Bosonisation

CMT Journal Club, given by Stefan Groha

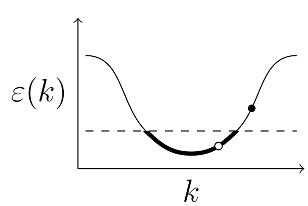
June 19th, 2015 Thomas Veness

References

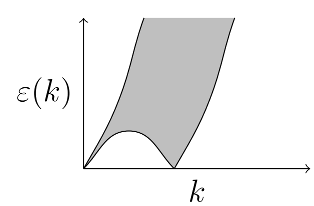
- "Bosonization for Beginners — ReFermionization for Experts" by J. von Delft, H. Schoeller, arXiv:cond-mat/9805275
- "An introduction to bosonization" by D. Sénéchal: arXiv:cond-mat/9908262

1 Motivation

Bosonisation is typically used to describe he low-energy properties of Fermionic systems in one dimension. We first want to consider *free* electrons in one dimension. One of the distinguishing properties of these systems is that we have a filled Fermi sea with two Fermi points:



(a) Free Fermion dispersion relation and a particle-hole excitation on top of the filled ground state



(b) The continuum for particle-hole excitations in a one-dimensional free Fermion model

Figure 1: Free Fermions in one dimension

The essence of bosonisation is to recognise that, in one dimension, at low energies, the particle-hole dispersion relation is essentially linear.

For a free electronic system with linear dispersion, we can easily calculate the dispersion relation to be

$$\langle \psi(x)\psi(0)^\dagger \rangle \sim \frac{1}{x} \tag{1}$$

We also know that for a free, massless bosonic system, the correlation function behaves as

$$\langle\phi(x)\phi(0)\rangle \sim -\ln x. \tag{2}$$

It is therefore plausible, but by no means obvious that an identification of the form $\psi(x) \sim e^{i\phi(x)}$ could give a sensible representation of a Fermionic operator in terms of bosonic operators.

2 Constructive approach

Prerequisites

In order to establish the discussion, we have Fermionic creation and annihilation operators of various species $\{\eta\}$, obeying the anti-commutation relations

$$\{c^\dagger_{k\eta}, c_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'} \tag{3}$$

We must also have that $k \in (-\infty, \infty)$, where $k = \frac{2\pi}{L}(n_k - \frac{1}{2}\delta_\eta)$, where $\delta_\eta = 0, 1$ depending on the choice of boundary conditions and $n_k \in \mathbb{Z}$.

$$\psi_\eta(x) = \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta} \tag{4}$$

We can see how δ_η effects the boundary conditions, as

$$\psi_\eta(x + L/2) = e^{i\pi\delta_\eta}\psi_\eta(x - L/2) \tag{5}$$

We will use the fact that the Fock space can be decomposed as the direct sum of N-particle Hilbert spaces i.e.

$$\mathcal{F} = \bigoplus_N \mathcal{H}_N. \tag{6}$$

We must also define a vacuum state for the theory $|0\rangle_0$ s.t.

$$c_{k\eta}|0\rangle_0 = 0, \qquad k > 0 \tag{7}$$

$$c^\dagger_{k\eta}|0\rangle_0 = 0, \qquad k \le 0 \tag{8}$$

i.e. a filled Fermi sea up to $k = 0$. We can also define a number operator which means the number of Fermions *relative* to the filled Fermi sea, i.e.

$$N_\eta := \sum_{k=-\infty}^{\infty} :c^\dagger_{k\eta}c_{k\eta}: = \sum_{k=-\infty}^{\infty} \left\{c^\dagger_{k\eta}c_{k\eta} - {}_0\langle 0|c^\dagger_{k\eta}c_{k\eta}|0\rangle_0\right\} \tag{9}$$

The N-particle ground state can therefore be defined as

$$|\mathbf{N}\rangle_0 = (C_1)^{N_1} \dots (C_M)^{N_M}|0\rangle_0 \tag{10}$$

where

$$(C_\eta)^{N_\eta} := \begin{cases} c^\dagger_{N_\eta \eta} \dots c^\dagger_{1\eta} & N_\eta > 0 \\ 1 & N_\eta = 0 \\ c_{N_\eta+1\eta} \dots c_{0\eta} & N_\eta < 0 \end{cases} \tag{11}$$

Bosonic operators

With this set up, we are now in a position to define a bosonic operator. This can be understood as being related to the Fermionic density operator, creating a superposition of particle-hole excitations:

$$b^\dagger_{q\eta} = \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k+q,\eta} c_{k,\eta} \tag{12}$$

This operator has a well-defined momentum and is normalised such that the operators obey

$$[b_{q,\eta}, b^\dagger_{q',\eta'}] = \delta_{\eta\eta'}\delta_{qq'} \tag{13}$$

And we also have, by construction,

$$b_{q,\eta}|\mathbf{N}\rangle_0 = 0 \tag{14}$$

Theorem 2.1. Completeness of states in $\mathcal{H}_N$ For every $|N\rangle_0$, there exists a function $f(\{b^\dagger, b\})$ such that a general state $|N\rangle$ can be reached i.e. there is an f satisfying

$$|\mathbf{N}\rangle = f(\{b, b^\dagger\})|\mathbf{N}\rangle_0 \tag{15}$$

The proof can be found in Schöller and von Delft.

The upshot of this is that the individual N-particle Hilbert spaces are now equivalent. The only part remaining is that the electronic operators connect the separate N-particle Hilbert spaces and by understanding these we will be able to construct a full correspondence of the Fock spaces.

The operators that do this are the so-called *Klein factors*: $F^\dagger_\eta$, F_η. These satisfy

$$[b_{q\eta}, F^\dagger_{\eta'}] = [b^\dagger_{q\eta}, F^\dagger_{\eta'}] = 0 \tag{16}$$

and behave as

$$F^\dagger_\eta|\mathbf{N}\rangle = f(b^\dagger)c^\dagger_{N_\eta+1,\eta}|N_1, \dots, N_\eta, \dots, N_M\rangle \tag{17}$$

and

$$F^\dagger_\eta|\mathbf{N}\rangle = f(b^\dagger)c_{N_\eta,\eta}|N_1, \dots, N_\eta, \dots, N_M\rangle \tag{18}$$

as well as obeying anti-commutation relations

$$\{F_\eta, F^\dagger_{\eta'}\} = 2\delta_{\eta,\eta'} \tag{19}$$

$$[N_\eta, F^\dagger_{\eta'}] = \delta_{\eta,\eta'}F^\dagger_\eta \tag{20}$$

The Fock spaces are therefore in one-to-one correspondence and thus equivalent. Before exhibiting the operator equivalence, we construct the bosonic fields

$$\varphi_\eta(x) := -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} b_{q,\eta} e^{-aq/2} \tag{21}$$

$$\phi_\eta(x) = \varphi_\eta(x) + \varphi_\eta^\dagger(x) = -\sum_{q>0} \frac{1}{\sqrt{n_q}} \left[e^{-iqx} b_{q,\eta} + e^{iqx} b6\dagger_{q,\eta}\right] e^{-aq/2} \tag{22}$$

with a a regulator, which can be removed at the end of any calculation. Using the mode expansion, it is clear that

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(x')] = 0 \tag{23}$$

and

$$[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] = -\delta_{\eta,\eta'} \ln\left[1 - e^{-\frac{2\pi i}{L}(x-x'-ia)}\right] \xrightarrow{L\to\infty} -\delta_{\eta,\eta'} \ln\left(\frac{2\pi i}{L}(x-x'-ia)\right) \tag{24}$$

And utilising the Baker-Campbell-Hausdorff formula, we can see that

$$e^{i\varphi_\eta^\dagger(x)} e^{i\varphi_\eta(x)} = e^{i(\varphi_\eta^\dagger(x)+\varphi_\eta(x))} e^{-[\varphi_\eta^\dagger(x),\varphi_\eta(x)]/2} = \sqrt{\frac{L}{2\pi a}} e^{i\phi_\eta(x)} \tag{25}$$

and

$$[\phi_\eta(x), \partial\varphi_{\eta'}(x')] \xrightarrow{L\to\infty, a\to 0} 2\pi i \left[\delta(x-x') - \frac{1}{L}\right] \tag{26}$$

$$[\phi_\eta(x), \varphi_{\eta'}(x')] \xrightarrow{L\to\infty, a\to 0} -\delta_{\eta,\eta'} i\pi\varepsilon(x-x') \tag{27}$$

If we look at the commutation relations of the bosonic operators with the Fermionic operators, we can see that

$$[b_{q,\eta'}, \psi_\nu(x)] = \delta_{\eta,\eta'} \alpha_q(x) \psi_\eta(x), \qquad \alpha_q(x) = \frac{i}{\sqrt{n_q}} e^{iqx} \tag{28}$$

This therefore means, that as we noted earlier $b_{q,\eta'}|\mathbf{N}\rangle_0 = 0$, that

$$b_{q\eta'} \psi_\eta(x)|\mathbf{N}\rangle_0 = \delta_{\eta,\eta'} \alpha_q(x) \psi_\eta(x)|\mathbf{N}\rangle_0 \tag{29}$$

That the state $\psi_\eta(x)|\mathbf{N}\rangle_0$ is a coherent state of the bosonic operator! Because the Fermionic operator links sectors of different particle-number, we will also need a Klein factor, as well as having some undetermined function of x multiplying the state, but this can be determined:

$$\psi_\eta(x)|\mathbf{N}\rangle_0 = \lambda_\eta(x) e^{\sum_{q>0} \alpha_q(x) b_{q\eta}^\dagger} F_\eta |\mathbf{N}\rangle_0 \tag{30}$$

The undetermined factor $\lambda_\eta(x)$ can be extracted by considering

$${}_0\langle\mathbf{N}|F_\eta^\dagger \psi_\eta(x)|\mathbf{N}\rangle_0 = \lambda_\eta(x) = \sqrt{\frac{2\pi}{L}} e^{-i2\pi/L(N_\eta - \delta_\eta/2)} \tag{31}$$

We have seen how $\psi_\eta(x)$ acts on an N-particle ground state, and we are not very far from establishing how it acts on a *general* N-particle state. Using the commutation relations, we can see that

$$e^{i\varphi_\eta(x)}|\mathbf{N}\rangle_0 = |\mathbf{N}\rangle_0 \tag{32}$$

Therefore, with a little algebra, one can establish that

$$\psi_\eta(x)|\mathbf{N}\rangle = F_\eta \lambda_\eta(x) e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)}|\mathbf{N}\rangle \tag{33}$$

And therefore obtain the bosonisation identity

$$\boxed{\psi_\eta(x) = F_\eta \lambda_\eta(x) e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\gamma(x)}} \tag{34}$$

or, employing BCH

$$\psi_\eta(x) = F_\eta a^{-1/2} e^{-i\Phi_\eta(x)} \tag{35}$$

$$\Phi_\eta(x) := \varphi_\eta^\dagger(x) + \varphi_\eta(x) + \frac{2\pi}{L}\left(N^\eta - \delta_\eta/2\right) \tag{36}$$

3 Field-theoretical approach

We start from the free massless Dirac Fermion, i.e.

$$H_e = iv_F \int \mathrm{d}x \left[L^\dagger(x)\partial_x L(x) - R^\dagger(x)\partial_x R(x)\right] \tag{37}$$

In this model, we can show that

$$\langle L^\dagger(x,t)L(0,0)\rangle = -\frac{i}{2\pi(x+vt)} \tag{38}$$

$$\langle R^\dagger(x,t)R(0,0)\rangle = \frac{i}{2\pi(x-vt)} \tag{39}$$

This can be compared to the free compact massless boson

$$H_0 = \frac{v}{2}\int \mathrm{d}x \left[(\partial_x\Phi)^2 + \Pi^2\right] \tag{40}$$

with

$$\Pi(x,t) = \frac{1}{v}\partial_t\Phi(x,t), \qquad [\Phi(x,t),\Pi(x',t)] = i\delta(x-x') \tag{41}$$

If we compactify the boson to a ring of radius R i.e. impose that

$$\Phi = \Phi + 2\pi R \tag{42}$$

then confining the x-coordinate to a ring of length L, examining the equation of motion for the fields

$$\left(\partial_t^2 - v^2\partial_x^2\right)\Phi(x,t) = 0 \tag{43}$$

suggests a mode expansion of the form

$$\Phi(x,t) = q + \pi_0 \frac{vt}{L} + \widetilde{\pi}_0 \frac{x}{L} + \sum_{n\neq 0} \frac{1}{\sqrt{4\pi|n|}} \left[\beta_n e^{i(k_n - i|vk_n|t)} + \text{h.c.}\right] \tag{44}$$

Where the $\{\beta_n\}$ obey bosonic commutation relations and $k_n = \frac{2\pi n}{L}$. Importantly, we have a zero-mode and q is an angular coordinate due to the compactification of the boson. $\widetilde{\pi}_0$ counts the number of times that the field winds around the circle. We ideally want to split this field into left- and right-moving pieces. This is not so simple due to the presence of the zero mode. We introduce the operator $\widetilde{q}$ conjugate to $\widetilde{\pi}_0$ such that

$$[\widetilde{q}, \widetilde{\pi}_0] = i \tag{45}$$

We can then write $\Phi = \varphi_R + \varphi_L$ where

$$\varphi_R(vt - x) = Q + \frac{P}{2L}(vt - x) + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left[\beta_n e^{ik_n(x-vt)} + \beta_n^\dagger e^{-ik_n(x-vt)}\right] \tag{46}$$

$$\varphi_L(vt + x) = \bar{Q} + \frac{\bar{P}}{2L}(vt + x) + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left[\beta_n e^{-ik_n(x+vt)} + \beta_{-n}^\dagger e^{-ik_n(x+vt)}\right] \tag{47}$$

where

$$Q = \frac{q - \tilde{q}}{2}, \qquad \bar{Q} = \frac{q + \tilde{q}}{2}, \qquad P = \frac{\pi_0 - \tilde{\pi}_0}{2}, \qquad \bar{P} = \frac{\pi_0 + \tilde{\pi}_0}{2} \tag{48}$$

and we can see that

$$[Q, P] = [\bar{Q}, \bar{P}] = i \tag{49}$$

If we consider the right-moving field and some exponential of this, we can show that $P \sim$ Fermion number. This already hints at the bosonisation relation we want to establish. We actually proceed here by comparing correlation functions. Consider a correlation function of the form

$$\langle e^{i\alpha\varphi_R(x)} e^{-i\beta\varphi_R(x')} \rangle = \delta_{\alpha,\beta} \left(\frac{ia_0}{x - x'}\right)^{\frac{\alpha^2}{4\pi}} \tag{50}$$

which can be evaluated by simply using the mode expansion. This suggests that if we pick $\alpha = \sqrt{4\pi}$, that we can reproduce the Fermionic operators by taking

$$R(x) = \frac{\eta}{\sqrt{2\pi a_0}} e^{-i\sqrt{4\pi}\varphi_R(x)} \tag{51}$$

$$L(x) = \frac{\bar{\eta}}{\sqrt{2\pi a_0}} e^{-i\sqrt{4\pi}\varphi_L(x)} \tag{52}$$

where the η, $\bar{\eta}$ are introduced to reproduce the correct commutation relations i.e.

$$\{\eta, \bar{\eta}\} = 0, \qquad \eta^2 = \bar{\eta}^2 = 1 \tag{53}$$

4 Interacting electrons

If we consider an interacting model

$$H = H_{\text{free}} + H_{\text{int}}, \tag{54}$$

$$H_{\text{free}} = -J/2 \sum_{j=1}^{L} c_j^\dagger c_{j+1} + \text{h.c.} + h' \sum_{j=1}^{L} n_j, \tag{55}$$

$$H_{\text{int}} = J\Delta \sum_j :n_j:\,:n_{j+1}:. \tag{56}$$

Then, taking the continuum limit and looking at the low-energy theory, we can expand the Fermionic operators in terms of the right- and left-moving modes

$$c_j \to \sqrt{a_0}\left[R(x)e^{ik_F x} + L(x)e^{-ik_F x}\right] \tag{57}$$

Plugging this into the free Hamiltonian, we recover the Dirac Hamiltonian. The derivative arises from expanding $L(x + a_0)$. The free theory is therefore, as established earlier, equivalent to the free compactified massless boson. We can to add the interaction piece, now, and we expect this description to be valid for small Δ. Doing this, we find that we can throw away rapidly oscillating pieces, in the appropriate filling regime, yielding

$$H = \frac{v_F}{2}\int \mathrm{d}x \left[\left(1 + \frac{4\Delta}{\pi}\sin k_F a_0\right):(\partial\Phi)^2: + :(\partial\Theta)^2: + \frac{J\Delta \sin 2k_F a_0}{\sqrt{\pi}}:\partial\Phi:\right] \tag{58}$$

where $\Theta = \varphi_R - \varphi_L$. This is just a free theory, again! Completing the square and rescaling Φ etc., we can write this as

$$H = \frac{\widetilde{v}_F}{2}\int \mathrm{d}x \left[:(\partial\widetilde{\Phi})^2: + :(\partial\widetilde{\Theta})^2:\right] \tag{59}$$

Where we have changed the Fermi velocity

$$\widetilde{v}_F = v_F\left(1 + \frac{2\Delta}{\pi}\sin 2k_F a_0\right) \tag{60}$$

$$\widetilde{\Phi} = \widetilde{\Phi} + 2\pi\widetilde{R} \tag{61}$$

$$\widetilde{R} = \frac{1}{\sqrt{4\pi}}\left(1 + \frac{4\Delta\sin k_F a_0}{\pi}\right)^{1/2} \tag{62}$$

Using the Bethe Ansatz for the XXZ model, we can see exactly in what regime of Δ this approximation is reasonable.