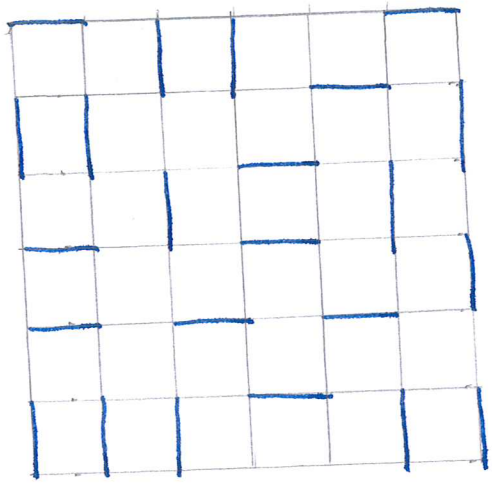
Quantum Dimer Models

A Dimer Configuration Looks Like The Following:



Each Dimer Connects Two Sites i.e. Lives on a Bond. No Site Can Be Part Of More Than One Dimer (Hard Constraint).

We Can Think Of The Dimer As An SU(2) Spin-½ Singlet Between Spins Living On The Two Sites.

The Hamiltonian Of Interest is

$$H = \sum -t\left[\left|\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right| + \left|\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right|\right]$$

$$+V\left[\left|\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right| + \left|\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}\right|\right].$$

t-Term: Describes A "Resonance" of Dimerisation On A Plaquette.

Where Does This Come From?

The (Hilbert) Space Is Defined By The Dimer Constraint. This Could Arise From, Say, Some Hubbard-U Term Defining A Large Energy Scale, Giving An Effective Model With Single Spins

$$H_{Hubb} \to -\frac{4t^2}{U}\sum_{\langle i,j\rangle} \underline{S}_i \cdot \underline{S}_j. \quad \text{At Half-Filling.}$$

: Non-Magnetic Parent State For Superconductor.

Minising A.F.M. Interactions → Singlet Pairs.

THE KEY FEATURES OF THE MODEL ARE:

(i) NON-TRIVIAL LOCAL CONSTRAINTS

(ii) LOCAL QUANTUM DYNAMICS.

ADDING QUANTUM DYNAMICS

WE NORMALLY ELEVATE THE CLASSICAL, CONSTRAINED CONFIGURATIONS TO A DYNAMICAL QUANTUM SYSTEM BY SIMPLY WRITING DOWN A MODEL. THIS IS DONE IN THREE EASY STEPS:

1. CLASSICAL CONFIGURATIONS → BASIS VECTORS OF A HILBERT SPACE.
2. SIMPLEST LOCAL REARRANGEMENTS → PLAQUETTE FLIPS.
3. ENDOW SUCH A LOCAL REARRANGEMENT W/ COHERENT QUANTUM DYNAMICS.

HILBERT SPACE

FROM A PREVIOUS JOURNAL CLUB, RECALL THAT FOR THE SQUARE LATTICE, THE NUMBER OF VALID DIMER CONFIGURATIONS IS MUCH SMALLER THAN 2^N. IT IN FACT GROWS AS $(1.3385)^N$.

THIS IS A REFLECTION OF THE FACT THAT THE D.O.F. ARE NOT INDEPENDENT.

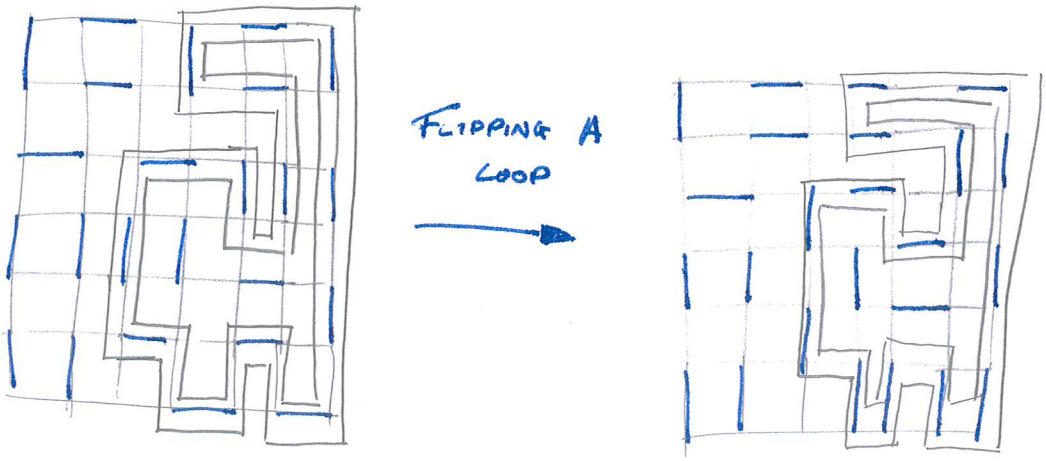
THIS, AND THE LOCAL REARRANGEMENTS, HAS TOPOLOGICAL IMPLICATIONS:

IF WE WERE JUST WORKING WITH A SPIN SYSTEM, BY A SERIES OF LOCAL MANIPULATIONS, WE COULD GET FROM ANY ONE STATE TO ANY OTHER. HOWEVER, BECAUSE OF THE DIMER CONSTRAINT, THIS IS NOT THE CASE.

THE SIMPLEST MOVE WE HAVE IS THE PLAQUETTE FLIP.

MORE GENERALLY, WE CAN IDENTIFY A LOOP WITH DIMERS ON EVERY OTHER LINK. WE CALL THIS A "FLIPPABLE LOOP".

THIS CAN BE FLIPPED BY EXCHANGING OCCUPIED AND VACANT LINKS ON THIS LOOP.



SOME QUANTITIES ARE LEFT UNCHANGED UNDER SUCH MOVES.

NON-BIPARTITE LATTICE: IF WE HAVE A NON-CONTRACTIBLE LINE SPANNING THE LATTICE, THEN THE PARITY OF # DIMERS CROSSING THIS IS CONSERVED.

BIPARTITE: ALTERNATELY CHOOSE VERTICAL LINES AS A, B.

FOR A LINE AS BEFORE, N_A IS # CROSSINGS ON A
N_B IS " " B.

$N_A - N_B$ IS CONSERVED.

THE ARGUMENT COMES FROM CROSSING THE LINE AN EVEN # OF TIMES WITH A LOOP.

If, However, We Are On A Torus/Higher Genus Surface, By Having A loop Around An Entire Cycle, We Can Intersect The Line Only Once.

This Non-Local Transformation Changes The "Winding #".

These Winding #s Are A Useful Way To Carve Up The Hilbert Space.

We Pick Two Loops And Get Winding #s (W_x, W_y).

The Sector Defined By These Contains States Which Can Be Connected Via Local Moves.

For More General Surfaces, We Need 2g Winding #s.

↳ The Labels $(W_x W_y)$ Depend On The Choice Of Coords.
The Sectors Themselves Are Only About The Topology.

⇒ The Name Topological Sectors.

To Define Our Hilbert Space, We Still Need To Define An Inner Product.

↳ We Choose The States For Distinct Dimer Coverings To Be Orthogonal.

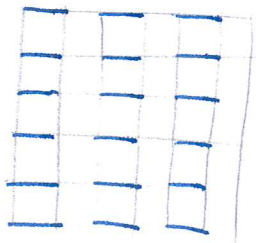
Because We Defined The QDM In Terms Of ~~The~~ Local Transformations, The Winding #s Are Good Quantum #s.

We Therefore Choose To Divide The Hilbert Space Into Dynamically Independent Topological Sectors

(4) It may still be possible to further subdivide into finer dynamical sectors. This is indeed the case for the 2D triangular lattice.

Special Sectors

Columnar Phase:



For $\frac{V}{t} \to -\infty$, we want to maximise the number of flippable plaquettes.

We can identify 4 "columnar states".

N.B. These break rotational and single-site translational invariance.

These are, a priori, only eigenstates strictly in the limit. However, columnar correlations persist to some

$$\frac{V}{t} = \left(\frac{V}{t}\right)_c < 1.$$

i.e. $-\infty < \frac{V}{t} < \frac{V}{t}_c$ is in the columnar phase.

As is a generic feature of such crystalline phases, if we allow monomer excitations, they are linearly confined.

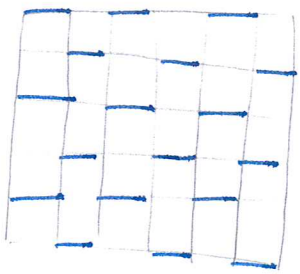
Staggered Phase

$\frac{V}{t} \to \infty \Rightarrow$ Minimise The Number Of Flippable Plaquettes.

It May Be Possible That The Lattice Permits No Flippable Plaquettes. In Such A Case, These States Are The Sole Inhabitants Of Their Topological Sectors.

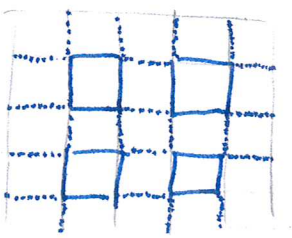
A non-trivial Fact is That These Are Still Ground States For $\left(\frac{V}{t}\right) > 1$. (This Follows From The Hamiltonian Being +ve Semi-Def For $\frac{V}{t} > 1$).

On The Square Lattice, There Are Exponentially Many States In The System Size With No Flippable Plaquettes. E.G.



(For The Triangular Lattice, It Remains Finite i.e. Does not Grow).

Plaquette Phase



This Picture Should Be Understood In A Mean Field Sense i.e. Weight Is On Plaquettes.

Four-Fold Degenerate.

This Breaks Different Symmetries To The Other Phases.

RK Point

$v = t$.

We can write the Hamiltonian as

$$H = \sum_{\square} t\left[|\square\rangle - |\,|\,|\rangle\right]\left[\langle\square| - \langle|\,|\,|\right] + (v-t)\left[|\square\times\square| + |\,|\,|\times|\,|\,|\right].$$

$v = t \Rightarrow$ $H = \sum_{\square} t\left[|\square\rangle - |\,|\,|\rangle\right]\left[\langle\square| - \langle|\,|\,|\right]$.

Each term, appropriately normalised, has either eigenvalue 0 or 1.

We write

$$|\Phi\rangle = \sum_c A_c |c\rangle.$$

We want this to have E.V. 0 for all plaquettes simultaneously.

$H|\Phi\rangle = 0$ iff the amplitude A_c is the <u>same</u> as all $\{A_{c'}\}$ differing by only one plaquette flip.

$\Rightarrow$ Within one topological sector

$$|\Phi\rangle = \sum_c |c\rangle.$$

Phase Diagram For The 2D Bipartite Model

??? : Honeycomb: 1st order transition between Col. + Plaq.

This is mainly based on R.G. studies of a continuum field theory.

RK Point, Again

For correlation functions diagonal in the dimer basis, ground state correlation functions are special:

$\eta_\mu(x)$:

$$G(x) := \frac{\langle GS|\, \eta_\mu(x)\, \eta_\nu(0)\, |GS\rangle}{\langle GS|GS\rangle} = \frac{\sum_{\{n\},\{n'\}} \langle n|\, \eta_\mu(x)\, \eta_\nu(0)\, |n'\rangle}{\sum_{\{n\}} \langle n|n\rangle}$$

$$= \frac{\sum_{\{n\}} \eta_\mu(x)\, \eta_\nu(0)}{N_d}.$$

This looks like a classical stat. mech. problem and can be evaluated via Pfaffian methods

$\sim \frac{1}{x^2}$.

THE RK POINT IS A CRITICAL PT. BETWEEN THE PLAQUETTE PHASE AND THE STAGGERED PHASE.

$\frac{V}{t} \to 1^-$: PLAQUETTE ORDER PARAMETER VANISHES CONTINUOUSLY.

$\frac{V}{t} = 1$: ALGEBRAICALLY DECAYING DIMER CORRELATIONS.

$\frac{V}{t} \to 1^+$: STAGGERED PHASE IS AT FULL STRENGTH.

$\Rightarrow$ G.S. E HAS A DISCONTINUITY.

INDEED, LANDAU SAYS THAT DIFFERENT SYMMETRIES $\Rightarrow$ 1ST ORDER TRANSITION (GENERICALLY).

BUT

THIS TRANSITION CAN BE MADE CONTINUOUS BY WEAKLY PERTURBING THE MODEL.

THIS CRITICAL POINT ALSO SUPPORTS DECONFINED MONOMER EXCITATIONS.

IF WE CONSIDER THE RK HAMILTONIAN WITH TWO MONOMERS HELD FIXED, THE GROUND STATE WILL STILL HAVE ZERO ENERGY i.e. THEY ARE DECONFINED.

~~WE~~ THIS IS KNOWN AS DECONFINED CRITICALITY, AS THE SCALE INV. IS RELATED TO THE EXISTENCE OF AN EMERGENT GAUGE FIELD SUPPORTING DECONFINED EXCITATIONS.

THESE HAVE BEEN PROPOSED AS A GENERIC MECHANISM FOR SUCH NON-LANDAU TRANSITIONS IN A QUANTUM SYSTEM.

WE CAN USE THE SINGLE-MODE APPROXIMATION TO PROVE GAPLESSNESS AT THE RK POINT.

$\sigma^z_\tau(x) = \pm 1$ FOR DIMER/NO-DIMER.

$$\tilde{\sigma}^x_\tau(q) = \sum_r e^{iq\cdot r}\,\sigma^z_\tau(r).$$

$$|q;\tau\rangle := \tilde{\sigma}^x_\tau(q)|0\rangle.$$

WE NEED

$$\langle q,\tau|0\rangle = 0 \qquad (q\neq 0).$$

$$\langle q,\tau|q,\tau\rangle \neq 0.$$

$$\Delta E(q,\tau) \leq \frac{\langle q|H|q\rangle}{\langle q|q\rangle} - \frac{\langle 0|H|0\rangle}{\langle 0|0\rangle}.$$

$$\leq \frac{\langle 0|\left[\tilde{\sigma}^z_\tau(-q),\left[H,\tilde{\sigma}^x_\tau(q)\right]\right]|0\rangle}{\langle 0|\,\tilde{\sigma}^z_\tau(q)\,\tilde{\sigma}^z_\tau(q)|0\rangle} := \frac{f(q)}{s(q)}.$$

FURTHER CALCULATION YIELDS $\sim q^2$, DEPENDING ON VARIOUS MORE COMPLICATED FEATURES.

Overlap Expansion

This motivates R.K. a bit more.

We can treat the overlap between states as a "small parameter".

We can pick a sign convention s.t. if states ~~overlap~~ differ by only one min. loop: $-2x^4$.

The overlap matrix is then

$$S_{ab} = \delta_{ab} - 2x^4 \square_{ab} + \mathcal{O}(x^6)$$

$$\delta H = J \sum_{\langle ij \rangle} \underline{S}_i \cdot \underline{S}_j + V \sum_{\square} \left((\underline{S}_1 \cdot \underline{S}_2)(\underline{S}_3 \cdot \underline{S}_4) + (\underline{S}_1 \cdot \underline{S}_3)(\underline{S}_2 \cdot \underline{S}_4) \right).$$

We would like to write this as an op acting on N-N VB manifold.

We first need $\perp$ basis:

$$|\alpha\rangle = \sum_i S^{-1/2}_{\alpha i} |i\rangle.$$

$$(S^{-1/2})_{ab} = \delta_{ab} + x^4 \square_{ab} + \mathcal{O}(x^6).$$

$\Rightarrow |\alpha\rangle$ can be labelled in terms of $\mathcal{O}(1)$ component.

$$H_{\alpha\beta} = (S^{-1/2} \delta H S^{-1/2})_{\alpha\beta} = \sum_{ij} S^{-1/2}_{\alpha i} \langle i | \delta H | j \rangle S^{-1/2}_{j\beta}.$$

$$= -t \square_{\alpha\beta} + V \eta_{\text{plaq}} \delta_{\alpha\beta} + \mathcal{O}(Vx^4, tx^4), \qquad t = Jx^4$$

The leading term is just RK-QDM!