Effective Collision Operator for Heat-Flux-Generated Whistler Turbulence

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13th Plasma Kinetics Working Meeting

August 3, 2022

Whistler Heat Flux Instability

Simulations

Collision Operator

Outline

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Background

Parallel whistler dispersion relation:

$$\omega = \left(k_{\|} d_e\right)^2 |\Omega_e|,$$

When $k_{\|}\rho_e \sim 1$

$$\omega \simeq \frac{|\Omega_e|}{\beta_e}, \quad v_p = \frac{\omega}{k_{\|}} \simeq \frac{v_{\mathrm{th},e}}{\beta_e}, \quad v_{\|,\mathrm{res}} = \mathrm{sign}(k_{\|})\left(\frac{1}{\beta_e} + n\right) v_{\mathrm{th},e}.$$

A distribution function with a heat flux can overcome cyclotron damping at $n = \pm 1$ resonances.

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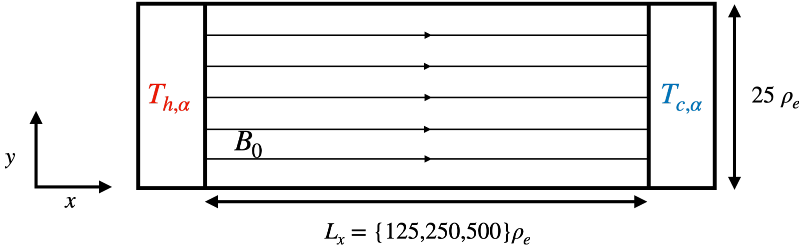
Setup



Using electromagnetic PIC code TRISTAN-MP (Anatoly Spitkovsky), following Komarov et al. 2018

Boundary Conditions:

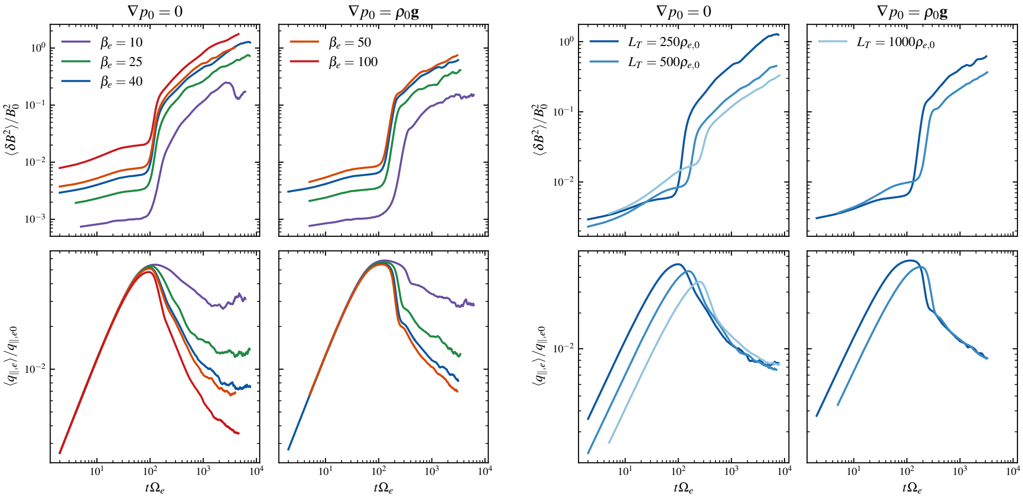
x: Absorbing E&M, re-sample particles with wall temperature

y: Periodic E&M, periodic particles

Linear temperature profile with two equilibria:

$$\nabla p_0 = 0 \qquad \text{and} \qquad \nabla p_0 = \rho_0 g$$

Results I



$\nabla p_0 = 0$
$\nabla p_0 = \rho_0 \mathbf{g}$
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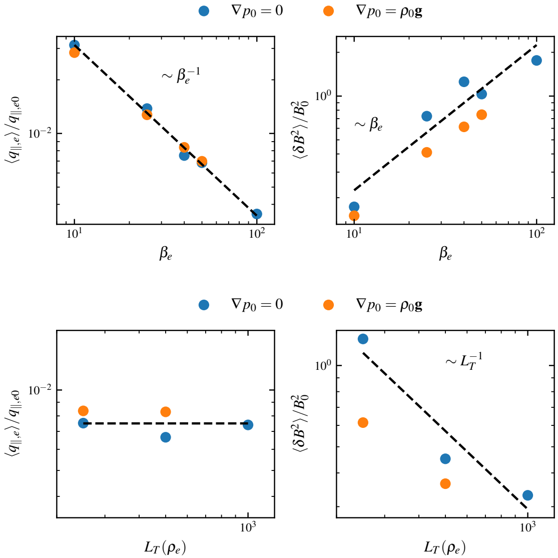
$\beta_e = 10$
$\beta_e = 25$
$\beta_e = 40$
$\beta_e = 50$
$\beta_e = 100$
$L_T = 250\rho_{e,0}$
$L_T = 500\rho_{e,0}$
$L_T = 1000\rho_{e,0}$
$\langle \delta B^2 \rangle / B_0^2$
$\langle q_{\parallel,e} \rangle / q_{\parallel,e0}$
$t\Omega_e$

Results II



$$\frac{q_{\|,e}}{q_{\|,e0}} \sim \beta_e^{-1} \quad \text{and} \quad \frac{\delta B^2}{B_0^2} \sim \frac{v_{\mathrm{th},e}\beta_e}{|\Omega_e| L_T}$$

$$q_{\|} \sim -n\frac{v_{\mathrm{th},e}^2}{\nu_{\mathrm{w}}}\nabla T$$

$$\Rightarrow \nu_{\mathrm{w}} \sim \frac{\beta_e v_{\mathrm{th},e}}{L_T} \sim \frac{\delta B^2}{B_0^2}|\Omega_e|.$$

Agrees with previous work:
Komarov et al. 2018,
Roberg-Clark et al. 2018

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Let's Take the Next Step

Can we construct a collision operator for whistler turbulence so we can perform a Chapman-Enskog-Braginskii like closure for this instability?

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Three Methods:

1. Fokker-Planck
2. Quasilinear
3. Chapman-Enskog

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Can we construct a collision operator for whistler turbulence so we can perform a Chapman-Enskog-Braginskii like closure for this instability?

Three Methods:

1. Fokker-Planck
2. Quasilinear
3. Chapman-Enskog

For each we can ask:

- Is the method self-consistent?
- If so, what model can we derive from it?

Fokker-Planck Method: Background I

$$\frac{\partial f(t,\boldsymbol{x},\boldsymbol{v})}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{A}(t,\boldsymbol{x},\boldsymbol{v}) f(t,\boldsymbol{x},\boldsymbol{v}) + \frac{\partial^2}{\partial \boldsymbol{v} \partial \boldsymbol{v}} : \boldsymbol{B}(t,\boldsymbol{x},\boldsymbol{v}) f(t,\boldsymbol{x},\boldsymbol{v}).$$

$$\boldsymbol{A}(t,\boldsymbol{x},\boldsymbol{v}) \doteq \lim_{\Delta t \to \text{“0”}} \frac{\langle \Delta \boldsymbol{v}(t,\boldsymbol{x},\boldsymbol{v},\Delta t)\rangle}{\Delta t}$$

and

$$\boldsymbol{B}(t,\boldsymbol{x},\boldsymbol{v}) \doteq \frac{1}{2} \lim_{\Delta t \to \text{“0”}} \frac{\langle \Delta \boldsymbol{v}(t,\boldsymbol{x},\boldsymbol{v},\Delta t) \Delta \boldsymbol{v}(t,\boldsymbol{x},\boldsymbol{v},\Delta t)\rangle}{\Delta t},$$

where

$$\Delta \boldsymbol{v}(t,\boldsymbol{x},\boldsymbol{v},\Delta t) = \boldsymbol{v}(t+\Delta t,\boldsymbol{x}) - \boldsymbol{v}(t,\boldsymbol{x})$$

$$\langle \dots \rangle \doteq \int \mathrm{d}\boldsymbol{v} \left(\ \dots \ \right) f(t,\boldsymbol{x},\boldsymbol{v}).$$

Fokker-Planck Method: Background II

Ornstein-Uhlenbeck proess

$$dv_t = -\nu(v - \bar{v})\mathrm{d}t + \sigma \mathrm{dW}_t$$

moments of Δv have evolve in time

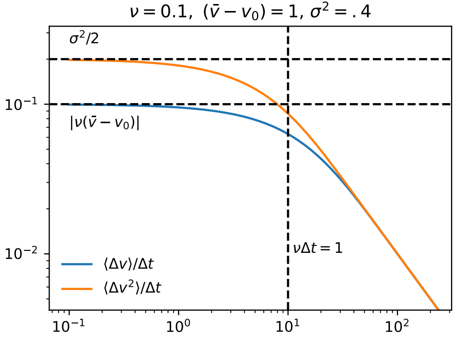
$$\langle \Delta v \rangle = \left(\bar{v} - v_0\right)\left(1 - e^{-\nu \Delta t}\right)$$

$$\langle \Delta v^2 \rangle - \langle \Delta v \rangle^2 = \frac{\sigma^2}{2\nu}\left(1 - e^{-2\nu \Delta t}\right).$$

Recover the Fokker-Planck equation for $\Delta t \ll \nu$:

$$A(v,t) = \frac{\langle \Delta v \rangle}{\Delta t} \simeq -\nu\left(v - \bar{v}\right)$$

$$B(v,t) = \frac{1}{2}\frac{\langle \Delta v^2 \rangle}{\Delta t} \simeq \frac{\sigma^2}{2}.$$



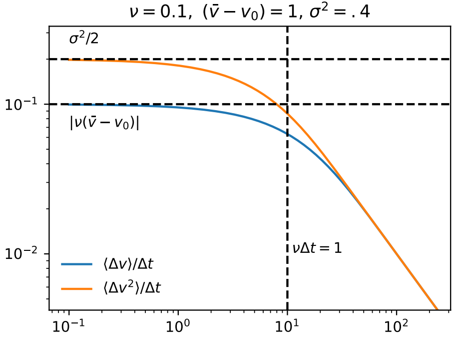
Fokker-Planck Method: Background II

Ornstein-Uhlenbeck proess

$$dv_t = -\nu(v - \bar{v})\mathrm{d}t + \sigma \mathrm{dW}_t$$

Assumptions:

- Stationary: $\nu(t) = \nu$, $\sigma(t) = \sigma$
- Markov: $\Delta t \gg \tau_{ac}$
- PDF of increments is Gaussian



Any deviation from these is non-FP behavior

A Note on Autocorrelation Time

For Fokker-Planck

$$\tau_{ac} \ll \Delta t \ll \nu^{-1}$$

But

$$\tau_{ac}^{lin} \sim (v_\| \cdot \Delta k_\|)^{-1}$$

Consider a wave packet centered at $k\rho_e \sim 1$ with $\Delta k/k \sim 1$.
For gyroresonaces $v_\| \sim v_{th,e}$:

$$\tau_{ac}^{lin} \sim \Omega_e^{-1}$$

But for $v_\| \to 0$:

$$\tau_{ac}^{lin} \to \infty$$

Fokker-Planck fomally invalid as $v_\| \to 0$

(Unless nonlinearities are involved)

Quasilinear Method: Background I

In its simplest form, a quasilinear diffusion coefficient follows the form

$$D \sim \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} W_B(\boldsymbol{k}) \delta\big(\omega(k) - k_{||} v_{||} - n\Omega_e\big).$$

Assuming $\omega(k) \ll k_{||} v_{||} \sim \Omega_e$ and $W_B(\boldsymbol{k}) = W_B(k) \sim k^a$,

$$\delta\big(\omega(k) - k_{||} v_{||} - n\Omega_e\big) \sim \frac{1}{|v_{||} \cos\theta|} \delta\big(k - k_{||,\mathrm{res}}\big), \qquad \text{where} \qquad k_{||,\mathrm{res}} = -\frac{n\Omega_e}{v_{||}\cos\theta}.$$

Therefore,

$$D \sim \frac{1}{(2\pi)^2} \int_{-1}^{1} d\cos\theta \frac{k_{||,\mathrm{res}}^{a+2}}{|v_{||}\cos\theta|} = \frac{1}{(2\pi)^2} \int_{-1}^{1} d\cos\theta \frac{1}{|v_{||}\cos\theta|} \left(-\frac{n\Omega_e}{v_{||}\cos\theta} \right)^{a+2}$$

will diverge when $v_{||} \to 0$, sending $k_{||,\mathrm{res}} \to \infty$.

Quasilinear Method: Background II

The singularity as $v_{\parallel} \to 0$ is the well-known 90-degree pitch angle problem.
Replace $\delta(x)$ with:

1. Laplacian

$$\delta(x) \to \frac{1}{\pi} \frac{\Delta\omega}{(x)^2 + \Delta\omega^2}$$

   - $\Delta\omega = \frac{1}{3} k_{\parallel}^2 D$ (Treumann & Baumjohann 1997)
   - $\Delta\omega \sim \omega_b$ (Meng et al. 2018)

2. Integral function of D (Dupree 1966)

$$\delta(x) \to R(x, D) = \int_0^{\infty} \mathrm{d}\tau \exp\left[i(x)\tau - \frac{1}{3} k_{\parallel}^2 D \tau^3\right]$$

3. Box distribution (Karimabadi et al. 1992)

$$\delta(x) \to \begin{cases} 1/4\omega_b, & |x| \leq 2\omega_b \\ 0, & |x| > \omega_b \end{cases}$$

etc...

Chapman-Enskog Method: Background I

Assume the form of the operator: pitch-angle scattering in the whistler frame $v'_{\|} = v_{\|} - v_w$, where v_w is the whistler phase speed:

$$C\left[f\right] = \frac{\partial}{\partial \xi'} \frac{1 - \xi'^2}{2} \nu(v', \xi') \frac{\partial f}{\partial \xi'}.$$

Utilize the correction equation from a Chapman-Enskog expansion:

$$\frac{Cf_1}{f_0} = \left(\frac{v^2}{v_{\mathrm{th},e}^2} - \frac{5}{2}\right) v\xi \nabla \ln T.$$

Transform back to lab coordinates:

$$C[f] = C_0[f] + \epsilon_{\mathrm{w}} C_1[f] + \cdots$$

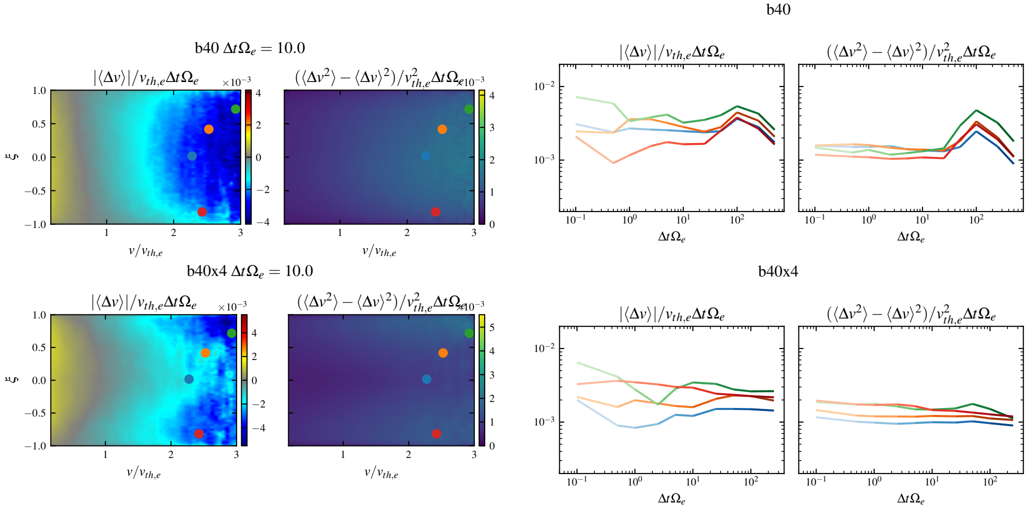
where $\epsilon_{\mathrm{w}} \sim v_{\mathrm{w}}/vthe$.

Chapman-Enskog Method: Background II

$\epsilon_{\rm w} \sim v_{\rm w}/vthe \sim 1/\beta_e$ is same order as diffusive flux (Drake et al. 2021).
Put all together and Invert to solve for ν:

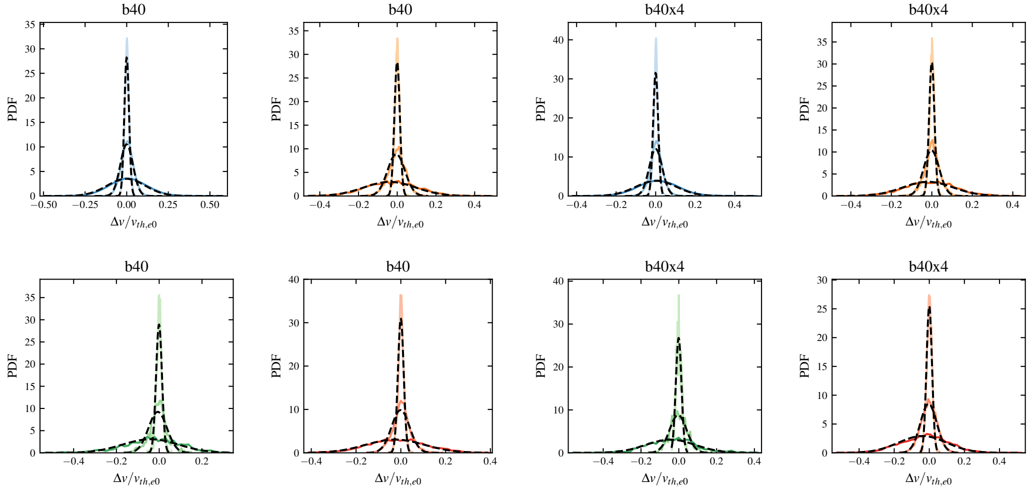
$$D_{\xi\xi}(v,\xi) = -f_0 \frac{1-\xi^2}{2}\left(\frac{v^2}{v_{\mathrm{th},e}^2} - \frac{5}{2}\right) v \nabla \ln T \Bigg/ \left(\frac{\partial f_1}{\partial \xi} + v_w \frac{\partial f_0}{\partial v}\right)$$

Fokker-Planck: Velocity Results I



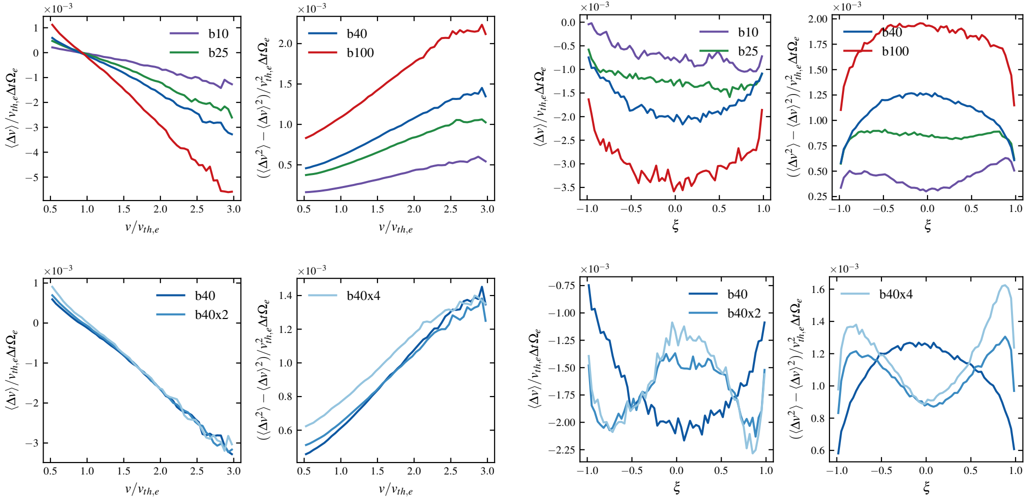
Fokker-Planck Method: Velocity Results II

PDFs of Δv for $\Delta t\Omega_e = \{1/10, 1, 10\}$

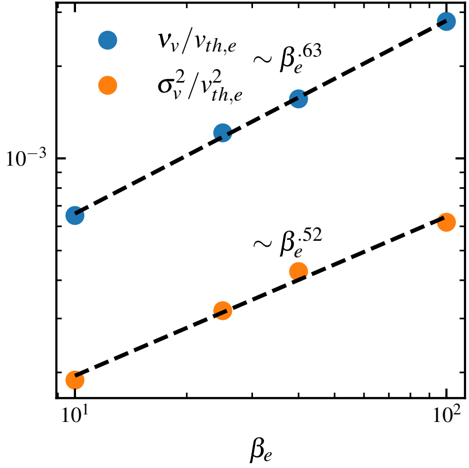


Fokker-Planck Method: Velocity Results III

$$\tau_{\text{ac}}^{\text{lin}} \sim \Omega_e^{-1} < \Delta t \Omega_e = 10 \ll \nu^{-1}$$



Fokker-Planck Method: Velocity Results IV



$$A(v,\xi) = -\nu_{\mathrm{v}}(\xi,\beta_e)\left(v - v_{\mathrm{th},e0}\right)$$

$$B(v,\xi) = \frac{\sigma_{\mathrm{v}}(\xi,\beta_e)^2}{2} v$$

$$\nu_{\mathrm{v}}(\xi,\beta_e) = \nu_{\mathrm{v},0}\beta_e^{.63} f_{\nu_{\mathrm{v}}}(\xi)$$

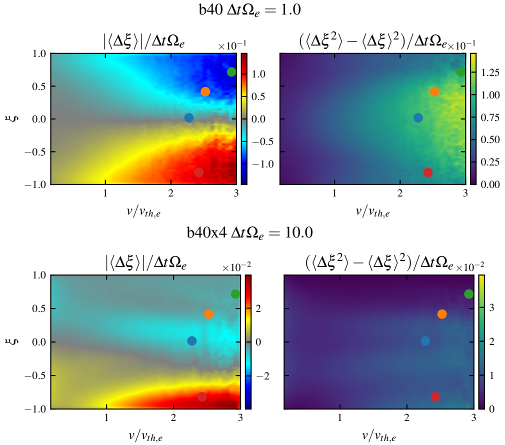
$$\sigma_{\mathrm{v}}^2(\xi,\beta_e) = \sigma_{\mathrm{v},0}^2\beta_e^{.52} f_{\sigma_{\mathrm{v}}^2}(\xi)$$

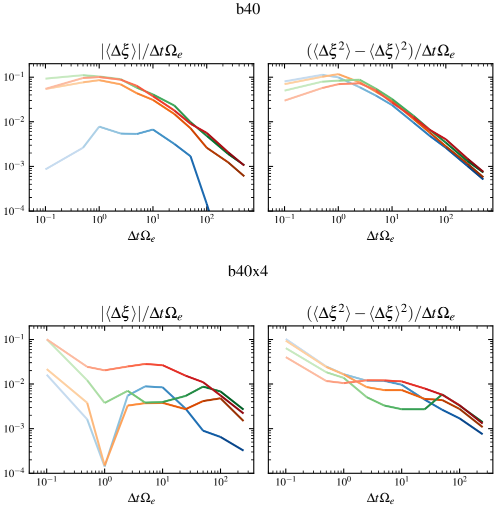
$f_{\nu_{\mathrm{v}}}(\xi)$ and $f_{\sigma_{\mathrm{v}}}(\xi)$ are nontrivial functions of $\delta B/B_0$

Does not explain $q_{\|} \sim 1/\beta_e$

Fokker-Planck Method: Pitch-Angle Results I

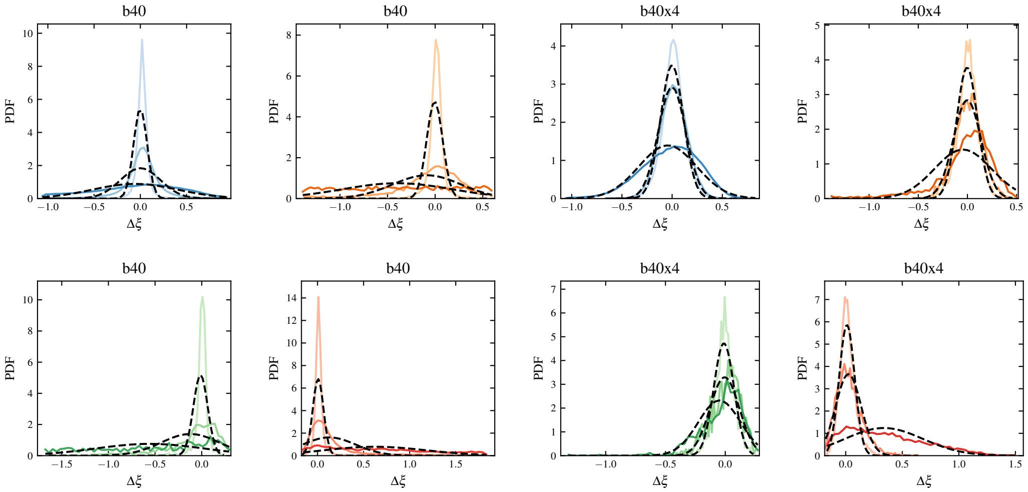
Pitch-angle scattering dominates velocity





Fokker-Planck Method: Pitch-Angle Results II

PDFs of $\Delta\xi$ for $t\Omega_e = \{1/10, 1, 10\}$



Fokker-Planck Method: Pitch-Angle Results III

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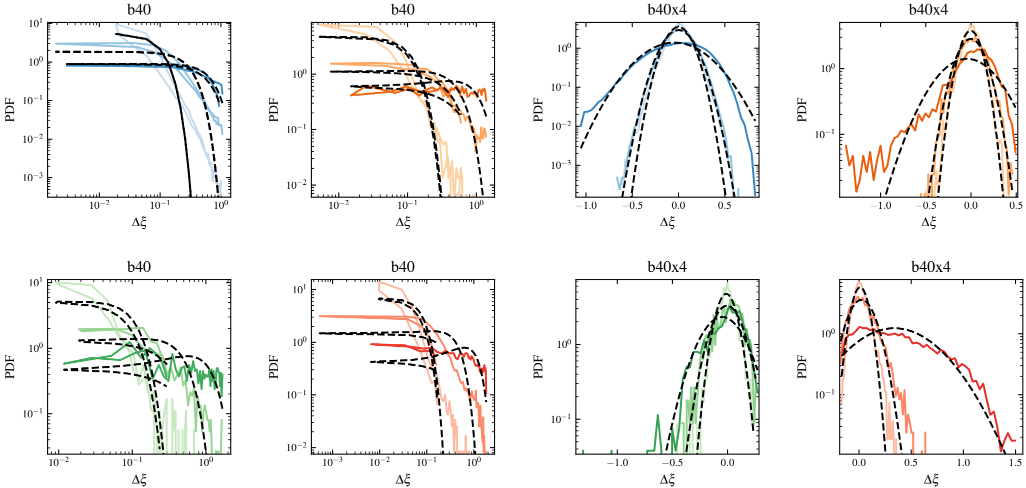
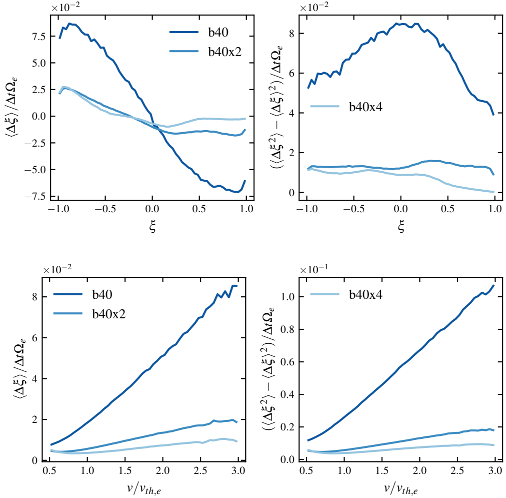


Issues:

Fokker-Planck Method: Pitch-Angle Results IV

ξ-average



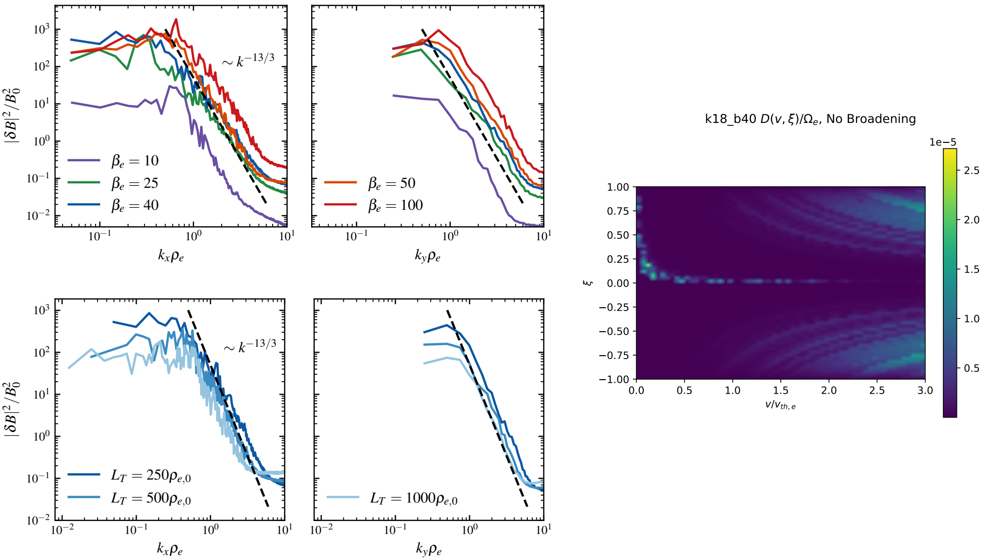
b40: $t\Omega_e = 1$
b40x2: $t\Omega_e = 10$
b40x4 : $t\Omega_e = 10$

b40x(2,4) flat because (?)

$$\langle \Delta\xi^2 \rangle^{1/2} \sim L_{B_{\xi\xi}} = \left(\frac{\partial \ln B_{\xi\xi}}{\partial \xi} \right)^{-1}$$

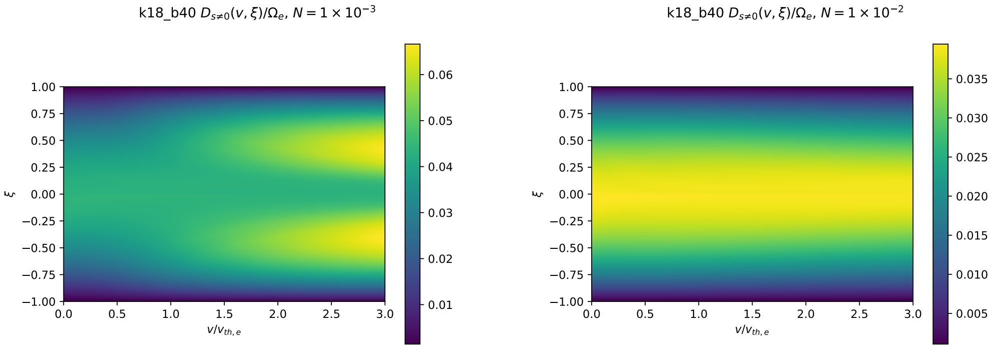
What is the asymptotic shape of the collision operator?

Quasilinear Method: Results I



Quasilinear Method: Results II

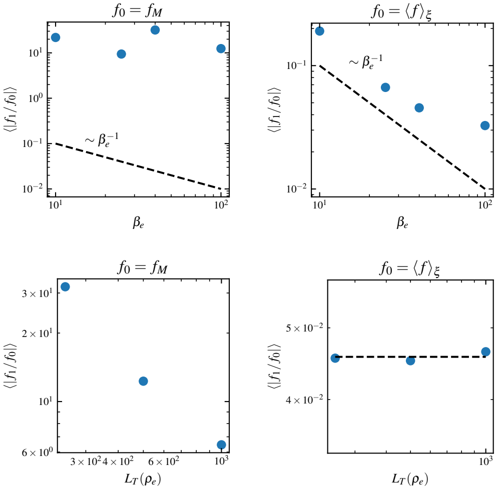
Sufficiently wide broadening gives large scattering at $v_{\|} = 0$.



Chapman-Enskog Method: Results I

Two methods to construct f_0:

- $f_0 = \langle f \rangle_\xi$
  - Simple pitch-angle average
  - Ensures $\int \mathrm{d}\boldsymbol{w}\boldsymbol{w}^{(0,1,2)} f_1 = 0$
  - v-dependence follows f - not guaranteed Maxwellian
- $f_0 = f_M$
  - Maxwellian constructed from moments
  - $\int \mathrm{d}\boldsymbol{w}\boldsymbol{w}^{(0,1,2)} f_1 \neq 0$
  - v-dependence guaranteed Maxwellian



Chapman-Enskog Method: Results II

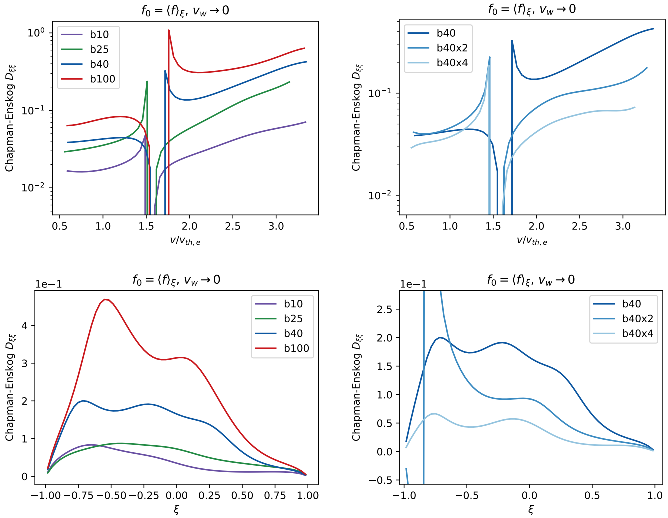
$\langle f \rangle_\xi$ follows expected scaling – f_M does not

Our expansion ordering is formally incorrect. Let's say you didn't check and try $f_0 = \langle f \rangle_\xi$ anyway. In the limit $v_{\rm w} \to 0$:

$$D_{\xi\xi}(v,\xi) = -f_0 \frac{1-\xi^2}{2}\left(\frac{v^2}{v_{\mathrm{th},e}^2} - \frac{5}{2}\right) v \nabla \ln T \left(\frac{\partial f_1}{\partial \xi}\right)^{-1}$$

and velocity dependence of f_0 doesn't matter

Chapman-Enskog Method: Results III



- Messy: 0s in the numerator do not exactly cancel singularities in the denominator
- Hotward-propagating electrons diffuse the most
- Exponential scaling in velocity (?)

Nonsense?

Drake 2021 Model

Pitch-angle scattering in the whistler frame $v'_{\parallel} = v_{\parallel} - v_w$

$$C[f] = \frac{\partial}{\partial \xi'} \frac{1-\xi'^2}{2} \nu_{\mathrm{w}}(v') \frac{\partial f}{\partial \xi'}.$$

$$\nu_{\mathrm{w}} = 0.1\Omega_e \left(\frac{\delta B}{B_0}\right)^2 \left(\frac{v}{v_{\mathrm{th},e}}\right)^{4/3}$$

- ν_{w} independent of ξ.
- 0.1 prefactor estimated from their simulations
- $v^{4/3}$ dependence comes from $k^{-7/3}$ electron-MHD cascade (Biskamp et a. 1999) and quasilinear argument.

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- ν_{w} independent of ξ. NOT true for quasilinear operators nor our FP operator
- 0.1 prefactor estimated from their simulations
- $v^{4/3}$ dependence comes from $k^{-7/3}$ electron-MHD cascade (Biskamp et a. 1999) and quasilinear argument. Our FP operator suggests closer to $\sim v$

Conclusions

1. We didn't have enough scale separation.
   - We tried to go larger L_T (lower $\delta B/B_0$), but PIC noise suppressed the instability.
   - $\delta B/B_0 \ll 1$ for Gaussian statistics
   - Presumably there are real collisionless systems where $\delta B/B_0 \geq 1/10$ where this is important

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2. Why does

   $$\nu_{\mathrm{w}} \sim \frac{\beta_e v_{\mathrm{th},e}}{L_T} \sim \frac{\delta B^2}{B_0^2}|\Omega_e|$$

   carry deep into large $\delta B/B_0$ where other methods fail?
   - Can this be used to build a generic nonlinear diffusion model?
   - Refine a resonance broadening model?

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3. What is the message here?