Tilings and statistical physics

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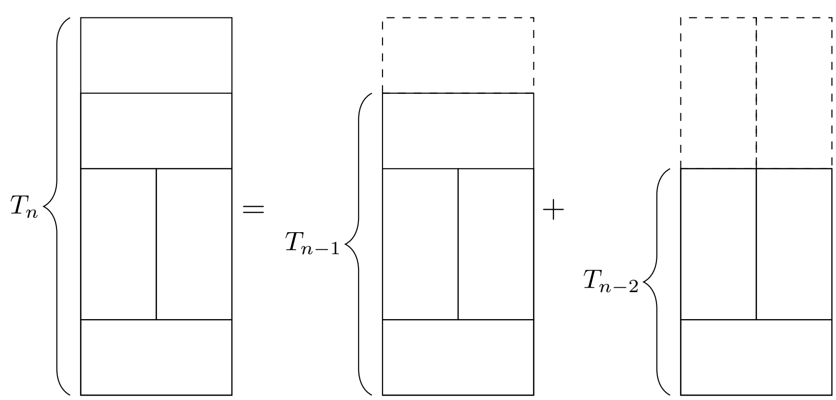
References

- P. W. Kasteleyn, Physica, Vol. 27, Issue 12 (1961), p. 1209-1225
- E. H. Lieb, Journal of Mathematical Physics, 8, 2339-2341 (1967) (transfer matrix solution)

1 Warm-up

We consider the case of an $n \times 2$ rectangle and ask how many different ways can we tile this with 2×1 dominoes? It is simple to identify a recursion relation for this case:



(1)

Which, with the boundary condition of $T_0 = 1$, $T_1 = 1$, trivially yields T_n as the nth Fibonacci number. We can also solve in a similar way for $2n \times 3$ rectangles, and we find the recurrence relation

$$T_{2n} = 4T_{2(n-1)} - T_{2(n-2)}, \tag{2}$$

and for $n \times 3$

$$T_n = T_{n-1} + 5T_{n-2} + T_{n-3} - T_{n-4} \tag{3}$$

the proof of which is left as an exercise. For larger widths this gets quite tricky. There are, however, other ways! Here we will cover the Pfaffian method, but it can also be attacked via the transfer matrix (see Lieb paper).

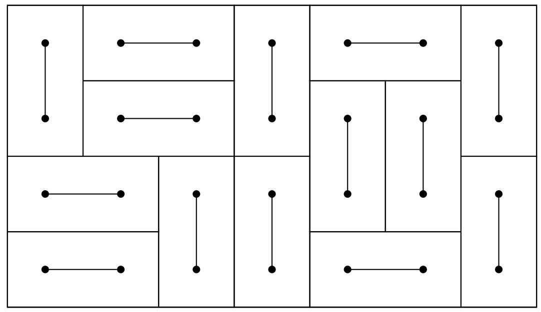


Figure 1: Domino tiling and lattice dimer coverings

2 Planar lattice dimer coverings

One can immediately see that tilings of dominos in an $n \times m$ rectangle a related to lattice dimer coverings Despite their playful appearance at first glance, problems such as domino tilings have a relevance to genuine physical scenarios. If one considers a system in which a regular lattice is covered with monomer, dimers etc., if the energy of mixing between the various bond lengths is zero, then one is interested in the entropic factor which is simply a game of counting. Covering only by dimers may be a drastically simplified version of this problem, but is at least a (tractable) step in the right direction.

The way we approach it is, as any good statistical physicist, too define a configuration generating function (partition function)

$$Z_{mn} = \sum_{N_2, N_2'}{}' g(N_2, N_2') z^{N_2} z'^{N_2'}, \tag{4}$$

here N_2 is the number of horizontal bonds and N_2' is the number of vertical bonds, we sum over configurations such that $2(N_2 + N_2') = mn$. At least one of m, n must be even: we take this to be m. It will be useful to have a way to denote a configuration. We choose

$$C = |p_1; p_2||p_3; p_4| \dots \tag{5}$$

To make this description unique, we "unfold" the lattice and take a point $(i, j) \to p = (j-1)m + i$ and can then use a canonical ordering

$$p_1 < p_2, p_3 < p_4, \dots \tag{6}$$

$$p_1 < p_3 < p_5 < \dots \tag{7}$$

2.1 What is a Pfaffian?

If we have a triangular array of coefficients $a(k, k')$, $k = 1, \dots, N$, $k' < k$, then the Pfaffian is defined as

$$\text{Pf}\{a(k, k')\} = \sum_{P \in S_N}{}' \delta_P a(P_1, P_2) \dots a(P_{N-1}, P_N) \tag{8}$$

Where the sum is restricted to configurations such that $P_1 < P_2$, $P_3 < P_4$, ... and $P_1 < P_3 < \dots$. We can now see why the lattice dimer covering problem may well be related to Pfaffians, due to the very similar nature of the sums.

2.1.1 Anti-symmetric matrices and the Pfaffian

We will need one intermediary theorem to get to the important result.

Theorem 2.1. If A is a complex, invertible, anti-symmetric $2n \times 2n$ matrix, then there exists an invertible $2n \times 2n$ matrix P such that

$$A = P^\top J P \tag{9}$$

where

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{10}$$

Proof. As A is anti-symmetric and invertible, it follows that it can be diagonalised. Applying elementary transformations to the diagonalised form will bring it to the form stated. □

We will also use an equivalent definition of the Pfaffian:

$$\mathrm{Pf}A = \frac{1}{2^n n!} \sum_{i \in S_{2n}} (-1)^P A_{i_1,i_2} \dots A_{i_{2n-1},i_{2n}} \tag{11}$$

We can now prove

Theorem 2.2.

$$\det A = (\mathrm{Pf}A)^2 \tag{12}$$

Proof. Firstly, it is evident that

$$\det A = (\det P)^2 . \tag{13}$$

Using the expression of the previous theorem, we have that

$$A_{ij} = \sum_{k,l} P_i^k J_{kl} P_j^l = P_i^1 P_j^2 - P_i^2 P_j^1 + P_i^3 P_j^4 - P_i^4 P_j^3 + \dots \tag{14}$$

We can now plug this into the expression for the Pfaffian and write

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{i \in S_{2n}} (-1)^i \left(P_{i_1}^1 P_{i_2}^2 - P_{i_1}^2 P_{i_2}^1 + \dots\right) \left(P_{i_3}^1 P_{i_4}^2 - P_{i_3}^2 P_{i_4}^1 + \dots\right) \dots \tag{15}$$

We can identify that a number of terms in this sum will vanish. Specifically, if we have something like

$$\sum_i (-1)^i P^1_{i_1} P^2_{i_2} P^1_{i_3} P^2_{i_4} (\dots) \tag{16}$$

Then we can see that by exchanging i_1 and i_3, we get the same sum but pick up a minus sign due to the permutation factor. Simply put, this means that the *upper indices must not be repeated*. It is simple enough to count how many of these terms we obtain: each bracket contributes a factor of 2 and we can rearrange each of the terms within them, giving a factor if $2^n n!$, i.e.

$$\text{Pf}(A) = \sum_{p \in S_{2n}} (-1)^P P^1_{i_1} P^2_{i_2} \dots P^{2n}_{i_{2n}} = \det P \tag{17}$$

That is

$$\det A = (\text{Pf} A)^2 \tag{18}$$

□

2.2 The punchline

Claim 2.1. It is possible to define a triangular array of elements $D(p; p')$ such that

$$Z_{mn}(z, z') = \text{Pf}\{D(p, p')\} \tag{19}$$

There are a few notes to make concerning this claim

Remark 2.1. Note that if $D(p, p') = 0$ for all pairs of sites *not* connected by a bond, then all terms in the Pfaffian *not* corresponding to a dimer configuration will vanish.

Remark 2.2. All coefficients corresponding to either a vertical or horizontal bond must, at least in magntiude, be equal to z', z respectively. This means that, if the claim is true, we will find a one-to-one correspondence of terms

Remark 2.3. We need to count all of these configurations *positively*. We must pick this such that the product $D(p_1; p_2) \dots$ has the same sign as the parity factor δ_P.

We can gain some insight into what Rk. 2.3 tells us by considering C_0, the standard configuration. The term corresponding to this have δ_P positive. We can now obtain any configuration from C_0 in the following way: we can draw the standard configuration thus If we now take a generic dimer packing, we can draw the bonds on top in dashed lines, viz.

We can consider what to do by looking at just a simple square. In the standard configuration, we have a term of the form

$$p_1 p_2 p_3 p_4 \tag{20}$$

And turning it into a square gives us

$$p_1 p_3 p_2 p_4 \tag{21}$$

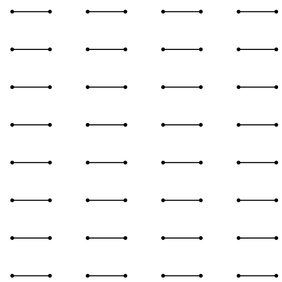


Figure 2: Standard configuration C_0

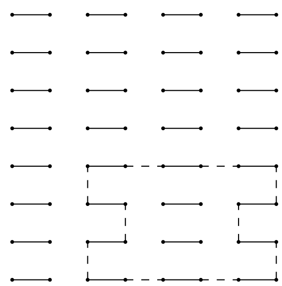


Figure 3: Generic configuration C

which is an *odd* permutation. Any configuration can be made up of performing these local transformations and therefore each polygon, comprising of an odd number of squares, will have a factor of -1 required in order to count positively in the sum. How can this be enacted in our expression for D? Upon reflection, as the vertical edges of the polygon are typically made of staggered lines, we can obtain the correct factor by including a (-1) for each vertical dashed bond on an odd coordinate i.e. we can account for all of the properties listed above uniquely by the choice

$$D(i,j;i+1,j) = z, \qquad D(i,j;i,j+1) = (-1)^i z', \qquad D(i,j;i'j') = 0 \,\text{otherwise}. \tag{22}$$

2.3 Solving the system

We can write this array in a particularly suggestive matrix form. If we define

$$Q_m = \underbrace{\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}}_{m} \tag{23}$$

$$F = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \tag{24}$$

Then

$$D = z\left(Q_m \otimes \mathbb{I}_n\right) + z'\left(F_m \otimes Q_n\right) \tag{25}$$

As Q appears in both of these expressions, diagonalising Q would be a big help. We can bring Q to diagonal form with

$$U_n(l, l') = \sqrt{\frac{2}{n+1}} i^l \sin\left(\frac{ll'\pi}{n+1}\right) \tag{26}$$

$$U_n^{-1}(l, l') = \sqrt{\frac{2}{n+1}} (-i)^l \sin\left(\frac{ll'\pi}{n+1}\right) \tag{27}$$

And this gives eigenvalues of the form $2i\cos\left(\frac{l\pi}{n+1}\right)$. This doesn't quite diagonalise D, but brings it into a simple enough form. That is, if we take

$$\widetilde{D} = (U_m^{-1} \otimes U_n^{-1}) D (U_n \otimes U_m) = 2iz\delta_{k,k'}\delta_{l,l'}\cos\left(\frac{k\pi}{m+1}\right) - 2iz'\delta_{k+k',m+1}\delta_{l,l'}\cos\left(\frac{l\pi}{n+1}\right) \tag{28}$$

i.e. we find 2×2 blocks along the diagonal. The determinant can be easily taken and we are left with the result that

$$Z_{mn}(z, z') = \prod_{k=1}^{m/2} \prod_{l=1}^{n} 2\left[z^2 \cos^2 \frac{k\pi}{m+1} + z'^2 \cos^2 \frac{l\pi}{n+1}\right]^{1/2} \tag{29}$$

And the total number of configurations is therefore simply $Z_{mn}(1, 1)$. It should also be noted that such a determinant form can be generated for many other lattices. Although these may be less readily amenable to direct analytic calculation, the calculation is still computationally far simpler: rather than having to count the exponentially many configurations, the computation can be done in polynomial time by calculating the appropriate determinant.