi_1}\, P(\psi|0)$$

MRK vs Flux Tube

- Basso, Caron-Huot and Sever: BFKL eigenvalue and impact factors for all values of the coupling by analytic continuation of the flux tub data!

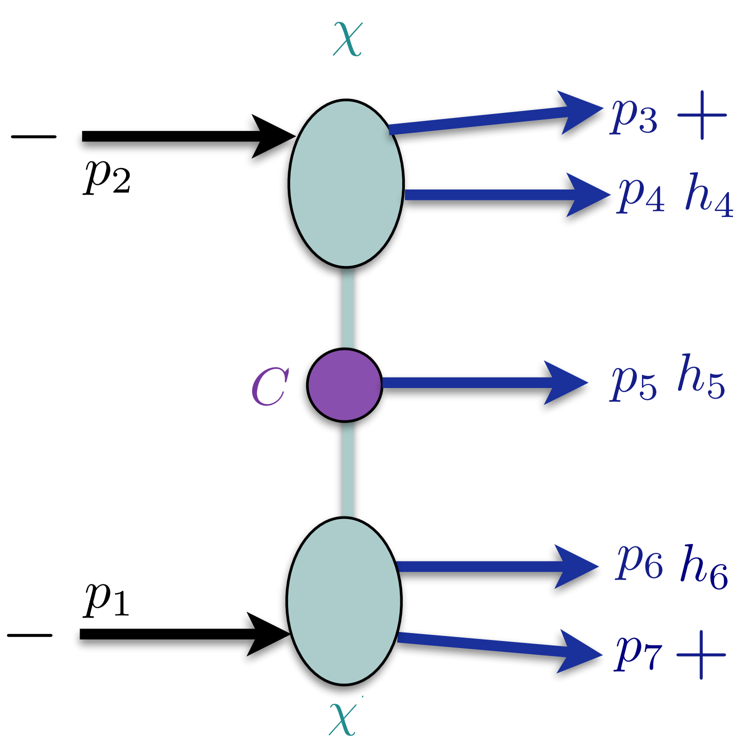
$$\sum_n \left(\frac{z}{\bar{z}}\right)^{n/2} \int \frac{d\nu}{2\pi} |z|^{2i\nu} \chi_1^{h_3} \tau_1^{\omega_1} \chi_1^{-h_4} \qquad \sum_\psi \int d\mu\, P(0|\psi)\, e^{-E\tau+ip+im\phi_1} P(\psi|0)$$

- We have recently also determined the central emission block to all orders in the coupling!

$$\sum_{n_1} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} \int \frac{d\nu_1}{2\pi} |z_1|^{2i\nu_1} \sum_{n_2} \left(\frac{z_2}{\bar{z}_2}\right)^{\frac{n_2}{2}} \int \frac{d\nu_2}{2\pi} |z_2|^{2i\nu_2} \chi_1^{h_3} \tau_1^{\omega_1} C_{12}^{h_4} \tau_2^{\omega_2} \chi_2^{-h_5}$$

Convolutions

- Next step: what happens in momentum space?



$$\sim \prod_{j=1,2} \sum_{n_j} \left(\frac{z_j}{\bar{z}_j}\right)^{n_j/2} \int \frac{d\nu_j}{2\pi} |z_j|^{2i\nu_j} \chi^{h_3} \tau_1^{aE_{\nu_1 n_1}} C^{h_4} \tau_2^{aE_{\nu_2 n_2}} \chi^{-h_5}$$

$$\sim \sum_{i_1,i_2} \frac{a^{i_1+i_2}}{i_1! i_2!} \log^{i_1} \tau_1 \ \log^{i_2} \tau_2 \ g^{(i_1,i_2)}_{h_3 h_4 h_5}(z_1, z_2)$$

[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygaryn, Schnitzer; Bartels, Lipatov, Kormilitzin, Prygaryn]

- Fourier-Mellin transform: $\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} F(\nu,n)$

- Which $F(\nu,n)$ can appear?

FM building blocks

- Integrability: In perturbation theory, integrand is a polynomial in multiple zeta values and

$$E = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)$$

$$V = \frac{i\nu}{\nu^2+\frac{n^2}{4}} \qquad N = \frac{n}{\nu^2+\frac{n^2}{4}} \qquad D_\nu = -i\partial_\nu$$

$$M = \psi\left(i(\nu_k-\nu_l)-\frac{n_k-n_l}{2}\right) + \psi\left(1-i(\nu_k-\nu_l)-\frac{n_k-n_l}{2}\right)$$

- Example: NLO BFKL eigenvalue: $\omega = -E - a\,E^{(1)} - a^2\,E^{(2)} + \dots$

$$E^{(1)} = -\frac{1}{4}D_\nu^2 E + \frac{1}{2}V\,D_\nu E - \zeta_2\,E - 3\,\zeta_3$$

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$$E = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)$$

$$F = -2\psi(1) + \psi\left(1+i\nu-\frac{n}{2}\right) + \psi\left(1-i\nu-\frac{n}{2}\right)$$

$$V = \frac{i\nu}{\nu^2+\frac{n^2}{4}}\,, \qquad N = \frac{n}{\nu^2+\frac{n^2}{4}}\,, \qquad D_\nu^n \equiv (-i)^n \partial_\nu^n$$

$$M = \psi\left(i(\nu_k-\nu_l) - \frac{n_k-n_l}{2}\right) + \psi\left(1 - i(\nu_k-\nu_l) - \frac{n_k-n_l}{2}\right)$$

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$$E^{(1)} = -\frac{1}{4}D_\nu^2 E + \frac{1}{2} V\,D_\nu E - \zeta_2\,E - 3\,\zeta_3$$

FM building blocks

- FM transform maps products into convolutions:

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g \;=\; \frac{1}{\pi}\int \frac{d^2w}{|w|^2}\, f(w)\; g\left(\frac{z}{w}\right)$$

➡ The building blocks have simple FM transforms, e.g.:

$$E \;=-\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}}+\psi\left(1+i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi(1)$$

$$\mathcal{F}[E] = -\frac{z+\bar{z}}{2|1-z|^2}$$

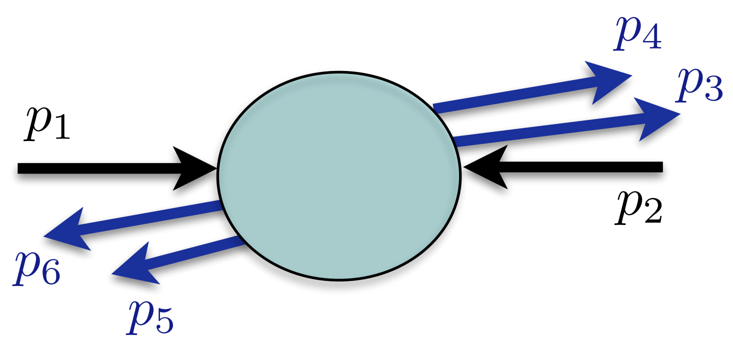
- How to evaluate the convolutions?

Doing the integrals

The geometry side of the story

Multi-Regge kinematics

- Non-trivial kinematical dependence in transverse plane.



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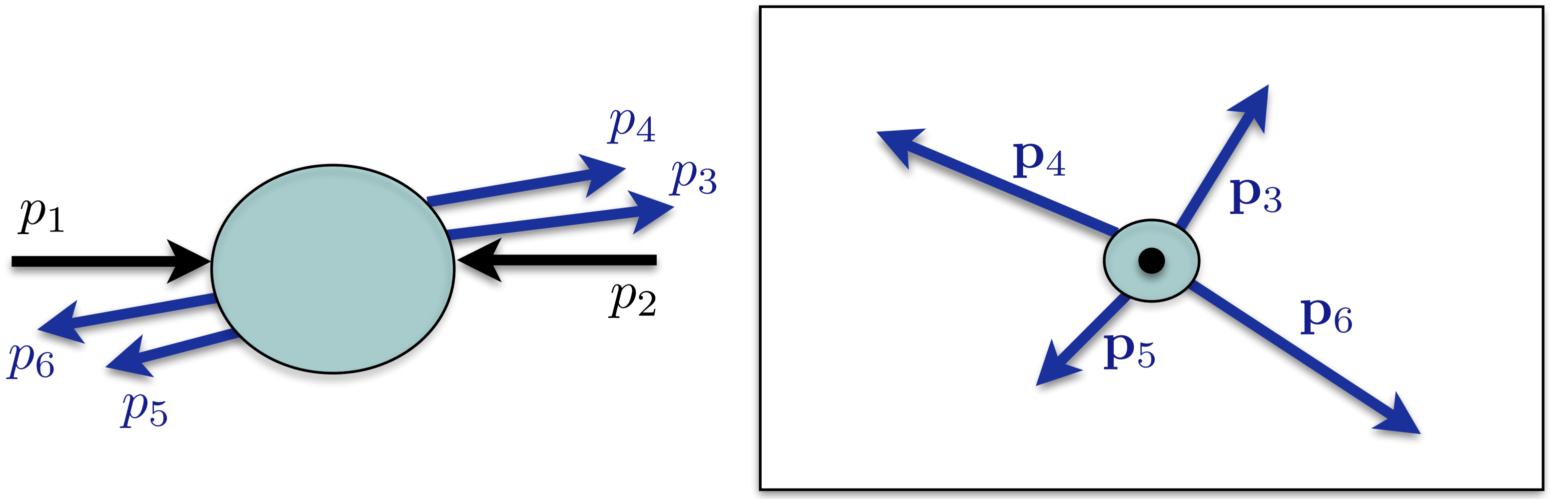

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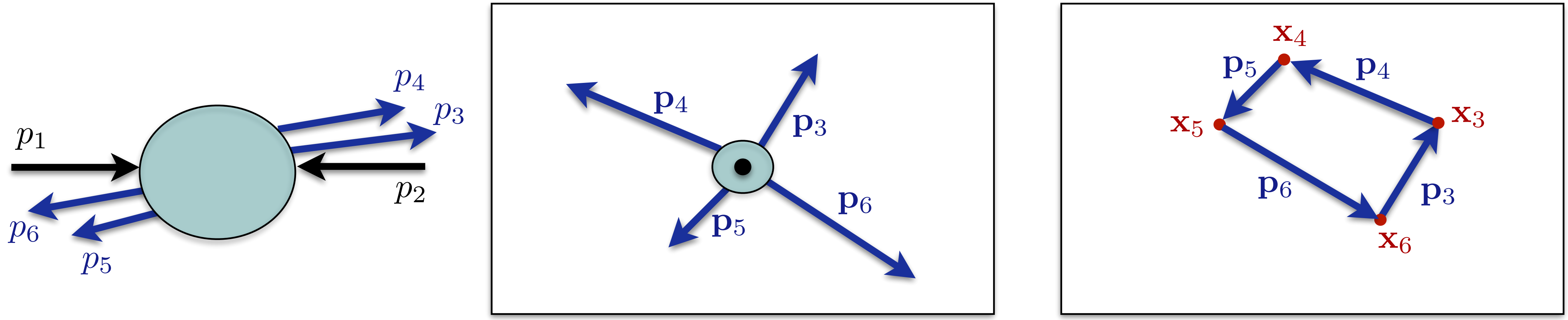


  ➡ Kinematics encoded into $N-2$ points in transverse plane.

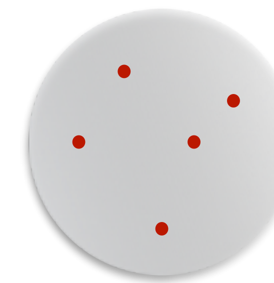
- Dual conformal invariance in transverse plane:

  ➡ Functional dependence only on $N-5$ cross ratios in transverse plane:

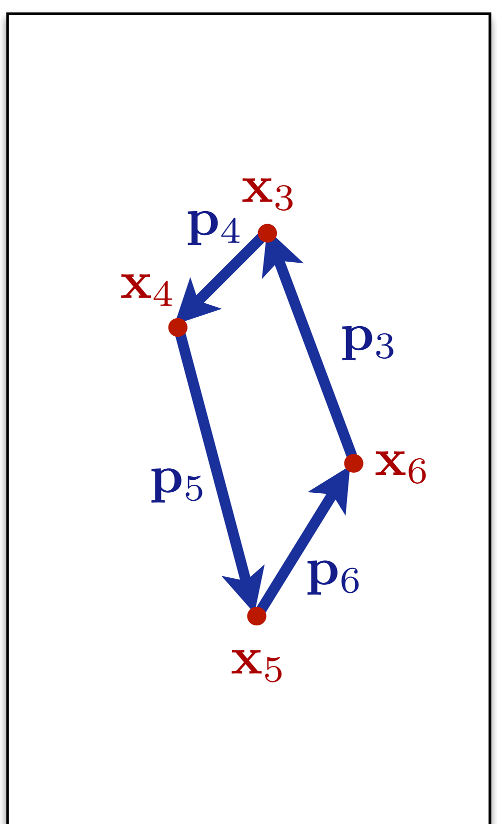
$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

The moduli space $\mathfrak{M}_{0,n}$

- $\mathfrak{M}_{0,n}$ = moduli space space of Riemann spheres with n marked points.
  = space of configurations of n points on the Riemann sphere.



- For $n = N-2 : \mathfrak{M}_{0,N-2}$ is 'phase space' of MRK.



➡ Fix 3 points to $0, 1, \infty$.

➡ $\dim_{\mathbb{C}} \mathfrak{M}_{0,n} = n-3$

➡ Coordinates are collection of $n-3 = N-5$ cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

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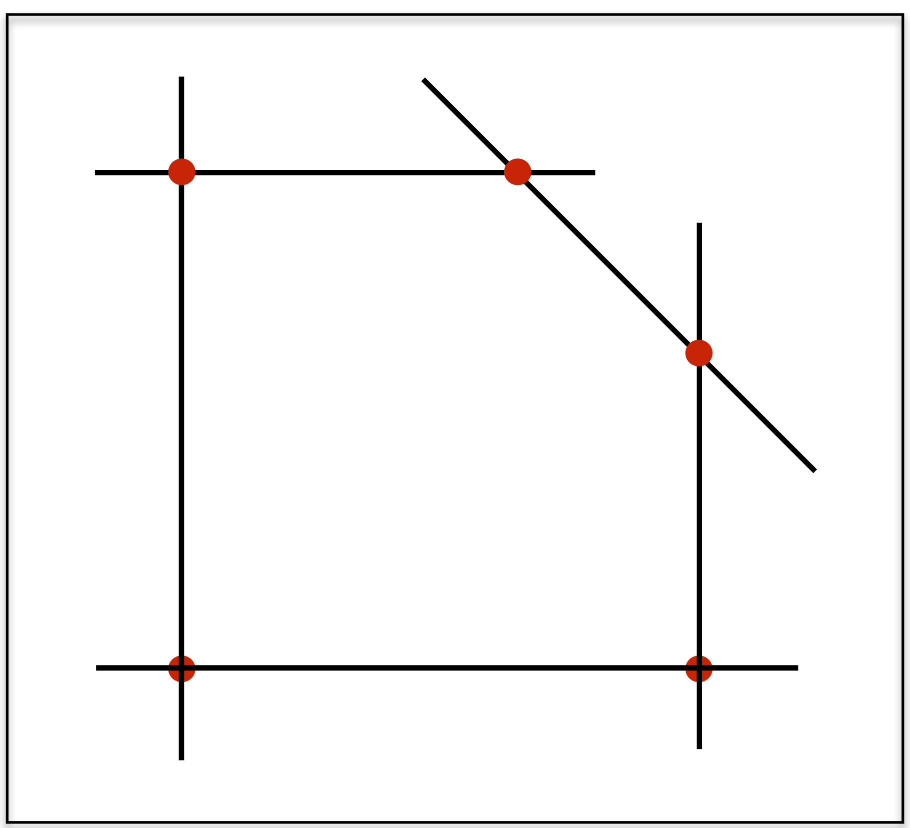
$\mathfrak{M}_{0,5}$



The moduli space $\mathfrak{M}_{0,n}$

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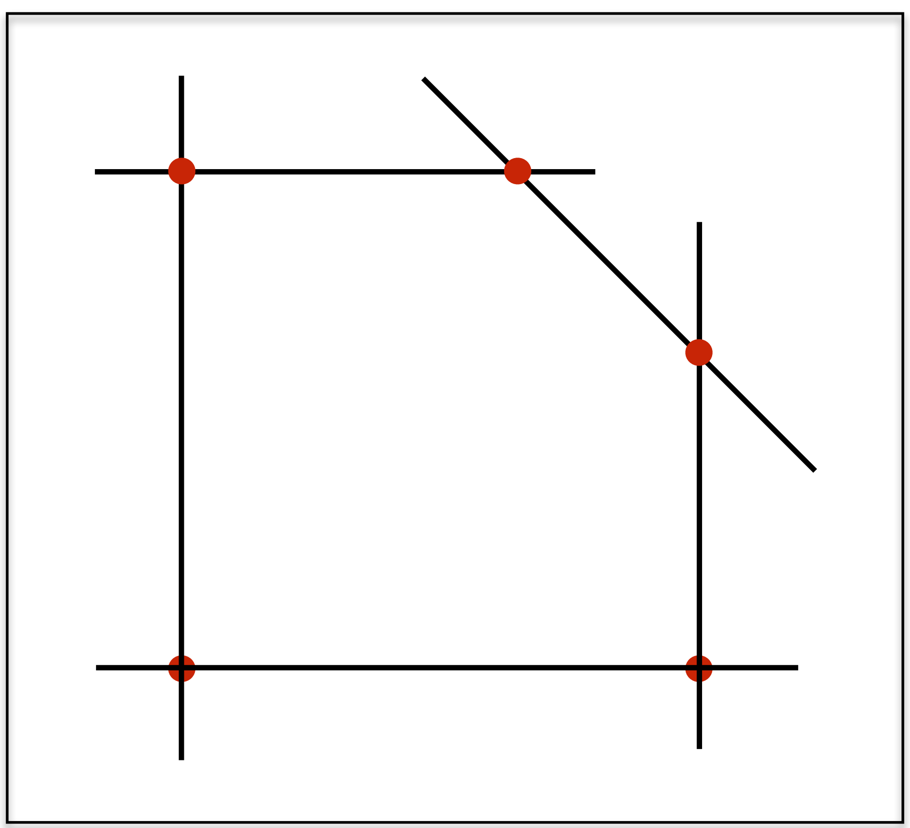
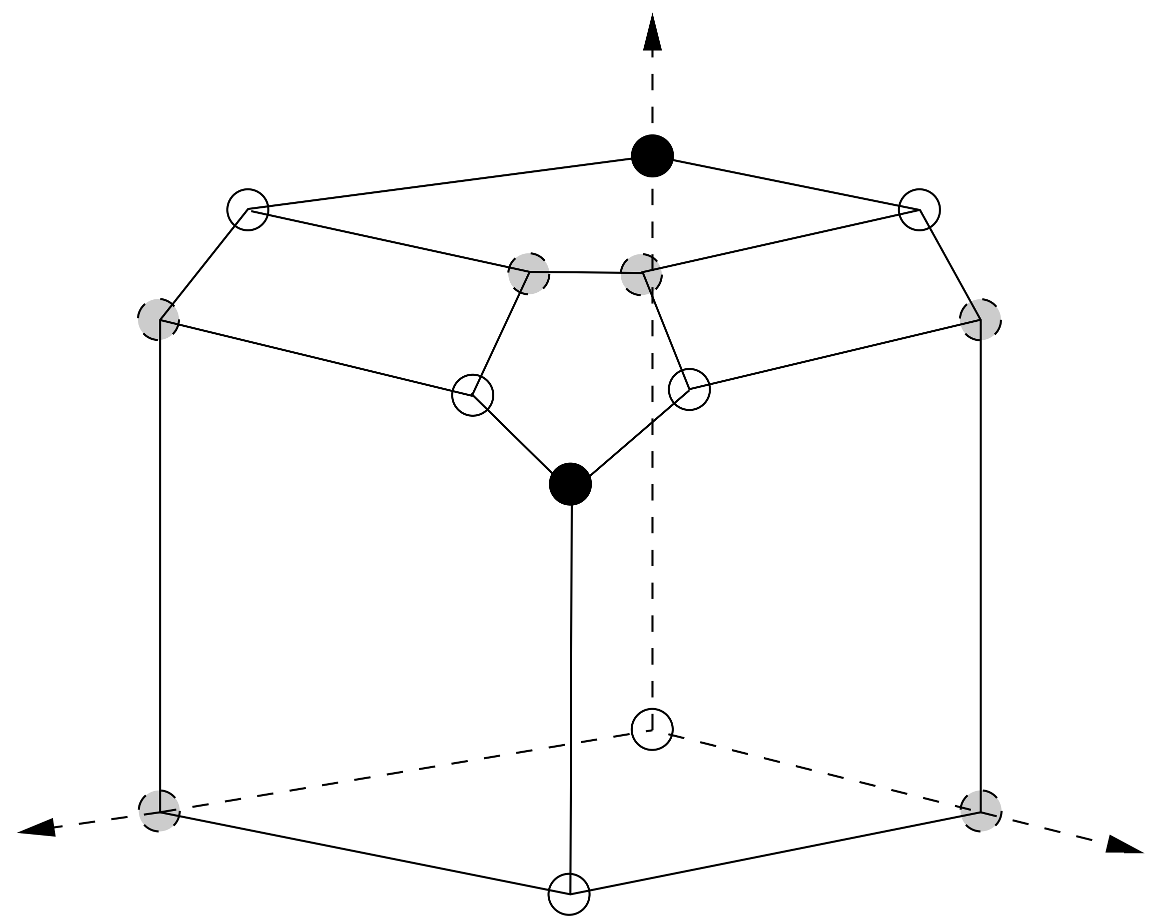
$\mathfrak{M}_{0,5}$



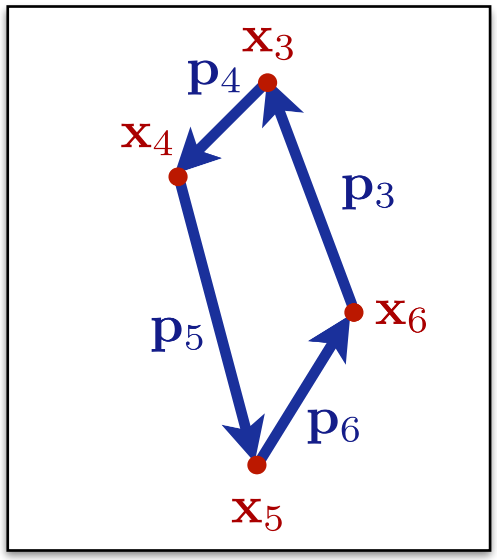
$\mathfrak{M}_{0,6}$



[Figure: F. Brown]

Iterated integrals on $\mathfrak{M}_{0,n}$

- Singularities: 'Degenerate' configurations of points.



= 2 points become equal.

➡ Physically: momentum is soft.

- What are 'natural integrals' on this space?

➡ Should have singularities at most when $\mathbf{x}_i = \mathbf{x}_j$.

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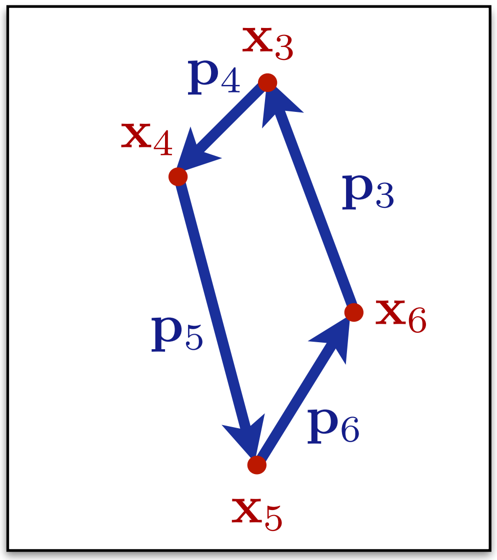

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- All iterated integrals on $\mathfrak{M}_{0,n}$ can be written in terms of polylogarithms. [Brown]

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

$$G(a_1; z) = \log\left(1 - \frac{z}{a_1}\right)$$

$$G(0, 1; z) = -\text{Li}_2(z)$$

- Consequence: Amplitudes in MRK can be written in terms of polylogarithms.

➡ Must have branch cuts dictated by unitarity!

Polylogarithms & Amplitudes

- Branch cuts of massless amplitudes:
  - ➡ Unitarity: The only branch cuts start at $(p_{i+1} + \ldots + p_j)^2 = 0$.
  - ➡ Example: $\log s_{12}$ is allowed, $\log(s_{12} - s_{34})$ is not.

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- In 2D transverse space: $\log\underbrace{|\mathbf{x}_i-\mathbf{x}_j|^2}_{>0,\text{ if } i\neq j}\,.$
  - ➡ Unitarity: There is no branch cut in 2D transverse plane!

Single-valued functions

- Single-valued polylogarithms = combinations of polylogarithms and their complex conjugates such that all branch cuts cancel.

- For every polylogarithm there is a single-valued analogue.
  - ➡ Can be constructed algorithmically. [Brown]
  - ➡ Example: $\log z \longrightarrow \log|z|^2$

$$\mathrm{Li}_2(z) \longrightarrow \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log|z|^2 \log(1-\bar{z})$$

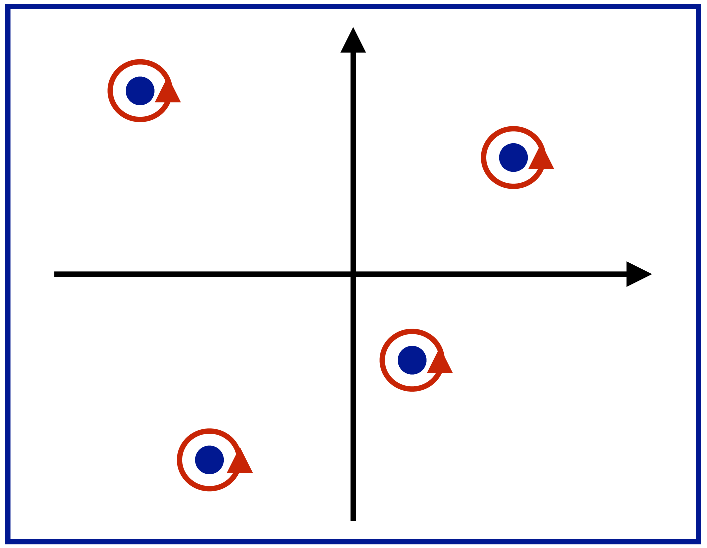
- Conclusion:

> N-point scattering amplitudes in planar N=4 SYM in MRK are single-valued iterated integrals on $\mathfrak{M}_{0,N-2}$.

[Dixon, Duhr, Pennington; Del Duca, Duhr, Dulat, Drummond, Druc, Marzucca, Papathanasiou, Verbeek]

Stokes' theorem & residues

- Single-valuedness implies that the integrals can easily be performed in terms of Stokes' theorem.
  - ➡ All singularities are isolated.
  - ➡ Singularity structure of $\mathfrak{M}_{0,n}$ known.
  - ➡ Can integrate over the boundary of the punctured complex plane.



$$\int \frac{d^2z}{\pi} f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z) \qquad \bar{\partial}_z F = f$$ [Schnetz]

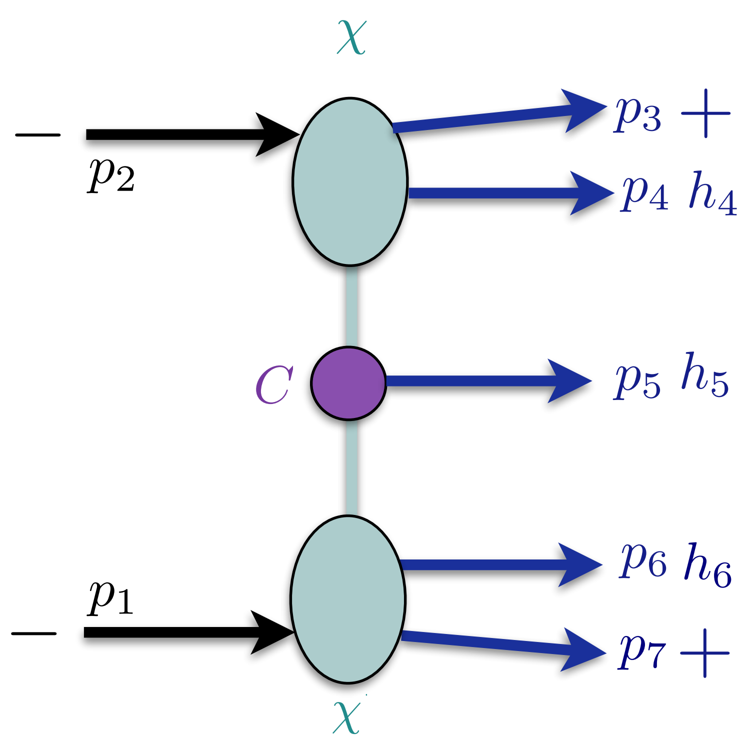
- Computation of FM integral reduces to a simple residue computation!

N = 4 SYM in the high-energy limit

From integrality & geometry to dynamics

Convolutions

- Dynamics described by an all-order factorisation formula.



$$\sim \prod_{j=1,2} \sum_{n_j} \left(\frac{z_j}{\bar{z}_j}\right)^{n_j/2} \int \frac{d\nu_j}{2\pi} |z_j|^{2i\nu_j}\, \chi^{h_3}\, \tau_1^{aE_{\nu_1 n_1}}\, C^{h_4}\, \tau_2^{aE_{\nu_2 n_2}}\, \chi^{-h_5}$$

$$\sim \sum_{i_1,i_2} \frac{a^{i_1+i_2}}{i_1! i_2!} \log^{i_1}\tau_1\, \log^{i_2}\tau_2\, g^{(i_1,i_2)}_{h_3h_4h_5}(z_1,z_2)$$

[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygaryn, Schnitzer; Bartels, Lipatov, Kormilitzin, Prygaryn]

- Fourier-Mellin transform: $\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} F(\nu,n)$

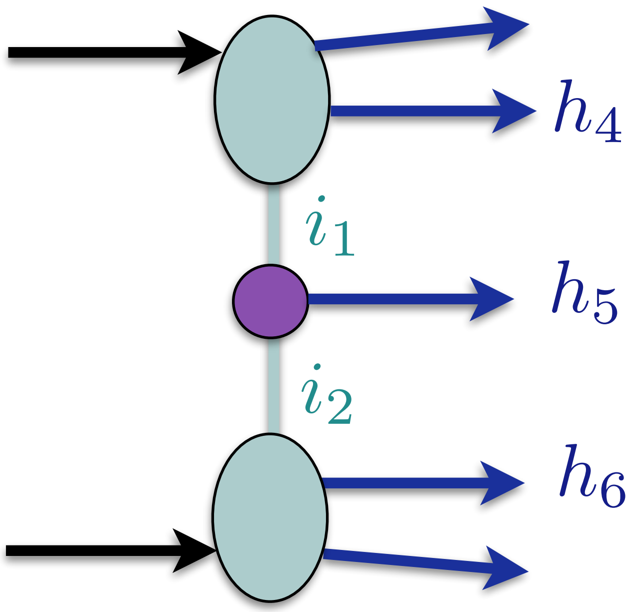
- FM transform maps products into convolutions:

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = f * g = \frac{1}{\pi}\int \frac{d^2w}{|w|^2} f(w)\, g\left(\frac{z}{w}\right)$$

Factorisation

- Consequence 1: Convolutions imply a factorisation theorem!

$$g^{(i_1,i_2)}_{h_4 h_5 h_6} \sim$$



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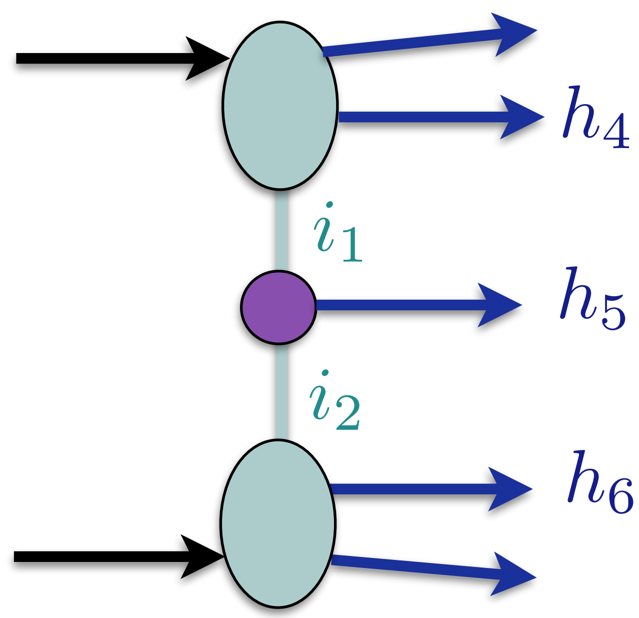
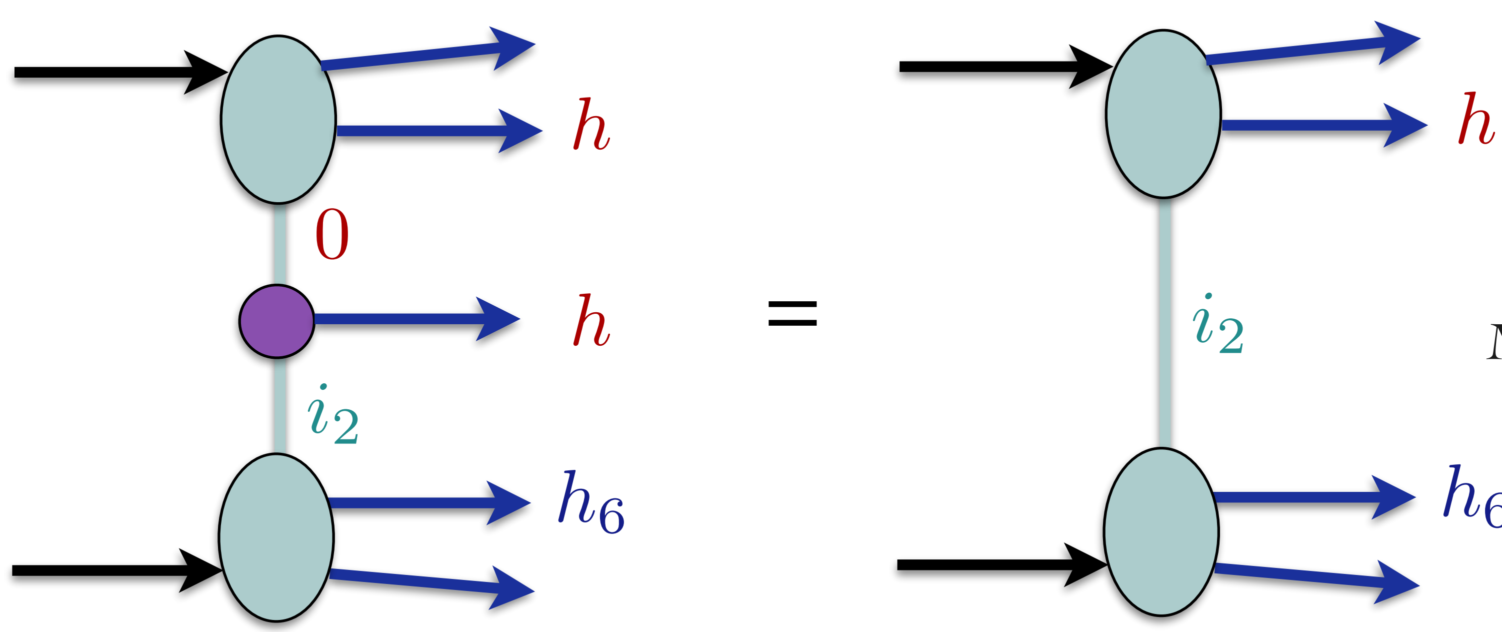


- Theorem:



[Del Duca, Druc, Drummond, CD ,Dulat, Marzucca, Papathanasiou, Verbeek; Bargheer]

➡ Implies relations between amplitudes with different numbers of legs.

Factorisation for MHV

- Consequence: At L loops an MHV amplitudes in MRK is determined by amplitudes with at most $L+4$ external legs.

- Two loops, LLA: [Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]

$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\le i\le N-5} \log \tau_i \, g^{(1)}_{++}(\rho_i)$$

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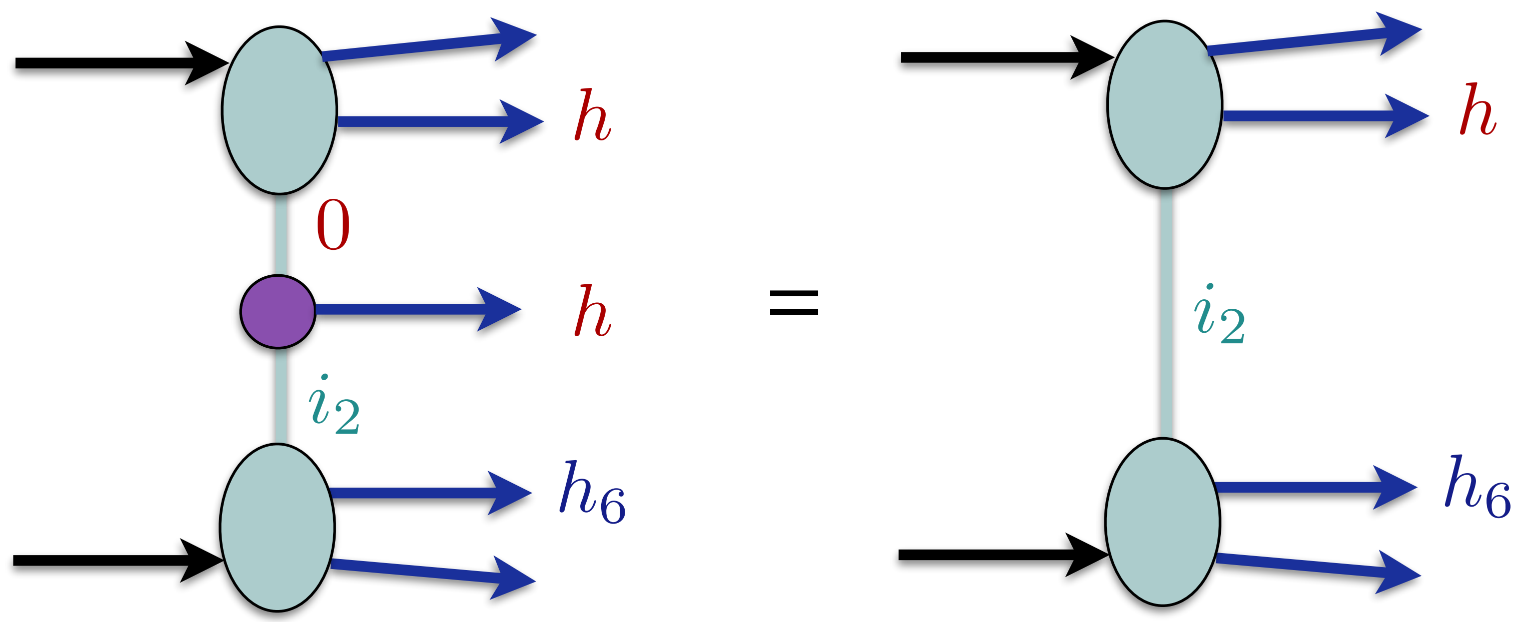
$$\mathcal{R}^{(2)}_{+\ldots+} = \sum_{1\leq i\leq N-5} \log\tau_i\, g^{(1)}_{++}(\rho_i)$$

- Three loops, LLA:

$$\mathcal{R}^{(3)}_{+\ldots+} = \frac{1}{2}\sum_{1\leq i\leq N-5} \log^2\tau_i\, g^{(2)}_{++}(\rho_i) + \sum_{1\leq i<j\leq N-5} \log\tau_i\, \log\tau_j\, g^{(1,1)}_{+++}(\rho_i,\rho_j)\,.$$

Factorisation for non-MHV

- Factorisation theorem still holds for non-MHV amplitudes.



➡ Unlike MHV: infinite number building blocks per loop.

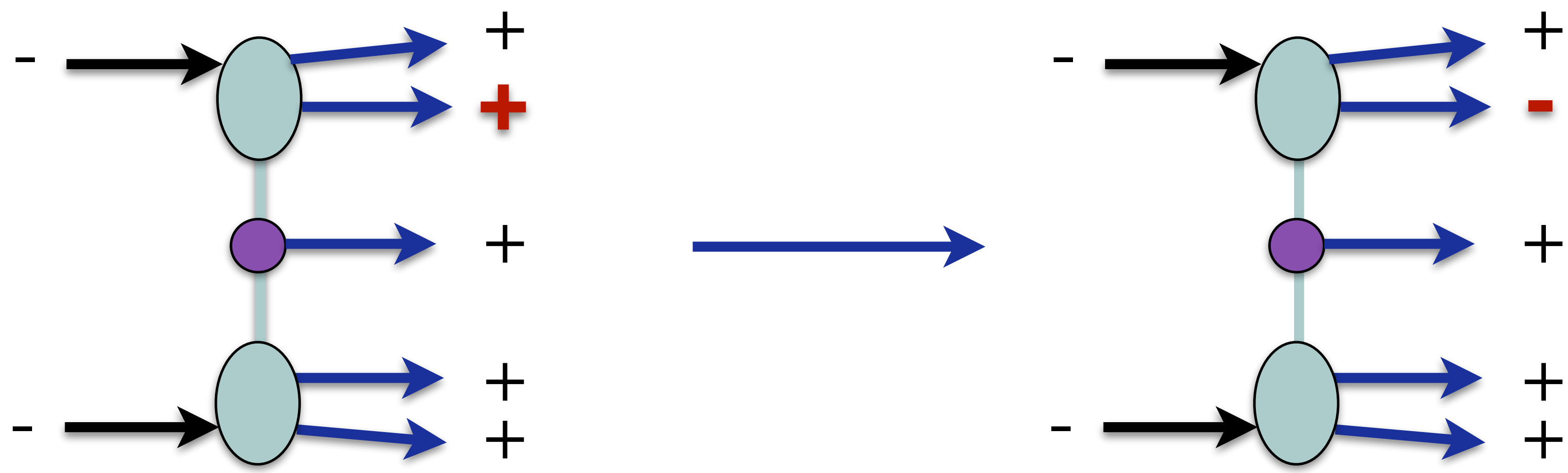
- Example:

$$\mathcal{R}^{(2)}_{-+\ldots} = \log\tau_1\, g^{(1)}_{-+}(\rho_1) + \sum_{j=2}^{N-5} \log\tau_j\, g^{(0,1)}_{-++}(\rho_1,\rho_j)$$

$$\mathcal{R}^{(2)}_{+-+\ldots} = \log\tau_1\, g^{(1,0)}_{+-+}(\rho_1,\rho_2) + \log\tau_2\, g^{(0,1)}_{+-+}(\rho_1,\rho_2) + \sum_{j=3}^{N-5} \log\tau_j\, g^{(0,0,1)}_{+-++}(\rho_1,\rho_2,\rho_j)$$

Helicity flips

- Consequence 2: Non-MHV amplitudes from MHV ones.



$$\mathcal{F}\left[\chi^{+}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

$$\mathcal{F}\left[\chi^{-}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

$$\sim \mathcal{F}\left[\chi^{-}/\chi^{+}\right] * \mathcal{F}\left[\chi^{+}\,\tau_1^{aE}\,C^{+}\,\tau_2^{aE}\,\chi^{-}\right]$$

- Helicity flip kernel: $\mathcal{F}\left[\chi^{-}/\chi^{+}\right] = -\frac{z}{(1-z)^2}$

- Helicity flips on central emission block are similar.

Transcendentality

- Consequence 3: Complete characterisation of the function space.

- Integrability: In perturbation theory, integrand is a polynomial in multiple zeta values and

$$E \qquad V \qquad N \qquad M \qquad D_\nu$$

- Example: NLO BFKL eigenvalue

$$E^{(1)} = -\frac{1}{4}\,D_\nu^2 E + \frac{1}{2}\,V\,D_\nu E - \zeta_2\,E - 3\,\zeta_3$$

- Theorem: If $\mathcal{A}(z)$ is a pure combination of SVMPLs of uniform weight n, then $\mathcal{A}(z) * \mathcal{F}[X]$, with $X \in \{E, V, N, M, D\}$, is a pure combination of SVMPLs of uniform weight $n+1$.

Transcendentality

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- Integrability: In perturbation theory, integrand is a polynomial **of uniform weight** in multiple zeta values and

$$E \qquad V \qquad N \qquad M \qquad D_\nu$$ weight 1

- Example: NLO BFKL eigenvalue

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Transcendentality

Theorem: All amplitudes in MRK in planar N=4 SYM are combinations of uniform weight of SVMPLs, (single-valued) multi zeta values and powers of $2\pi i$.

In addition:

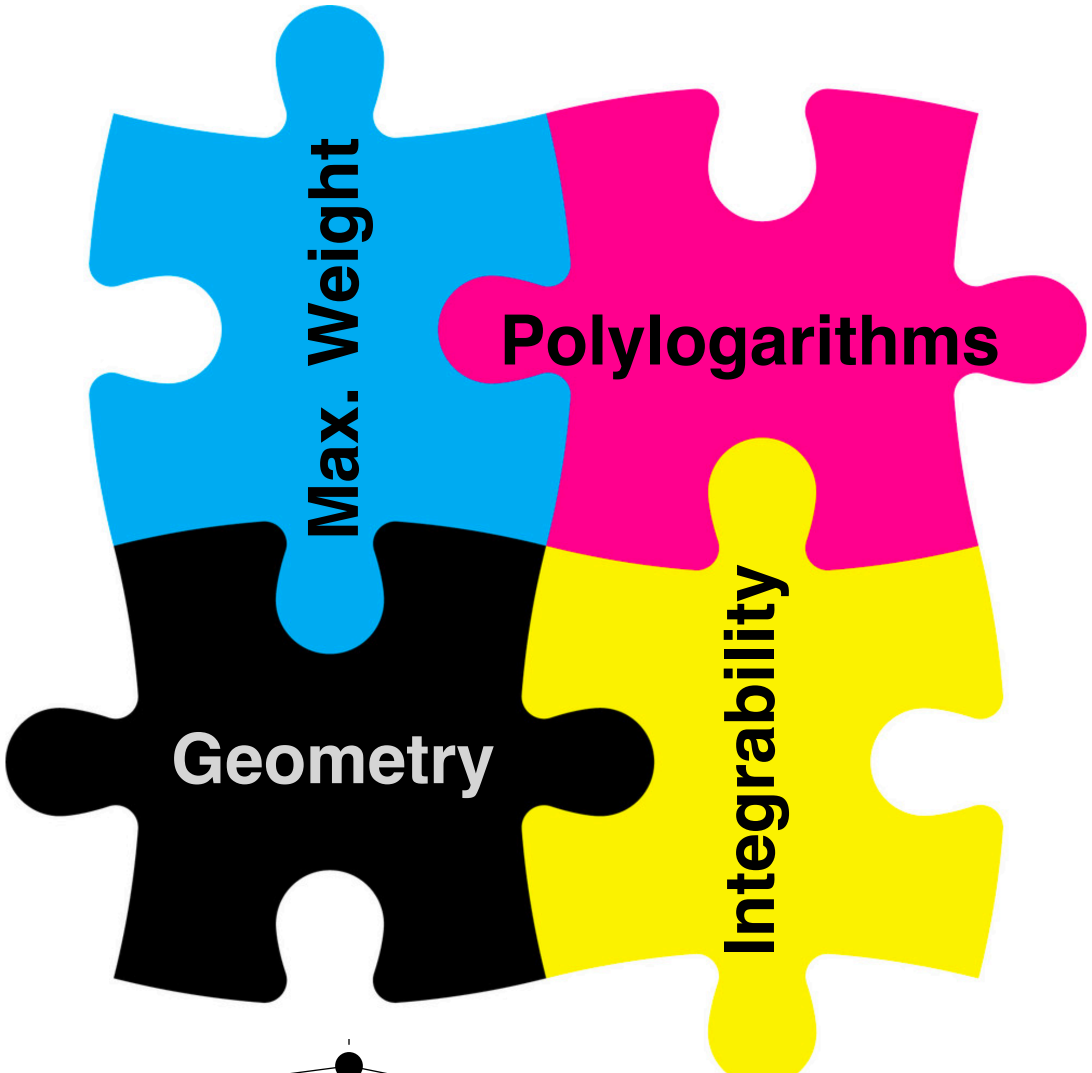
- MHV amplitudes are pure functions (no rational prefactors).
- Non-MHV amplitudes are not pure.

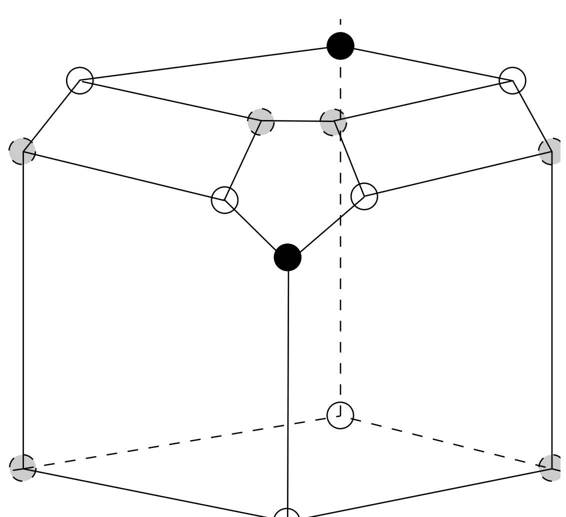
[Del Duca, Druc, Drummond, CD, Dulat, Marzucca, Papathanasiou, Verbeek]

➡ First proof that an infinite class of amplitudes can be expressed in terms of polylogarithms, for arbitrary number of legs, loops and helicity configurations.

Conclusion

High-energy limit





$$\sum_n \left(\frac{z}{\bar{z}}\right)^{n/2} \int \frac{d\nu}{2\pi} |z|^{2i\nu} \chi_1^{h_3} \tau_1^{\omega_1} \chi_1^{-h_4}$$

Full kinematics

$$\sum_\psi \int d\mu\, P(0|\psi)\, e^{-E\tau+ip+im\phi_1}\, P(\psi|0)$$