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Factorized S-Matrices in Two Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Theory Models

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The general properties of the factorized S-matrix in two-dimensional space-time are considered. The relation between the factorization property of the scattering theory and the infinite number of conservation laws of the underlying field theory is discussed. The factorization of the total S-matrix is shown to impose hard restrictions on two-particle matrix elements: they should satisfy special identities, the so-called factorization equations. The general solution of the unitarity, crossing and factorization equations is found for the S-matrices having isotopic $O(N)$-symmetry. The solution turns out to have different properties for the cases $N = 2$ and $N \geqslant 3$. For $N = 2$ the general solution depends on one parameter (of coupling constant type), whereas the solution for $N \geqslant 3$ has no parameters but depends analytically on N. The solution for $N = 2$ is shown to be an exact soliton S-matrix of the sine-Gordon model (equivalently the massive Thirring model). The total S-matrix of the model is constructed. In the case of $N \geqslant 3$ there are two "minimum" solutions, i.e., those having a minimum set of singularities. One of them is shown to be an exact S matrix of the quantum $O(N)$-symmetric nonlinear σ-model, the other is argued to describe the scattering of elementary particles of the Gross-Neveu model.

1. INTRODUCTION

The general two-dimensional relativistic S-matrix (not to mention higher space-time dimensionalities) is a very complicated object. In two space-time dimensions, however, a situation is possible in which the total S-matrix being nontrivial is simplified drastically. This is the case of factorized scattering. Generally, the factorization of a two-dimensional S-matrix means a special structure of the multiparticle S-matrix element: it is factorized into the product of a number of two-particle ones as if an arbitrary process of multiparticle scattering would be a succession of space-time separated elastic two-particle collisions, the movement of the particles in between being free.

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The factorized S-matrix has been first discovered in the nonrelativistic problem of one-dimensional scattering of particles interacting through the δ-function pair potential [1–3]. Furthermore, the factorization is typical for the scattering of solitons of the nonlinear classical field equations completely integrable by the inverse scattering method [4–6]. Note, that all the dynamical systems leading to the factorized S-matrix possess, as a common feature, an infinite set of "close to free" conservation laws.[1] This set of conservation laws is considered to be a necessary and sufficient condition for the S-matrix factorization [7–11]. Some speculations about this point are presented in Section 2.

The expressibility of the multiparticle S-matrix in terms of two-particle ones provides an essential simplification and enables one to construct in many cases the total S-matrix up to the explicit calculation of the two-particle matrix elements themselves. In the present paper we construct a certain class of the relativistic factorized S-matrices being invariant under $O(N)$ isotopic transformations. We use the method first suggested by Karowski, Thun, Truong and Weisz [12] (in the sine-Gordon context). The selfconsistency of the factorized structure of the total S-matrix turns out to impose special cubic equations (the factorization equations in what follows) on the two-particle S-matrix elements (see Section 2). Therefore, the factorization, unitarity and crossing symmetry provide a nontrivial system of equations which is basic for the method mentioned above. The general solution of these equations has an ambiguity of CDD type: there is a "minimum solution" (i.e., the solution having minimum set of singularities); one obtains the general one adding an arbitrary number of auxiliary CDD poles.

Are there any two-dimensional quantum field theory (QFT) models that lead to these S-matrices? Most of the nonlinear classical field equations have evident QFT versions. The problem of factorizing quantum S-matrices of these models (which is closely connected with that of "surviving" classical conservation laws under quantization) is nontrivial and requires special investigation in each case. In this paper we consider three models, the aim is to show that they lead to $O(N)$-symmetric factorized S-matrices.

(1) The quantum sine-Gordon model, i.e., the model of a single scalar field $\phi(x)$, which is defined by the Lagrangian density:

$$\mathscr{L}_{SG} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m_0^2}{\beta^2}\cos(\beta\phi), \tag{1.1}$$

where m_0 is a mass-like parameter and β is a coupling constant.

It is well-known that the classical sine-Gordon equation is completely integrable [6]. The structure of the quantum theory has been also studied in detail. The mass

[1] The meaning of this term is as follows. In the asymptotic states where all particles are far enough from each other these conservation laws tend to those of the theory of free particles. The latter laws lead to conservation of the individual momentum of each particle and can be formulated, e.g., as the conservation of sums of the entire powers of all particle momenta $\sum_a p_a^n$; $n = 1, 2,\ldots$ [7].

spectrum of this model has been found by a quasiclassical method [13–15]. It contains particles carrying the so-called "topological charge"[2]—quantum solitons and corresponding antisolitons—and a number of neutral particles (quantum doublets) which can be thought of as the soliton-antisoliton bound states; the "elementary particle" corresponding to field ϕ turns out to be one of these bound states. Some of the quasiclassical results (the mass formula for the doublets) proved to be exact [15].

The other exact result has been obtained by Coleman [16] (see also Refs. [17, 19]). The quantum sine-Gordon model is equivalent to the massive Thirring model, i.e., the model of charged fermion field, defined by the Lagrangian density

$$\mathscr{L}_{MTM} = i\bar{\psi}\gamma_\mu \partial_\mu \psi - m\bar{\psi}\psi - g/2(\bar{\psi}\gamma_\mu\psi)^2 \tag{1.2}$$

provided the coupling constants are connected by

$$g/\pi = 4\pi/\beta^2 - 1. \tag{1.3}$$

Fundamental fermions of (1.2) are identical to quantum solitons of (1.1).

There is a considerable amount of results in support of the factorization of the quantum sine-Gordon S-matrix; these results are mentioned in Section 4.

(2) The quantum chiral field on the sphere S^{N-1} ($N = 3, 4,\ldots$) ($O(N)$ symmetric nonlinear σ-model) defined by the Lagrangian density and the constraint

$$\mathscr{L}_{CF} = \frac{1}{2g_0} \sum_{i=1}^{N} (\partial_\mu n_i)^2; \qquad \sum_{i=1}^{N} n_i^2 = 1, \tag{1.4}$$

where g_0 is a (bare) coupling constant. This model is $O(N)$ symmetric, renormalizable and asymptotically free [20, 12]. The infrared charge singularity of this model seems to cause the disintegration of the Goldstone vacuum [22]. True vacuum is $O(N)$ symmetric and nondegenerate; all particles of the model are massive and form $O(N)$-multiplets. This situation is surely the case when N is large enough [23, 24] and we suppose it is valid for $N \geqslant 3$.

In Section 5 some arguments in favour of S-matrix factorization in the model (1.4) are presented. The first evidence of this phenomenon is based on the properties of the $1/N$ expansion of the model [23, 24]. Namely, the absence of $2 \to 4$ production amplitude and the factorization of $3 \to 3$ amplitude can be shown to the leading order in $1/N$ [25]. A more rigorous proof of the σ-model S-matrix factorization follows from the recently discovered infinite set of quantum conservation laws [26, 27]. In Section 5 we review briefly the results of Ref. [26].

[2] In model (1.1) the topologic charge q is connected with the asymptotic behaviour of the field $\phi(x, t)$ as $x \to \pm\infty$:

$$q = \beta/2\pi \int_{-\infty}^{\infty} \frac{d\phi}{dx}\, dx = \beta/2\pi[\phi(\infty) - \phi(\infty)].$$

(3) The Gross–Neveu model, i.e., the model of N-component self-conjugated Fermi-field $\psi_i(x)$; $i = 1, 2, \ldots, N$ ($N \geqslant 3$) with four-fermion interaction

$$\mathscr{L}_{GN} = \frac{i}{2} \sum_{i=1}^{N} \bar{\psi}_i \gamma_\mu \partial_\mu \psi_i + \frac{g_0}{8} \left[\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right]^2, \tag{1.5}$$

where $\bar{\psi}_i = \psi_i \gamma_0$. Like the chiral field this model is renormalizable, asymptotically free and explicitly $O(N)$ symmetric.

Model (1.5) has been studied by Gross and Neveu in the limit of $N \to \infty$ [28]. They have found a spontaneous breakdown of discrete γ_5-symmetry (the field $\sum_{i=1}^{N} \bar{\psi}_i \psi_i$ acquires a nonzero vacuum expectation value) leading to the dynamical mass transmutation. Using quasiclassical method Dashen, Hasslacher and Neveu [29] have studied the model in the same $N \to \infty$ limit. These authors have found a rich spectrum of bound states of the fundamental fermions of this model and determinded their masses.

We support the factorization of the Gross–Neveu S-matrix by arguments which are quite analogous to those for model (1.4).

The paper is arranged as follows. In Section 2 general properties of factorized scattering are considered, factorization equations are introduced and their meaning is cleared up. Furthermore, a convenient algebraic representation of the factorized S-matrix is suggested. Sect. 3 contains the general solution of analyticity, unitarity and factorization equations for the S-matrix having $O(N)$ isotopic symmetry. The "minimum" solutions of these equations turn out to be essentially different for $N = 2$ and $N \geqslant 3$. The solution for $N = 2$ depends on one parameter of coupling constant type. As it is shown in Section 4 this solution turns out to be the exact S-matrix of quantum sine-Gordon solitons. In this Section we construct the total sine-Gordon S-matrix too, which includes all bound states (doublets). For the case $N \geqslant 3$ "minimum" solutions of Section 3 do not depend on any free parameters. They correspond to asymptotically free field theories with the dynamical mass transmutation. In Sections 5 and 6 one of these solutions is shown to be the S-matrix of model (1.4) and the other—to be an exact one of elementary fermions of (1.5).

2. Factorized Scattering, General Properties, Factorization Equations

Consider a two-dimensional scattering theory and suppose that the underlying dynamical theory is governed by the infinite set of conservation laws, the corresponding conserving charges Q_n ; $n = 1, 2, \ldots, \infty$ being diagonal in one-particle states:

$$Q_n \mid p^{(a)} \rangle = \omega_n^{(a)}(p) \mid p^{(a)} \rangle. \tag{2.1}$$

In (2.1) p is the particle momentum and (a) letters the kind of particle (if the theory contains more than one kind). Suppose, furthermore, that the eigenvalues $\omega_n(p)$ form the set of independent functions. In fact, all the known systems with the infinite number of conservation laws permit such a choice of the set Q_n that $\omega_n(p)$, speaking roughly, should be the entire powers of the momentum p. E.g., for the sine-Gordon case they are:

$$\omega_{2n+1}^{(a)}(p) = p^{2n+1}; \qquad \omega_{2n}^{(a)}(p) = (p^2 + m_a^2)^{1/2} p^{2n}, \tag{2.2}$$

where m_a is the mass of the particle (a). The laws described above are said to be "deformation of free laws". If the theory is governed by laws of this type the corresponding scattering theory satisfies strong selection rules (first noticed by Polyakov). Namely:

(i) Let $\{m_a\}$ be the mass spectrum of the theory. Then the number of particles of the same mass m_a remains unchanged after collision.

(ii) The final set of the two-momenta of particles is the same as the initial one.[3]

These two selection rules become evident if one takes into account that:

(a) $$Q_n \mid p_1^{(a_1)}, p_2^{(a_2)},..., p_k^{(a_k)}, \text{in(out)}\rangle = [\omega_n^{(a_1)}(p_1) + \cdots + \omega_n^{(a_k)}(p_k)] \mid p_1^{(a_1)},..., p_k^{(a_k)} \cdot \text{in(out)}\rangle. \tag{2.3}$$

(b) $dQ_n/dt = 0$

and hence

$$\sum_{j\in\text{in}} \omega_n^{(a_j)}(p_j) = \sum_{j\in\text{out}} \omega_n^{(a_j)}(p_j). \tag{2.4}$$

Note that all the intermediate states where the particles are far enough from each other should satisfy both selection rules (i) and (ii) too. This note, together with the special properties of the two-dimensional kinematics, gives an impression that if the theory is governed by an infinite set of conservation laws the multiparticle S-matrix elements can be expressed in terms of two-particle ones.

To clarify this point consider the space of configurations of the system of N particles. There are $N!$ disconnected domains in this space where all the particles are far enough from each other and one can neglect interaction between them. Let $\{x\} = \{x_1 ,..., x_N\}$ be the coordinates of the particles and R—the interaction range (we suppose the latter to be finite). Then each domain can be identified by the succession of inequalities $x_{P1} < x_{P2} < \cdots < x_{PN}$ where $|x_{Pi+1} - x_{Pi}| \gg R$ and P is any permutation of the integers $1, 2,..., N$. We denote this domain by X_P.

The free motion of particles in these domains can be described in terms of the wave function $\Psi(x_1 ,..., x_N)$; $\{x\} \in X_P$ Selection rules (i) and (ii) mean now that if the incident particles are of momenta $p_1 > p_2 > \cdots > p_N$, the wave function in each domain

[3] It may appear that (i) and (ii) mean that the S-matrix is diagonal in the momentum representation. It is not true if the theory contains different particles (having different internal quantum numbers, e.g., particle and antiparticle) of the same mass. In this case the exchanges of momenta between these particles and other nondiagonal processes are possible (see Sect. 3).

should be a superposition of waves, the set of wave vectors being selected by these rules:

$$\Psi_P(x_1, ..., x_N) = \sum_{P'} C(P, P') \exp\{ip_{P1}x_{P'1} + \cdots + ip_{PN}x_{P'N}\}; \qquad \{x\} \in X_P. \quad (2.5)$$

Here summation is carried out over all permutation P' of $p_1, ..., p_N$, permitted by (i) and (ii). Symmetrization (antisymmetrization) in the coordinates of identical particles is implied in (2.5). The coefficients $C(P, P')$ are functions of the domain X_P and of the permutation P'. In particular, the coefficient $C(P, I)$ describes the inci- incident wave in the domain X_P; to obtain the scattering wave function one puts $C(P, I) = 0$ if $P \neq I$ and $C(I, I) = 1$ (here I is identical transposition). The coefficients $C(P, \tilde{I})$ ($\tilde{I}$ is the inverse transposition $\tilde{I}(1, 2, ..., N) = (N, N-1, ..., 1)$) describe outgoing waves in these domains and thus they are elements of the N-particle S-matrix. For example, in the case of two particles of the same mass the wave function becomes

$$\begin{aligned} \Psi_{x_1 \ll x_2}(x_1, x_2) &= e^{ip_1x_1}e^{ip_2x_2} + S_R(p_1, p_2)\, e^{ip_2x_1}e^{ip_1x_2}; \\ \Psi_{x_1 \gg x_2}(x_1, x_2) &= S_T(p_1, p_2)\, e^{ip_1x_1}e^{ip_2x_2}. \end{aligned} \quad (2.6)$$

In Eq. (2.6) S_R and S_T are two-particle S-matrix elements corresponding to backward scattering (reflection) and forward scattering (transition).

It is convenient to picture the situation as the scattering of the N-dimensional plane wave in the system of semipenetrable hypersurfaces $x_i = x_j$ (for any i and j). Far enough from these hypersurfaces the wave is described by (2.5); near them the motion is more complicated because of the interaction between the particles. Moreover, if the relativistic problem is taken into consideration the motion in the interaction region cannot be treated in terms of the wave function of a finite number of variables (because the virtual pair creation is possible). The determination of the coefficients $C(P, P')$ in (2.5) requires the extrapolation of the wave function from one domain of free motion to another through the boundary between them, where the particles are in interaction. The solution of the problem of interacting particles is, in general, a very complicated task. Note, however, that the extrapolation of the wave function can pass through the region of the boundary, where two particles are close and others are arbitrary far from them and each other (e.g., $|x_1 - x_2| \lesssim R$, $|x_i - x_1| \gg R$, $|x_2 - x_i| \gg R$ and $|x_i - x_j| \gg R$; $i, j = 3, 4, ...$). These regions describe two-particle collisions and there the extrapolation conditions are the same as in the two-particle problem. Therefore, in this case the knowledge of the two-particle S-matrix elements provides one with a sufficient information to determine all the coefficients $C(P, P')$ and, therefore, to obtain the multiparticle S-matrix. N-particle S-matrix element turns out to be a product of $N(N-1)/2$ two particle ones. Such a structure is spoken about as the factorized S-matrix. Note that the possibility of this structure is due to the fact that the wave function in each domain X_P is a superposition of a finite number of waves, the latter being a consequence of the infinite set of conservation laws.

Of course, this consideration connecting the factorization and the existance of infinite number of conservation laws is not a rigorous proof; a complete evidence can be found in a recent paper [11]. All the considerations presented in the above paragraph are of exact sense in the case of the one dimensional problem of nonrelativistic particles interacting via the δ-function potential [1–3].

The factorized S-matrix corresponds to the following simple scattering picture. In the infinite past particles of momenta $p_1 > p_2 > \cdots > p_N$ were spatially arranged in the opposite order: $x_1 < x_2 < \cdots < x_N$. In the interaction region the particles successively collide in pairs; they move as free real (not virtual) particles in between. The set of momenta of particles is conserved in each pair collision; if the particles are of different mass the transition is possible only, the collision of particles of the same mass may result in the reflection too. After $N(N-1)/2$ pair collisions the particles are arranged along the x axis in the order of momenta increasing. This corresponds to the final state of scattering—outgoing particles.

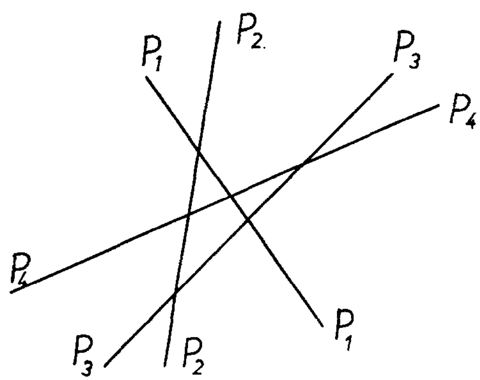


FIG. 1. The space-time picture illustrating the multiparticle factorized scattering.

The space-time picture of the multiparticle factorized scattering can be represented by a spacial diagram; an example is drawn in Fig. 1. Each straight line in the diagram corresponds to any value of momentum, obviously connected with the slope of the line (in this diagram time is assumed to flow up). Two-particle collisions are represented by the vertices where the lines cross each other; the corresponding two particle amplitude should be attached to each cross. The total multiparticle S-matrix element of the process drawn in the diagram is given by a sum of products of all the $N(N-1)/2$ two-particle amplitudes corresponding to each vertex. The summation mentioned above should be carried out over all possible kinds of particles flowing through the internal lines of the diagram and resulting in a given final state.

The following is to be mentioned. The same scattering process can be represented by a number of different diagrams in which some of the lines are translated in parallel (e.g., see Figs. 2a and 2b). The amplitudes of these diagrams should not be added in the multiparticle S-matrix element. In terms of the wave function in sectors X_p amplitudes drawn in Figs. 2a and 2b correspond to different semifronts of the same

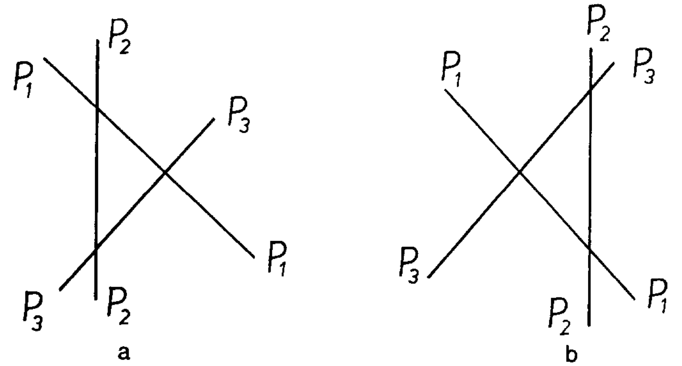


FIG. 2. Two possible ways of the three-particle scattering.

outgoing wave. Both should have the same amplitudes and phases (because of (i) and (ii)), i.e., be coherent. This requirement makes two-particle matrix elements satisfy special cibic equations, the latter being necessary conditions of the factorization. In what follows these equations play an essential role and we shall call them the factorization equations.[4]

In the present paper the relativistic scattering is mainly considered. The following notations are convenient in this case. We shall use rapidities θ_a instead of momenta p_a of particles (of mass m_a)

$$p_a^0 = m_a \operatorname{ch} \theta_a ; \qquad p_a^1 = m_a \operatorname{sh} \theta_a \tag{2.7}$$

Two-particle amplitudes $S(p_a, p_b)$ become functions of the rapidity difference of colliding particles $\theta_{ab} = \theta_a - \theta_b$, the latter being simply connected with the s-channel invariant $s_{ab} = (p_a^\mu + p_b^\mu)^2$

$$s_{ab} = m_a^2 + m_b^2 + 2m_a m_b \operatorname{ch} \theta_{ab} \tag{2.8}$$

(m_a and m_b are masses of the particles).

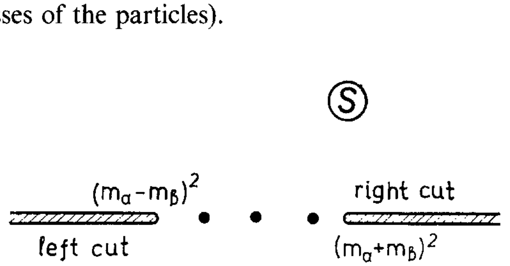


FIG. 3. The analytical structure of two-particle amplitudes in the physical sheet of the s-plane.

[4] Factorization equations and their physical sense in the problem of nonrelativistic particles interacting via the δ-function potential have been considered in Ref. [2]; in the case of the sine-Gordon problem they were obtained in Refs. [30, 31] and used in Ref. [12].

Two-particle amplitudes $S(s)$ are the analytical functions in the complex s-plane with two cuts along the real axis $s \leqslant (m_a - m_b)^2$ and $s \geqslant (m_a + m_b)^2$ (see Fig. 3). The points $s = (m_a - m_b)^2$ and $s = (m_a + m_b)^2$, being the two-particle thresholds, are square root branching point of $S(s)$. In the case of the factorized scattering there is only the two-particle unitarity and it is natural to suppose functions $S(s)$ not to exibit other branching points. If it is the case, the functions $S(\theta)$ should be meromorphic. Mapping (2.8) transforms physical sheet of the s-plane into the strip $0 < \operatorname{Im} \theta < \pi$ (if it cannot lead to a misunderstanding we shall drop subindices $\theta \equiv \theta_{ab}$) in the θ-plane, the edges of the right and the left cuts of the s-plane physical sheet being mapped on the axes $\operatorname{Im} \theta = 0$ and $\operatorname{Im} \theta = \pi$, respectively (see Fig. 4). The axes $\operatorname{Im} \theta = l\pi$; $l = -1, \pm 2, \ldots$ correspond to the edges of cuts of the other complex s-plane sheets.

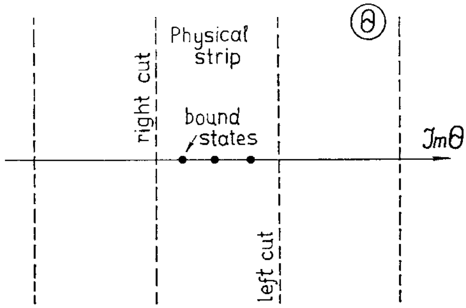


FIG. 4. The structure of the θ-plane.

The functions $S(\theta)$ are real at the imaginary axis of the θ-plane (real analyticity). In particular, at $\operatorname{Im} \theta = 0$ the relation $S(-\theta) = S^*(\theta)$ is valid. Crossing symmetry transformation $s \to 2m_a^2 + 2m_b^2 - S$ corresponds in terms of the variable θ to substitution $\theta \to i\pi - \theta$.

In the nonrelativitic limit $p_a^1 \ll m_a$ rapidities can be replaced by the nonrelativistic velocities $\theta_a \to v_a = p_a/m_a$. All the following expressions (except those connected with the crossing relations) can be applied to the case of nonrelativistic S-matrices after replacement $\theta_a \to v_a$, $\theta_b \to v_b$, $\theta_{ab} \to v_a - v_b$.

It is convenient to describe a general structure of the factorized S-matrix by means of a special algebraic construction [30, 25]. Consider a factorized scattering theory containing several kinds of particles (A, B, C and so on; particles of the same kind are supposed to be identical; statistics is not important for our consideration). These particles are represented in our construction by the special noncommutative symbols $A(\theta)$, $B(\theta)$, $C(\theta)$,..., the variable θ being the rapidity of the correponding particle. These symbols are frequently called the particles.

The scattering theory is stated as follows. Identify asymptotical states of the scattering theory with the products of all the particles in the state. The arrangement of the symbols in the product corresponds to that of particles along the spatial axis x: in-states should be identified with the products arranged in the order of decreasing rapidities of particles while out-states with those arranged in the order of increasing rapidities. For example, in-state of three particles A, A and B having rapidities θ_1, θ_2 and θ_3, respectively ($\theta_1 > \theta_2 > \theta_3$), acquires the form $A(\theta_1)\,A(\theta_2)\,B(\theta_3)$.

Any product can be rearranged by means of a number of subsequent commutations of neighbour particles (the associativity of the symbol multiplication is supposed). Each commutation corresponds to the certain two-particle collision; this leads to commutation rules for the symbols $A(\theta)$, $B(\theta)$,.... For example, if particles A and B are of different mass, one writes

$$A(\theta_1)\,B(\theta_2) = S_T^{AB}(\theta_{12})\,B(\theta_2)\,A(\theta_1), \tag{2.9}$$

where $S_T^{AB}(\theta_{12})$ is the transition amplitude for the reaction $AB \to AB$ (remind, that in the case of different masses, reflection is forbidden by (ii)). If particles of differerent kinds (say A and C) but of the same mass are under consideration the reflection is permitted and we should write

$$A(\theta_1)\,C(\theta_2) = S_T^{AC}(\theta_{12})\,C(\theta_2)\,A(\theta_1) + S_R^{AC}(\theta_{12})\,A(\theta_2)\,C(\theta_1). \tag{2.10}$$

Reflection and transition are indistinguishable in the case of identical particles, therefore

$$A(\theta_1)\,A(\theta_2) = S^{AA}(\theta_{12})\,A(\theta_2)\,A(\theta_1). \tag{2.11}$$

As it was mentioned above (see footnote 3) if there are different particles of the same mass one of them is permitted to turn into the other in the process of two-particle scattering. It means that additional channels in the two-particle scattering are open and, hence, corresponding terms should be added into the right hand sides of Eqs. (2.9), (2.10) and (2.11). We shall not discuss this point here, there are some examples of such situation in the next section.

The consistency of the commutation relations of type (2.9), (2.10), (2.11) in the calculation of symbols $A(\theta)$, $B(\theta)$ and so on and their associativity requires certain equations for the two-particle amplitudes to be satisfied. The latter are of two kinds. The identities of the first kind arise when one performs the opposite transposition of symbols after the direct one, and requires the result to be equal to the initial combination; these identities coincide with the two-particle unitarity relations. The multiparticle in-states may be rearranged into out-states in many possible successions of pair commutations but the result should be the same. This leads to the identities of the second kind. Clearly, it is sufficient to consider three particle states only and require the same result of permutations in two possible successions. One obtains all the required identities which coincide, of course, with the conditions ensuring the equality of triangle diagrams (see Figs. 2a, 2b), and so they are the factorization equations.

If identities of both kinds are satisfied the commutation relations permit one to rearrange unambiguously any in-state into a superposition of out-states and then this construction represents the total factorized S-matrix. Its unitarity is trivial. One obtains the matrix S^{-1} after the rearrangement of out-states into in-states; it differs from the S-matrix in the signs of the arguments of all two-particle amplitudes $\theta_{ab} \to -\theta_{ab}$. This change of signs leads to the complex conjugation of the two-particle matrix elements. Taking into account the symmetry of the S-matrix one obtains $S^{-} = S^{-1}$.

3. Relativistic S-Matrix with $O(N)$-Isosymmetry. General Solution

Following the general consideration of the previous section we treat now the class of relativistic factorized S-matrices characterized by the isotopic $O(N)$ symmetry. To introduce the $O(N)$ symmetry we assume the existance of isovector N-plet of particles A_i ; $i = 1, 2,..., N$ with equal masses m and require the $O(N)$ symmetry of the two-particle scattering (this ensures $O(N)$ symmetry of the total S-matrix due to the factorization). Namely, we assume for two-particle S-matrix the form:

$$\begin{aligned} {}_{ik}S_{jl} &= \langle A_j(p_1') A_l(p_2'), \text{out} \mid A_i(p_1) A_k(p_2), \text{in}\rangle \\ &= \delta(p_1 - p_1')\, \delta(p_2 - p_2')[\delta_{ik}\delta_{jl}S_1(s) + \delta_{ij}\delta_{kl}S_2(s) + \delta_{il}\delta_{jk}S_3(s)] \\ &\quad \pm (i \leftrightarrow k, p_1 \leftrightarrow p_2), \end{aligned} \tag{3.1}$$

where $s = (p_1{}^{\mu} + p_2{}^{\mu})^2$ and the $+(-)$ refers to bosons (fermions). The functions $S_2(s)$ and $S_3(s)$ are the transition and reflection amplitudes, respectively, while $S_1(s)$ describes the "annihilation" type processes: $A_i + A_i \to A_j + A_j$ $(i \neq j)$.

The S-matrix (3.1) will be cross-symmetric provided the amplitudes $S(s)$ satisfy equations $S_2(s) = S_2(4m^2 - s)$ and $S_1(s) = S_3(4m^2 - s)$. After introducing the rapidity variables (2.7), (2.8) we deal with meromorphic functions $S_1(\theta)$, $S_2(\theta)$ and $S_3(\theta)$, where $s = 4m^2 \operatorname{ch}^2(\theta/2)$, and the cross-symmetry relations become

$$S_2(\theta) = S_2(i\pi - \theta), \tag{3.2a}$$

$$S_1(\theta) = S_3(i\pi - \theta). \tag{3.2b}$$

To describe now the factorized total S-matrix let us introduce, following the general method of Section 2, symbols $A_i(\theta)$; $i = 1, 2,..., N$. The commutation rules corresponding to (3.1) are

$$\begin{aligned} A_i(\theta_1)\, A_j(\theta_2) &= \delta_{ij}S_1(\theta_{12}) \sum_{k=1}^{N} A_k(\theta_2)\, A_k(\theta_1) \\ &\quad + S_2(\theta_{12})\, A_j(\theta_2)\, A_i(\theta_1) + S_3(\theta_{12})\, A_i(\theta_2)\, A_j(\theta_1). \end{aligned} \tag{3.3}$$

It is straightforward to obtain the unitarity conditions for two-particle S-matrix (3.1)

$$S_2(\theta)\, S_2(-\theta) + S_3(\theta)\, S_3(-\theta) = 1, \tag{3.4a}$$

$$S_2(\theta)\, S_3(-\theta) + S_2(-\theta)\, S_3(\theta) = 0, \tag{3.4b}$$

$$\begin{aligned} &NS_1(\theta)\, S_1(-\theta) + S_1(\theta)\, S_2(-\theta) + S_1(\theta) S_3(-\theta) \\ &\quad + S_2(\theta)\, S_1(-\theta) + S_3(\theta)\, S_1(-\theta) = 0. \end{aligned} \tag{3.4c}$$

Obviously, Eqs. (3.2) and (3.4) are not sufficient to determine the functions $S(\theta)$. Further restrictions arise from the factorization equations (see Section 2). One obtains the factorization equations considering all possible three-particle in-products $A_i(\theta_1)\,A_j(\theta_2)\,A_k(\theta_3)$, reordering them to get out-products by means of (3.3) and requiring the results obtained in two possible successions of two-particle commutations to be equal. The equations arising are evidently different for the cases $N = 2$ and $N \geqslant 3$ (fewer different three-particle products are possible at $N = 2$). Therefore it is convenient to make a notational distinction between these two cases. Dealing with the case $N \geqslant 3$ we redenote the amplitudes S_1, S_2 and S_3 by σ_1, σ_2 and σ_3, respectively, reserving the original notations for the case $N = 2$.

The factorization equations have the form (the derivation is straightforward but somewhat cumbersome)

$$S_2S_1S_3 + S_2S_3S_3 + S_3S_3S_2 = S_3S_2S_3 + S_1S_2S_3 + S_1S_1S_2\,, \tag{3.5a}$$

$$\begin{aligned} S_3S_1S_3 + S_3S_2S_3 &= S_3S_3S_1 + S_3S_3S_2 + S_2S_3S_1 \\ &\quad + S_2S_3S_3 + 2S_1S_3S_1 + S_1S_3S_2 + S_1S_3S_3 + S_1S_2S_1 + S_1S_1S_1 \end{aligned} \tag{3.5b}$$

for $N = 2$ and

$$\sigma_2\sigma_3\sigma_3 + \sigma_3\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\,, \tag{3.6a}$$

$$\sigma_2\sigma_1\sigma_1 + \sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_1\sigma_2\,, \tag{3.6b}$$

$$\begin{aligned} &N\sigma_1\sigma_3\sigma_1 + \sigma_1\sigma_3\sigma_2 + \sigma_1\sigma_3\sigma_3 + \sigma_1\sigma_2\sigma_1 \\ &\quad + \sigma_2\sigma_3\sigma_1 + \sigma_3\sigma_3\sigma_1 + \sigma_1\sigma_1\sigma_1 = \sigma_3\sigma_1\sigma_3 \end{aligned} \tag{3.6c}$$

for $N \geqslant 3$. For each term in (3.5) and (3.6) the argument of the first, the second and the third S (σ in (3.6)) is implied to be θ, $\theta + \theta'$ and θ', respectively.

The factorization equations turn out to be rather restrictive. They allow one to express explicitly all the amplitudes in terms of one function.

General solutions for both systems (3.5) and (3.6) satisfying the real-analyticity condition (all the amplitudes are real if θ is purely imaginary) are derived in Appendix A. For system (3.5) (i.e., for $N = 2$) this solution is

$$S_3(\theta) = i \operatorname{ctg}\left(\frac{4\pi\delta}{\gamma}\right) \operatorname{cth}\left(\frac{4\pi\theta}{\gamma}\right) S_2(\theta) \tag{3.7a}$$

$$S_1(\theta) = i \operatorname{ctg}\left(\frac{4\pi\delta}{\gamma}\right) \operatorname{cth}\left(\frac{4\pi(i\delta - \theta)}{\gamma}\right) S_2(\theta) \tag{3.7b}$$

with arbitrary real γ and δ. The general solution for (3.6) contains only one free parameter λ and have the form:

$$\sigma_3(\theta) = -\frac{i\lambda}{\theta}\,\sigma_2(\theta) \tag{3.8a}$$

$$\sigma_1(\theta) = -\frac{i\lambda}{i[(N-2)/2]\,\lambda - \theta}\,\sigma_2(\theta). \tag{3.8b}$$

The restrictions on the amplitudes $S_2(\theta)$ and $\sigma_2(\theta)$ come from the unitarity conditions (3.4). The equations (3.4b) and (3.4c) are satisfied by (3.7) and (3.8) identically, while equation (3.4a) gives

$$S_2(\theta)\, S_2(-\theta) = \frac{\sin^2\left(\frac{4\pi\delta}{\gamma}\right) \mathrm{sh}^2\left(\frac{4\pi\theta}{\gamma}\right)}{\sin^2\left(\frac{4\pi\delta}{\gamma}\right) \mathrm{sh}^2\left(\frac{4\pi\theta}{\gamma}\right) + \cos^2\left(\frac{4\pi\delta}{\gamma}\right) \mathrm{ch}^2\left(\frac{4\pi\theta}{\gamma}\right)} \tag{3.9}$$

for $N = 2$ and

$$\sigma_2(\theta)\, \sigma_2(-\theta) = \frac{\theta^2}{\theta^2 + \lambda^2} \tag{3.10}$$

for $N \geqslant 3$.

Until now we have deliberately avoided the use of the cross-symmetry relations (3.2). Although the above consideration concerns the relativistic case, the unitarity conditions (3.4) and factorization equations (3.5), (3.6) are valid for any nonrelativistic $O(N)$ symmetric factorized S-matrix as well, under the substitution:

$$\theta \to \frac{k}{m} = \frac{k_1 - k_2}{m}, \tag{3.11}$$

where k_1 and k_2 are momenta of the colliding particles. Therefore, the general solutions (3.7), (3.9) and (3.8), (3.10) are still valid (after the substitution (3.11)) in a nonrelativistic case. This will be used at the end of Section 4.

Equations (3.2) are especially relativistic. They turn out to give restrictions on free parameters in (3.7) and (3.8). It is easy to see that (3.2) is satisfied only if

$$\delta = \pi \tag{3.12}$$

in (3.7), (3.9) and

$$\lambda = \frac{2\pi}{N - 2} \tag{3.13}$$

in (3.8), (3.10). Thus, the formulas for $N \geqslant 3$ do not actually contain any free parameter. This circumstance will be important in Section 5.

Equation (3.2a) (which is certainly valid for $\sigma_2(\theta)$ as well as for $S_2(\theta)$) together with (3.9) and (3.10) will be used to determine $S_2(\theta)$ and $\sigma_2(\theta)$. In both cases $N = 2$ and $N \geqslant 3$ the solution admits the CDD-ambiguity only [32]: an arbitrary solution can be obtained multiplying some "minimum" solution by a meromorphic function of the type

$$f(\theta) = \prod_{k=1}^{L} \frac{\mathrm{sh}\,\theta + i \sin \alpha_k}{\mathrm{sh}\,\theta - i \sin \alpha_k}, \tag{3.14}$$

where α_1, α_2,..., α_L are arbitrary real[5] numbers. It is the "minimum" solutions, i.e., the solutions having a minimum set of singularities in the θ plane, that will be of most interest below. For $N = 2$ such a solution can be represented in the form

$$S_2(\theta) = \frac{2}{\pi} \sin\left(\frac{4\pi^2}{\gamma}\right) \operatorname{sh}\left(\frac{4\pi\theta}{\gamma}\right) \operatorname{sh}\left[\frac{4\pi(i\pi - \theta)}{\gamma}\right] U(\theta), \tag{3.15}$$

where

$$U(\theta) = \Gamma\left(\frac{8\pi}{\gamma}\right) \Gamma\left(1 + i\frac{8\theta}{\gamma}\right) \Gamma\left(1 - \frac{8\pi}{\gamma} - i\frac{8\theta}{\gamma}\right) \prod_{n=1}^{\infty} \frac{R_n(\theta)\, R_n(i\pi - \theta)}{R_n(0)\, R_n(i\pi)},$$

$$R_n(\theta) = \frac{\Gamma\left(2n\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right) \Gamma\left(1 + 2n\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)}{\Gamma\left((2n+1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right) \Gamma\left(1 + (2n-1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)}. \tag{3.16}$$

In the case $N \geqslant 3$ there are, in general, two different "minimum" solutions (the exceptional cases are $N = 3$ and $N = 4$, when these two solutions coincide). We denote these solutions $\sigma_2^{(+)}(\theta)$ and $\sigma_2^{(-)}(\theta)$; they can be written in the form

$$\sigma_2^{(\pm)}(\theta) = Q^{(\pm)}(\theta)\, Q^{(\pm)}(i\pi - \theta), \tag{3.17}$$

where

$$Q^{(\pm)}(\theta) = \frac{\Gamma\left(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right) \Gamma\left(-i\frac{\theta}{2\pi}\right)}. \tag{3.18}$$

The difference between these two solutions is of the CDD type (3.14)

$$\sigma_2^{(-)}(\theta) = \frac{\operatorname{sh}\theta + i\sin\lambda}{\operatorname{sh}\theta - i\sin\lambda}\, \sigma_2^{(+)}(\theta). \tag{3.19}$$

In the following Sections we point out the relation between the solutions (3.15) and (3.17) and certain two-dimensional quantum field theory models. Namely, we show that (3.15) together with (3.7a, b) is an exact S-matrix of quantum sine-Gordon solitons, while the solutions $\sigma_2^{(+)}(\theta)$ and $\sigma_2^{(-)}(\theta)$ for $N \geqslant 3$ give the exact S-matrices for the quantum chiral field (1.4) and for the "fundamental" fermions of Gross–Neveu model (1.5), respectively.

[5] We consider the solutions having singularities on the imaginary θ axis only, i.e., the solutions exhibiting bound and virtual states only.

4. Exact S-Matrix of the Quantum Sine-Gordon Model

Quantum sine-Gordon model (1.1) is the most known example of relativistic quantum field theory leading to the factorized scattering. There are various results ensuring the factorization of the sine-Gordon S-matrix. Complete integrability of the classical sine-Gordon equation [6] means the existance of an infinite number of conservation laws in the classical theory which are "deformation of free ones". The analogous set of conservation laws is also present in the "classical" massive Thirring model (which corresponds to the "tree" approximation for the Lagrangian (1.2) [33, 34]). The important problem of the conservation laws in quantum theory has been treated in Ref. [34] where such conservation laws were shown to survive after quantization by the perturbation theory approach (in all perturbational orders). The absence of particle production and factorization of multiparticle quantum S-matrix which are the consequenses of conservation laws has been previously demonstrated applying the direct sine-Gordon perturbative calculations by Arefyeva and Korepin [8].[6] The same result can be obtained in perturbation theory of massive Thirring model, i.e., for the soliton scattering [35, 36]. The semiclassical arguments for the soliton S-matrix factorization are also possible [14]. We use here the results mentioned above and treat the total sine-Gordon S-matrix as a factorized one.

The bound states of quantum solitons (the quantum doublets) and the soliton scattering have been investigated by a semiclassical approach in Refs. [13–15, 37, 38]. We represent here some semiclassical formulas which will be necessary below.

The two-particle scattering amplitude $S(\theta)$ for the solitons of the same sign and the transition amplitude $S_T(\theta)$ for soiton-antisoliton scattering calculated in the main semiclassical approximation have the form [13, 14, 37]

$$
\begin{aligned}
S_T^{(\mathrm{sem})}(\theta) &= S^{(\mathrm{sem})}(\theta)\exp\left\{i\,\frac{8\pi}{\beta^2}\right\},\\
S^{(\mathrm{sem})}(\theta) &= \exp\left\{\frac{8}{\beta^2}\int_0^{\pi}\ln\left[\frac{e^{\theta-i\eta}+1}{e^{\theta}+e^{-i\eta}}\right]d\eta\right\}
\end{aligned}
\tag{4.1}
$$

(where β is introduced in (1.1)) which is connected in a simple way with parameters of the classical soliton scattering. Calculation of one-loop correction leads to the change $\beta^2 \to \gamma'$ in (4.1) (see Ref. [46]) where[7]

$$
\gamma' = \beta^2\left[1 - \frac{\beta^2}{8\pi}\right]^{-1}.
$$

[6] Besides the explicit expressions for the factorized sine-Gordon "elementary" particle S-matrix has been first proposed in [8] (see also [9, 10]).

[7] The singularity of the sine-Gordon theory at $\beta^2 = 8\pi$ has been discussed by Coleman [16]. As shown in [16], the Hamiltonian of the theory becomes unbounded from below at $\beta^2 \geqslant 8\pi$, provided the standard renormalization technique is used; the phenomenon is of the ultraviolet nature. This scarcely means the failure of the theory with $\beta^2 \geqslant 8\pi$, but rather indicates a lack of superrenormalizability property and suggests that another renormalization prescription is necessary at $\beta^2 \geqslant 8\pi$. Throughout this paper we restrict our consideration to the case $\beta^2 < 8\pi$.

The semiclassical soliton-antisoliton reflection amplitude (which takes into account an imaginary time classical trajectory, see Ref. [38]) is

$$S_R^{(\mathrm{sem})}(\theta) = i \sin\left(\frac{8\pi^2}{\gamma'}\right) \exp\left(-\frac{8\pi}{\gamma'} \mid \theta \mid\right) S_T^{(\mathrm{sem})}(\theta). \tag{4.2}$$

The derivation of the semiclassical mass spectrum of the quantum doublets was carried out in papers [13–15]. In the first two semiclassical approximations it is

$$m_n^{(\mathrm{sem})} = 2m \sin\left(\frac{n\gamma'}{16}\right); \qquad n = 1, 2, \ldots < \frac{8\pi}{\gamma'}, \tag{4.3}$$

where m is a soliton mass. The authors of [15] have presented some arguments for formula (4.3) to be not only semiclassical but exact. Independent supports for this hypothesis have been given in [10, 39, 40]. The exact solution for the S-matrix which is derived in this section also confirms the exactness of spectrum (4.3).

We begin constructing the quantum sine-Gordon S-matrix stressing that the model exhibits an $O(2)$ isotopic symmetry. In terms of the massive Thirring fields ψ this symmetry is quite obvious; it corresponds to the phase invariance $\psi \to e^{i\alpha}\psi$ of (1.2). From the view-point of the sine-Gordon Lagrangian $O(2)$ symmetry is of a more delicate nature; it is the rotational symmetry of the disorder parameter (see Ref. [41] for the concept of the disorder parameter). The detailed discussion of the last point is beyond the scope of this paper. For our purpose it is sufficient to note only that the soliton and antisoliton of model (1.1) can be incorporated into an isovector $O(2)$ doublet. Following the convention of Section 3 we denote real components of this doublet by symbols $A_i(\theta)$; $i = 1, 2$. Then the soliton and antisoliton themselves will be the combinations

$$A(\theta) = A_1(\theta) + iA_2(\theta); \qquad \bar{A}(\theta) = A_1(\theta) - iA_2(\theta). \tag{4.4}$$

In terms of the particles $A(\theta)$ and $\bar{A}(\theta)$ commutation rules (3.3) take the form

$$\begin{aligned} A(\theta_1)\, \bar{A}(\theta_2) &= S_T(\theta_{12})\, \bar{A}(\theta_2)\, A(\theta_1) + S_R(\theta_{12})\, A(\theta_2)\, \bar{A}(\theta_1), \\ A(\theta_1)\, A(\theta_2) &= S(\theta_{12}) A(\theta_2)\, A(\theta_1), \\ \bar{A}(\theta_1)\, \bar{A}(\theta_2) &= S(\theta_{12})\, \bar{A}(\theta_2)\, \bar{A}(\theta_1). \end{aligned} \tag{4.5}$$

In (4.5) $S_T(\theta)$ and $S_R(\theta)$ are transition and reflection amplitudes for the soliton-antisoliton scattering while $S(\theta)$ is the scattering amplitude for identical solitons. They are connected in a simple way with amplitudes $S_1(\theta)$, $S_2(\theta)$ and $S_3(\theta)$ from (3.3)

$$\begin{aligned} S(\theta) &= S_3(\theta) + S_2(\theta), \\ S_T(\theta) &= S_1(\theta) + S_2(\theta), \\ S_R(\theta) &= S_1(\theta) + S_3(\theta). \end{aligned} \tag{4.6}$$

It is seen from (3.2) that

$$S(\theta) = S_T(i\pi - \theta); \qquad S_R(\theta) = S_R(i\pi - \theta). \tag{4.7}$$

The factorization and $O(2)$ symmetry of the sine-Gordon soliton S-matrix allows one to apply immediately the results of the previous section. It follows from (3.71, b), (3.12) and (3.15) that

$$S_T(\theta) = -i \frac{\operatorname{sh}\left(\frac{8\pi}{\gamma}\theta\right)}{\sin\left(\frac{8\pi^2}{\gamma}\right)} S_R(\theta), \tag{4.8a}$$

$$S(\theta) = -i \frac{\operatorname{sh}\left(\frac{8\pi}{\gamma}(i\pi - \theta)\right)}{\sin\left(\frac{8\pi^2}{\gamma}\right)} S_R(\theta), \tag{4.8b}$$

where (with arbitrariness of the CDD-type (3.14) only)

$$S_R(\theta) = \frac{1}{\pi}\sin\left(\frac{8\pi^2}{\gamma}\right) U(\theta) \tag{4.9}$$

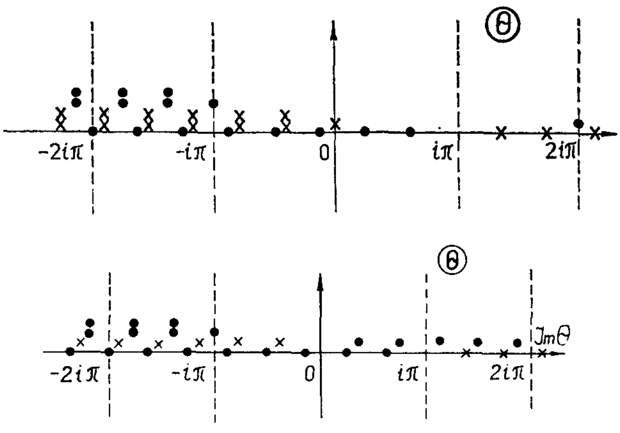


FIG. 5. The soliton-antisoliton scattering amplitudes. Location of poles (dots) and zeroes (crosses) in the θ-plane. (a) Transition amplitude $S_R(\theta)$. (b) Reflection amplitude $S_T(\theta)$. Some of the dots and crosses are displaced from imaginary axis for the sake of transparancy; actually all the singularities are at $R_e\theta = 0$.

and $U(\theta)$ is given by (3.16). The location of zeroes and poles of functions $S_T(\theta)$ and $S_R(\theta)$ (4.8), (4.9) is shown in Fig. 5. Note the equidistant (with separation $\gamma/8$) positions of $S_T(\theta)$ poles in the physical strip $0 < \operatorname{Im}\theta < \pi$. Such positions are in accord with the semiclassical mass spectrum (4.3). The correspondence is exact if

$$\gamma = \gamma'. \tag{4.10}$$

Therefore, the whole bound state spectrum (4.3) is already contained in the "minimum" solution (4.9), (3.16) and CDD poles need not be added. This solution automatically stisfies also another necessary requirement for the exact sine-Gordon S-matrix. If $\gamma = 8\pi(\beta^2 = 4\pi)$ the massive Thirring-model coupling vanished and the S-matrix should become unity. In fact when $\gamma = 8\pi$ one has from (4.8), (4.9) and (3.16)

$$S_T(\theta) = S(\theta) = 1; \qquad S_R(\theta) = 0.$$

These two remarkable properties of the "minimum" solution (together with its obvious aesthetic appeal) may serve as initial arguments to choose it as the exact S-matrix of quantum sin-Gordon solitons. We present below a number of checks which confirm such a choice.

If $\gamma \to 8\pi$ the formulas (4.8), (4.9) and (3.16) can be expanded in powers of $2g/\pi = 8\pi/\gamma - 1$ and the expansion coefficients can be compared with the results of diagrammatic calculus in massive Thirring model (1.2). Such a comparison has been carried out in [42] up to g^3 and the coincidence has been found.

Another check is a comparison with semiclassical formulas (4.1), (4.2). The semiclassical limit for Lagrangian (1.1) corresponds to $\beta^2 \to 0$. At θ fixed and $\beta^2 \to 0$ exact relation (4.8a) converts into semiclassical one (4.2). Furthermore, it can be easily verified that asymptotics of the exact amplitudes $S(\theta)$ and $S_T(\theta)$ as $\gamma \to 0$ coincide with (4.1). To do this one represents the exact $S_T(\theta)$ from (4.8a), (4.9) in the form

$$S_T(\theta) = \prod_{l=1}^{\infty} \frac{\Gamma\left(\frac{l\gamma}{16\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{l-1}{16\pi}\gamma - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{l\gamma}{16\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-\frac{1}{2} + \frac{l-1}{16\pi}\gamma - i\frac{\theta}{2\pi}\right)}$$
$$\times \frac{\Gamma\left(\frac{3}{2} + \frac{l\gamma}{16\pi} + i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{l-1}{16\pi}\gamma + i\frac{\theta}{2\pi}\right)}{\Gamma\left(1 + \frac{l\gamma}{16\pi} + i\frac{\theta}{2\pi}\right)\Gamma\left(1 + \frac{l-1}{16\pi}\gamma + i\frac{\theta}{2\pi}\right)} \qquad (4.11)$$

Changing in (4.4) the infinite product by a sum in the exponent and then replacing at $\gamma \to 0$ the summation by the integration one reproduced (4.1) exactly.

The larger the coupling parameter, the larger the mass of each bound state (4.3) (in units of the soliton mass). The n-th bound state acquires the soliton-antisoliton threshold when $\gamma = 8\pi/n$, and when $\gamma \geqslant 8\pi/n$ it disappears from the spectrum converting into the virtual state. At $\gamma \geqslant 8\pi$ all bound states (4.3) including the "elementary" particle of sine-Gordon Lagrangian (1.1) become unbound (remind that "elementary" particle is one of states (4.3), corresponding to $n = 1$ [15, 14]). Thus, at $\gamma \geqslant 8\pi$ the spectrum contains soliton and antisoliton only. The values $\gamma \geqslant 8\pi$ correspond to $g \leqslant 0$ in (1.2), i.e., to the repulsion between soliton and antisoliton.

Note that at $\gamma = 8\pi/n$ the reflection amplitude (4.9) vanishes identically (this property appears already in semi-classical formula (4.2)) while transition apmplitude $S_T(\theta)$ acquires, as a result of special cancellation of poles and zeroes in Fig. 5, a simple form:

$$S_T(\theta) = e^{in\pi} \prod_{k=1}^{n-1} \frac{e^{\theta - i(\pi k/n)} + 1}{e^{\theta} + e^{-i(\pi k/n)}} \tag{41.2}$$

This expression together with the hypothesis of its exact nature at $\gamma = 8\pi/n$ has been first presented by Korepin and Faddeev [14]. The general formulas (4.8), (4.9), (3.16) for arbitrary γ were given in [43][8].

Commutation rules (4.5) together with explicit expressions (4.8), (4.9) and (3.16) for the two-particle amplitudes represent S-matrix for an arbitrary number of solitons and antisolitions. To obtain the total sine-Gordon S-matrix one should supplement it with elements describing the scattering of any number of solitons and bound states (4.3). We denote the latter particles as B_n ; $n = 1, 2, ... < 8\pi/\gamma$.

Particles B_n with even (odd) values of n turn out to have positive (negative) C-parity. This can be seen if one considers the soliton-antisoliton ammplitudes with the definite s-channel C-parity:

$$S_+(\theta) = \tfrac{1}{2}[S_T(\theta) + S_R(\theta)], \tag{4.14a}$$

$$S_-(\theta) = \tfrac{1}{2}[S_T(\theta) - S_R(\theta)]. \tag{4.14b}$$

The amplitude $S_+(\theta)$ has even subset $n = 2, 4, ...$ of bound state poles $\theta = i\pi - in(\gamma/8)$ only, while $S_-(\theta)$ exhibits only odd subset $n = 1, 3, ...$ (these poles of $S_\pm(\theta)$ have positive residues as it should be). In particular, the sine-Gordon "elementary" particle B_1 is C-odd.

Since the particles B_n appear as poles of soliton-antisoliton amplitudes, an arbitrary S-matrix element involving these particles can be calculated as a residue of an appropriate multiparticle soliton amplitude. In terms of the algebraic formalism described in Section 2 algebra (3.5) of particles $A(\theta)$ and $\bar{A}(\theta)$ should be supplemented with new symbols $B_n(\theta)$; $n = 1, 2, ... < 8\pi/\gamma$ and commutation rules of B_n with A and $\bar{A}$ and of B_n with B_m should be specified. The procedure for the residue calculation mentioned above corresponds to the following definition of symbols $B_n(\theta)$ in terms of A and $\bar{A}$

$$B_n\left(\frac{\theta_1 + \theta_2}{2}\right) = \lim_{\theta_1 - \theta_2 \to i(n\gamma/8)} [A(\theta_2)\, \bar{A}(\theta_1) + \bar{A}(\theta_2)\, A(\theta_1)] \tag{4.15a}$$

for n even, and

$$B_n\left(\frac{\theta_1 + \theta_2}{2}\right) = \lim_{\theta_1 - \theta_2 \to i(n\gamma/8)} [A(\theta_2)\, \bar{A}(\theta_1) - \bar{A}(\theta_2)\, A(\theta_1)] \tag{4.15b}$$

for n odd.

[8] The derivation presented in [43] is based on certain special assumptions such as exactness of mass spectrum (4.3) and vanishing of reflection at $\gamma = 8\pi/n$. A derivation relying on the factorization equations and not referring to these assumptions was first given in [12].

This definition has a formal character and should be used to derive the rules for commutation of B_n with A and of B_n with B_m. Considering, for instance, in-product $A(\theta_1)\,\bar{A}(\theta_2)\,A(\theta_3)$, using (4.5) and taking the limit $\theta_1 - \theta_2 \to in(\gamma/8)$ by means of (4.15) one obtains

$$\begin{aligned} A(\theta_1)\,B_n(\theta_2) &= S^{(n)}(\theta_{12})\,B_n(\theta_2)\,A(\theta_1), \\ \bar{A}(\theta_1)\,B_n(\theta_2) &= S^{(n)}(\theta_{12})\,B_n(\theta_2)\,\bar{A}(\theta_1), \end{aligned} \tag{4.16}$$

where

$$S^{(n)}(\theta) = \frac{\operatorname{sh}\theta + i\cos\dfrac{n\gamma}{16}}{\operatorname{sh}\theta - i\cos\dfrac{n\gamma}{16}} \prod_{l=1}^{n-1} \frac{\sin^2\left(\dfrac{n-2l}{32}\gamma - \dfrac{\pi}{4} + i\dfrac{\theta}{2}\right)}{\sin^2\left(\dfrac{n-2l}{32}\gamma - \dfrac{\pi}{4} - i\dfrac{\theta}{2}\right)} \tag{4.17}$$

is the amplitude of two-particle scattering $A + B_n \to A + B_n$. Analogous consideration leads to commutation rules

$$B_n(\theta_1)\,B_m(\theta_2) = S^{(n,m)}(\theta_{12})\,B_m(\theta_2)\,B_n(\theta_1) \tag{4.18}$$

where $S^{(n,m)}(\theta)$ is the two-particle amplitude for $B_n + B_m \to B_n + B_m$ scattering. Its explicit form is

$$S^{(n,m)}(\theta) = \frac{\operatorname{sh}\theta + i\sin\left(\dfrac{n+m}{16}\gamma\right)}{\operatorname{sh}\theta - i\sin\left(\dfrac{n+m}{16}\gamma\right)}\;\frac{\operatorname{sh}\theta + i\sin\left(\dfrac{n-m}{16}\gamma\right)}{\operatorname{sh}\theta - i\sin\left(\dfrac{n-m}{16}\gamma\right)} \times \prod_{l=1}^{m-1} \frac{\sin^2\left(\dfrac{m-n-2l}{32}\gamma + i\dfrac{\theta}{2}\right)\cos^2\left(\dfrac{m+n-2l}{32}\gamma + i\dfrac{\theta}{2}\right)}{\sin^2\left(\dfrac{m-n-2l}{32}\gamma - i\dfrac{\theta}{2}\right)\cos^2\left(\dfrac{m+n-2l}{32}\gamma - i\dfrac{\theta}{2}\right)}; \quad n \geqslant m. \tag{4.19}$$

Amplitudes (4.17) and (4.18) turn out to be $2\pi i$-periodic functions of θ (in fact, this property is dictated by the cross-symmetry and the two-particle unitarity of $S^{(n)}(\theta)$ and $S^{(n,m)}(\theta)$). The location of poles and zeroes of these amplitudes is shown in Fig. 6. Note the set of double poles $\theta_l = i(\pi/2) + [(2l - n)/16]\,\gamma$; $l = 1, 2, \ldots, n - 1$ of $S^{(n)}(\theta)$ for $n \geqslant 2$; these "redundant" poles do not correspond to any bound states. Single poles $\theta = i(\pi/2) + in(\gamma/16)$ and $\theta = i(\pi/2) - in(\gamma/16)$ are the s-channel and u-channel soliton poles, respectively: in the s-plane these poles are at $s = m^2$ and $u = m^2$ (m is the soliton mass).

In amplitude $S^{(n,m)}(\theta)$ only the poles $\theta = i[(n+m)/16]\,\gamma$ and $\theta = i\pi - i[(n+m)/16]\,\gamma$ correspond to the real particle B_{n+m}, all other poles are redundant. The appearance of poles B_{n+m} in the amplitude $S^{(n,m)}(\theta)$ allows one to interpret

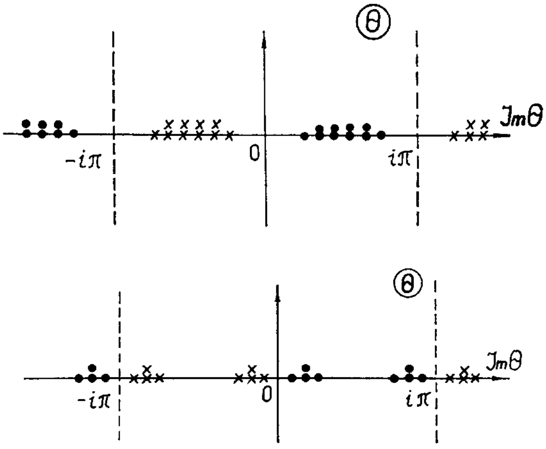


FIG. 6. Poles and zeroes of the soliton-bound state and the bound state — bound state scattering amplitudes. (a) $S^{(n)}(\theta)$ amplitude for $n = 5$. (b) $S^{(n,m)}(\theta)$ amplitude for $n = 4$, $m = 2$.

any particle B_l for $l \geqslant 2$ as a bound state $B_n + B_m$ with $n + m = l$[9] and, consequently, to interpret B_l as a bound state of l "elementary" particles B_1. A possibility of such interpretation was mentioned in Ref. [15].

In the case $m = n = 1$ Eq. (4.19) gives the two-particle amplitude of "elementary" particles

$$S^{(1,1)}(\theta) = \frac{\text{sh}\,\theta + i\sin(\gamma/8)}{\text{sh}\,\theta - i\sin(\gamma/8)}. \tag{4.20}$$

This expression can be expanded in powers of β^2 and compared with β^2-perturbation theory results for Lagrangian (1.1). Formula (4.20) together with its perturbation verification was presented in [8–10] as a solution of analyticity and unitarity for particles B_1.

Formulas (4.16–4.19) solve the bound state problem of the sine-Gordon model. Together with (4.5), (4.8–4.10) and (3.16) they represent the total quantum sine-Gordon S-matrix.

To conclude this section let us consider the nonrelativistic version of $O(2)$-symmetric S-matrix. After substitution (3.11) the general solution of the factorization equations becomes, instead of (4.8), (see Appendix A)

[9] One can verify that definition

$$B_{n+m}\left(\frac{\theta_1 + \theta_2}{2}\right) = \lim_{\theta_1 - \theta_2 \to i[(n+m/16)]\gamma} B_m(\theta_2) B_n(\theta_1)$$

is consistent with (4.14) and completely self-consistent [31].

$$S_T(k) = -i \frac{\operatorname{sh}\left(\frac{8\pi k}{\gamma m}\right)}{\sin(\pi\kappa)} S_R(k), \tag{4.21a}$$

$$S(\kappa) = -i \frac{\operatorname{sh}\left(i\pi\kappa - \frac{8\pi k}{\gamma m}\right)}{\sin(\pi\kappa)} S_R(k), \tag{4.21b}$$

where $\kappa = 8\delta/\gamma$. Since we do not require any crossing-symmetry now, γ and κ are independent parameters. The unitarity condition, then, gives the following equation

$$S_R(k)\, S_R(-k) = \frac{\sin^2(\pi\kappa)}{\operatorname{sh}^2\left(\frac{8\pi k}{\gamma m}\right) + \sin^2(\pi\kappa)}. \tag{4.22}$$

The "minimum" solution of (4.22) is

$$S_R(k) = \frac{\sin(\pi\kappa)}{i \operatorname{sh}\left(\frac{8\pi k}{\gamma m}\right)} \frac{\Gamma\left(-i\frac{8k}{\gamma m} - \kappa\right) \Gamma\left(-i\frac{8k}{\gamma m} + \kappa + 1\right)}{\Gamma\left(-i\frac{8k}{\gamma m}\right) \Gamma\left(1 - i\frac{8k}{\gamma m}\right)}. \tag{4.23}$$

Formulas (4.21–4.23) clearly give the nonrelativistic limit of the sine-Gordon soliton S-matrix. Furthermore, amplitudes (4.23) and (4.21a) are just reflection and transition ones for the scattering on the potential

$$V_{A\bar{A}}(x) = -\frac{m}{64} \frac{\gamma^2 G}{\operatorname{ch}^2\left(\frac{m\gamma}{8} x\right)} \tag{4.24}$$

while amplitude (4.21b) describes the scattering on the potential

$$V_{AA}(x) = \frac{m}{64} \frac{\gamma^2 G}{\operatorname{sh}^2\left(\frac{m\gamma}{8} x\right)} \tag{4.25}$$

where $G = \kappa^2 - \kappa + \frac{3}{4}$. It is known that a system of $N + M$ nonrelativistic particles of two different kinds described by the Hamiltonian

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} - \frac{1}{2m} \sum_{j=1}^{M} \frac{d^2}{dy_j^2} + \sum_{i<i'}^{N} 2V_{AA}(x_i - x_{i'})$$
$$+ \sum_{j<j'}^{M} 2V_{AA}(y_j - y_{j'}) + \sum_{i=1}^{N} \sum_{j=1}^{M} 2V_{A\bar{A}}(x_i - y_j) \tag{4.26}$$

is completely integrable, i.e., it possesses an infinite number of conservation laws, and its S-matrix is factorized [44, 45]. So, system (4.26) describes just the nonrelativistic dynamics of quantum sine-Gordon solitons. The analogy between the sine-Gordon soliton scattering and the one on potential $\sim -1/ch^2 x$ was noticed in Ref. [38].

5. $O(N)$-Symmetric Nonlinear σ-Model with $N \geqslant 3$

Now we consider the problem what type of quantum field theory can serve as a dynamical background of the $O(N)$-symmetric factorized S-matrix of Section 3 with $N \geqslant 3$.

First of all, remind the essential difference between a general solution of Section 3 with $N \geqslant 3$ and that with $N = 2$. For $N = 2$ the solution depends on a free parameter which could be interpreted as a coupling constant (in particular, a "weak coupling regime" can be achieved by a special choice of this parameter), while for $N \geqslant 3$ no free parameter enters the solution (the only umbiguity is (3.14)), but it depends analytically on the symmetry group rank N[10] and can be expanded in powers of $1/N$.

The coupling constant independence of all observable properties but the overall mass scale is the phenomenon characteristic for asymptotically free theories with dynamical mass transmutation (actually, it is a consequence of renormalizability [28]). One can believe, therefore, that a theory of this very type describes the dynamics of factorized scattering of Section 3 with $N \geqslant 3$.

To ensure the fact that a certain quantum field theory really leads to the S-matrix of Section 3 one should reveal its following properties:

(a) The model is one of the massive particles with $O(N)$-symmetric spectrum. The spectrum contains the isovector N-plet.

(b) The total S-matrix of this theory is factorized.

If these properties turn out to be true the problem of CDD-umbiguity (3.14) should be solved to obtain an exact S-matrix of the theory.

Consider $O(N)$-symmetric chiral field model (1.4). The usual g-perturbation theory of (1.4) is based on the Goldstone vacuum and leads in two dimensions to infrared catastrophe. Therefore, it is unlikely applicable to elucidate observable properties such as the spectrum and the S-matrix. However, there is another powerful approach to this model, namely, the $1/N$-expansion. This method for the model (1.4) has been developed in Refs. [23, 24] (see Appendix B). It is based on the exact solution at $N \to \infty$ which obviously satisfies the requirement (a): at $N \to \infty$ the model contains an isovector N-plet of free massive particles only.[11] The interaction of these particles is of the order of $1/N$ and $1/N$ expansion is just the perturbation theory in this interaction. The property (a) is still valid in any order of this perturbation theory. Thus, it is obviously true as N is sufficiently large. It is not ultimately clear whether the

[10] In this case the weak coupling limit is achieved as $N \to \infty$.

[11] The absence of other particles at large N is shown in [48].

situation is the same if N becomes not large, say $N = 3, 4$. However, we shall assume that the situation characteristic of large N is still valid at all $N \geqslant 3$. The result of this section confirms this assumption to some extent.

Let us turn to the scattering properties of this model. The existence of an infinite number of conservation laws for classical model (1.4) has been discovered by Pohlmeyer [47]. However, since the quantum vacuum of the model appears to be crucially different from the classical one, the relation between the classical conservation laws and quantum ones cannot be straightforward. In particular, the conformal invariance of the classical theory which is of essential use in Pohlmeyer's derivation is surely broken in a quantum case due to coupling constant renormalization.

The presence of higher conservation laws in quantum model (1.4) has been shown by Polyakov [26]. Here we present briefly Polyakov's derivation.

The equations of motion corresponding to Lagrangian (1.4) are:

$$n^i_{,\sigma\tau} + \omega n^i = 0; \qquad \sum_{i=1}^{N} (n^i)^2 = 1, \tag{5.1}$$

where ω is a Lagrange field (see Appendix B) and the indices σ and τ mean derivation with respect to

$$\sigma = x^0 + x^1; \qquad \tau = x^0 - x^1. \tag{5.2}$$

Equations (5.1) in the classical theory imply

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right]_{\tau} = \left[\sum_{i=1}^{N} (n^i_{,\tau})^2\right]_{\sigma} = 0. \tag{5.3}$$

These equations, which are of essential use in Pohlmeyer's derivation [47], mean both the conservation of energy-momentum and the conformal invariance of classical theory. In the quantum case the conformal symmetry is broken by stress-energy tensor anomaly and instead of (5.3) one has

$$\begin{aligned} \left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right]_{\tau} &= b\omega_{,\sigma}, \\ \left[\sum_{i=1}^{N} (n^i_{,\tau})^2\right]_{\sigma} &= b\omega_{,\tau}, \end{aligned} \tag{5.4}$$

where b is a constant which can be easily related to the Gell–Mann–Low function. Of course, relations (5.4) imply the energy-momentum conservation in the quantum theory since they are just of the divergence-zero type.

To obtain the next conservation law let us consider, following Polyakov, the derivatives $[\sum_{i=1}^{N} (n^i_{,\sigma\sigma})^2]_{\tau}$ and $[\sum_{i=1}^{N} (n^i_{,\tau\tau})^2]_{\sigma}$. In a classical theory one has, for instance,

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma})^2\right]_{\tau} = -\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2 \,\omega\right]_{\sigma} + 3 \sum_{i=1}^{N} (n^i_{,\sigma})^2 \,\omega_{,\sigma}. \tag{5.5}$$

In the quantum case this relation is deformed by anomalies. It is easy to see that the most general quantum variant of (5.5) is

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma})^2\right]_\tau = (3 + \alpha)\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right] \omega_{,\sigma} + (\cdots)_\sigma\,, \tag{5.6}$$

where the term proportional to α is the anomaly. Furthermore, one can consider the derivative $[\sum_{i=1}^{N} (n^i_{,\sigma})^2]_\tau$ and obtain in a quantum case, quite analogously to (5.6),

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right]_\tau = (2b + \alpha')\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right] \omega_{,\sigma} + (\cdots)_\sigma\,. \tag{5.7}$$

Now it is straightforward to construct a new conservation law

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma})^2 - \frac{3 + \alpha}{2b + \alpha'}\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right]^2\right]_{,\tau} = (\cdots)_{,\sigma}\,. \tag{5.8}$$

No difficulties arise in constructing a further conservation law. The equation

$$\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma\sigma})^2 + a_1\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma})^2\right]\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right] + a_2\left[\sum_{i=1}^{N} (n^i_{,\sigma})^2\right]^3 + a_3\left[\sum_{i=1}^{N} (n^i_{,\sigma\sigma} n^i_{,\sigma})\right]^2\right]_{,\tau} = (\cdots)_{,\sigma} \tag{5.9}$$

can be satisfied, in the same manner as (5.8), by an appropriate choice of parameters a_1, a_2 and a_3. Higher conservation laws of the infinite set require more delicate investigation; they have been constructed in a recent paper [27]. We do not discuss all the infinite set here. As shown in [26], the first two conservation laws (5.4) and (5.8) are already sufficient to restrict the S-matrix to the processes satisfying the selection rules (i) and (ii) pointed in Section 2. According to the general consideration of Section 2, this implies the S-matrix factorization for model (1.4).

It is instructive to observe the last property of the chiral field S-matrix in the $1/N$-perturbation theory [25]. Let us do this in the order of $1/N^2$ (in the order of $1/N$ these properties are trivial, being determined by kinematics). In this order we are

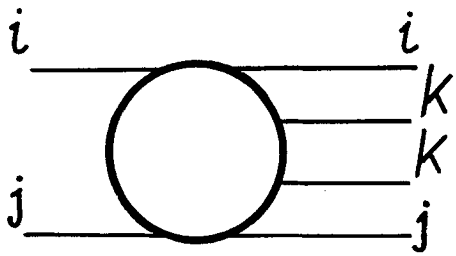


FIG. 7. The $2 \to 4$ amplitude.

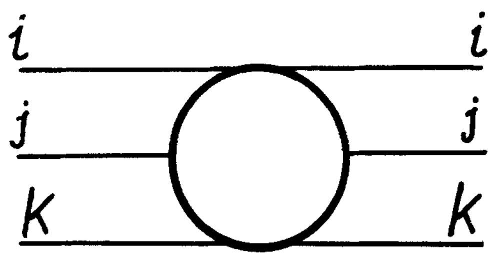


FIG. 8. The 3 → 3 amplitude.

interested in the amplitudes 2 → 4 (Fig. 7) and in connected amplitudes 3 → 3 (Fig. 8).

Using the diagrammatic technique of $1/N$-expansion (see Appendix B) one can represent the amplitudes 2 → 4 by a sum of diagrams shown in Fig. 9 (we consider only the case $i \neq j \neq k \neq i$ for simplicity; a general case includes more diagrams, but the result is the same). To demonstrate the total cancellation among the diagrams in Fig. 9 it is useful to take into account the explicit expression for arbitrary two dimensional one-loop diagram [8]. The extession is shown schemtatically in Fig. 10: an arbitrary boson loop is the sum of terms, each corresponding to any division of the

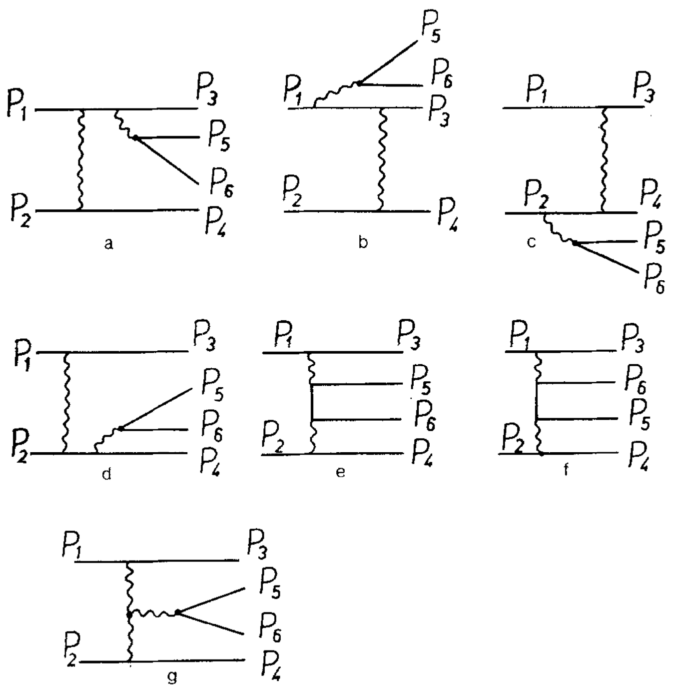


FIG. 9. Diagrams of the order of $1/N^2$ contributing to the amplitude in Fig. 7 in the case $i \neq j \neq k \neq i$.

loop through two lines. The contribution of each division is equal to the product of two "tree" diagrams separated by a dashed line in Fig. 10 by the function

$$i\Phi(s_{ab}) = \frac{1}{(2\pi)^2} \int \frac{d^2p}{(p^2 - m^2 + i\epsilon)((p + k_a + k_b)^2 - m^2 + i\epsilon)} \tag{5.10}$$

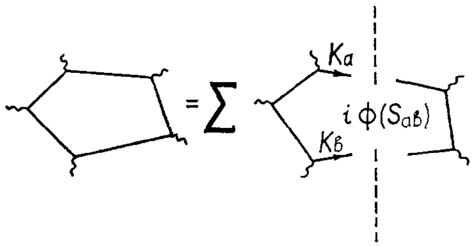


FIG. 10. The "division rule" for calculation of an arbitrary one-loop diagram.

where $s_{ab} = (k_a + k_b)^2$, the momenta k_a and k_b of cut lines being determined by the condition $K_a^2 = k_b^2 = m^2$. At s_{ab} fixed this equation has two solutions connected by the exchange $k_a \leftrightarrow k_b$, both should be taken into account in Fig. 10.

It is easy to see that all possible divisions of triangle loop in diagram in Fig. 9g cancel exactly the other diagrams in Fig. 9. Consider, for example, the division shown in Fig. 11. Two solutions of equation $k_1^2 = k_2^2 = m^2$ are $k_1 = p_5$; $k_2 = p_6$ and $k_1 = p_6$; $k_2 = p_5$. The factor $i\Phi(s_{56})$ in this division is just $-1/D(s_{56})$, where D is a ω' field propagator (wavy line). Therefore, the division in Fig. 11 cancels diagrams in Fig. 9e and Fig. 9f. The other divisions of the loop cancel diagrams in Fig. 9a–d.

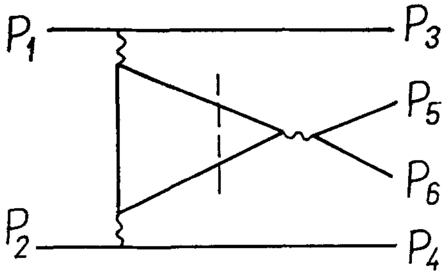


FIG. 11. A division of the three ω' vertex in Fig. 9g.

The factorization of connected amplitudes 3 → 3 in the order of $1/N^2$ can be shown analogously. Corresponding diagrams are listed in Fig. 12 (we again consider the the case $i \neq j \neq k \neq i$ only). It is easily seen, that in the kinematical regions, where the solid-line propagators in diagrams in Fig. 12a–f are nonsingular, all possible divisions of the loop in Fig. 12g cancel out contributions of other diagrams. Mass-shell singularities of the solid-line propagators in Fig. 12a–f require special consideration. For example, if $p_1' \to p_3$, $p_2' \to p_1$, $p_3' \to p_2$ the intermediate solid lines of diagrams

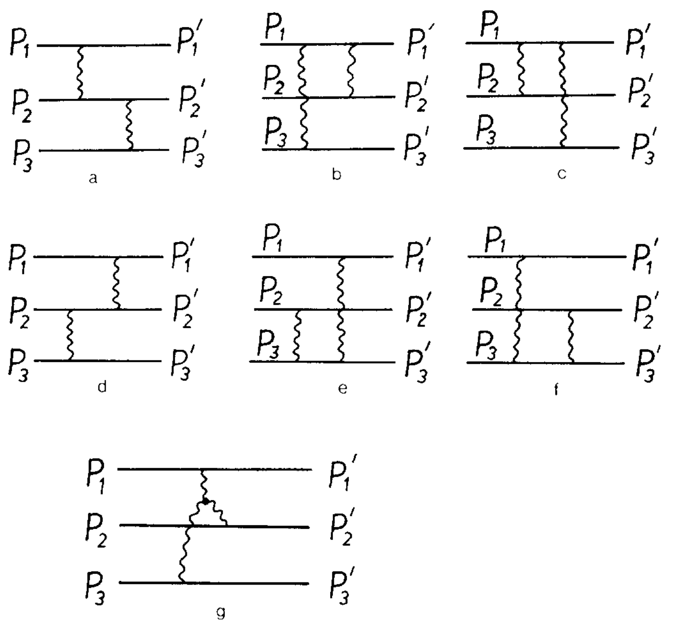


FIG. 12. Diagrams of the order of $1/N^2$ contributing to the amplitude in Fig. 8 in the case $i \neq j \neq k \neq i$.

in Figs. 12c, d and f acquire their mass shell poles. These propagators can be written in the form

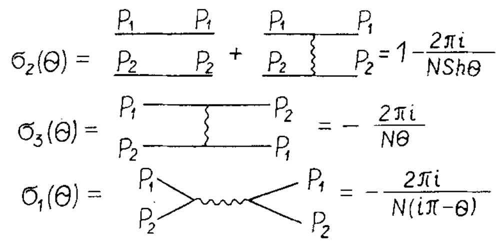
$$\frac{1}{p^2 - m^2 + i\epsilon} = P \frac{1}{p^2 - m^2} + i\pi\, \delta(p^2 - m^2) \tag{5.11}$$

It can be shown that the principal parts of these singularities are cancelled among three diagrams in Figs. 12c, d and f, leaving the δ-function terms only, corresponding to intermediate particles on the mass shell. The diagram 12g cannot cancel these δ-function terms, being nonsingular in the region under consideration (where all the momenta transferred are space-like). So, only the terms with mass-shell δ-functions remain in the sum of diagrams in Fig. 12. These δ-functions ensure the factorized structure of the amplitude in Fig. 8.

We have shown that quantum model (1.4) satisfies properties (a) and (b). Hence, one of the solutions (3.8), (3.13), (3.17) specified by a choice of CDD poles (3.14) can be used to describe the scattering of this model. We do not know any way to remove the CDD-ambiguity rigorously, but the choice of "minimum" solution, namely $\sigma_2(\theta) = \sigma_2^{(+)}(\theta)$ from (3.17), appears to be the most natural. Below we present some arguments in support of this choice.

At first let us note that CDD-poles (3.14), if added, in general, result in additional poles in all three channels of two-particle scattering: isoscalar, antisymmetric-tensor and symmetric tensor[12]. Such a strong isospin degenerasy of states seems to be unnatural. The "minimum" solution $\sigma_2(\theta) = \sigma_2^{(+)}(\theta)$ possess no poles in the physical strip $0 < \text{Im}\,\theta < \pi$ and therefore "elementary" isovector particles A_i of (1.4) produce no bound states.

Furthermore, a calculation of two-particle amplitudes for model (1.4) by $1/N$-expansion technique (see Appendix B) in the order of $1/N$ leads to the result:



(5.12)

The signs in (5.12) mean that the interaction between A_i is of repulsive type (at least for large N). Hence, A_i is unlikely to form bound states. It is easy to verify that expressions (5.12) really coincide with the first terms of $1/N$-expansion of exact solution (3.8), (3.13), (3.17) with $\sigma_2(\theta) = \sigma_2^{(+)}(\theta)$. Thus, the latter choice is in accordance with $1/N$-expansion of (1.4).

It is interesting to compare the solution of Section 3 with the results of the ordinary g-perturbation theory of model (1.4). Adopting the S-matrix (3.8), (3.17) to correspond to some renormalizable asymptotically free field theory, one can expand the scattering amplitudes, which are the functions of variable

$$\ln \frac{s}{m^2} = \ln \frac{s}{\mu^2} + \int^{g(\mu)} \frac{dg}{\beta(g)} \tag{5.13}$$

in the asymptotic series in powers of $g(\mu)$. Using the first term of σ-model Gell–Mann–Low function $\beta(g)$ [20]

$$\beta(g) = -\frac{N-2}{4\pi} g^2 + O(g^3) \tag{5.14}$$

one obtains up to g^2 ($g \equiv g(\mu)$)

$$\begin{aligned} \sigma_2(s) &= 1 - i\frac{g^2}{8} + O(g^3), \\ \sigma_3(s) &= -i\frac{g}{2} + i\frac{N-2}{8\pi} g^2 \ln\frac{s}{\mu^2} + O(g^3), \\ \sigma_1(s) &= i\frac{g}{2} - i\frac{N-2}{8\pi} g^2 \ln\frac{s}{\mu^2} - \frac{N-2}{8} g^2 + O(g^3). \end{aligned} \tag{5.15}$$

[12] The only exceptional case is that of single CDD-pole $\alpha_1 = \lambda$ added to $\sigma_2^{(+)}(\theta)$, where bound states appear in isoscalar and antisymmetric tensor channels only. This case corresponds to $\sigma_2(\theta) = \sigma_2^{(-)}(\theta)$ (see (3.19)) and is under consideration in the further Section.

In (5.15) the asymptotics $s \to \infty$ is written down and power terms in s are dropped.

The usual perturbation theory (expansion in g) is based on Goldstone vacuum and deals with $N - 1$ plet of Goldstone particles instead of N-plet of massive particles A_i. In this perturbation theory the loop diagram calculation leads to infrared divergencies. However, all the infrared divergencies are cancelled among the diagrams contributing to the scattering amplitude of Goldstone particles in the order of g^2. Hence, one can believe that calculation of these diagrams results in the correct ultraviolet asymptotics of the reeal A_i particle scattering amplitude. This calculation is straightforward and does lead to (5.15).

6. S-MATRIX OF THE GROSS-NEVEU "ELEMENTARY" FERMIONS

Another example of an asymptotically free field theory exhibiting the properties (a) and (b) of Section 5 and therefore leading to the factorized S-matrix of Section 3 with $N \geqslant 3$ is Gross-Neveu model (1.5).

An infinite set of nontrivial conservation laws for a classical version of model (1.5) has been found in a recent paper [49]. These classical conservation laws are quite analogous in their structure to those of the nonlinear σ-model found by Pohlmeyer. Again the conformal invariance of the classical theory (1.5) (which is broken in a quantum case) plays the crucial role in the derivation of these conserved currents. However, it is natural to expect that higher conservation laws are present in quantum theory (1.5) as well. Dashen, Hasslacher and Neveu [29] have investigated the classical field equation which determine the stationary phase points of the effective action (B.5′) (see Appendix B). They have been able to find out explicitly a series of time-dependent solutions. It means almost surely the complete integrability of the system determined by these equations.

To make sure that quantum theory (1.5) really possesses higher conservation laws let us derive the first nontrivial law following Polyakov's method [26]. All the considerations will be quite parallel to those applied in Section 5 to the case of the σ-model.

It is convenient to use the motion equations of (1.5) explicitly in terms of right and left-handed components of Majorana "bispinors" $\psi_i(x) = (\psi_i^r(x), \psi_i^l(x))$

$$\begin{aligned} i\psi^r_{i,\tau} &= \omega\psi_i^l, \\ i\psi^l_{i,\sigma} &= -\omega\psi_i^r; \qquad i = 1, 2,\ldots, N, \end{aligned} \tag{6.1}$$

where $\omega = g_0 \sum_{i=1}^N \psi_i^r\psi_i^l$. The momentum-energy conservation and conformal invariance of equations in classical case imply analogously to (5.3)

$$\left(\sum_{i=1}^N \psi_i^r\psi^r_{i,\sigma}\right)_{,\tau} = \left(\sum_{i=1}^N \psi_i^l\psi^l_{i,\tau}\right)_{,\sigma} = 0 \tag{6.2}$$

which should be, of course, replaced in the quantum theory

$$\left(\sum_{i=1}^{N} \psi_i{}^r \psi^r_{i,\sigma}\right)_{,\tau} = -b\omega^2_{,\sigma}\,,$$
$$\left(\sum_{i=1}^{N} \psi_i{}^l \psi^l_{i,\tau}\right)_{,\sigma} = -b\omega^2_{,\tau}\,. \tag{6.3}$$

Using these quantum equations one can easily verify that the following equations can be satisfied by an appropriate choice of the parameter C

$$\left[\sum_{i=1}^{N} \psi^r_{i,\sigma}\psi^r_{i,\sigma\sigma} + C\left(\sum_{i=1}^{N} \psi_i{}^r\psi^r_{i,\sigma}\right)^2\right]_{,\tau} = (\cdots)_{,\sigma}\,,$$
$$\left[\sum_{i=1}^{N} \psi^l_{i,\tau}\psi^l_{i,\tau\tau} + C\left(\sum_{i=1}^{N} \psi_i{}^l\psi^l_{i,\tau}\right)^2\right]_{,\sigma} = (\cdots)_{,\tau}\,. \tag{6.4}$$

These equations are just the first nontrivial conservation law of the Gross–Neveu model.

The existence of higher conservation laws implies that the Gross–Neveu S-matrix satisfies property (b) of Section 5 [26].

Alternatively one could discover this property of the Gross–Neveu S-matrix in $1/N$-expansion [50]. The $1/N$-expansion technique for this model has been developed in Ref. [28] (it is described briefly in Appendix B). It is similar in the main to that used in the case of the nonlinear σ-model. In particular, the diagrammatic consideration of previous section can be repeated word for word in the Gross–Neveu case.

There is an important difference, however, between the σ-model $1/N$ technique and that of the Gross–Neveu model. Note the additional minus sign in (B.9′) against (B.9) which is connected with the fermion nature of ψ_i-fields (see Appendix B). This leads to the significant difference between two scattering theories. For instance, for model (1.5) one has, instead of (5.12),

$$\sigma_2^{GN}(\theta) = 1 + \frac{2\pi i}{N\,\mathrm{sh}\,\theta} + O\left(\frac{1}{N^2}\right);$$
$$\sigma_3^{GN}(\theta) = \frac{2\pi i}{N\theta} + O\left(\frac{1}{N^2}\right); \qquad \sigma_1^{GN}(\theta) = \frac{2\pi i}{N(i\pi-\theta)} + O\left(\frac{1}{N^2}\right). \tag{6.5}$$

The signs in Eq. (6.5) correspond to attractive interaction of "elementary" fermions (which we denote again by A_i; $i = 1, 2,\ldots, N$). Therefore, bound states of A_i should exist.

The bound state problem in model (1.5) has been investigated by a semiclassical large N method in Ref. [29]. The rich spectrum of $O(N)$-multiplets of bound states has been found. There are isoscalar, isovector multiplets and a number of higher rank antisymmetric-tensor ones. The semiclassical spectrum possesses a strong isospin degenracy: different isospin multiplets are gathered into supermultiplets defined by the "main quantum number" n which corresponds to the number of bounded "elementary" fermions. Semiclassical masses depend on this number only

$$m_n^{(\mathrm{sem})} = m \frac{\sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}; \qquad n = 1, 2, \ldots < \frac{N}{2}, \tag{6.6}$$

where m is the mass of A_i.

There are other particles apart from A_i—bound states—"kinks" of the $\omega(x)$ field [29]. Their semiclassical masses are

$$M_{\mathrm{kink}} = \frac{m}{2 \sin\left(\frac{\pi}{N}\right)} \tag{6.7}$$

and the existence of these "kinks" is just an explanation of the upper bound for n in Eq. (6.6).

The qualitative structure of semiclassical bound state spectrum (which becomes exact as $N \to \infty$) makes one choose solution (3.8), (3.13), (3.17) with $\sigma_2(\theta) = \sigma_2^{(-)}(\theta)$ as an exact S-matrix of "elementary" Gross–Neveu fermions. The $1/N$-expansion of this solution turns out to coincide with (6.5). Furthermore, let us consider two particle amplitudes of A_i-scattering with the definite s-channel isospin

$$\sigma_{(\mathrm{isocalar})} = N\sigma_1 + \sigma_2 + \sigma_3 = -\frac{(\theta + i\lambda)(\theta + i\pi)}{\theta(i\pi - \theta)} \sigma_2^{(-)}(\theta), \tag{6.8a}$$

$$\sigma_{(\mathrm{antisymm})} = \sigma_2 - \sigma_3 = \frac{\theta + i\lambda}{\theta} \sigma_2^{(-)}(\theta), \tag{6.8b}$$

$$\sigma_{(\mathrm{symm})} = \sigma_2 + \sigma_3 = \frac{\theta - i\lambda}{\theta} \sigma_2^{(-)}(\theta). \tag{6.8c}$$

As it is seen from (6.8), bound states exist only in isoscalar and antisymmetric-tensor channels. We denote these particles B and B_{ij}. Their masses are

$$m_B = m_{B_{kj}} \equiv m_2 = m \sin\left(\frac{2\pi}{N-2}\right)\left[\sin\left(\frac{\pi}{N-2}\right)\right]^{-1}. \tag{6.9}$$

Higher bound states appear as poles in multiparticle amplitudes. The investigation of these poles (quite parallel to that made in Ref. [10] for the bound states of sine-Gordon "elementary" particles) leads to the spectrum of multiparticle A_i-bound states which agrees qualitatively with the semiclassical spectrum of Dashen, Hasslacher and Neveu [29]. Semiclassical isospin degeneracy turns out to be exact while the exact mass formula is

$$m_n = m \sin\left(\frac{\pi n}{N-2}\right)\left[\sin\left(\frac{\pi}{N-2}\right)\right]^{-1}; \qquad n = 1, 2, \ldots < \frac{N-2}{2}, \tag{6.10}$$

which differs from the semiclassical one (6.6) by substitution $N \to N-2$ only. It is natural to suppose that all the qualitative picture of semiclassical spectrum remains unchanged in the exact solution provided the substituion $N \to N-2$ is made. In particular, there are "kink" particles at any N and formula (6.10) in terms of kink mass becomes

$$m_n = 2M_{\text{kink}} \sin\left(\frac{\pi n}{N-2}\right); \qquad n = 1, 2, \ldots < \frac{N-2}{2}. \tag{6.11}$$

It follows from Eq. (6.11) that the bound states subsequently disappear from the spectrum with the decrease of N and no particles but the "kinks" remain in the system at $N \leqslant 4$. In particular, there are no "fundamental" fermions A_i at $N = 3, 4$. Therefore, the exact S-matrix of Gross–Neveu "fundamental" fermions presented above has a direct physical meaning at $N > 4$ only being quite fictitious at $N = 3, 4$.

To construct the total Gross–Neveu S-matrix for any $N \geqslant 3$ one should calculate the factorized S-matrix for the "kinks". The essential problem arises in this way: what representation of internal symmetry group do the "kinks" belong to? There are some arguments that these particles form $O(N)$ isospinor multiplets (E. Witten, Private communication). In any event the problem of obtaining the total Gross–Neveu S-matrix remains open.

For the further development of the subject reviewed in this paper see Refs. [51–54].

APPENDIX A

In this Appendix we derive solutions of Eqs. (3.5) and (3.6) [12, 25].

(1) Consider system (3.5). It is convenient to introduce the ratios

$$h(\theta) = \frac{S_2(\theta)}{S_3(\theta)}; \qquad g(\theta) = \frac{S_1(\theta)}{S_3(\theta)}. \tag{A.1}$$

Then Eqs. (3.5a, b) become

$$\begin{aligned} h(\theta) + h(\theta') - h(\theta+\theta') &= g(\theta')\, h(\theta+\theta') \\ &\quad - h(\theta')\, g(\theta+\theta') + g(\theta')\, g(\theta+\theta')\, h(\theta), \end{aligned} \tag{A.2}$$

$$\begin{aligned} &[1 + h(\theta+\theta') + g(\theta+\theta')][1 - g(\theta)\, g(\theta')] + h(\theta)\, h(\theta') \\ &\quad = (1 + g(\theta) + h(\theta))(1 + g(\theta') + h(\theta')). \end{aligned} \tag{A.3}$$

Substituting $\theta = 0$ or $\theta' = 0$ into (A.3) and (A.2) one obtains the following relations

$$\begin{aligned} [1 - g^2(\theta)]\, h(0) &= 0, \\ [1 + g(\theta)][(1 + g(\theta) + h(\theta))\, g(0) + h(0)] &= 0, \\ [1 + g(\theta)][h(0) - g(0)\, h(\theta)] &= 0. \end{aligned}$$

These equations can be satisfied in three possible ways: (a) $g(\theta) \equiv 1$; $h(\theta) \equiv -1$ (b) $g(\theta) \equiv -1$; $h(\theta)$ is arbitrary, (c) $g(0) = h(0) = 0$. The first two cases are not interesting for us since possibility (a) is against unitarity (3.4b) and possibility (b) cannot satisfy crossing relation (3.2b). Therefore,

$$g(0) = h(0) = 0. \tag{A.4}$$

Differentiating (A.2) and (A.3) with respect to θ' and then setting $\theta' = 0$ one gets

$$h'(\theta) = (1 + g(\theta)((\alpha - \beta h(\theta)); \tag{A.5}$$

$$h'(\theta) + g'(\theta) = (1 + g(\theta))[\alpha + \beta h(\theta) + \beta(1 + g(\theta))], \tag{A.6}$$

where $\alpha = h'(0)$ and $\beta = g'(0)$. These equations can be easily turned to the form

$$g(\theta) = \beta h(\theta) \frac{h(\theta) + 1}{\alpha - \beta h(\theta)}; \tag{A.7}$$

$$h'(\theta) = \beta h^2(\theta) + \alpha. \tag{A.8}$$

The solution of (A.8) and (A.7) is

$$h(\theta) = -i \operatorname{tg}\left(\frac{4\pi\delta}{\gamma}\right) \operatorname{th}\left(\frac{4\pi\theta}{\gamma}\right); \tag{A.9}$$

$$g(\theta) = \operatorname{th}\left(\frac{4\pi\theta}{\gamma}\right) \operatorname{cth}\left[\frac{4\pi}{\gamma}(i\delta - \theta)\right], \tag{A.10}$$

where $\alpha = -i(4\pi/\gamma)\operatorname{tg}(4\pi\delta/\gamma)$; $\beta = -i(4\pi/\gamma)\operatorname{ctg}(4\pi\delta/\gamma)$. The real analyticity condition for the scattering amplitudes requires $h(\theta)$ and $g(\theta)$ to be real at $\operatorname{Re}\theta = 0$. Hence, γ and δ are real parameters. Formulas (A.9) and (A.10) are equivalent to (3.7).

(2) Let us turn to the system (3.6). Using the notation $h(\theta) = \sigma_2(\theta)/\sigma_3(\theta)$ one reduces (3.6a) to the form

$$h(\theta) + h(\theta') = h(\theta + \theta'). \tag{A.11}$$

Hence

$$\sigma_3(\theta) = -i\frac{\lambda}{\theta}\sigma_2(\theta), \tag{A.12}$$

where λ is an arbitrary real parameter. Substitution of (A.12) into (3.6b) leads to the following equation

$$\rho(\theta + \theta')\,\rho(\theta') = \frac{i\lambda}{\theta}[\rho(\theta') - \rho(\theta + \theta')], \tag{A.13}$$

where $\rho(\theta) = \sigma_1(\theta)/\sigma_2(\theta)$. The solution of (A.13) is

$$\rho(\theta) = -\frac{i\lambda}{i\kappa - \theta}, \tag{A.14}$$

where κ is the other real parameter. Now Eq. (3.6c) leads to the restriction

$$\kappa = i\frac{N-2}{2}\lambda \tag{A.15}$$

and we obtain (3.8).

Appendix B

This Appendix is intended for the derivation of the diagrammatic technique of $1/N$ expansion of models (1.4) and (1.5) [23, 24, 28].

All the following calculations will be performed for both models (1.4) and (1.5) simultaneously. To avoid any confusion, relative variables corresponding to models (1.4) and (1.5) are marked by subindices *CF* (chiral field) and *GN* (Gross–Neveu). Furthermore, all the numbers of formulas relating to model (1.5) are primed.

Following [23, 28] introduce the auxiliary Lagrange field and write

$$\mathscr{L}'_{CF} = \frac{1}{2g_0}\sum_{i=1}^{N}[(\partial_\mu n_i)^2 + \omega(x)\, n_i^2] - \frac{\omega(x)}{2g_0}; \tag{B.1}$$

$$\mathscr{L}'_{GN} = \frac{1}{2}\sum_{i=1}^{N}[i\bar{\psi}_i\, \partial_\mu\gamma^\mu\psi_i - \omega(x)\, \bar{\psi}_i\psi_i] - \frac{\omega^2(x)}{2g_0}. \tag{B.1$'$}$$

The generating functional for the Green functions of the field $n_i(x)(\psi_i(x)$ in the case of (1.5)) can be written in the form

$$Z[J] = I[J]/I[0], \tag{B.2}$$

where

$$I_{CF}[J] = \int \prod_x \Big[d\omega(x) \prod_i dn_i(x)\Big]$$
$$\times \exp\Big\{i\int d^2x\Big[\mathscr{L}'_{CF}[n_i, \omega] + (g_0)^{1/2}\sum_{i=1}^{N} J_i(x)\, n_i(x)\Big]\Big\}, \tag{B.3}$$

$$I_{GN}[J] = \int \prod_x \Big[d\omega(x) \prod_i d\psi_i(x)\Big]$$
$$\times \exp\Big\{i\int d^2x\Big[\mathscr{L}'_{GN}[\psi_i, \omega] + \sum_{i=1}^{N} \bar{J}_i(x)\, \psi_i(x)\Big]\Big\}. \tag{B.3$'$}$$

The integration over $n_i(x)$ in expression (B.3) (over $\psi_i(x)$ in (B.3′)) may be carried out explicitly, and results in (irrelevant factor which is cancelled in (B.2) is dropped):

$$I_{CF}[J_i] = \int \prod_x d\omega(x) \times \exp\left\{iS_{CF}^{(\text{eff})}[\omega] + \frac{i}{2}\int d^2x\, d^2x'\, J_i(x)\, J_i(x')\, G_{CF}(x, x' \mid \omega)\right\}; \quad \text{(B.4)}$$

$$I_{GN}[J_i] = \int \prod_x d\omega(x) \times \exp\left\{iS_{GN}^{(\text{eff})}[\omega] + \frac{i}{2}\int d^2x\, d^2x'\, \bar{J}_i(x)\, G_{GN}(x, x' \mid \omega)\, J_i(x')\right\}, \quad \text{(B.4′)}$$

where

$$S_{CF}^{(\text{eff})} = i\,\frac{N}{2}\,\text{tr}\ln[\partial_\mu{}^2 - \omega(x)] - \int d^2x\,\frac{\omega(x)}{2g_0}; \quad \text{(B.5)}$$

$$S_{GN}^{(\text{eff})} = -i\,\frac{N}{2}\,\text{tr}\ln[\gamma_\mu\partial^\mu - \omega(x)] - \int d^2x\,\frac{\omega^2(x)}{2g_0}; \quad \text{(B.5′)}$$

and $G_{CF}(x, x' \mid \omega)$ and $G_{GN}(x, x' \mid \omega)$ are Green functions of differential operators

$$\partial_\mu{}^2 - \omega(x), \quad \text{(B.6)}$$

$$\gamma_\mu\partial^\mu - \omega(x), \quad \text{(B.6′)}$$

respectively.

One obtains the $1/N$-series of model (1.4)((1.5)) calculating integral (B.4)((B.4′)) perturbatively. The stationary phase point of this integral at $\omega = \tilde{\omega}$

$$\tilde{\omega}_{CF} = m_{CF}^2 = \Lambda^2 \exp\left\{-\frac{4\pi}{Ng_0}\right\}, \quad \text{(B.7)}$$

$$\tilde{\omega}_{GN} = m_{GN} = \Lambda \exp\left\{-\frac{2\pi}{Ng_0}\right\} \quad \text{(B.7′)}$$

should be taken into account and functionals $S_{CF}^{(\text{eff})}[\omega]$ and $G_{CF}(x, x' \mid \omega)(S_{GN}^{(\text{eff})}[\omega]$ and $G_{GN}(x, x' \mid \omega))$ should be expanded in $\omega' = \omega - \tilde{\omega}$.[13]

It is easy to follow from the integrals (B.4) and (B.4′) to the simple diagrammatic technique with elements drawn in Fig. 13a, b, where the sign $+(-)$ in multileg vertices (Fig. 13b) corresponds to the case of the chiral field (Gross–Neveu) model Constructing any diagram from these elements one should not draw closed solidline loops since they are already taken into account by multileg vertices in Fig. 13b.

[13] In fact, there are two symmetrical stationary phase points $\tilde{\omega} = \pm m_{GN}$ in (B.4′). The system is settled in one of them by the Higgs effect. This corresponds to spontaneous breakdown of the discrete γ_5-symmetry [28].

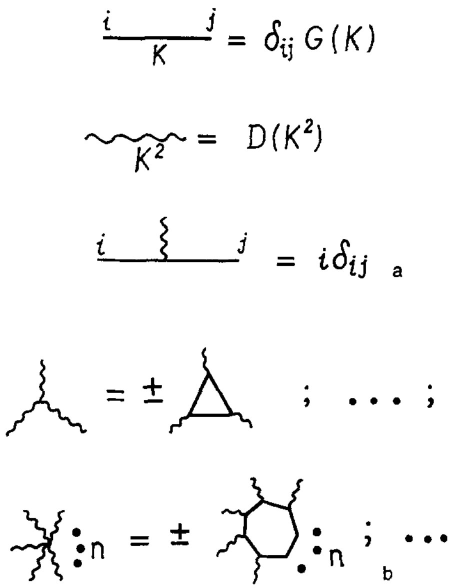


FIG. 13. Elements of the $1/N$-diagrammatic technique for chiral field and Gross-Neveu models.

Functions corresponding to the solid an wavy lines in Fig. 13 are different for the cases of (1.4) and (1.5):

$$G_{CF}(k) = \frac{i}{k^2 - m^2 + i\epsilon}, \tag{B.8}$$

$$G_{GN}(k) = i\,\frac{\hat{k} + m}{k^2 - m^2 + i\epsilon}, \tag{B.8$'$}$$

and

$$[D_{CF}(k^2)]^{-1} = \frac{1}{(2\pi)^2} \int \frac{d^2p}{(p^2 - m^2 + i\epsilon)((p+k)^2 - m^2 + i\epsilon)}; \tag{B.9}$$

$$[D_{GN}(k^2)]^{-1} = -\frac{1}{(2\pi)^2}\,\mathrm{tr} \int \Big[\frac{d^2p}{(\hat{p} - m + i\epsilon)(\hat{p} + \hat{k} - m + i\epsilon)} - \frac{d^2p}{(\hat{p} - m + i\epsilon)^2}\Big]. \tag{B.9$'$}$$

In formulas (B.9) and (B.9′) and in the general part of the paper indices *CF* and *GN* near the masses of particles are dropped.

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