Notes on gauge kinematics

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These notes are mainly based on the following two papers:

- Alfred Shapere and Frank Wilczek, *Gauge Kinematics of Deformable Bodies*, in *Geometric Phases in Physics*, World Scientific (1989);
- Richard Montgomery, *Gauge Theory of the Falling Cat*, Fields Institute Communications **1**, 193 (1993),

This talk is on the kinematics of deformable bodies. It illustrates nicely the importance of finding the right language. In this case, once realizing the gauge theory is the appropriate language, everything follows naturally from it.

I won't discuss dynamics, i.e., how much work is required for deformation or what the optimal deformation sequence is. Instead, I will take the sequence of deformation as given.

On a side note, there are a few problems that share similar mathematical structures. This includes swimming in a highly viscous environment and parallel parking.

1 Rigid body kinematics

We review rigid body kinematics. Consider a mass element rotates along with the body:

$$\frac{d\mathbf{r}}{dt} = \Omega\mathbf{r} = (\sum_{i=1,2,3}\omega_i L_i)\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}, \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity in the *lab* frame. The matrix Ω is an element of the *real* $so(3)$ Lie algebra. L_i are the canonical generators of the $so(3)$ algebra:

$$(L_i)_{jk} = -\epsilon_{ijk}; \quad [L_i, L_j] = \sum_k \epsilon_{ijk}L_k. \tag{2}$$

To specify the orientation of a rigid body, we attach a right-handed frame $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$ to the body, which we call the *body* frame. The body frame may be expressed as a 3×3 orthogonal matrix $R = (\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$. We may relate the angular velocity to R:

$$\frac{dR}{dt} = \Omega R \Rightarrow \Omega = \frac{dR}{dt}R^T. \tag{3}$$

We will use the angular velocities in the *body* frame later (exercise):

$$\widetilde{\boldsymbol{\omega}} = R^T\boldsymbol{\omega}, \quad \widetilde{\Omega} = R^T\Omega R = R^T\frac{dR}{dt}. \tag{4}$$

Finally, given $\widetilde{\Omega}(t)$, we may relate the frame before and after rotation as:

$$R(t_f) = R(t_i)\mathcal{P}\exp(\int\widetilde{\Omega}(t)dt). \tag{5}$$

$\mathcal{P}$ stands for path (time) ordering.

2 Deformable body kinematics

We move on to deformable body. We parametrize the shape of a deformable body by a set of numbers $\sigma \equiv \{\sigma_1, \sigma_2, \cdots\}$. It is essential to require that σ is independent of the choice of the lab frame. Abstractly, we may think of σ as the coordinates of the *shape space*. A sequence of deformation is thus a path in the shape space.

We orient the shapes by attaching to a shape σ a right-handed frame $R(\sigma)$. Note $R(\sigma)$ depends on σ in general. With the attached frames, we may compute the angular velocity in the body frame:

$$\widetilde{\Omega} = R^T \frac{dR}{dt} = \sum_\alpha R^T \frac{\partial R}{\partial \sigma_\alpha} \frac{d\sigma_\alpha}{dt} = \sum_\alpha A(\sigma)_\alpha \frac{d\sigma_\alpha}{dt}, \quad A(\sigma)_\alpha \equiv R^T \frac{\partial R}{\partial \sigma_\alpha}. \tag{6}$$

We have defined a vector field $A(\sigma)_\alpha$ that relates the angular velocity in physical space to thee shape space velocity. The number of its components is the same as the dimension of the shape space. $A(\sigma)_\alpha$ takes value in the $so(3)$ algebra. If we write $A(\sigma)_\alpha = \sum_i a_\alpha^{(i)}(\sigma) L_i$, then (exercise),

$$\widetilde{\boldsymbol{\omega}} = \sum_\alpha \mathbf{a}(\sigma)_\alpha \frac{d\sigma_k}{dt}. \tag{7}$$

Crucially, the frames attached to two different shapes are completely unrelated: it doesn't make a lot of sense to compare the orientation of an apple and an orange. Therefore, the angular velocity of a deformable body is inherently ambiguous. This ambiguity is reflected in the freedom of choosing $R(\sigma)$ for a given shape σ. Suppose we chose another frame $R'(\sigma) = R(\sigma)G(\sigma)$, where $G(\sigma) = I + \sum_i g_i(\sigma) L_i$ is an infinitesimal $SO(3)$ rotation. We may evaluate $A(\sigma)_k$ for this new choice:

$$A'_\alpha = G^T A_\alpha G + G^T \frac{\partial G}{\partial \sigma_\alpha} = A_\alpha + \sum_i \frac{\partial g_i}{\partial \sigma_\alpha} L_i + \sum_i g_i [A_\alpha, L_i]. \tag{8}$$

$$a'^{(i)}_\alpha = a_\alpha^{(i)} + \frac{\partial g_i}{\partial \sigma_\alpha} + \sum_{jk} \epsilon_{ijk} a_\alpha^{(j)} g_k. \tag{9}$$

We recognize the above as the gauge transformation of a Yang-Mills theory if we interpret A_k as the gauge potential. Its "color" (Latin) index labels the physical directions, whereas its "spatial" (Greek) index labels the tangential directions in the shape space. Choosing a specific frame $R(\sigma)$ is amount to fixing a gauge.

We now compute the total rotation due to a cycle of deformation. Since the shape at the initial and final stage of deformation is the same, the frames are identical. It is therefore meaningful to compute the total rotation as:

$$R^T(t_i) R(t_f) = \mathcal{P} \exp\left(\int \widetilde{\Omega} dt\right) = \mathcal{P} \exp\left(\int_{\mathcal{C}} A d\sigma\right). \tag{10}$$

The right hand side is the gauge flux picked up by the closed path $\mathcal{C}$, which is a gauge invariant quantity. This agrees with the expectation that self rotation is independent of frame choice.

3 Calculating the gauge potential

The last piece of the theory is an explicit expression for $A(\sigma)_\alpha$. We use the conservation of angular momentum (we omit gravity all together). The angular momentum is 0 in the lab frame. It follows that the angular momentum in the body frame is 0. We write:

$$0 = \mathbf{J}_{\text{app}} dt + \widetilde{I}(\sigma) \widetilde{\boldsymbol{\omega}} dt = \mathbf{J}_{\text{app}} dt + \sum_\alpha \widetilde{I}(\sigma) \mathbf{a}_\alpha d\sigma_\alpha, \tag{11}$$

Here, $\mathbf{J}_{\text{app}}$ is the apparent angular momentum due to an infinitesimal change of shape. It is defined as the angular momentum the body would have had the frame been frozen. $\widetilde{I}$ is the moment of inertia matrix in the body frame. The above equation says the apparent angular momentum must be balanced by an overall rotation of the body. Solving it for $\mathbf{a}_\alpha$, we obtain:

$$\mathbf{a}_\alpha = -\widetilde{I}^{-1}(\sigma) \frac{\partial (\mathbf{J}_{\text{app}} dt)}{\partial \sigma_\alpha}. \tag{12}$$

This concludes our discussion on the formal aspects of the theory.

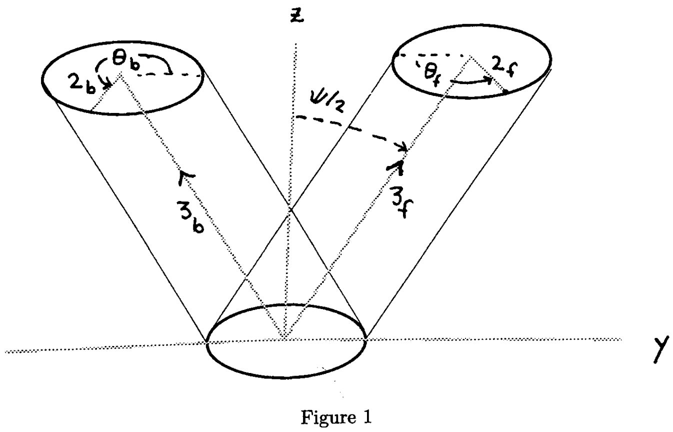

Figure 1: Kane-Scher cat. Taken from Montgomery's paper.

4 Example: turning table problem

Consider a cat walking on the edge of a free-rotating turning table. Let R be the radius of the table, I the moment of inertia of the table, and m the mass of the cat. We want to compute the total angle of rotation Φ if the cat walks around the table once.

We solve this problem following Shapere and Wilczek's prescription. We attach to the turning table an xOy frame. With this frame, we use the azimuthal angle θ of the cat as the shape coordinate. Clearly, the shape space is a circle. The cat walking around the table once maps to a loop of winding number 1 in the shape space.

As the shape space is one-dimensional, the gauge potential has only one "spatial" component. Furthermore, since the rotation is around a fixed axis, only one colour component is non-zero. We thus write: $a_\theta^{(3)} = a(\theta)$ and $a_\theta^{(1)} = a_\theta^{(2)} = 0$.

The moment of inertia of the whole system is $I + mR^2$. The apparent angular momentum is $mR^2\dot{\theta}$. We thus find,

$$a(\theta) = -\frac{mR^2}{I + mR^2}. \tag{13}$$

This is the gauge potential due to an A-B flux, whose value is precisely the total angle of rotation Φ:

$$\Phi = \int_0^{2\pi} a(\phi)d\phi = -\frac{mR^2}{I + mR^2}2\pi. \tag{14}$$

Shapere and Wilczek worked out an analogous 3D problem. There, the shape space is a 2-sphere and the gauge potential is generated by a Yang-Mills monopole.

5 Example: Falling cat problem

We consider the Kane-Scher cat ("cylindrical cat") shown in Figure 1. The two identical cylinders represent the front and back of the cat's body. They are connected by a ball-and-socket joint. The triplet $(\psi, \theta_f, \theta_b)$ are the shape coordinates of the Kane-Scher cat.

We restrict ψ to be in the range of $[0, \pi]$. Any shape with $\psi > \pi$ is equivalent to another shape with $\psi < \pi$ by a global rotation. We also impose the no-twist condition: $\theta_f = -\theta_b = \theta$. It means the spine can't be twisted. The coordinate is thus (ψ, θ).

There are coordinate singularity at $\psi = 0$ or π. At $\psi = \pi$, (π, θ) describes the same shape for any θ. At $\psi = 0$, $(0, \theta)$ and $(0, \pi - \theta)$ describe the same shape.

We visualize the shape space as a unit disk with $r = \cos(\psi/2)$ being the radius and θ being the azimuthal angle. The coordinates singularity at $\psi = \pi$ is the usual ambiguity of the polar coordinates at $r = 0$. On the other hand, the ambiguity at $\psi = 0$ means a point on the edge of the disk ($r = 1$) and its antipodal are the same point. Thus, the shape space is homeomorphic to RP^2.

Since the shape space is two dimensional, the gauge potential carries two spatial components, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$.

The moment of inertia with respect to the centre of mass is given by:

$$\widetilde{I} = 2\begin{bmatrix} I_1 + ml^2\sin^2(\psi/2) & 0 & 0 \\ 0 & I_1\cos^2(\psi/2) + I_3\sin^2(\psi/2) & 0 \\ 0 & 0 & (I_1 + ml^2)\sin^2(\psi/2) + I_3\cos^2(\psi/2)\end{bmatrix}. \tag{15}$$

Here, I_3 and I_1 are respectively the moment of inertia of the cylinder about and transverse to the symmetric axis. l is the distance between the cylinder centre to the joint. Note the joint is *not* the centre of the cat's centre of mass. One can quick check its correctness by considering $\psi = 0$ and π. This also shows the result given in Montgomery's paper contains typos.

We compute the apparent angular momentum due to deformation. Changing ψ doesn't produce any apparent angular momentum as the angular momenta due to the front and back cylinders cancel. Changing θ, on the other hand, yields an apparent angular momentum in the $\hat{\mathbf{y}}$ direction:

$$J_1 dt = 0; \quad J_2 dt = 2I_3 \sin\frac{\psi}{2} d\theta; \quad J_3 dt = 0. \tag{16}$$

We thus find:

$$a_\psi^{(1,2,3)} = 0; \quad a_\theta^{(1)} = 0; \quad a_\theta^{(2)} = -\frac{I_3\sin(\psi/2)}{I_1\cos^2(\psi/2) + I_3\sin^2(\psi/2)}; \quad a_\theta^{(3)} = 0. \tag{17}$$

The above result implies the cat rotation is always along the y axis. This is just what we need: for the cat to flip, it has to rotate with respect to the y axis.

For the sake of simplicity, let's assume the cat is Garfield. We set $I_1 = I_3$, and yield:

$$a_\theta^{(2)} = -\sin\frac{\psi}{2}. \tag{18}$$

In the disk representation of the shape space, the gauge potential is 0 at the edge (hence no total flux over the entire shape space) but singular at the centre ($\psi = \pi$). An infinitesimal contour around the disk centre picks up a flux of -2π.

To flip the cat, we need to rotate by π. This is done by considering the following non-contractible loop in the shape space: it starts from somewhere near the centre, goes around the centre by half a circle, moves along the radial direction, hits the edge, jumps to the antipodal point, and returns. It is obvious that the contour integral is π. Alternatively, we may consider a circle with constant ψ_0. The flux is $-2\pi\sin(\psi_0/2) = -\pi$. Solving for ψ_0, we find $\psi_0 = 2\pi/3$.

6 Final remarks

I have avoided some subtle points related to the global properties of the gauge field. For a full three-dimensional rotation problem, the gauge group is $SO(3)$. $SO(3)$ is double connected, which would produce some interesting topological effects. In the falling cat problem, the rotation is about $\hat{\mathbf{y}}$ axis, which reduced the gauge group to $O(2)$ (*not* $SO(2)$!). In addition, the shape space of the Kane-Scher cat is homeomorphic to RP^2, a non-orientable manifold. One therefore naturally worries to what extent a gauge field is well defined. These subtle issues are discussed by Montgomery in details.