Cross Scale Interaction Mechanisms: Suppression of the ETG linear instability by ion gyroradius scale turbulence

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Outline

- Motivation for studying multi-scale turbulence
- Brief introduction to theory of scale-separated ion scale (IS) and electron scale (ES) turbulence
- Insights gained by simulations probing the effect of IS turbulence on ES instabilities

Introduction

Anomalous transport is driven by turbulence,

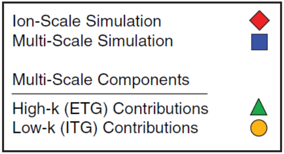
- IS : at scales where $k\rho_i \lesssim 1$
- ES : at scales where $k\rho_e \lesssim 1 \ll k\rho_i$
- N.B. $\rho_e/\rho_i \sim (m_e/m_i)^{1/2} \simeq 1/60 \ll 1$

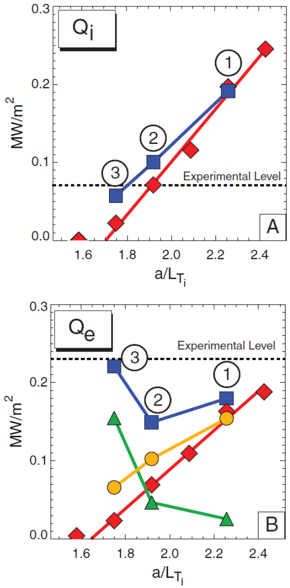
- do all scales matter?
- is cross scale coupling important?

- To answer these questions we take a scale separated approach
- $(m_e/m_i)^{1/2} \to 0$

Introduction: do all scales matter?

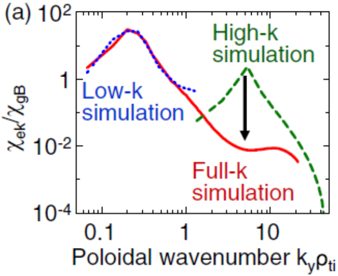
- simulation evidence where $Q_e \sim 10 Q_{e\mathrm{gB}} \sim (?) Q_{i\mathrm{gB}}$ e.g. Jenko and Dorland (2002)
- recent experimental evidence on NSTX Ren et al. (2017)
- Howard et al. (2016) Fig 3:



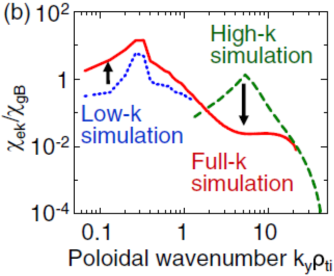


Introduction: is cross scale coupling important?

- Fig 2 from Maeyama et al. (2015):



(a) $\beta = 0.04\%$



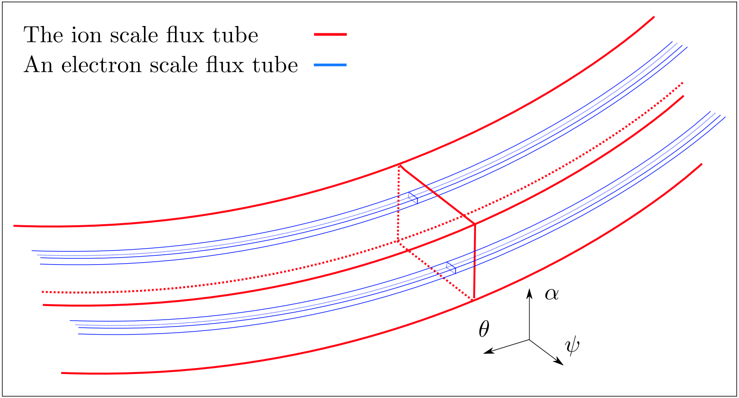
(b) $\beta = 2.0\%$

- See also

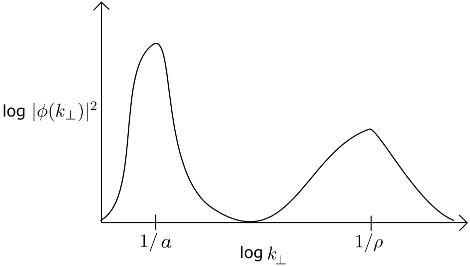
  Maeyama et al. (2017); Howard et al. (2016); Bonanomi et al. (2018)

  Görler and Jenko (2008); Candy et al. (2007); Waltz et al. (2007)

Introduction: a scale separated approach



A Quick Reminder: Scale separation in local δf turbulence



- scale separation: $\rho_* = \rho/a \to 0 \Rightarrow \quad f = F + \delta f$
- statistical periodicity: $\langle \delta f \rangle_{\text{turb}} = 0$
- gyro average: $\langle \cdot \rangle|_{\mathbf{R}}^{\text{gyro}}$
- orderings:

$$\delta f \sim \rho_* F$$

$$\nabla F \sim \nabla_\perp \delta f \sim \rho_*^{-1} \nabla_\| \delta f$$

$$\partial_t \delta f \sim (v_t/a)\delta f \sim \rho_* \Omega \delta f$$

$$\partial_t F \sim \rho_*^3 \Omega F$$

A Quick Reminder: The local δf Gyrokinetic Equation

The gyrokinetic equation for $h = \delta f + (Ze\phi/T)F_0$:

$$\frac{\partial h}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial h}{\partial\theta} + (\mathbf{v}^M + \mathbf{v}^E)\cdot\nabla h + \mathbf{v}^E\cdot\nabla F_0 = \frac{ZeF_0}{T}\frac{\partial\varphi}{\partial t}, \tag{1}$$

where,

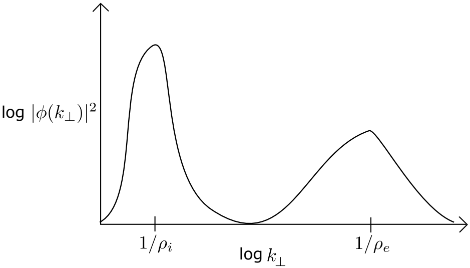
$$\varphi = \langle\phi\rangle|_{\mathbf{R}}^{\mathrm{gyro}}, \quad \mathbf{v}^E = \frac{c}{B}\mathbf{b}\wedge\nabla\varphi. \tag{2}$$

Closed by quasi-neutrality,

$$\sum_\alpha Z_\alpha \int d^3\mathbf{v}|_{\mathbf{r}} h_\alpha = \sum_\alpha \frac{Z_\alpha^2 e n_\alpha}{T_\alpha}\phi(\mathbf{r}). \tag{3}$$

- electrostatic approximation
- zero equilibrium toroidal rotation
- (h for compactness – we later find $g = \langle\delta f\rangle|_{\mathbf{R}}^{\mathrm{gyro}} = h - (Ze\varphi/T)F_0$ is more convenient)

Separating IS and ES Turbulence



- scale separation: $\rho_e/\rho_i \sim v_{ti}/v_{te} \sim \sqrt{m_e/m_i} \to 0$, $\Rightarrow \delta f = \overline{\delta f} + \widetilde{\delta f}$
- ES statistical periodicity: $\left\langle \widetilde{\delta f} \right\rangle^{\mathrm{ES}} = 0$
- orderings:

$$\nabla_\perp \overline{\delta f} \sim \rho_i^{-1} \overline{\delta f}, \quad \frac{\partial \overline{\delta f}}{\partial t} \sim \frac{v_{ti}}{a} \overline{\delta f}$$

$$\nabla_\perp \widetilde{\delta f} \sim \rho_e^{-1} \widetilde{\delta f}, \quad \frac{\partial \widetilde{\delta f}}{\partial t} \sim \frac{v_{te}}{a} \widetilde{\delta f}.$$

Separating IS and ES Turbulence: Size of the Fluctuations – Possible impacts of cross-scale interaction

We show in Hardman et al. (2019) that the only ordering which allows saturated dominant balance is

$$\nabla F_0 \sim \nabla_\perp \overline{\delta f} \sim \nabla_\perp \widetilde{\delta f}, \tag{4}$$

resulting in the usual gyro-Bohm ordering,

$$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\widetilde{\phi}}{T} \sim \rho_{e*} \tag{5}$$

$$\frac{\overline{h}_i}{F_{0i}} \sim \frac{\overline{h}_e}{F_{0e}} \sim \frac{e\overline{\phi}}{T}, \quad \frac{\widetilde{h}_e}{F_{0e}} \sim \frac{e\widetilde{\phi}}{T}, \quad \frac{\widetilde{h}_i}{F_{0i}} \sim \left(\frac{m_e}{m_i}\right)^{1/4} \frac{e\widetilde{\phi}}{T} \tag{6}$$

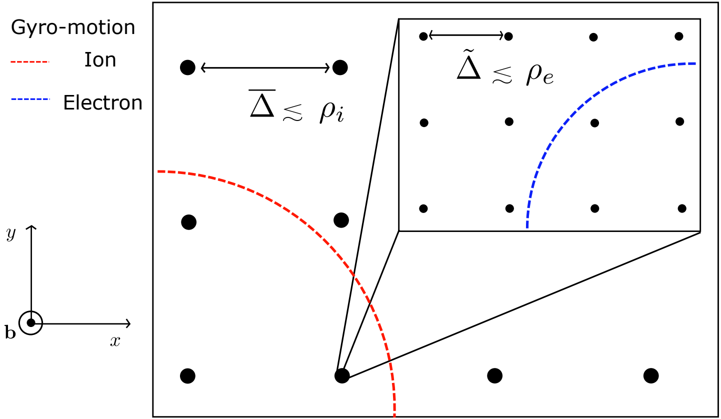
(4) $\Rightarrow$ ES eddies can be driven or suppressed by gradients of the IS distribution function

(4) $\Rightarrow \nabla\widetilde{\phi} \sim \nabla\overline{\phi} \Rightarrow$ eddy E×B drifts $v_{\mathrm{E\times B}}$, are comparable at all scales
Critical balance $\Rightarrow$ parallel correlation lengths are the same for IS and ES eddies

$\Rightarrow$ ES eddies can be sheared by the IS $E \times B$ drift in the direction parallel to the magnetic field

Separating IS and ES Turbulence: Difficult Points

- electrons at IS – fast electron streaming timescales are removed by the orbital average $\langle\cdot\rangle^{\mathrm{o}}$ – $\overline{h}_e = 0$ (adiabatic response) for non-zonal passing electrons
- ions at ES – non-locality of the gyro average – adiabatic response
- the parallel boundary condition for the ES flux tubes



Separating IS and ES Turbulence – The Coupled Equations

- IS equations, where the leading-order cross-scale terms are small

$$\frac{\partial \overline{h}_i}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla\theta \frac{\partial \overline{h}_i}{\partial \theta} + (\mathbf{v}_i^M + \overline{\mathbf{v}}_i^E)\cdot\nabla\overline{h}_i + \overline{\mathbf{v}}_i^E\cdot\nabla F_{0i} = \frac{Z_i e F_{0i}}{T_i}\frac{\partial \overline{\varphi}_i}{\partial t}, \tag{7}$$

$$\frac{\partial \overline{h}_e}{\partial t} + \left\langle \mathbf{v}_e^M\cdot\nabla\alpha\right\rangle^{\mathrm{o}} \frac{\partial \overline{h}_e}{\partial \alpha} + \left\langle \overline{\mathbf{v}}_e^E\cdot\nabla\overline{h}_e\right\rangle^{\mathrm{o}} + \left\langle \overline{\mathbf{v}}_e^E\cdot\nabla F_{0e}\right\rangle^{\mathrm{o}} = -\frac{eF_{0e}}{T_e}\frac{\partial \left\langle\overline{\varphi}_e\right\rangle^{\mathrm{o}}}{\partial t}, \tag{8}$$

$$\int d^3\mathbf{v}|_{\mathbf{r}}(Z_i\overline{h}_i - \overline{h}_e) = \left(\frac{eZ_i^2 n_i}{T_i} + \frac{en_e}{T_e}\right)\overline{\phi}, \tag{9}$$

- ES equations, with the new advection and drive terms

$$\frac{\partial \widetilde{h}_e}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla\theta\frac{\partial \widetilde{h}_e}{\partial \theta} + (\mathbf{v}_e^M + \widetilde{\mathbf{v}}_e^E + \overline{\mathbf{v}}_e^E)\cdot\nabla\widetilde{h}_e + \widetilde{\mathbf{v}}_e^E\cdot(\nabla\overline{h}_e + \nabla F_{0e}) = -\frac{eF_{0e}}{T_e}\frac{\partial\widetilde{\varphi}_e}{\partial t}. \tag{10}$$

$$-\int d^3\mathbf{v}|_{\mathbf{r}}\widetilde{h}_e = \left(\frac{eZ_i^2 n_i}{T_i} + \frac{en_e}{T_e}\right)\widetilde{\phi}, \tag{11}$$

Separating IS and ES Turbulence – The Coupled Equations

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$$\frac{\partial \overline{h}_i}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla\theta \frac{\partial \overline{h}_i}{\partial \theta} + (\mathbf{v}_i^M + \overline{\mathbf{v}}_i^E)\cdot\nabla\overline{h}_i + \overline{\mathbf{v}}_i^E \cdot \nabla F_{0i} = \frac{Z_i e F_{0i}}{T_i}\frac{\partial \overline{\varphi}_i}{\partial t}, \tag{7}$$

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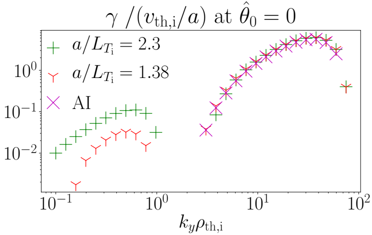
$$\frac{\partial \widetilde{h}_e}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla\theta \frac{\partial \widetilde{h}_e}{\partial \theta} + (\mathbf{v}_e^M + \widetilde{\mathbf{v}}_e^E + \overline{\mathbf{v}}_e^E)\cdot\nabla\widetilde{h}_e + \widetilde{\mathbf{v}}_e^E\cdot(\nabla\overline{h}_e + \nabla F_{0e}) = -\frac{eF_{0e}}{T_e}\frac{\partial \widetilde{\varphi}_e}{\partial t}. \tag{10}$$

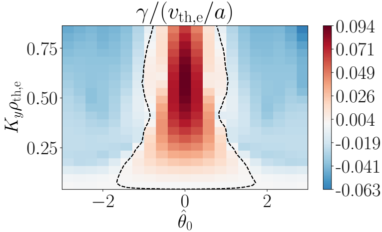
$$-\int d^3\mathbf{v}|_{\mathbf{r}} \widetilde{h}_e = \left(\frac{eZ_i^2 n_i}{T_i} + \frac{en_e}{T_e} \right) \widetilde{\phi}, \tag{11}$$

N.b. writing (10) in terms of $\overline{g}_e$ and $\widetilde{g}_e$ shows that the constant in θ piece of $\overline{\mathbf{v}}_e^E$ can be removed by a rotation $\Rightarrow$ The parallel-to-the-field variation of $\overline{\mathbf{v}}_e^E$ matters

The Effect of Cross Scale Interaction on the ES ETG Instability

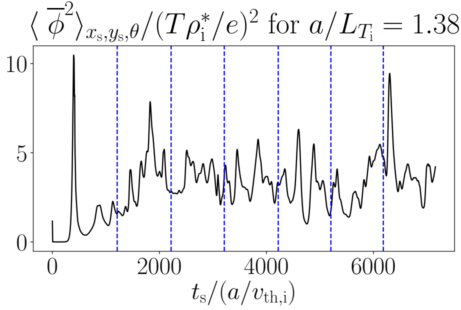
- The coupled equations capture the $O(1)$ effects of IS turbulence on ES fluctuations
- We pick Cyclone Base Case like (CBC) parameters where there is a separation of scales:

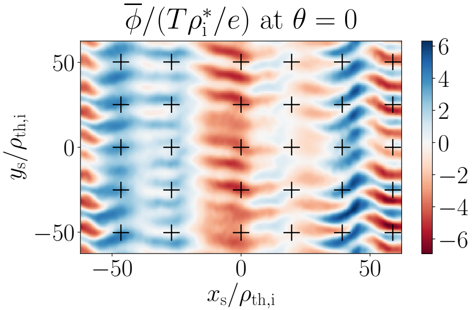




- We simulate the IS turbulence to obtain a sample of $\overline{\mathbf{v}}_e^E$ and $\nabla\overline{g}_e$
- We observe the effect on the ES ETG instability:
  Strongly driven ETG ($a/L_{T_\mathrm{e}} = 2.3$)
  - persists in the presence of weakly driven ($a/L_{T_\mathrm{i}} = 1.38$) IS turbulence
  - is suppressed by strongly driven ($a/L_{T_\mathrm{i}} = 2.3$) IS turbulence

Sampling IS Turbulence with $a/L_{T_\mathrm{i}} = 1.38$





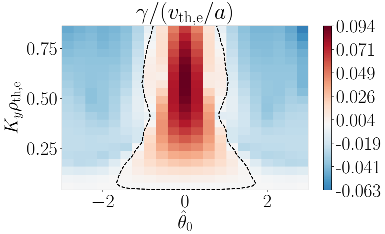
- Saturate IS turbulence
- Calculate $\nabla \overline{g}_e$
- Calculate $\overline{\mathbf{v}}_e^E$
- At 6 IS t_s times (blue dashes)
- At 6 radial (x_s) $\times$ 5 binormal (y_s) IS positions (crosses)

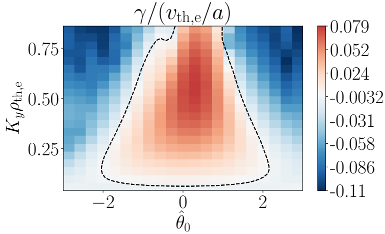
Simulations: modification of ES linear physics: CBC $a/L_{T_i} = 1.38$

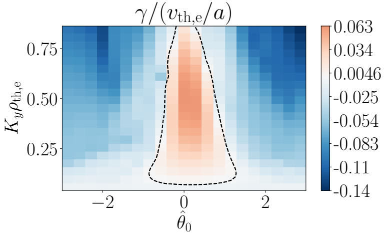
Top Right: No IS gradients.

Below: IS gradients from different IS (x_s, y_s) locations

Moderate suppression







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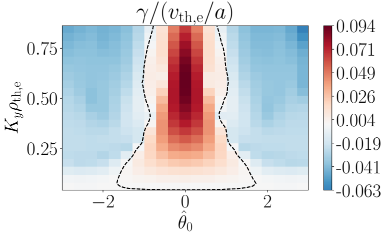
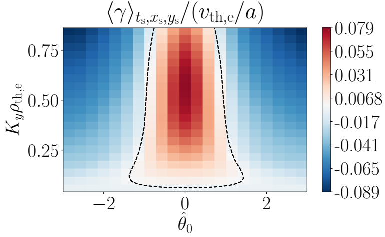
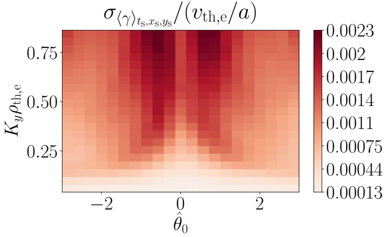


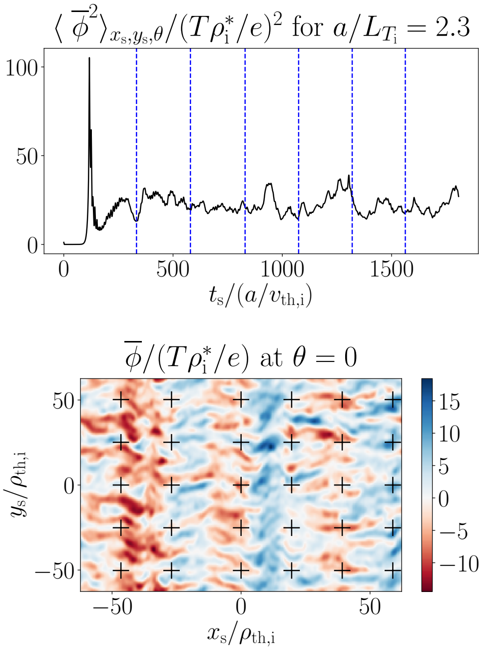
Below: average ETG growth rate across all sampled IS (x_s, y_s) locations and t_s times

Weak suppression!





Sampling IS Turbulence with $a/L_{T_\mathrm{i}} = 2.3$



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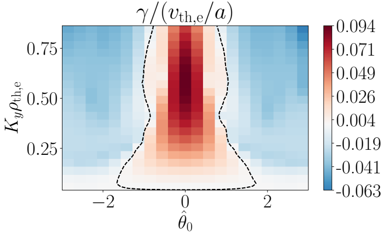

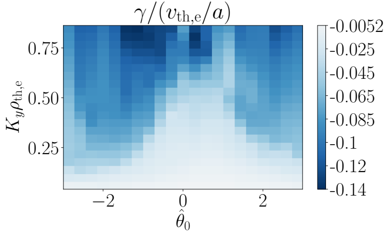

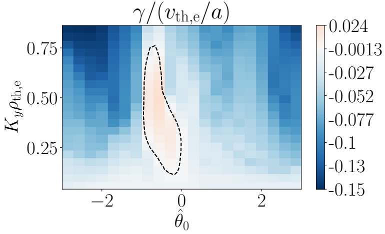

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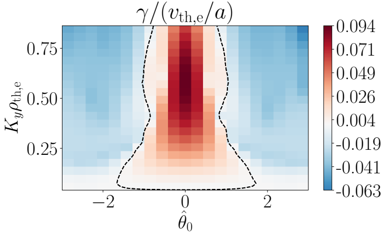
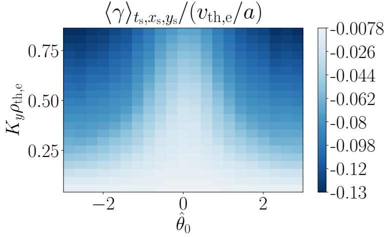
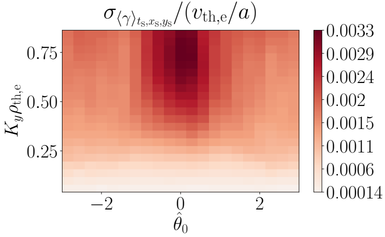


Below: average ETG growth rate across all sampled IS (x_s, y_s) locations and t_s times

Strong suppression!





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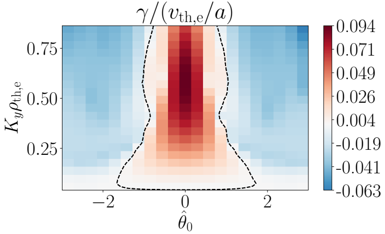

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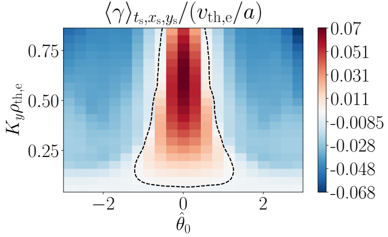
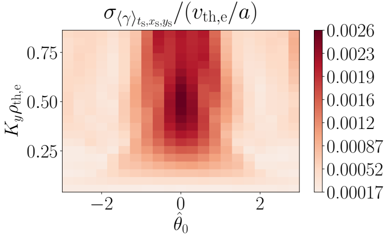
INCLUDING ONLY $\nabla \overline{g}_e$ (with $\overline{\mathbf{v}}_e^E = 0$)

weak suppression!





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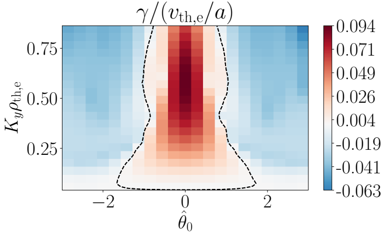
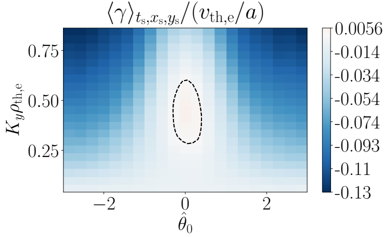
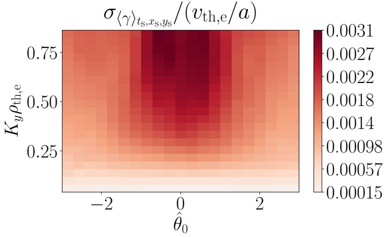
INCLUDING ONLY $\overline{\mathbf{v}}_e^E$ (with $\nabla \overline{g}_e$=0)

Strong suppression!







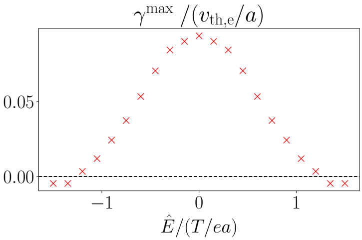
Simulations: A simple model of parallel-to-be-field shear in $\overline{\mathbf{v}}_e^E$

$$\overline{\mathbf{v}}_e^E \cdot \mathbf{k}_f = \hat{\omega}_E \theta, \quad (12)$$

- Simplest possible form for $\overline{\mathbf{v}}_e^E$ with local parallel-to-the-field shear (consistent with flux tube $\parallel$ b.c. Beer et al. (1995))
- (13) leads to $\overline{\mathbf{v}}_e^E \cdot \mathbf{k}_f$ with linear variation e.g. (12) for $K_x = 0$ (and our parameters)
- maximum ETG growth rate $\gamma^{\max}(\hat{E})$ shows suppression for all $\hat{E} \neq 0$
- $\Rightarrow$ Qualitative explanation of ETG behaviour in the presence of IS turbulence
- suppression when $\hat{\omega}_E = 0.4 \times 0.5 \times 1.0 \left(\frac{v_{\mathrm{th,e}}}{a}\right) \sim \gamma^{\max}(\hat{E} = 0) \simeq 0.1$

$$\left.\frac{\partial \overline{\phi}}{\partial y_{\mathrm{s}}}\right|_{x_{\mathrm{s}}} = -\hat{E}, \quad \left.\frac{\partial \overline{\phi}}{\partial x_{\mathrm{s}}}\right|_{y_{\mathrm{s}}} = -\hat{s}\theta\hat{E}, \quad (13)$$

$$\hat{\omega}_E = 0.4 \left(\frac{v_{\mathrm{th,e}}}{a}\right)(K_y \rho_{\mathrm{th,e}})\left(\frac{\hat{E}}{T/ea}\right). \quad (14)$$

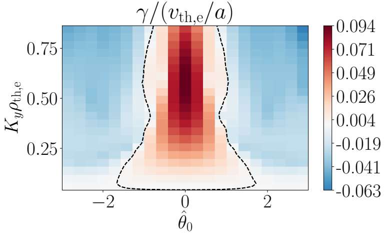


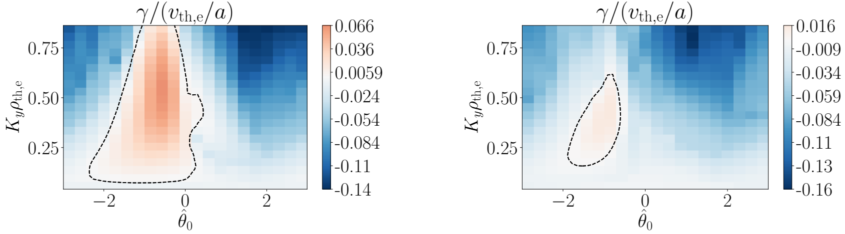
Simulations: A simple model of parallel-to-be-field shear in $\overline{\mathbf{v}}_e^E$

Top Right: No IS gradients.

Below: ETG growth rate with model $\overline{\mathbf{v}}_e^E$; (left) $\hat{E} = 0.5T/ea$ (right) $\hat{E} = 1.0T/ea$

Strong suppression!





Conclusions

We have derived coupled, scale-separated equations for IS and ES turbulence.

We assumed

- $(m_e/m_i)^{1/2} \to 0$; space and time separation; no other small parameters
- spatial isotropy – IS $l_\perp \sim \rho_{\mathrm{th,i}}$; ES $l_\perp \sim \rho_{\mathrm{th,e}}$
- negligible direct cascade; separation in the fluctuation spectrum

The model

- efficiently captures cross-scale interactions which persist as $(m_e/m_i)^{1/2} \to 0$
- is simulated in a system of coupled ES flux tubes nested in an IS flux tube

We found that

- strongly driven ETG ($a/L_{T_\mathrm{e}} = 2.3$)
    - persists in the presence of weakly driven ($a/L_{T_\mathrm{i}} = 1.38$) IS turbulence
    - is suppressed by strongly driven ($a/L_{T_\mathrm{i}} = 2.3$) IS turbulence
- the primary mechanism responsible for the suppression is parallel-to-the-field variation in $\overline{\mathbf{v}}_e^E$
- a simple model of local parallel-to-the-field variation in the flow qualitatively explains the result

Questions for Future Work

- Can we retain the effect of ES turbulence on IS fluctuations by taking other parameters to be small?
  - distance to marginal stability
  - zonal to non-zonal amplitude
- By taking other parameters to be small, can we find scalings where the ES fluctuation amplitude is comparable to the IS fluctuation amplitude?
- What is the perpendicular scale of an ETG streamer? Dorland et al. (2000); Jenko et al. (2000); Jenko and Dorland (2002); Guttenfelder and Candy (2011)
- Is it possible to enforce time scale separation if ETG turbulence saturates slowly? Colyer et al. (2017); Nakata et al. (2010)
- What is the effect of IS turbulence on non-linear ETG saturation?
- What changes in this picture with electromagnetic fluctuations?

Thank You for Listening!

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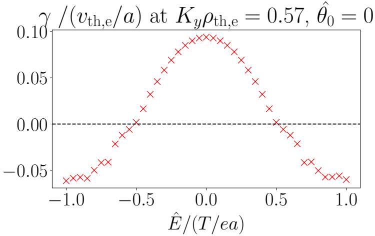
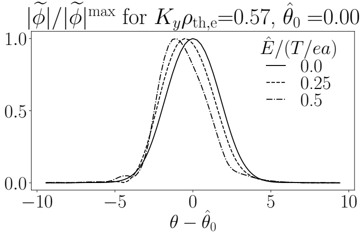
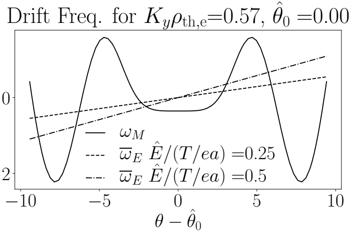
Simulations: A simple model of parallel-to-be-field shear in $\overline{\mathbf{v}}_e^E$

Top Right: ETG growth rate $\gamma(\hat{E})$ for $K_y \rho_{\mathrm{th,e}} = 0.57$, $\hat{\theta}_0 = 0.0$.

Below: (left) corresponding eigenmodes (right) corresponding drift coefficients







Separating IS and ES Turbulence: Technicalities

- We introduce a fast spatial variable $\mathbf{r}_f$ and a slow spatial variable $\mathbf{r}_s$ and the fast and slow times t_f, t_s
- In the gyrokinetic equation we send,

$$\delta f(t, \mathbf{r}) \rightarrow \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f), \quad \nabla \rightarrow \nabla_s + \nabla_f, \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}, \tag{15}$$

- then asymptotically expand in the mass ratio $(m_e/m_i)^{1/2}$
- remembering $\nabla_s \sim (m_e/m_i)^{1/2}\nabla_f$, and $\partial/\partial t_s \sim (m_e/m_i)^{1/2}\partial/\partial t_f$
- explicitly define the ES average,

$$\overline{\delta f}(t_s, \mathbf{r}_s) = \left\langle \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f) \right\rangle^{\mathrm{ES}} = \frac{1}{\tau_c A} \int_{t_s - \tau_c/2}^{t_s + \tau_c/2} dt_f \int_{A, \mathbf{r}_s} d^2\mathbf{r}_f \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f), \tag{16}$$

- We assume that,

$$\delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f) = \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f + n\Delta_{cx}\hat{\mathbf{x}} + m\Delta_{cy}\hat{\mathbf{y}}), \tag{17}$$

- This enforces $\left\langle \widetilde{\delta f} \right\rangle^{\mathrm{ES}} = 0$.

Splitting the Quasi-Neutrality Relation

- We split the guiding centre into a slow $\mathbf{R}_s$ and a fast $\mathbf{R}_f$ part.
- $\mathbf{R} = \mathbf{r} - \rho(\mathbf{r})$, where $\rho(\mathbf{r})$ is the vector gyroradius
- Thus using the periodicity property equation (17) the ES average may be taken over guiding centre or real space coordinates.
- This observation allows us to note that the ES average commutes with the gyro average,

$$\left\langle \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\phi(\mathbf{r}_s,\mathbf{r}_f)\right\rangle^{\mathrm{ES}} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\left\langle\phi(\mathbf{r}_s,\mathbf{r}_f)\right\rangle^{\mathrm{ES}} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\overline{\phi}(\mathbf{r}_s), \tag{18}$$

The splitting of the quasi neutrality relation follows directly,

$$\sum_\alpha Z_\alpha \int d^3\mathbf{v}|_{\mathbf{r}}\overline{h}_\alpha(\mathbf{R}_s) = \sum_\alpha \frac{Z_\alpha^2 e n_\alpha}{T_\alpha}\overline{\phi}(\mathbf{r}_s), \tag{19}$$

$$\sum_\alpha Z_\alpha \int d^3\mathbf{v}|_{\mathbf{r}}\widetilde{h}_\alpha(\mathbf{R}_s,\mathbf{R}_f) = \sum_\alpha \frac{Z_\alpha^2 e n_\alpha}{T_\alpha}\widetilde{\phi}(\mathbf{r}_s,\mathbf{r}_f). \tag{20}$$

Addressing the Non-Locality of the Gyro Average

- Taking the gyro average at fixed guiding centre $\langle\cdot\rangle|^{\mathrm{gyro}}_{\mathbf{R}}$, couples multiple $\mathbf{r}_s$ points.
- but we aim to find scale separated equations!
- Expanding both the slow and the fast spatial variable in Fourier series we note that,

$$\begin{aligned}
\widetilde{\varphi}(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) = \langle\widetilde{\phi}(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f)\rangle|^{\mathrm{gyro}}_{\mathbf{R}} &= \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\widetilde{\phi}(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}} \sum_{\mathbf{k}_s,\mathbf{k}_f} \widetilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{r}_s} e^{i\mathbf{k}_f\cdot\mathbf{r}_f} \\
&= \sum_{\mathbf{k}_s,\mathbf{k}_f} \widetilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{R}_s} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho),
\end{aligned} \tag{21}$$

for electrons:

- $|\mathbf{k}_f|\rho_e \sim 1$ and $|\mathbf{k}_s|\rho_e \sim (m_e/m_i)^{1/2}$
- we can expand the Bessel function to return to a local picture in the slow variable with $O(m_e/m_i)^{1/2}$ error.
- We will exploit this in scale separation.

for ions:

- $|\mathbf{k}_s|\rho_i \sim 1$ and $|\mathbf{k}_f|\rho_i \sim (m_e/m_i)^{-1/2}$.
- we are unable to expand the Bessel function
- we are unable to avoid the coupling of multiple $\mathbf{r}_s$ in the equations for ions at ES

Addressing the Non-Locality of the Gyro Average: continued

- we can neglect the ion contribution to ES quasi neutrality
  - ion gyroradius >> ES fluctuation scale length → ion can only respond to a large-scale average of ES potential
  - $J_0(|\mathbf{k}_f|\rho_i) \sim (m_e/m_i)^{1/4} << 1$
- Hence,

$$\widetilde{\varphi}_e(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) = \sum_{\mathbf{k}_s,\mathbf{k}_f} \widetilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{R}_s} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho)$$

$$= -\frac{T_e}{n_e e}\sum_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{R}_s} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho)\int d^3\mathbf{v}\, \widetilde{h}_{e,\mathbf{k}_s,\mathbf{k}_f} J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho) \quad (22)$$

- now we use that,

$$J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho_e) = J_0(|\mathbf{k}_f|\rho_e) + O\left(\mathbf{k}_s\cdot\mathbf{k}_f\rho_e^2\frac{dJ_0(z)}{dz}|_{z=|\mathbf{k}_f|\rho_e}\right), \quad (23)$$

- exploit that $|\mathbf{k}_s|\rho_e \sim (m_e/m_i)^{1/2}$ to bring $\mathbf{R}_s$ under the velocity integral
- regard $\mathbf{R}_s$ as a fixed parameter in the integration, to find,

$$\widetilde{\varphi}_e(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) = -e\left(\sum_\nu \frac{Z_\nu^2 n_\nu e^2}{T}\right)^{-1}\sum_{\mathbf{k}_f} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho)$$

$$\times\int d^3\mathbf{v}|_{\mathbf{R}_s}\widetilde{h}_{e\mathbf{k}_f}(\mathbf{R}_s)J_0(|\mathbf{k}_f|\rho_e)\left(1+O\left((m_e/m_i)^{1/2}\right)\right) \quad (24)$$

- we can evaluate quasi-neutrality purely locally in the slow variable.

Splitting the Gyrokinetic Equation

- we apply the ES average to the gyrokinetic equation
- we neglect terms which are small by $(m_e/m_i)^{1/2}$

Ion scale equation:

$$\frac{\partial \overline{h}}{\partial t_s}+v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial \overline{h}}{\partial \theta}+(\mathbf{v}^M+\overline{\mathbf{v}}^E)\cdot\nabla_s\overline{h}+\nabla_s\cdot\left\langle\frac{c}{B}\widetilde{h}\widetilde{\mathbf{v}}^E\right\rangle^{\mathrm{ES}}+\overline{\mathbf{v}}^E\cdot\nabla F_0=\frac{ZeF_0}{T}\frac{\partial\overline{\varphi}}{\partial t_s}. \quad (25)$$

- we subtract the IS equation from the full equation and neglect terms
- The electron equation is orbital averaged to remove fasts electron streaming timescales
- We consistently take $\overline{h}_e=0$ for non-zonal passing electrons, for which $\overline{h}_e\sim(m_e/m_i)^{1/2}$

ES equation:

$$\frac{\partial \widetilde{h}}{\partial t_f}+v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial \widetilde{h}}{\partial \theta}+(\mathbf{v}^M+\widetilde{\mathbf{v}}^E+\overline{\mathbf{v}}^E)\cdot\nabla_f\widetilde{h}+\widetilde{\mathbf{v}}^E\cdot(\nabla_s\overline{h}+\nabla F_0)=\frac{ZeF_0}{T}\frac{\partial\widetilde{\varphi}}{\partial t_f}, \quad (26)$$

where

$$\overline{\mathbf{v}}^E=\frac{c}{B}\mathbf{b}\wedge\nabla_s\overline{\varphi},\quad \widetilde{\mathbf{v}}^E=\frac{c}{B}\mathbf{b}\wedge\nabla_f\widetilde{\varphi}. \quad (27)$$

Note that,

- there are two additional terms on the ES, $\widetilde{\mathbf{v}}^E\cdot\nabla_f\widetilde{h}$ and $\widetilde{\mathbf{v}}^E\cdot\nabla_s\overline{h}$
- there is one new term at the IS, $\nabla_s\cdot\left\langle\frac{c}{B}\widetilde{h}\widetilde{\mathbf{v}}^E\right\rangle^{\mathrm{ES}}$
- $\overline{\mathbf{v}}^E$ cannot be removed with the boost or a solid body rotation because of the θ dependence of $\overline{\varphi}$

Critical balance

Note $\nabla\widetilde{\phi} \sim \nabla\overline{\phi}$

$\Rightarrow$eddy E×B drifts $v_{\mathrm{E\times B}}$, are comparable at all scales

- applying the critical balance argument
- $v_{te}/\widetilde{l}_{\|} \sim \widetilde{\tau}_{nl}^{-1} \sim \widetilde{v}_{\mathrm{E\times B}}/\widetilde{l}_{\perp}$
- $v_{ti}/\bar{l}_{\|} \sim \overline{\tau}_{nl}^{-1} \sim \overline{v}_{\mathrm{E\times B}}/\bar{l}_{\perp}$
- $\widetilde{l}_{\|} \sim \bar{l}_{\|}$

$\Rightarrow$ parallel correlation lengths are the same for IS and ES eddies

$\Rightarrow$ parallel correlation length are set by the system size – distance between stabilising inboard midplane regions parallel to the field

$\Rightarrow \widetilde{l}_{\|} \sim \bar{l}_{\|} \sim a$

$\Rightarrow$ ES eddies are long enough to be differentially advected by $\overline{v}_{\mathrm{E\times B}}$

Scaling Work: the Relative Size of the Fluctuations

- The usual gyro-Bohm ordering is the only ordering which results in non-linearly saturated balance

$$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\widetilde{\phi}}{T} \sim \rho_{e*} \tag{28}$$

$$\frac{\overline{h}_i}{F_{0i}} \sim \frac{\overline{h}_e}{F_{0e}} \sim \frac{e\overline{\phi}}{T}, \quad \frac{\widetilde{h}_e}{F_{0e}} \sim \frac{e\widetilde{\phi}}{T}, \quad \frac{\widetilde{h}_i}{F_{0i}} \sim \left(\frac{m_e}{m_i}\right)^{1/4} \frac{e\widetilde{\phi}}{T} \tag{29}$$

- We can show that the following orderings are inconsistent with dominant balance under our assumptions
  - $$\frac{e\overline{\phi}}{T} \gg \rho_{i*} \tag{30}$$
  - $$\frac{e\widetilde{\phi}}{T} \gg \rho_{e*} \tag{31}$$
  - $$\frac{e\overline{\phi}}{T} \ll \rho_{i*}, \quad \frac{e\widetilde{\phi}}{T} \sim \rho_{e*} \tag{32}$$
- The following ordering is possible only when the ES fluctuations are stabilised by the IS turbulence
  - $$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\widetilde{\phi}}{T} \ll \rho_{e*} \tag{33}$$

Scaling Work: Neglecting Ions at ES

note that:

- $J_0(\mathbf{k}_f \rho_i) \sim (m_e/m_i)^{1/4}$
- so:

$$\int d^3\mathbf{v}|_{\mathbf{r}} \widetilde{h}_i \sim \left(\frac{m_e}{m_i}\right)^{1/4} \left(\frac{m_e}{m_i}\right)^{1/4} \frac{en\widetilde{\phi}}{T} \tag{34}$$

Ions at ES can be neglected to $O\left((m_e/m_i)^{1/2}\right)$ in the ES equations!

note that:

- $\nabla_s \cdot \left\langle \frac{c}{B} \widetilde{h}_i \widetilde{\mathbf{v}}_i^E \right\rangle^{\mathrm{ES}} \sim O\left((m_e/m_i) \overline{\mathbf{v}}_i^E \cdot \overline{h}_i\right)$

Ions at ES can be neglected to $O\left(m_e/m_i\right)$ in the IS equations!

Scaling Work: which multiscale terms do we keep?

The only remaining multiscale terms are in electron species equations:

note that:

- $\widetilde{\mathbf{v}}_e^E \cdot \nabla_s \overline{h}_e \sim \overline{\mathbf{v}}_e^E \cdot \nabla_f \widetilde{h}_e \sim \widetilde{\mathbf{v}}_e^E \cdot \nabla_f \widetilde{h}_e$
- IS gradients contribute at $O(1)$ to the ES
- IS perpendicular shear in $\overline{\mathbf{v}}_e^E$ can be neglected to $O((m_e/m_i)^{1/2})$ at the ES

- $\nabla_s \cdot \left\langle \frac{c}{B} \widetilde{h}_e \widetilde{\mathbf{v}}_e^E \right\rangle^{\mathrm{ES}} \sim O((m_e/m_i)^{1/2} \overline{\mathbf{v}}_e^E \cdot \overline{h}_e)$
- back reaction contributes at $O((m_e/m_i)^{1/2})$ to the electron equation at IS
- small and therefore neglected along with the effect of non-zonal passing electrons

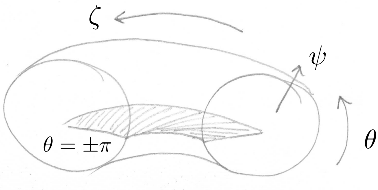
The Parallel Boundary Condition

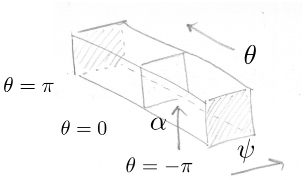
- ψ: radial, α: field line label,
  θ: poloidal angle, ζ: toroidal angle
- $\alpha(\zeta, \theta, \psi) = \alpha_0 + \zeta - q_0(\psi)\theta = \alpha_0 + \zeta - q_0\theta + q_0'(\psi - \psi_0)\theta$
- $\alpha(\zeta, \theta + 2\pi, \psi) - \alpha(\zeta, \theta, \psi) = -2\pi q_0 - 2\pi q_0'(\psi - \psi_0)$

$$A(\theta + 2\pi, \alpha(\zeta, \theta + 2\pi, \psi), \psi) = A(\theta, \alpha(\zeta, \theta, \psi), \psi) \tag{35}$$

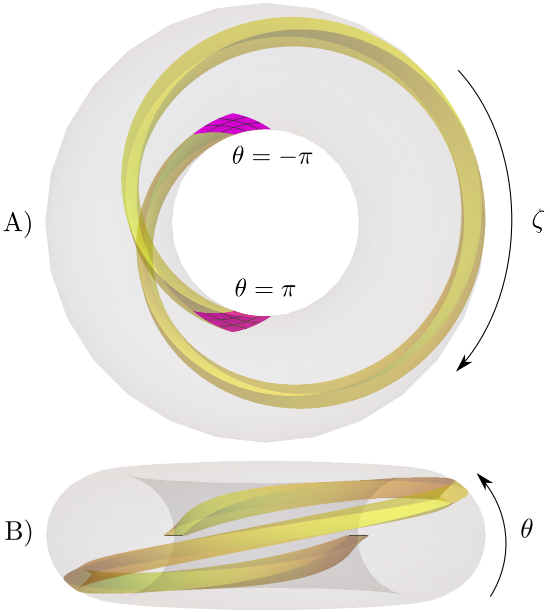
Beer et al. (1995)

$\Rightarrow$ b.c. enforces statistical periodicity on a (ψ, ζ) plane
$\Rightarrow$ b.c. couples in α





The Parallel Boundary Condition



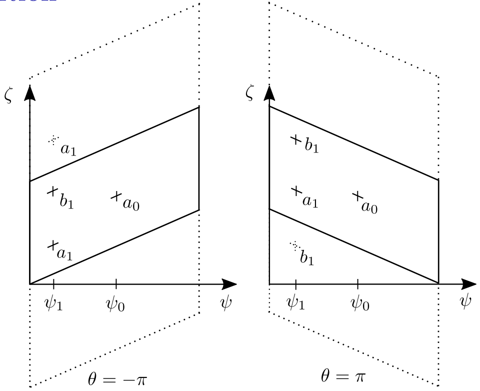
- A) view along the toroidal symmetry axis, of a flux tube, in yellow, with parallel ends in magenta. The flux surface is in grey.

- B) flux tube viewed perpendicular to the toroidal symmetry axis.

The Parallel Boundary Condition

Notation for ES fluctuations

$$\widetilde{A}(\theta{+}2\pi \underbrace{,\ \alpha_f,\ \psi_f}_{\text{ES coords}} \overbrace{;\ \alpha_s,\ \psi_s\ )}^{\text{IS coords}}$$



⇒ ES boundary condition

$$\widetilde{A}(\theta, \alpha(\zeta, \theta, \psi_f), \psi_f; \alpha(\zeta, \theta, \psi_s), \psi_s)$$
$$= \widetilde{A}(\theta{+}2\pi, \alpha(\zeta, \theta{+}2\pi, \psi_f), \psi_f; \alpha(\zeta, \theta{+}2\pi, \psi_s), \psi_s) \tag{36}$$

Electrons at IS