Kosterlitz-Thouless

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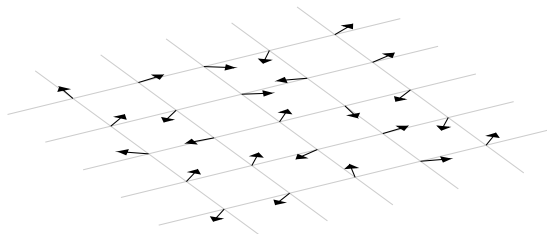
1 The XY Model

We consider the classical XY model in two-dimensions. By the Mermin-Wagner theorem, this phase can have no conventional order at any non-zero temperature as thermal fluctuations wipe it out. However, the system can still show order in the form of vortices, an order which cannot be classified by a typical order parameter and is more global in nature. It is therefore a key example of a topological phase.



1.1 Lowest Energy Solution

We begin with the Hamiltonian for spins on a dD square lattice with L sites in each direction and a lattice constant of a,

$$H = -J\sum_{\langle i,j\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + JN.$$

Considering a smooth solution in a continuum limit $a \to 0$,

$$\begin{aligned} H &= \frac{J}{2}\sum_{\langle i,j\rangle}(\theta_i - \theta_j)^2 \\ &= \frac{J}{a^{d-2}}\int \mathrm{d}^d\boldsymbol{r}\frac{1}{2}\left(\nabla\theta(\boldsymbol{r})\right)^2. \end{aligned}$$

Note that this is invariant under dilations in 2D. We then consider local minima,

$$\frac{\delta H}{\delta\theta(\boldsymbol{r})} = -\nabla^2\theta(\boldsymbol{r}) = 0,$$

an equation which admits two kind of solutions in 2D. The first is the one in which all spins point along one axis, $\theta(\boldsymbol{r}) = \text{const}$, breaking the U(1) symmetry.

We know that this order is killed by thermal fluctuations because of Mermin-Wagner and we see this by considering the expectation value of S_x. We start by quoting the form of the generating functional

$$\begin{aligned} \mathcal{Z}[h] &= \int \mathcal{D}\theta(\boldsymbol{r})\exp\left(-\beta H + \int \mathrm{d}^d\boldsymbol{r}\, h(\boldsymbol{r})\theta(\boldsymbol{r})\right) \\ &= \exp\left(\frac{1}{2}\int \mathrm{d}^d\boldsymbol{r}_1\mathrm{d}^d\boldsymbol{r}_2 h(\boldsymbol{r}_1)h(\boldsymbol{r}_2)\left[\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}}{Ka^{2-d}\boldsymbol{k}^2}\right]\right) \end{aligned}$$

where $K = \beta J$. We can use this somewhat unconventionally by setting $h(\boldsymbol{r}) = i\delta^{(d)}(\boldsymbol{r})$. At this point $\mathcal{Z}[h] = \langle e^{i\theta(\boldsymbol{0})}\rangle$ and, taking the real part,

$$\langle S_x\rangle = \exp\left(-\frac{a^{d-2}}{2K}\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{\boldsymbol{k}^2}\right).$$

This exponent is finite for all $d > 2$, allowing order, and infinitely negative for $d = 1$ prohibiting any order. In $d = 2$ the situation is slightly different with

$$\langle S_x \rangle = \left(\frac{L}{a}\right)^{-\frac{1}{4\pi K}}$$

but this still kills order for all $T > 0$. We can also consider the correlation function

$$\langle \boldsymbol{S}(\boldsymbol{0}) \cdot \boldsymbol{S}(\boldsymbol{r}) \rangle = \langle e^{i\theta(\boldsymbol{0})} e^{-i\theta(\boldsymbol{r})} \rangle.$$

We then set $h(\boldsymbol{r}') = i(\delta(\boldsymbol{r}') - \delta(\boldsymbol{r} - \boldsymbol{r}'))$ and find

$$\langle \boldsymbol{S}(\boldsymbol{0}) \cdot \boldsymbol{S}(\boldsymbol{r}) \rangle = \exp\left(-G(\boldsymbol{r}) + G(\boldsymbol{0})\right).$$

Thus, we want to find the long range form of this. Therefore, set $\boldsymbol{r}$ to large and then

$$\begin{aligned} G(\boldsymbol{r}) &= \frac{1}{\beta J} \int \frac{\mathrm{d}k \mathrm{d}\phi}{(2\pi)^2} \frac{e^{ikr\cos\phi}}{k} \\ &= \frac{1}{K} \int_0^{\frac{\pi r}{a}} \frac{\mathrm{d}(kr)}{2\pi} \frac{J_0(kr)}{kr} \\ &\to \frac{1}{2\pi K} \mathrm{Ln} \left| \frac{r}{r_0} \right| \end{aligned}$$

for large enough r. Therefore, we find that

$$\langle \boldsymbol{S}(\boldsymbol{0}) \cdot \boldsymbol{S}(\boldsymbol{r}) \rangle \sim r^{-\frac{1}{2\pi K}}.$$

This is a kind of quasi-ordered state, but doesn't contain the full order we see in higher dimensions.

1.2 Vortices

The second local minimum of the energy is a vortex solution, satisfying the boundary condition

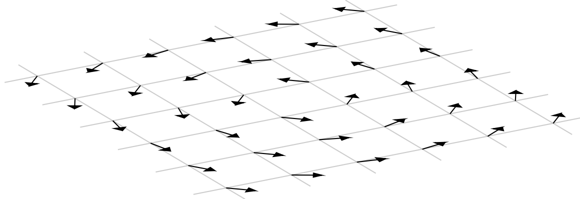
$$\oint_{\boldsymbol{x}_i} \nabla\theta \cdot \mathrm{d}\boldsymbol{l} = 2\pi n_i$$

where n_i refers to the charge of the vortex located at $\boldsymbol{x}_i$. This boundary condition is necessary for the spins to be continuous and admits solutions of the form $\theta(r, \phi) = n\phi + \theta_0$ when $\boldsymbol{x}_i = (0, 0)$. Therefore, $|\nabla\theta| = \frac{n}{r}$, and so the vortices have energy

$$\begin{aligned} E_{\text{vor}} &= \frac{J}{2} \int_0^{2\pi} \mathrm{d}\phi \int_a^L r \mathrm{d}r \frac{n^2}{r^2} \\ &= J n^2 \pi \mathrm{Ln} \left(\frac{L}{a}\right). \end{aligned}$$

The lower integration limit here reflects the fact that our continuum theory is only an approximation. This diverges in a macroscopically large system, suggesting single vortices are increasingly unfavourable. However, we could consider a vortex-anti-vortex pair, which could produce a gradient-free field far from their centers but maintaining vortex patterns approximately up to their separation, R. Therefore, we might expect the energy of such a configuration to be

$$E_{2\times\text{vor}} = 2E_c + E_1 \mathrm{Ln} \left(\frac{R}{a}\right).$$



One can also argue the existence of vortices with entropy. The vortex could sit in any one of the plaquettes of the lattice, so it can arrange itself in any of $\frac{L^2}{a^2}$ ways. Therefore, the free energy is

$$F = \left(Jn^2\pi - 2T\right) \mathrm{Ln}\left(\frac{L}{a}\right)$$

suggesting that vortices may become favourable at temperatures $T \geq \frac{J\pi}{2}$. Whilst this calculation is clearly approximate it does show that the logarithmic nature of both the energy and entropy being combined is the key to thermal activation of free vortices.

1.3 Following the Original Paper

So consider a distribution of vortices

$$\rho(\boldsymbol{r}) = \sum_i n_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$$

such that $\nabla^2\theta = 2\pi\rho(\boldsymbol{r})$. This has the solution

$$\begin{aligned}\theta(\boldsymbol{r}) &= -\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\tilde{\rho}(\boldsymbol{k})\frac{2\pi}{\boldsymbol{k}^2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= -2\pi\int \mathrm{d}^2\boldsymbol{r}'\rho(\boldsymbol{r}')\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{\boldsymbol{k}^2}.\end{aligned}$$

This integral over $\boldsymbol{k}$ is then $g(\boldsymbol{r} - \boldsymbol{r}')$, the Green's function,

$$\theta(\boldsymbol{r}) = -2\pi\int \mathrm{d}^2\boldsymbol{r}\rho(\boldsymbol{r}')g(\boldsymbol{r} - \boldsymbol{r}')$$

We then consider the spinwaves about this local minimum by setting $\theta = \bar{\theta} + \psi$ where ψ is the spinwave contribution. Then

$$H = \frac{J}{2}\int \mathrm{d}^2\boldsymbol{r}\left[(\nabla\bar{\theta})^2 + 2(\nabla\bar{\theta})\cdot(\nabla\psi) + (\nabla\psi)^2\right]$$

and the middle term is zero because $\oint \nabla\psi\cdot\mathrm{d}\boldsymbol{l} = 0$ along all paths. I think the point is that the $\nabla\psi$ term therefore turns up as an odd power that this term vanishes. But I don't have a full proof of that. Thus, inserting the form above for $\bar{\theta}(\boldsymbol{r})$,

$$H = \frac{J}{2}\int \mathrm{d}^2\boldsymbol{r}(\nabla\psi)^2 + 2\pi^2 J\int \mathrm{d}^2\boldsymbol{r}'\mathrm{d}^2\boldsymbol{r}''\rho(\boldsymbol{r}')\rho(\boldsymbol{r}'')\int \mathrm{d}^2\boldsymbol{r}\nabla g(\boldsymbol{r}-\boldsymbol{r}')\cdot\nabla g(\boldsymbol{r}-\boldsymbol{r}'').$$

This can be simplified using the definition of the Green's function to find

$$\begin{aligned}\int \mathrm{d}^2\boldsymbol{r}\nabla g(\boldsymbol{r}-\boldsymbol{r}')\cdot\nabla g(\boldsymbol{r}-\boldsymbol{r}'') &= \int \mathrm{d}^2\boldsymbol{r}\nabla\cdot\left(g(\boldsymbol{r}-\boldsymbol{r}')\nabla g(\boldsymbol{r}-\boldsymbol{r}'')\right) - \int \mathrm{d}^2\boldsymbol{r} g(\boldsymbol{r}-\boldsymbol{r}')\nabla^2 g(\boldsymbol{r}-\boldsymbol{r}') \\ &= \oint \mathrm{d}\boldsymbol{l}\cdot\left(g(\boldsymbol{r}-\boldsymbol{r}')\nabla g(\boldsymbol{r}-\boldsymbol{r}'')\right) - g(\boldsymbol{r}''-\boldsymbol{r}').\end{aligned}$$

Here we use the asymptotic form of the Green's function to approximate the long-range behaviour of vortices. This is

$$g(r) \simeq \frac{1}{2\pi}\mathrm{Ln}\left|\frac{r}{r_0}\right|$$

where r_0 is not really that important but comes out to be $\simeq 0.2a$. This is not something I've shown myself. Therefore,

$$\begin{aligned}\int \mathrm{d}^2\boldsymbol{r}\nabla g(\boldsymbol{r}-\boldsymbol{r}')\cdot\nabla g(\boldsymbol{r}-\boldsymbol{r}'') &= \oint L\mathrm{d}\phi\frac{1}{2\pi}\mathrm{Ln}\left|\frac{L}{r_0}\right|\frac{1}{2\pi L} - \frac{1}{2\pi}\mathrm{Ln}\left|\frac{\boldsymbol{r}''-\boldsymbol{r}'}{r_0}\right| \\ &= \mathrm{Ln}\left|\frac{L}{r_0}\right| - \frac{1}{2\pi}\mathrm{Ln}\left|\frac{\boldsymbol{r}''-\boldsymbol{r}'}{r_0}\right|.\end{aligned}$$

We can insert this relation into the expression for H along with ρ as a function of the n's to see

$$H = \frac{J}{2}\int \mathrm{d}^2\boldsymbol{r}(\nabla\psi)^2 - \pi J\sum_{ij} n_i n_j \mathrm{Ln}\left|\frac{\boldsymbol{r}_i-\boldsymbol{r}_j}{r_0}\right| - 2\pi^2 J\sum_i n_i^2 g(\mathbf{0}) + \pi J\left(\sum_i n_i\right)^2 \mathrm{Ln}\frac{L}{r_0}.$$

We have left $g(\mathbf{0})$ for the 'self-interaction' terms because the asymptotic form only holds for $|\boldsymbol{r}| \sim a$.

Thus, we have a Hamiltonian which describes 2D charged particles interacting via 2D Coulomb interactions. Each particle then has some energy cost associated with it and then the system as a whole has an energy depending on the total charge of the system. This final term demands charge neutrality, $\sum_i n_i \to 0$ in the thermodynamic limit, $L \to \infty$.

1.4 RG Calculation

We wish to derive the RG equations. So if we consider neutral states, ignoring spinwave excitations made entirely of charge-1 vortices then

$$H_n = -\pi J\sum_{i\neq j=1}^{n} n_i n_j \mathrm{Ln}\left|\frac{\boldsymbol{r}_i-\boldsymbol{r}_j}{r_0}\right|$$

gives the interactions and then each vortex has a chemical potential μ. We will then be working in the dilute limit where there aren't many vortices, which as is probably already clear is a low-temperature limit. The partition function is then

$$\begin{aligned}\mathcal{Z} &= \mathrm{Tr}\left(e^{-\beta(H_n+2n\mu)}\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(e^{-\beta\mu}\right)^{2n}\int_{|\boldsymbol{r}_i-\boldsymbol{r}_j|\geq r_0}\prod_{i=1}^{n}\frac{\mathrm{d}^2\boldsymbol{r}_i^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_i^-}{r_0^2}e^{-\beta H_n}\end{aligned}$$

where $\boldsymbol{r}_i^{\pm}$ refers to vortices of positive/negative charge and we impose the small-$\boldsymbol{r}$ cutoff because of the finite size of the vortices. We then call $y = e^{-\beta\mu}$ the fugacity. The factors of $n!$ come from the indistinguishability of vortices.

Now we perform the RG calculation, removing the small-scale behaviour by integrating out all vortex-anti-vortex pairs within r_0 and $\tilde{r}_0 = r_0(1+\delta l)$ of each other. The remaining system is then that for vortices of size $\tilde{r}_0$. Note that we only consider oppositely charged pairs as the similarly charged pairs will repel strongly at short distances so there isn't much probability for that to happen and it doesn't contribute much to $\mathcal{Z}$. Therefore, we can ignore it. So, splitting the sum in $\mathcal{Z} = \sum_p \mathcal{Z}_p(r_0)$ into parts, the first correction due to the integration is

$$\mathcal{Z}_p(r_0) = \mathcal{Z}_p(\tilde{r}_0) + \frac{y^{2p}}{(p!)^2}p^2 \int_{|\boldsymbol{r}_i - \boldsymbol{r}_j| \geq \tilde{r}_0} \prod_{i=1}^{p-1} \frac{\mathrm{d}^2\boldsymbol{r}_i^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_i^-}{r_0^2} \int_{r_0 \leq |\boldsymbol{r}_p^+ - \boldsymbol{r}_p^-| \leq \tilde{r}_0} \frac{\mathrm{d}^2\boldsymbol{r}_p^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_p^-}{r_0^2} e^{-\beta H_n} + \mathcal{O}(\delta l^2)$$

where the factor of p^2 arises because we can pair any vortex with any anti-vortex. This $\mathcal{O}(\delta l)$ term is then a $\mathcal{Z}_{p-1}(\tilde{r}_0)$ out front followed by the residual bits and pieces (throwing out anything $\mathcal{O}(\delta l^2)$), so

$$\begin{aligned}\mathcal{O}(\delta l) = \mathcal{Z}_{p-1}(\tilde{r}_0)y^2 \int_{r_0 \leq |\boldsymbol{r}_p^+ - \boldsymbol{r}_p^-| \leq \tilde{r}_0} \frac{\mathrm{d}^2\boldsymbol{r}_p^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_p^-}{r_0^2} \exp\Bigg(2\pi K \mathrm{Ln}\frac{|\boldsymbol{r}_p^+ - \boldsymbol{r}_p^-|}{r_0} + \\ 2\pi K \sum_{i=1}^{2(p-1)} n_i \left(\mathrm{Ln}\frac{|\boldsymbol{r}_i - \boldsymbol{r}_p^+|}{r_0} - \mathrm{Ln}\frac{|\boldsymbol{r}_i - \boldsymbol{r}_p^-|}{r_0}\right)\Bigg)\end{aligned}$$

where we define $K = \beta J$. The first term here is zero as $\mathrm{Ln}\frac{r_0}{r_0} = 0$. We then define the relative position by

$$\boldsymbol{r}_p^\pm = \boldsymbol{R} \pm \frac{1}{2}\boldsymbol{r}$$

at which point we take $|\boldsymbol{r}|$ to be small relative to other spacings to find

$$\mathrm{Ln}\frac{|\boldsymbol{r}_i - \boldsymbol{r}_p^+|}{r_0} - \mathrm{Ln}\frac{|\boldsymbol{r}_i - \boldsymbol{r}_p^-|}{r_0} = -\frac{\boldsymbol{r}\cdot(\boldsymbol{r}_i - \boldsymbol{R})}{(\boldsymbol{r}_i - \boldsymbol{R})^2} + \mathcal{O}(|\boldsymbol{r}|^3).$$

Therefore, the above expression simplifies to

$$\mathcal{O}(\delta l) = \mathcal{Z}_{p-1}(\tilde{r}_0)y^2 \int_{r_0 \leq |\boldsymbol{r}| \leq \tilde{r}_0} \frac{\mathrm{d}^2\boldsymbol{R}\mathrm{d}^2\boldsymbol{r}}{r_0^4} \exp\left(2\pi K \sum_{i=1}^{2(p-1)} n_i \boldsymbol{r}\cdot\frac{\boldsymbol{R} - \boldsymbol{r}_i}{(\boldsymbol{R} - \boldsymbol{r}_i)^2}\right).$$

We expand the exponential, defining $\boldsymbol{E}(\boldsymbol{R}) = \sum_i n_i \frac{\boldsymbol{R} - \boldsymbol{r}_i}{(\boldsymbol{R} - \boldsymbol{r}_i)^2}$ as the 'electric field' experienced by the charges and keeping only even powers of $|\boldsymbol{r}|$ (as the others integrate to zero)

$$\begin{aligned}\mathcal{O}(\delta l) &= \mathcal{Z}_{p-1}(\tilde{r}_0)y^2 \int_{r_0 \leq |\boldsymbol{r}| \leq \tilde{r}_0} \frac{\mathrm{d}^2\boldsymbol{R}\mathrm{d}^2\boldsymbol{r}}{r_0^4}\left(1 + \frac{(2\pi K)^2}{2}(\boldsymbol{r}\cdot\boldsymbol{E})^2 + \dots\right) \\ &= \mathcal{Z}_{p-1}(\tilde{r}_0)y^2 \int \mathrm{d}^2\boldsymbol{R}\left(\frac{2\pi\delta l}{r_0^2} + 2\pi\delta l(\pi K)^2 E^2 + \dots\right).\end{aligned}$$

Now, we want to find the integral of the electric field over space.

Derivation 1

So consider that the area is the material excluding regions of size a surrounding the vortices. Thus, converting to a surface integral, with $\boldsymbol{E} = \nabla V$ where $V = -\sum_i n_i \mathrm{Ln}\frac{|\boldsymbol{R} - \boldsymbol{r}_i|}{r_0}$, we have

$$W = \int \mathrm{d}^2\boldsymbol{R}(\nabla V)^2 = \int_S V\nabla V.\mathrm{d}\boldsymbol{l}$$

given that $\nabla^2 V = 0$ in the region (as it ignores where the vortices are located). The surface, S is then the boundary of the sample and the boundaries of all the circles of radius r_0 we cut out. Considering just one of these surfaces,

$$\oint_{S_k} V\nabla V\cdot\mathrm{d}\boldsymbol{l} = \oint_{S_k}\left(\sum_i n_i \mathrm{Ln}\frac{|\boldsymbol{R} - \boldsymbol{r}_i|}{r_0}\right)\left(\sum_i n_i \frac{\boldsymbol{R} - \boldsymbol{r}_i}{|\boldsymbol{R} - \boldsymbol{r}_i|^2}\right)\cdot\mathrm{d}\boldsymbol{l}.$$

Now, given $|\boldsymbol{R}-\boldsymbol{r}_k| = r_0$, the $i=k$ term is absent from the first term. We then perform a contour integral, which keeps only the $i=k$ part of the second term, to find

$$W_k = -2\pi n_k \sum_{i\neq k} n_i \mathrm{Ln}\frac{|\boldsymbol{r}_k-\boldsymbol{r}_i|}{r_0}$$

which means that in total,

$$\int \mathrm{d}^2\boldsymbol{R}E^2 = -2\pi \sum_{i\neq j=1}^{p-1} n_i n_j \mathrm{Ln}\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{r_0}.$$

We now separate this $\mathcal{O}(\delta l)$ term in this $\mathcal{Z}_p$ term and combine it with the $\mathcal{Z}_{p-1}$ term. This leaves us with

$$\frac{y^{2(p-1)}}{(p-1)!}\int_{|\boldsymbol{r}_i-\boldsymbol{r}_j|\leq\tilde{r}_0}\prod_{i=1}^{p-1}\frac{\mathrm{d}^2\boldsymbol{r}_i^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_i^-}{r_0^2}e^{-\beta H_{p-1}}\left[1+\mathrm{const}+2\pi\delta l(\pi K y)^2\int \mathrm{d}^2\boldsymbol{R}E^2\right].$$

We then say that $1+\mathrm{small} = e^{\mathrm{small}}$ and so

$$=\frac{y^{2(p-1)}}{(p-1)!}\int_{|\boldsymbol{r}_i-\boldsymbol{r}_j|\leq\tilde{r}_0}\prod_{i=1}^{p-1}\frac{\mathrm{d}^2\boldsymbol{r}_i^+}{r_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_i^-}{r_0^2}\exp\left((\pi K-(2\pi)^2(\pi K y)^2\delta l)\sum_{i\neq j} n_i n_j \mathrm{Ln}\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{r_0}+\mathrm{const}\right).$$

Therefore, we see that as we integrate out the smaller scales, the form of the partition function is the same except for

$$K\to\tilde{K} = K-4\pi^3K^2y^2\delta l.$$

The final step is to rescale all the r_0's into $\tilde{r}_0$'s. The first place this occurs is in

$$\sum_{i\neq j} n_i n_j \mathrm{Ln}\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{r_0} = \sum_{i\neq j} n_i n_j \mathrm{Ln}\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{\tilde{r}_0} + \sum_{i\neq j} n_i n_j \mathrm{Ln}\frac{\tilde{r}_0}{r_0}.$$

The second term is then just equal to

$$\delta l \sum_{i\neq j} n_i n_j = \delta l\left(\cancel{\left(\sum_i n_i\right)^2} - \sum_i n_i^2\right) = -2(p-1)\delta l.$$

The other place they appear is in the $\frac{1}{r_0^2}$ factors which brings about a $\left(\frac{\tilde{r}_0}{r_0}\right)^2 = e^{2\delta l}$ terms. So in total, the $\mathcal{Z}_{p-1}(\tilde{r}_0)$ term is

$$=\frac{y^{2(p-1)}e^{2(p-1)\delta l}e^{-2(p-1)\pi\tilde{K}\delta l}}{(p-1)!}\int_{|\boldsymbol{r}_i-\boldsymbol{r}_j|\leq\tilde{r}_0}\prod_{i=1}^{p-1}\frac{\mathrm{d}^2\boldsymbol{r}_i^+}{\tilde{r}_0^2}\frac{\mathrm{d}^2\boldsymbol{r}_i^-}{\tilde{r}_0^2}\exp\left(\tilde{K}\sum_{i\neq j} n_i n_j \mathrm{Ln}\frac{|\boldsymbol{r}_i-\boldsymbol{r}_j|}{\tilde{r}_0}+\mathrm{const}\right).$$

and so for everything to be of the same analytic form, but with some shifted coefficients, we must have

$$\mathrm{Ln}\tilde{y} = \mathrm{Ln}y + (2-\pi K)\delta l.$$

Note that the constant term has just been thrown away because it doesn't depend in any way on the position or on p so its form is the same for all $\mathcal{Z}_p$ and therefore just rescales the partition function as a whole.

1.5 RG Flows

Thus, we have our RG equations which, in differential form, read

$$\frac{\mathrm{d}K}{\mathrm{d}l} = -4\pi^3 K^2 y^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}l} = (2 - \pi K)y.$$

These equations are particularly accurate in the low temperature phase where the vortices are sparse. The reason for this is that we made some hidden approximations about the system being a dilute system.

So consider this near the main fixed point at $K = \frac{2}{\pi}$ by defining $x = \frac{1}{K} - \frac{\pi}{2}$. This turns the RG equations into

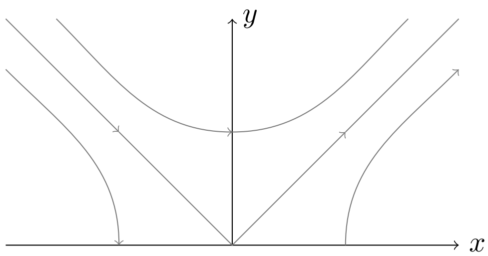
$$\frac{\mathrm{d}x}{\mathrm{d}l} = 4\pi^3 y^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}l} = \frac{4}{\pi} xy.$$

Physically, these variables are the energy cost of a core in $y = e^{-\frac{\mu}{T}}$ and temperature in $x + \frac{\pi}{2} = \frac{1}{K} = \frac{T}{J}$. These are equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}l}(x^2 - \pi^4 y^2) = 0$$

which traces out hyperbolae.



Thus, we find the critical temperature setting

$$x^2 - \pi^4 y^2 = 0$$

as this describes the flow towards the critical point. Thus,

$$x = -\pi^2 y$$

$$\frac{T_c}{J} - \frac{\pi}{2} = -\pi^2 \exp\left(-\frac{\mu}{T_c}\right).$$

The solution of this is shown in the figure. On the low temperature side of this transition the long range physics is described by $y = 0$ and $x < 0$. Thus, μ_{eff} is large, scaling vortices out of the problem. But they are there at small scales, just bound into close pairs.

In the other phase the long range behaviour is described by a significant number of vortices as $y \to \infty$, suggesting μ is small. However, note that the RG procedure doesn't really apply as soon as we get near $y \sim 1$ as we assumed vortices were dilute. But we can guess the correlation length. Firstly consider

$$x^2 - \pi^4 y^2 = C$$

where $C \simeq C(T_c) + b(T_c - T) + \ldots$. We take this form because we know that $C < 0$ when $T > T_c$. Furthermore, we know that at T_c we have $C = 0$ so $C(T_c) = 0$. We can now use this to solve for $x(l)$.

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \frac{4}{\pi}\pi^4 y^2 = \frac{4}{\pi}(x^2 - C)$$

which has the solution

$$l = \frac{\pi}{4\sqrt{|C|}}\left(\arctan\frac{x}{\sqrt{|C|}} - \arctan\frac{x_0}{\sqrt{|C|}}\right)$$

where x_0 is the starting position of the flow. We then use that at $x \sim 1$ things break down. Taking x_0 and $|C|$ to be small this is roughly as

$$l = \frac{\pi}{4\sqrt{|C|}}\frac{\pi}{2} = \frac{\pi^2}{8\sqrt{b(T - T_c)}}.$$

Now, we assume that this broke down at the correlation length, where $\xi = ae^l$. Therefore, we approximate

$$\xi \sim \exp\left(\frac{\text{const}}{\sqrt{T - T_c}}\right),$$

which diverges as we approach T_c.

Finally we consider the universal exponent at the critical point

$$\langle \boldsymbol{S}(\boldsymbol{0}) \cdot \boldsymbol{S}(\boldsymbol{r})\rangle \sim r^{-\frac{1}{2\pi K}}.$$

We see that $K_{\text{eff}} = \frac{2}{\pi}$ and so where

$$\langle \boldsymbol{S}(\boldsymbol{0}) \cdot \boldsymbol{S}(\boldsymbol{r})\rangle \sim r^{-\eta} \qquad \text{then} \qquad \eta = \frac{1}{4}.$$

This solution is exact thanks to the fact that one of our major assumptions, that vortices are dilute, is perfectly satisfied here as the critical point sits at $y = 0$.

1.6 Other pieces

This (Berezinskii)-Kosterlitz-Thouless transition occurs in a variety of systems. For example, consider a superfluid with the wavefunction

$$\Psi(\boldsymbol{r}, t) = A(\boldsymbol{r}, t)e^{i\phi(\boldsymbol{r},t)}$$

for which we define the velocity

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla\phi.$$

This defines an irrotational flow ($\nabla \times \boldsymbol{v} = 0$) with the quantisation condition

$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \frac{\hbar}{m}2\pi n$$

where n is some integer. Finally, this has kinetic energy

$$E_{\text{flow}} = \int \frac{1}{2}\rho\boldsymbol{v}^2\mathrm{d}^2\boldsymbol{r}$$

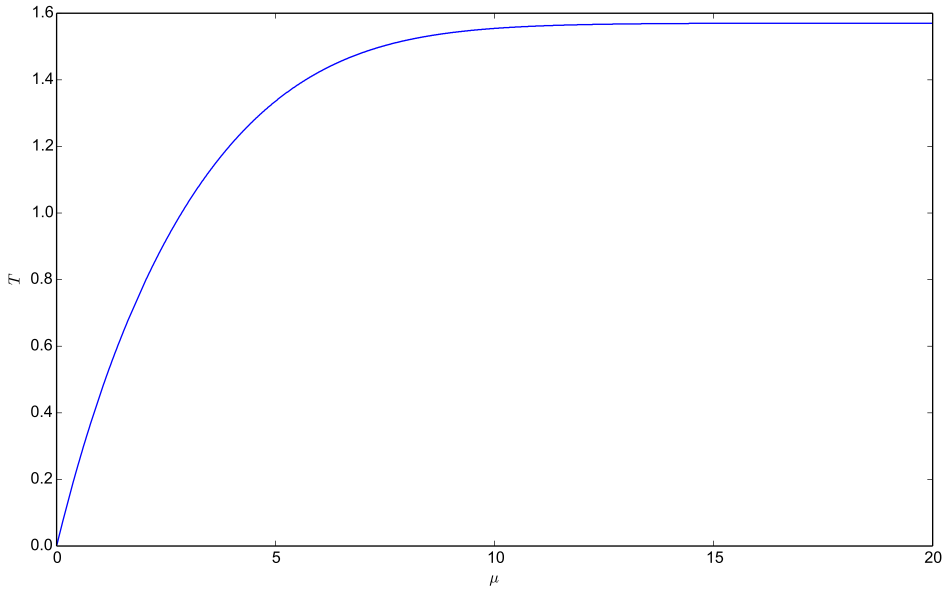


Figure 1.1: The KT transition temperature as a function chemical potential, μ, of a vortex. The units of both quantities are J. Note that is only really applies in the large-μ case where y is small (as these are linearised equations).

in two dimensions where we have defined ρ by the mass current

$$\boldsymbol{j}(\boldsymbol{r}) = \rho_s(T)\boldsymbol{v}(\boldsymbol{r}).$$

Thus, our energy for the flow is

$$E = \frac{\rho\hbar^2}{m^2}\int \mathrm{d}^2\boldsymbol{r}\frac{1}{2}(\nabla\phi)^2$$

subject to the same quantisation condition that we had in the XY case where $\phi \to \phi + 2\pi$ is the same value. Therefore, we can play all the same tricks in mapping to a 2D coulomb gas and solving from there. In this case then the density is what gets renormalised

Other models equivalent to the 2D Coulomb gas at large scales include the Sine-Gordon model,

$$H = \int \mathrm{d}^2\boldsymbol{r}\left(\frac{1}{2}(\partial_\mu\phi)^2 - \lambda\cos\beta\phi\right).$$

In this case the RG equations can be derived in a different way, but by the equivalence of all the models together they'll still be the same. Therefore, people will often derive the RG equations for the Kosterlitz-Thouless transition using this method instead. It's also seen in superconductivity, some defect models, etc.

1.7 Experimental Observation

So if we reconsider the superfluid case, it turns out that we cannot form a true Bose-Einstein condensate in 2D, but similar to the XY case we expect a quasi-condensate phase up to a critical temperature at which point vortices will proliferate, killing the superfluid. There are some good recent measurements of this. A group in South Korea image the vortices directly by trapping a large region of fluid and these squeezing the whole thing radially so see the pairs of vortices. Another method has been used by Zoran Hadzibabic. This one is less clear to me but basically they take two sets of 2D superfluid, far enough away that they don't affect each other and then relax the traps to allow the two fluids to expand into one another. This causes an interferance pattern which can be used to imply to presence of vortices due to sharp dislocations in the pattern. I can't say I fully understood why.

1.8 Sources

- 'The critical properties of the two-dimensional *xy* model', JM Kosterlitz
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- 'The Kosterlitz-Thouless Transition', HJ Jensen
- 'The Berezinskii-Kosterlitz-Thouless transition', AJ Leggett