Waves in Turbulent, Active Media

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Aug 8, 2019

Overview

- What is a plasma?
- Equations of plasma physics (Newton, Maxwell, Boltzmann)
- Mathematical problems in quest for magnetic confinement fusion
    - Adjoint optimization techniques
    - Multiscale expansions
    - Nonlinearly interacting waves in plasma
    - Invention of new numerical algorithms

Solid to liquid to gas to plasma

- Consider ice: There is little kinetic energy compared to the magnitude of the potential energy
- Heat is required to break H bonds. Ice melts. Molecules in water have more kinetic energy, but still a lot of H bonds
- Heat is required to break the rest of the H bonds. Water boils. Gas molecules have a lot of kinetic energy but there are still charges bound together
- Heat is required to separate electrons and nuclei.
- This makes plasma: A gas of charged particles with $K \gg |U|$
- Here on Earth, we are attempting to fuse H to release energy
- The typical conditions under which H fuses involve plasmas

Are the laws of physics compatible with controlled thermonuclear fusion?

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right) \qquad E = mc^2$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \mathbf{H}\Psi \qquad \mathbf{F} = -\frac{G m_1 m_2}{r^2}\hat{\mathbf{r}} \qquad \mathbf{F} = \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

$$\frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}\right) = \nabla \times \mathbf{B} \qquad \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$

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There is some help from quantum mechanics

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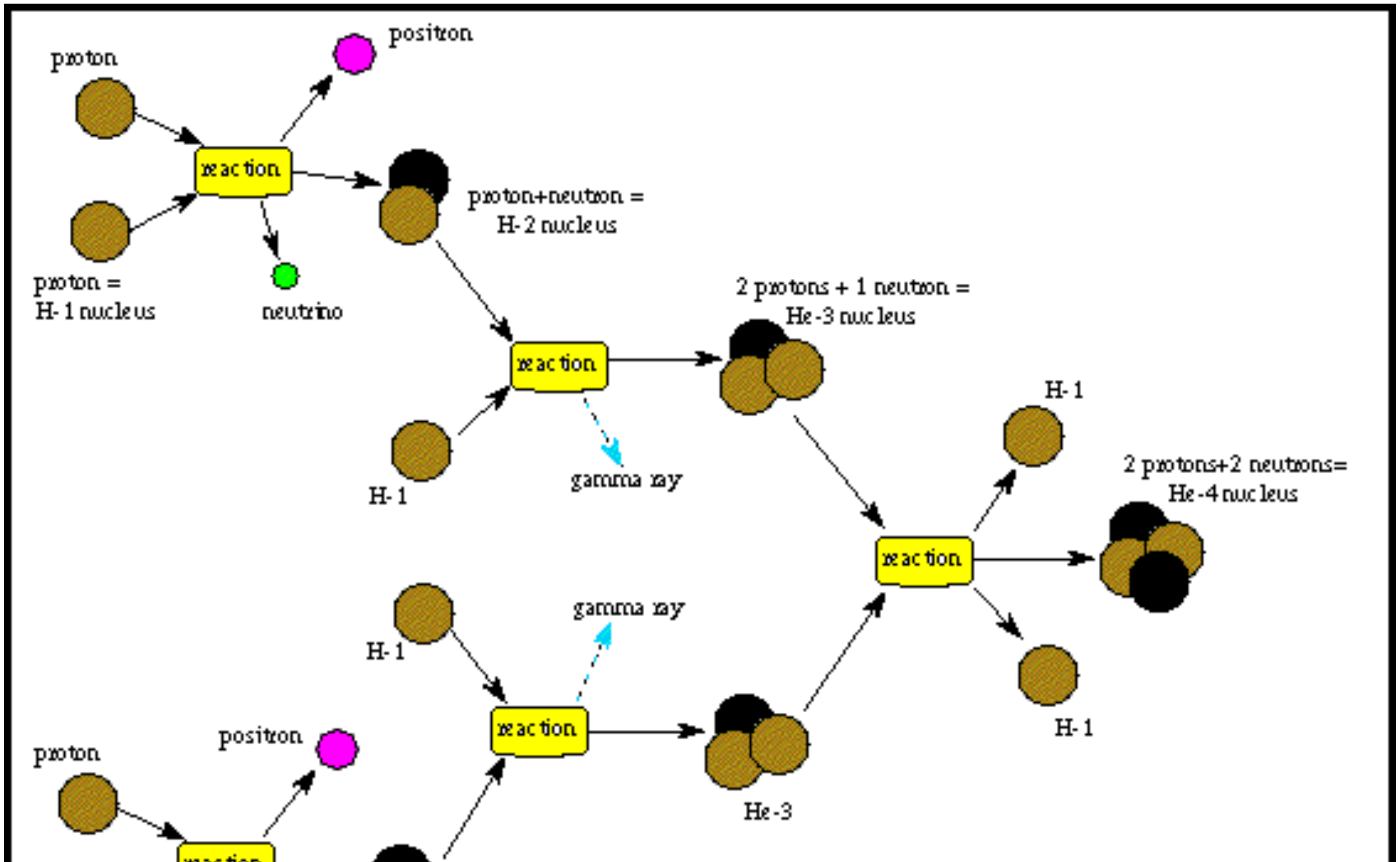

The proton-proton cycle. 4 protons (Hydrogen-1 nuclei) are fused to create one Helium-4 nucleus plus energy. The positron is the anti-matter counterpart of an electron. Adapted from a diagram in *Horizons* by M. Seeds (1991).

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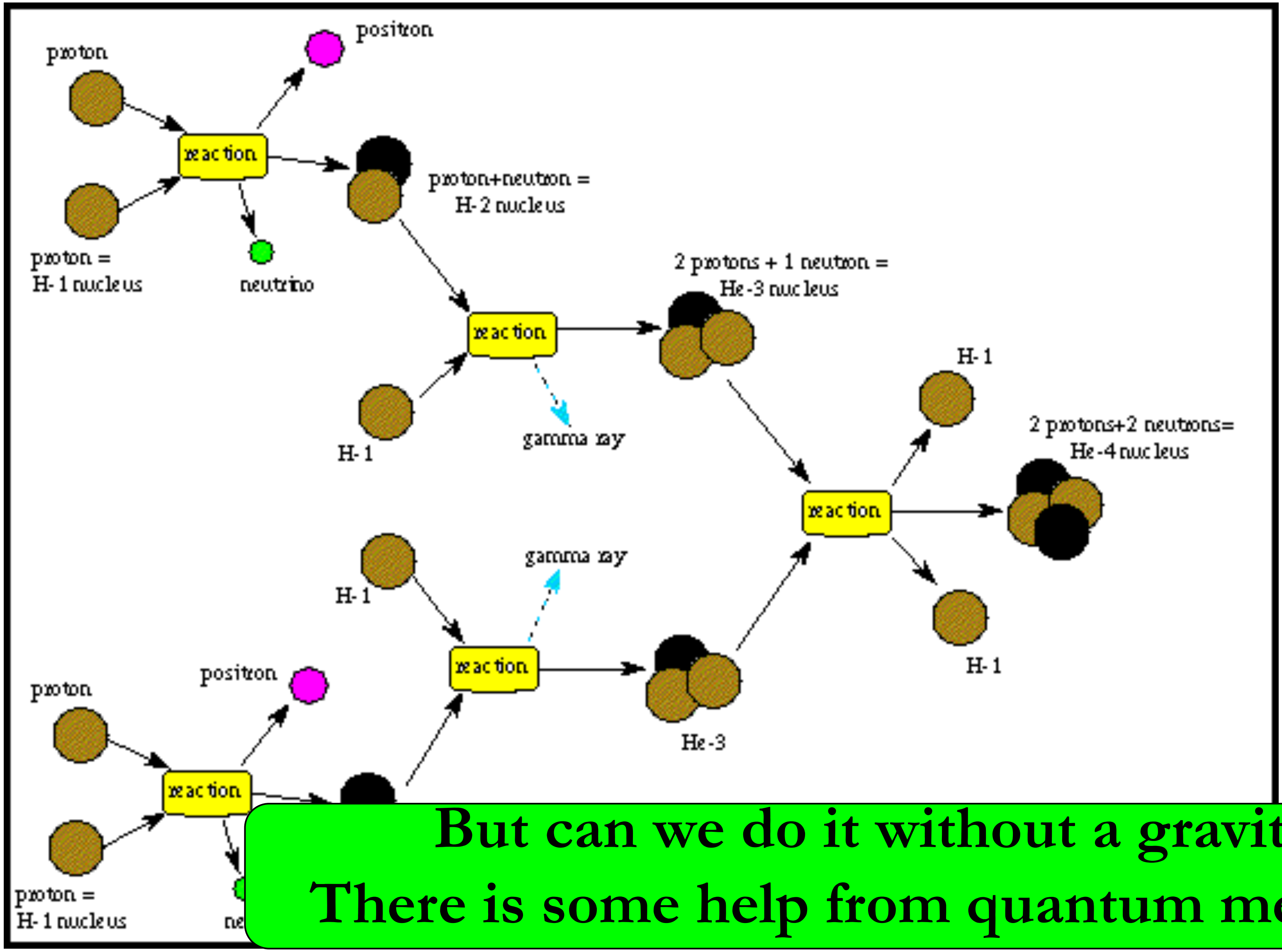

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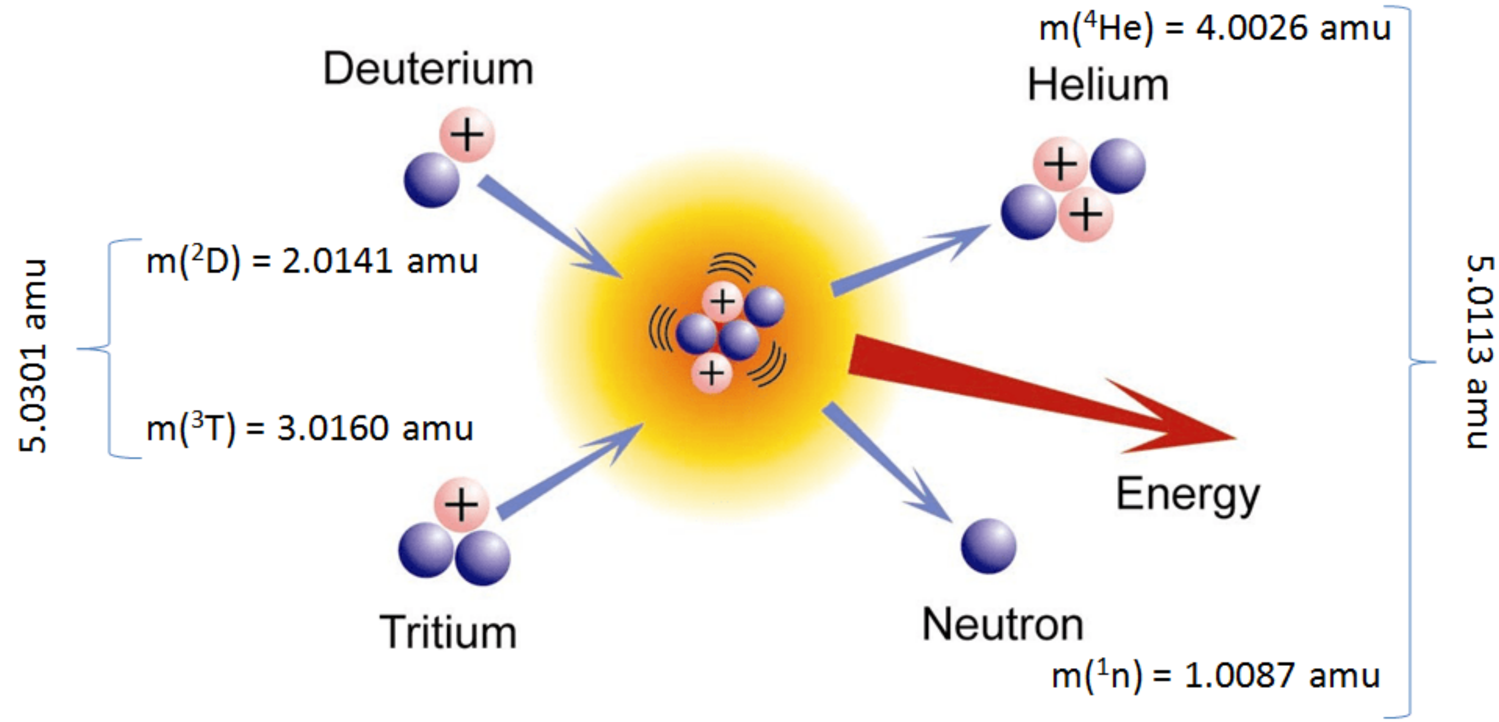

Q = 0.0188 amu x 931.481 MeV/amu = 17.5 MeV

$$E = mc^2$$

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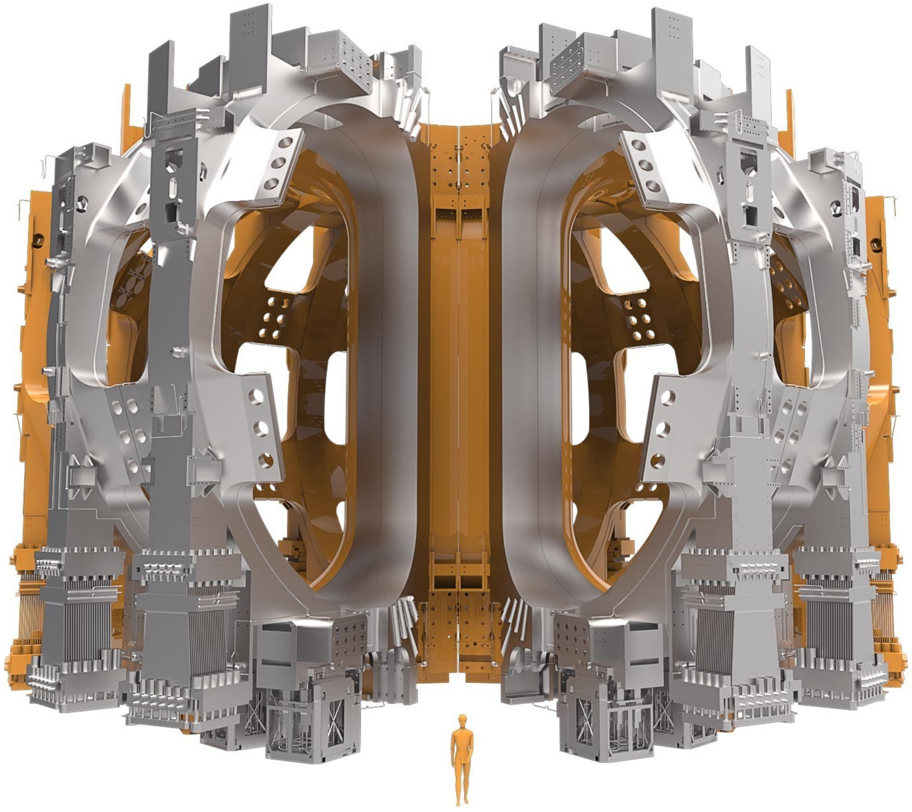
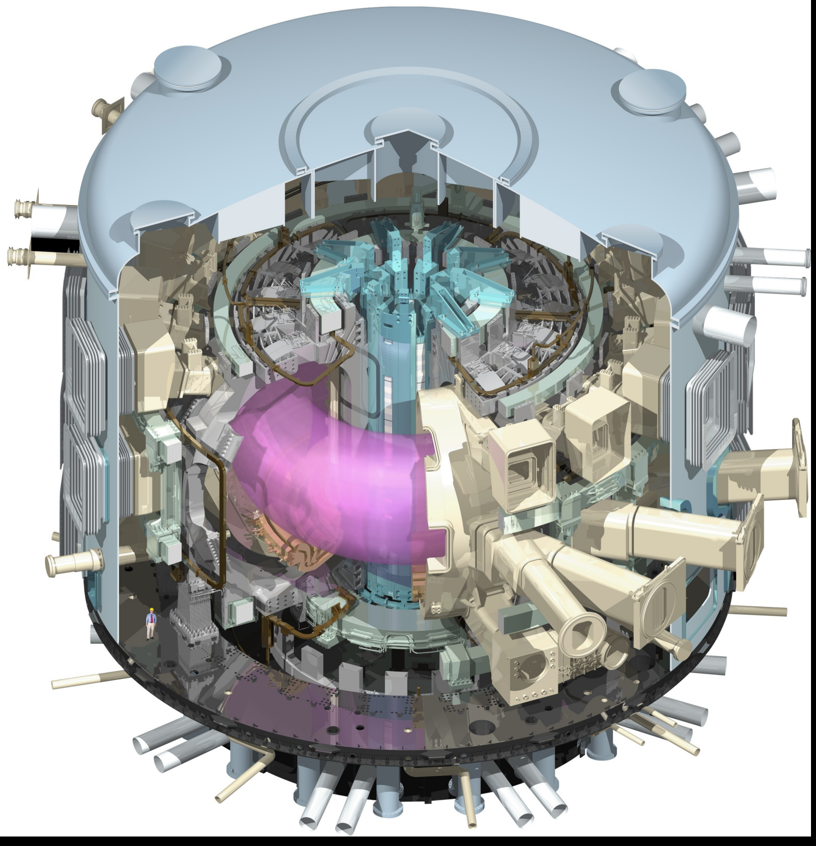
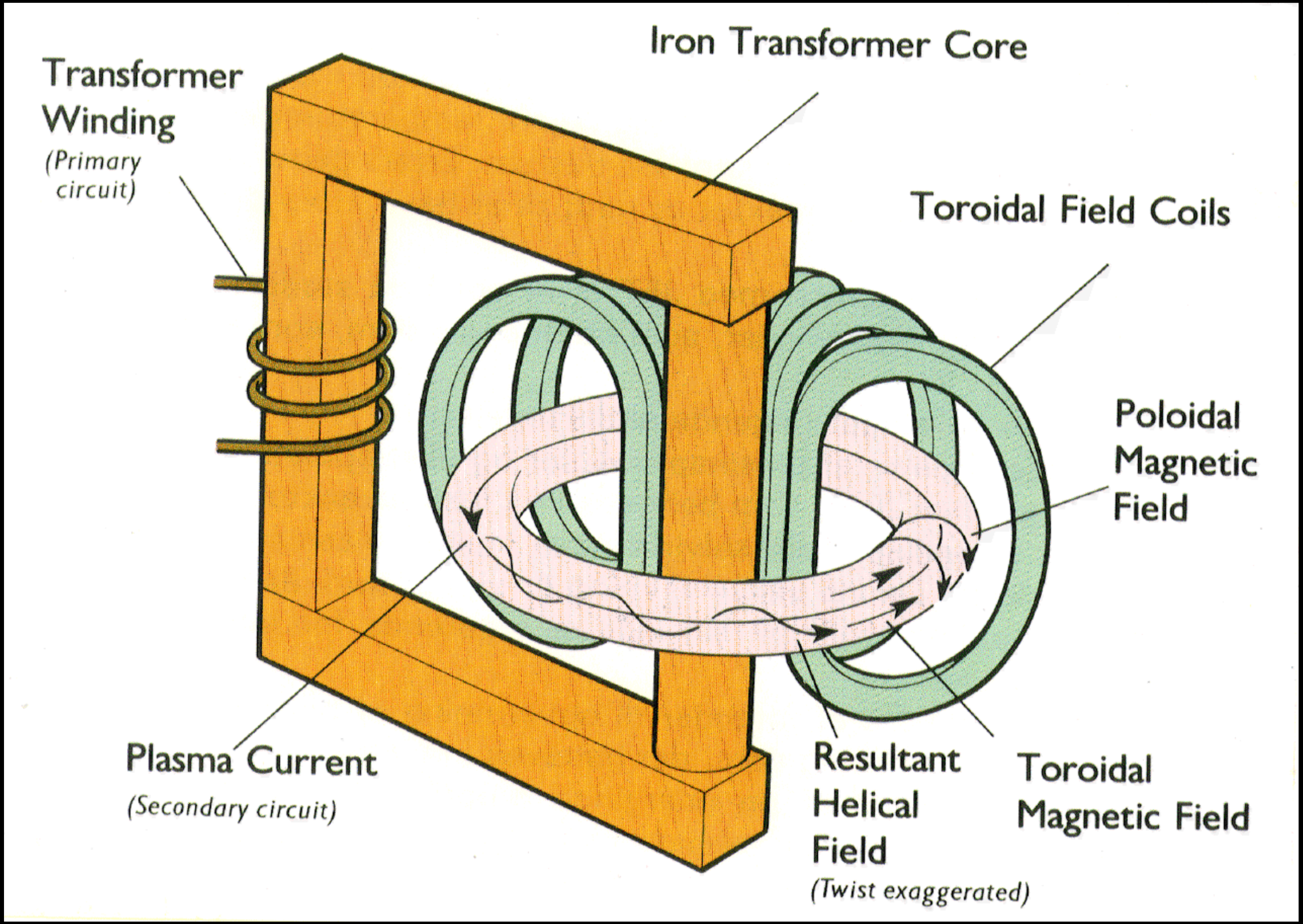
ITER is under construction in France

One way forward is to use magnetic fields instead of gravity, to confine and insulate the reacting plasma.

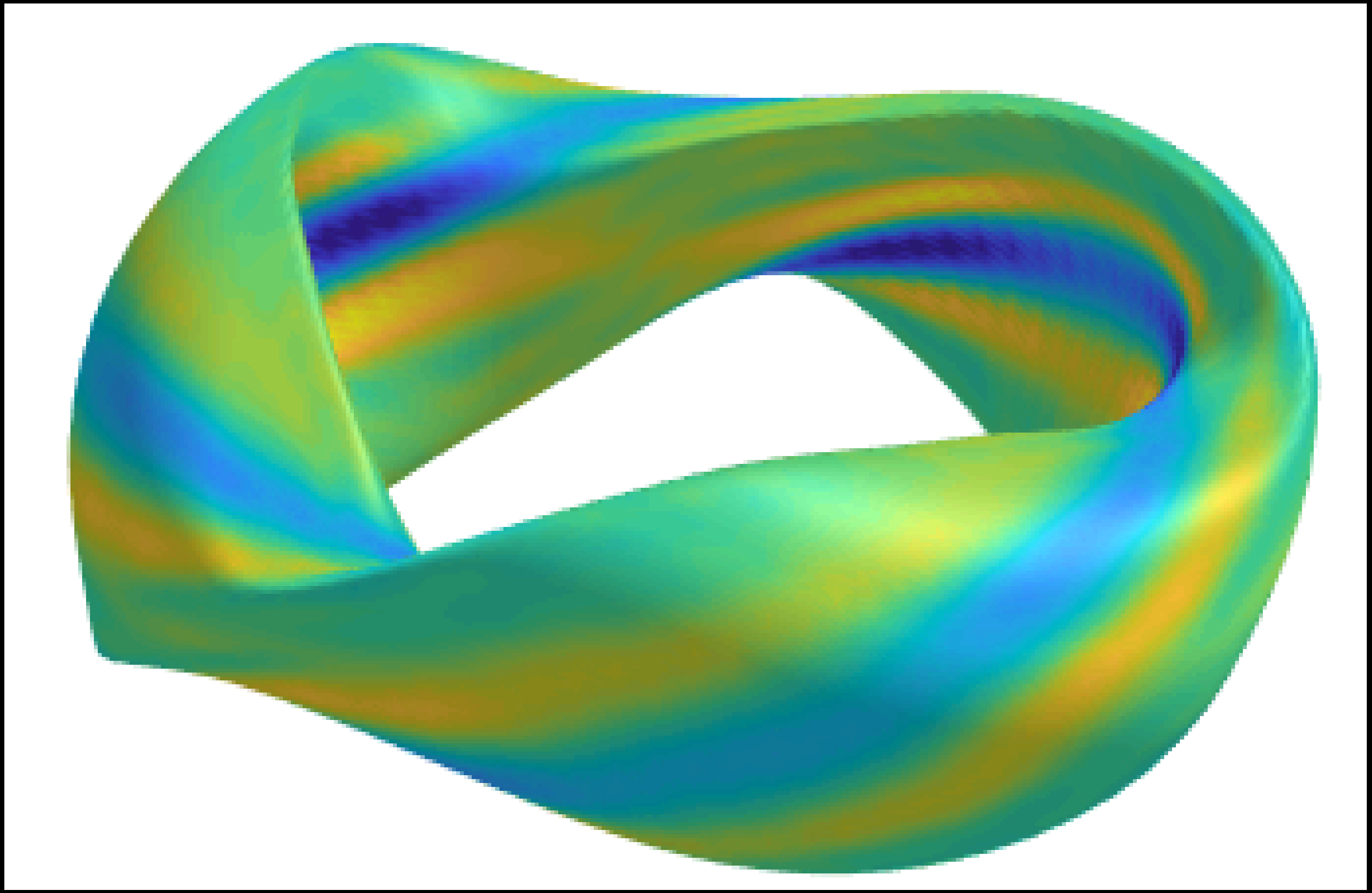




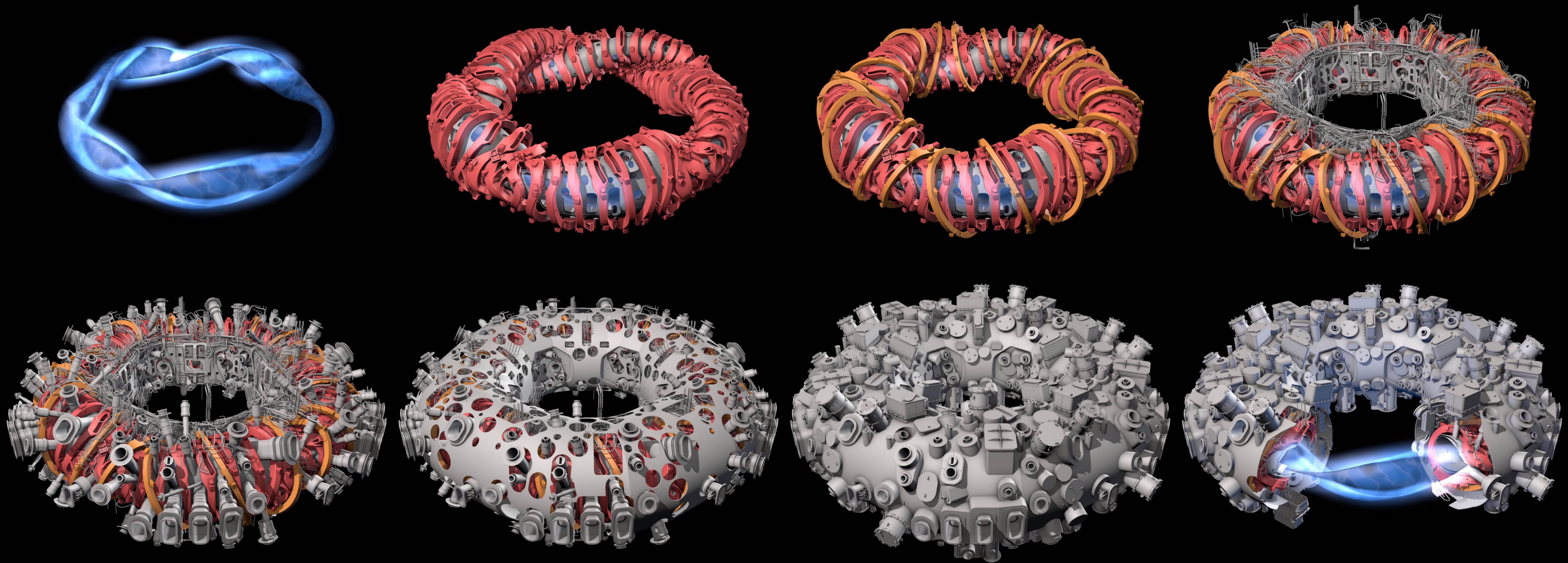
Magnetic confinement: Tokamak



Magnetic confinement: Stellarator



The stellarator approach…



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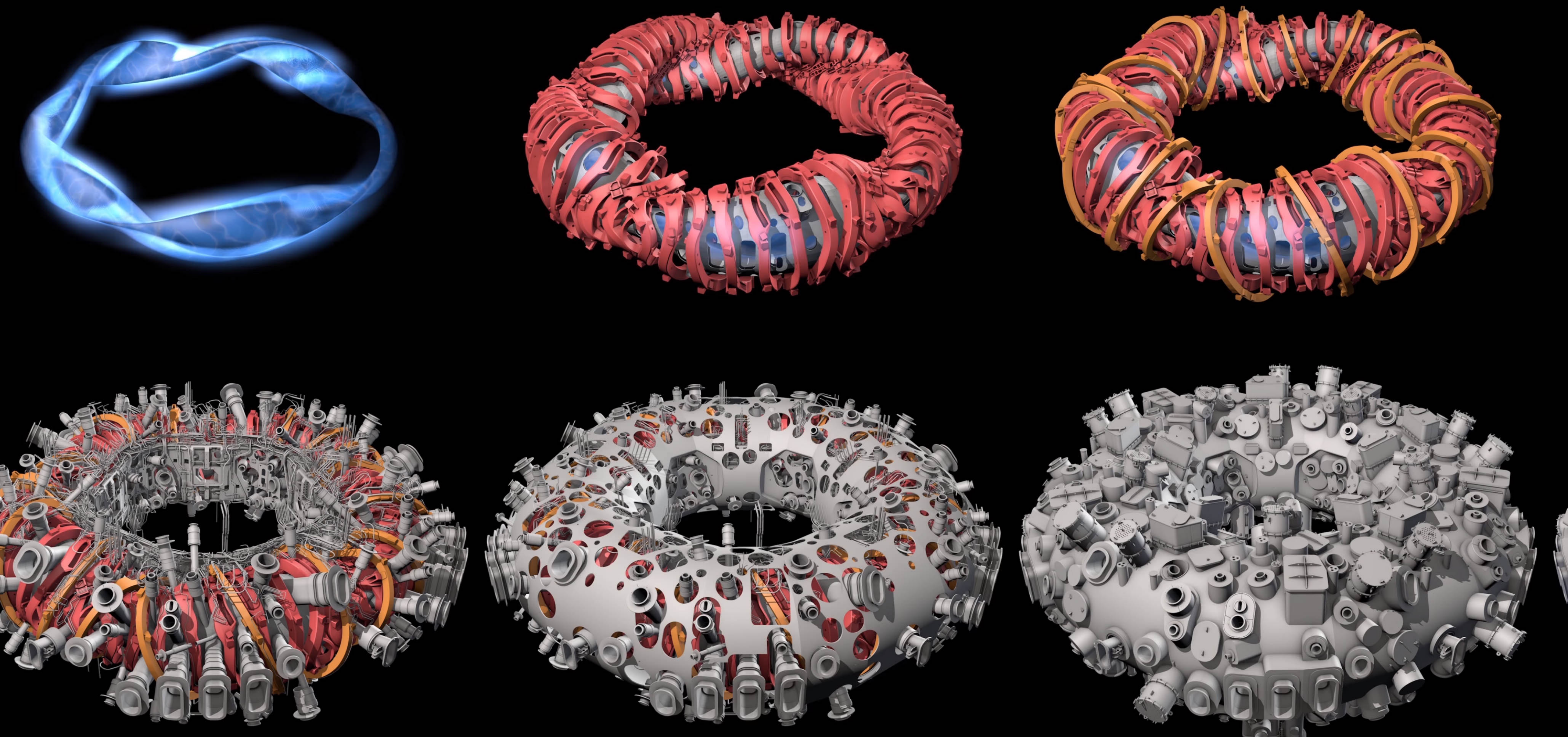

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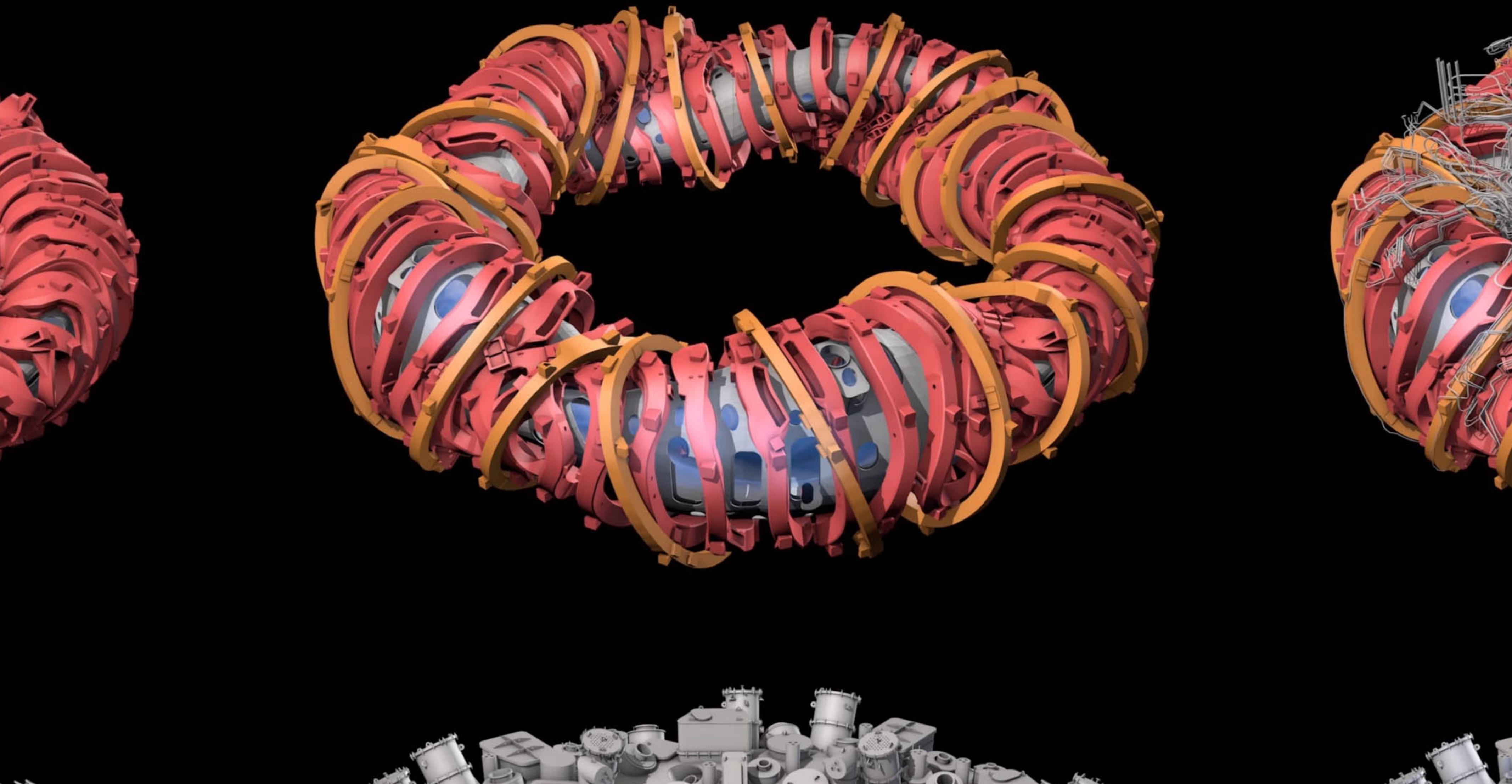
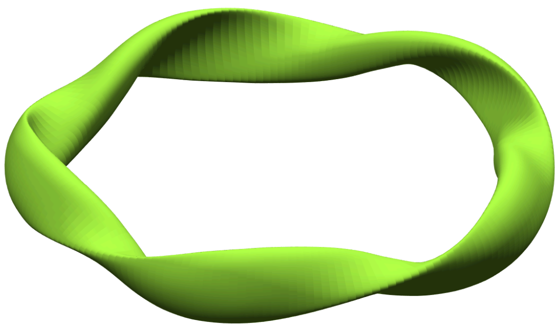


Shape optimization problems in stellarator design

MHD equilibrium

- Given outer boundary shape, S_P, and 2 free functions, $\boldsymbol{B}$ determined everywhere in confinement region
- Figures of merit describing configuration are a function of boundary shape, $f(S_P)$ (e.g. neoclassical transport)

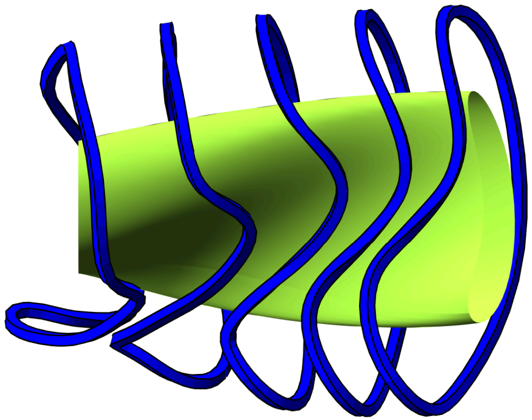
How should one deform S_P to obtain an optimal configuration?



Coil design

- Given desired boundary shape, S_P, where should one position electromagnetic coils such that S_P is a magnetic surface?
- Figures of merit are a function of coil shape or winding surface shape

How should one deform coils to obtain desired plasma surface?
How sensitive is a figure of merit to coil displacements?



Describing derivatives with respect to shape

- Consider $f(\Gamma)$, a functional of some surface, Γ
- For displacement of surface, $\Gamma_\epsilon = \{\boldsymbol{r}_0 + \epsilon\delta\boldsymbol{r} : \boldsymbol{r}_0 \in \Gamma\}$, *shape derivative* is

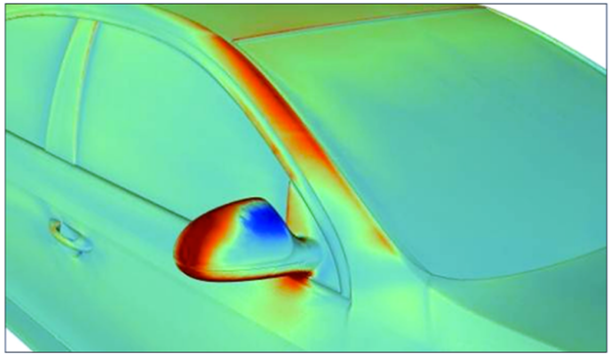
$$\delta f(\Gamma, \delta\boldsymbol{r}) = \lim_{\epsilon \to 0} \frac{f(\Gamma_\epsilon) - f(\Gamma)}{\epsilon}$$

- Differential change to f can be written as

$$\delta f(\Gamma, \delta\boldsymbol{r}) = \int_\Gamma d^2x \, \delta\boldsymbol{r} \cdot n \, S$$

- The *shape gradient*, S, describes the differential contribution of local perturbations to the surface, $\delta\boldsymbol{r}$, to changes in the the function, δf

C. Othmer, J. Math. Industry 4, 6 (2014).

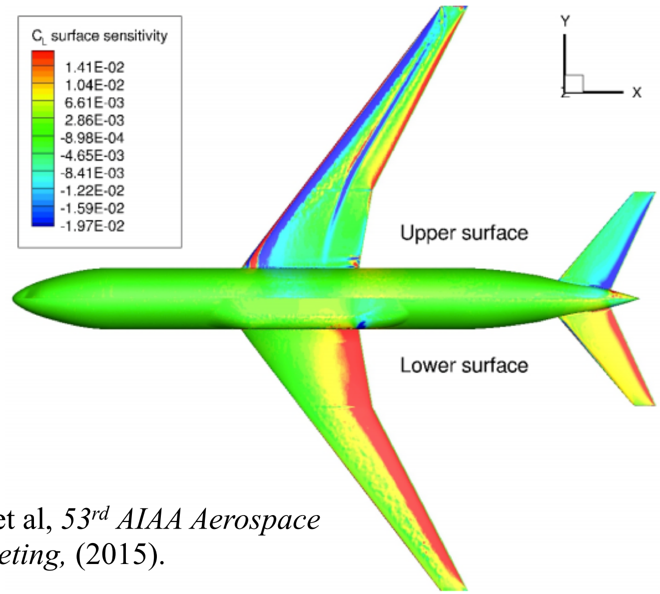


Why is S useful?

- Gradient-based optimization
- Local sensitivity analysis
- Quantifying engineering tolerances

Adjoint methods – the big picture

- Adjoint methods allow *gradient* of a function of the solution to a system of equations to be computed efficiently
- Useful for optimization within high-dimensional spaces with gradient-based methods
- Efficient computation of shape gradient
- Widely used in aerodynamic engineering



Palacios, F. et al, *53rd AIAA Aerospace Sciences Meeting,* (2015).

A linear algebra example

- Consider linear $M\times M$ system
$$\overleftrightarrow{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{b}$$
- Interested in linear dependence of inner product with $\boldsymbol{x}$ on parameters, $\Omega = \{\Omega_i\}_{i=1}^N$
$$F = \boldsymbol{x}^T\boldsymbol{c}$$
- Derivative expensive ($\mathcal{O}(M^3N)$) to compute direct way
$$\frac{\partial F}{\partial \Omega_i} = \boldsymbol{x}^T\left(\frac{\partial \boldsymbol{c}}{\partial \Omega_i}\right) + \left(\frac{\partial \boldsymbol{x}}{\partial \Omega_i}\right)^T \boldsymbol{c}$$
- Compute $\partial \boldsymbol{x}/\partial \Omega_i$ from perturbed linear system
$$\overleftrightarrow{\boldsymbol{A}}\frac{\partial \boldsymbol{x}}{\partial \Omega_i} = \left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \overleftrightarrow{\boldsymbol{A}}}{\partial \Omega_i}\boldsymbol{x}\right)$$
- Instead, solve additional adjoint equation
$$\overleftrightarrow{\boldsymbol{A}}^T\boldsymbol{q} = \boldsymbol{c}$$
- Compute derivative with **2 solutions of $M\times M$ system** $(\boldsymbol{x}, \boldsymbol{q})$
$$\frac{\partial F}{\partial \Omega_i} = \boldsymbol{x}^T\left(\frac{\partial \boldsymbol{c}}{\partial \Omega_i}\right) + \boldsymbol{q}^T\left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \overleftrightarrow{\boldsymbol{A}}}{\partial \Omega_i}\boldsymbol{x}\right)$$

Newton and Maxwell *(and Coulomb, and …)*

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right) \qquad E = mc^2$$

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Too much information; we need Boltzmann!

What is kinetic theory? A quick peek…

Consider a collection of a large number of charged particles which are neither created nor destroyed as time goes forward.

The particles move under the influence of their electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, which may be calculated from Maxwell's equations.

Under the influence of these fields, each particle moves along a classical trajectory $\mathbf{x}_i(t), \mathbf{v}_i(t)$. None disappear.

Define a function that describes the probability of finding this collection of *N* particles in a specific state:

$$f_N = f_N(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N; t)$$

According to the laws of classical physics, we have

$$\frac{Df_N}{Dt} = 0 \qquad \textit{(Liouville equation)}$$

Weak coupling

$$\frac{Df_N}{Dt} = 0$$

This equation looks simple, but remember that f_N is *(6N+1) dimensional*

Important simplification:

* *Weakly coupled* system of indistinguishable particles if

$$\Lambda \equiv 4\pi N \lambda_D^3 \gg 1$$

"plasma parameter"

Collective effects dominate behavior at scales larger than the screening radius

Mediated by EM fields

Weak coupling: No high-order correlations

$$\nu \sim \frac{\ln \Lambda}{\Lambda}\omega_p \ll \omega_p$$

Frequency of two-body collisions

"Plasma frequency"

Mathematical machinery of this reduction is "BBGKY" theory

Reduces dimensionality of problem radically, to Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f, f)$$

EM fields

Two-particle collisions

Fluid theory sometimes suffices

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = C(f,f) = 0$$

Solution of this equation is Maxwell-Boltzman distribution:

$$f \propto \frac{n}{T^{3/2}} \exp\frac{-m(\mathbf{v}-\mathbf{u})^2}{2T}$$

Need only keep track of density, momentum and temp: plasma as fluid

Mean-free-path is simply $\lambda_{\mathrm{mfp}} \equiv \dfrac{v_t}{\nu}$. Take L to be size of interest.

If $\lambda_{\mathrm{mfp}} \ll L$, then structures of size L are Maxwellian in v-space.

Fluid-like phenomena can be described by 3-D theory (*e.g.*, MHD)

Kinetic theory when $\lambda_{\mathrm{mfp}} \gg L$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f, f)$$

EM fields

Two-particle collisions

Boltzmann equation + Maxwell's equations = kitchen sink

Very frequently, the largest term in the equation is the acceleration due to the (self-consistent and/or imposed) magnetic field: leads to rapid gyration.

Take advantage of this, and work out asymptotically rigorous equations that describe all dynamics slower than the gyration: theory is known as "gyrokinetics".

Basic idea: Asymptotic, multiscale expansion

Expand Boltzmann and Maxwell equations in powers of epsilon, where

$$\epsilon \equiv \frac{\omega}{\Omega_c}, \qquad \Omega_c \equiv \frac{qB}{mc}.$$

Identify relevant physics at each order in epsilon (gyration, turbulence, thermodynamics) taking place at different space and time scales.

Key result: for processes that are slow compared to the gyration, equations are reduced in dimensionality (from 6 to 5) but are integro-differential.

Conceptually, particles are replaced by rings whose radii are time-varying.

Interesting turbulent phenomena exist with eddies both large and small, compared to a typical "gyroradius". Challenging to study!

GK equations describe evolution of guiding centers

- The order-by-order reduction of the Boltzmann equation is achieved by repeatedly orbit-averaging the equations, because the orbit-average annihilates the largest term at each order -- the gyration.
- One finds an equation for the part of the distribution function which is independent of gyro-angle. The gyro-angle dependent part of the distribution function yields a "polarization density".
- To find the actual currents and charge densities for Maxwell's equations, one must keep both contributions. Results in algebraic clutter.
- Key result: the gyroaveraging operation smooths over perturbations that are small compared to the gyroradius. Particles respond to and produce small-scale electromagnetic fields, but both are tempered by the spatial averaging coming from the rapid (instantaneous, in the theory) gyration.

Kinetic theory when $\lambda_{\mathrm{mfp}} \gg L$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f, f)$$

EM fields

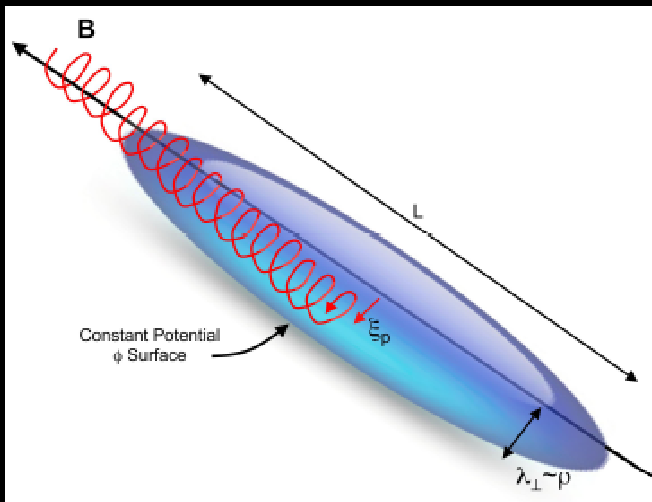
Two-particle collisions

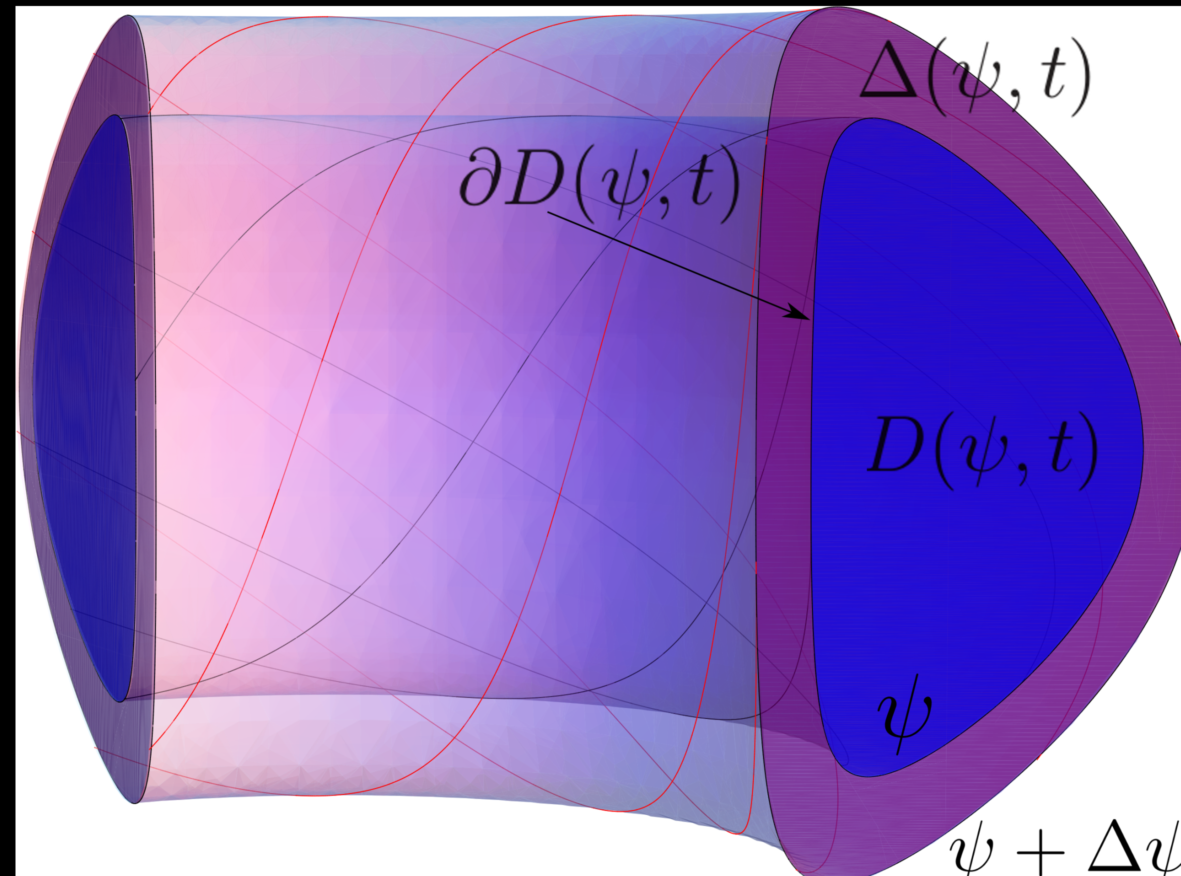
$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \nabla f_s + \frac{Z_s e}{m_s} \left(\widetilde{\boldsymbol{E}} + \frac{1}{c} \boldsymbol{v} \times \widetilde{\boldsymbol{B}} \right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s$$

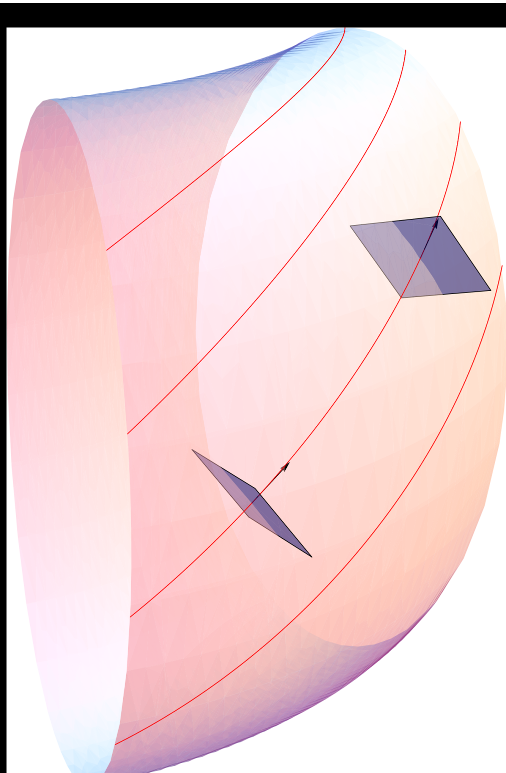
Choose good coordinates in velocity space…

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s$$

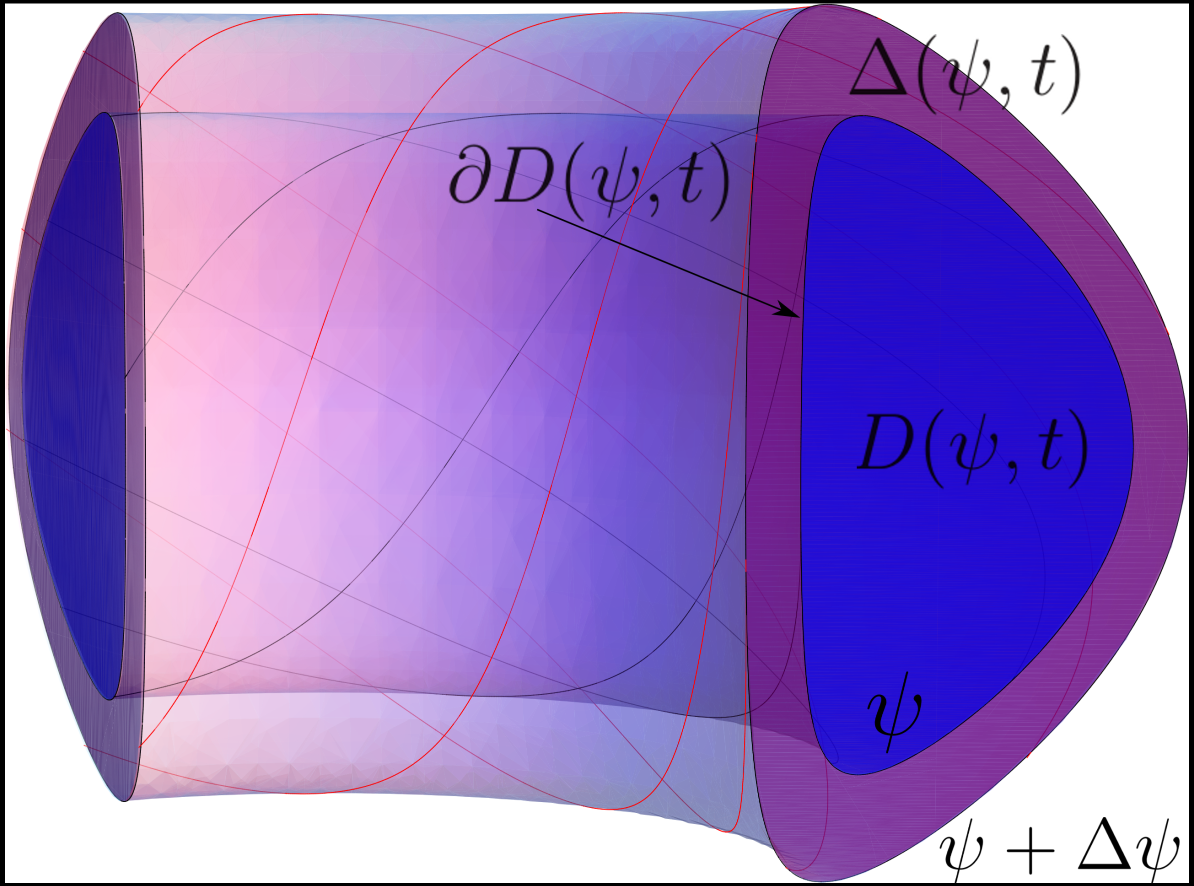
Expand in a small parameter and average in three different ways



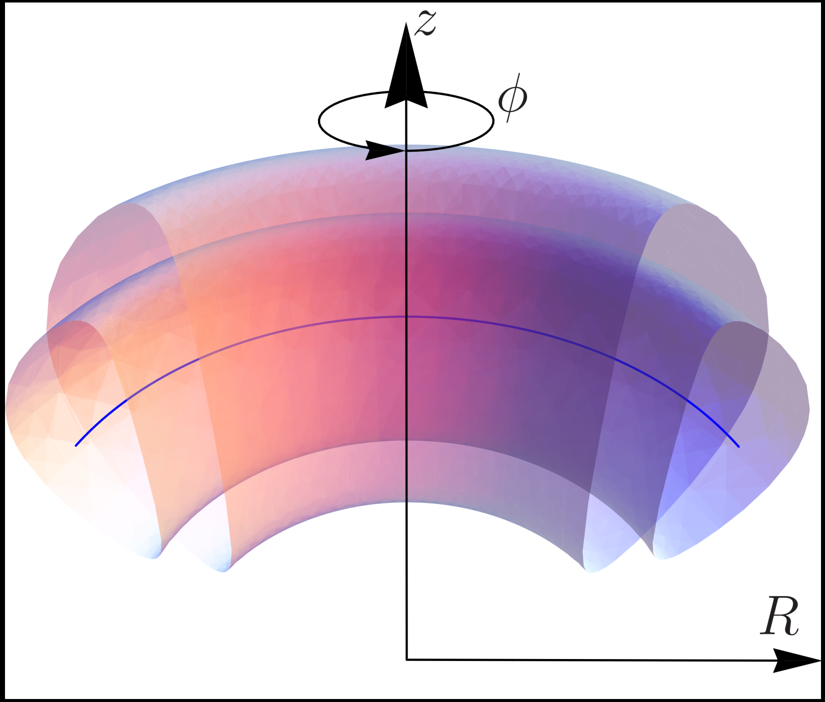




$$F_{0s} = n_s(\boldsymbol{r}) \left[\frac{m_s}{2\pi T_s(\boldsymbol{r})}\right]^{3/2} \exp\left\{-\frac{m_s\left[w^2 - 2m_s w_\parallel \hat{u}_{\parallel s}(\boldsymbol{r}) + \hat{u}^2_{\parallel s}(\boldsymbol{r})\right]}{2T_s(\boldsymbol{r})}\right\}$$



$$F_{0s} = N_s(\psi(\boldsymbol{R}_s)) \left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s))}$$



$$\Delta^* \psi = -4\pi R^2 \sum_s n_s \left\{ T_s \frac{d \ln N_s}{d\psi} + \left[Z_s e \varphi_0 - \frac{1}{2} m_s \omega^2(\psi) R^2 + T_s \right] \frac{d \ln T_s}{d\psi} \right\}$$
$$- 4\pi R^2 \left(\sum_s m_s n_s R^2 \right) \omega(\psi) \frac{d\omega}{d\psi} - I(\psi) \frac{dI}{d\psi},$$

$$\Delta^* \psi = \left(\frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2} \right) \psi$$

To find the shape, density, temperature, etc, we need to go to higher order

$$
\begin{aligned}
f_s &= F_s + \delta f_s, \\
F_s &= F_{0s}\left(\psi(\boldsymbol{R}_s), \varepsilon_s\right) + F_{1s}\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \mathrm{O}\left(\epsilon^2 f\right), \\
\delta f_s &= -\frac{Z_s e}{T_s}\delta\varphi'(\boldsymbol{r}) F_{0s} + h_s\left(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma\right) + \mathrm{O}\left(\epsilon^2 f\right), \\
F_{0s} &= N_s(\psi(\boldsymbol{R}_s))\left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))}\right]^{3/2} e^{-\varepsilon_s/T_s(\psi(\boldsymbol{R}_s))},
\end{aligned}
$$

Result is a 5-D, nonlinear, integro-differential system of equations. Remarkably, they can be solved.

$$\left[\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right]h_s+w_\parallel \boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{R}_s}(F_{1s}+h_s)+\left(\boldsymbol{V}_{\mathrm{D}s}+\langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\right)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}(F_{0s}+h_s)$$

$$=\frac{Z_s e F_{0s}}{T_s}\left[\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right]\langle\chi\rangle_{\boldsymbol{R}}-\frac{m_s F_{0s}}{T_s}\left[\frac{I w_\parallel}{B}+\omega(\psi)R^2\right]\frac{d\omega}{d\psi}\langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\cdot\nabla\psi$$

$$-\frac{Z_s e}{T_s c}w_\parallel F_{0s}\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{b}+\langle C[F_{0s}+F_{1s}+h_s]\rangle_{\boldsymbol{R}},$$

Some definitions:

$$\chi=\delta\varphi-\frac{1}{c}\boldsymbol{v}\cdot\delta\boldsymbol{A}=\delta\varphi'-\frac{1}{c}\boldsymbol{w}\cdot\delta\boldsymbol{A}.$$

EM potentials

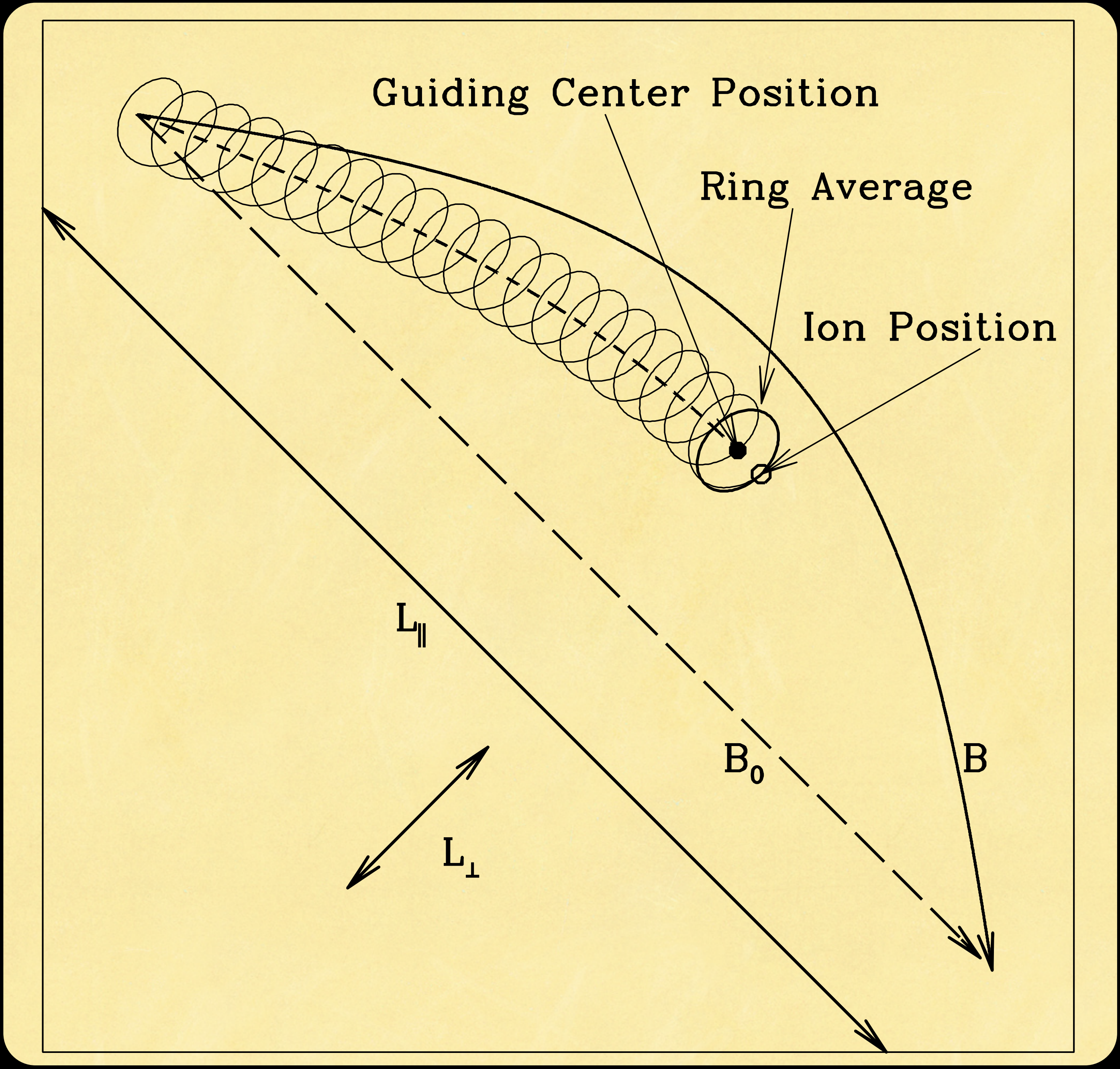
$$\boldsymbol{V}_\chi=\frac{c}{B}\boldsymbol{b}\times\nabla\chi,\qquad \langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}=\frac{c}{B}\boldsymbol{b}\times\frac{\partial\langle\chi\rangle_{\boldsymbol{R}}}{\partial \boldsymbol{R}_s}$$

Nonlinearities

$$\boldsymbol{V}_{\mathrm{D}s}=\frac{\boldsymbol{b}}{\Omega_s}\times\left[w_\parallel^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}+\frac{1}{2}w_\perp^2\nabla\ln B\right.$$
$$\left.-\omega^2(\psi)R\nabla R-2w_\parallel\omega(\psi)\boldsymbol{b}\times\nabla z+\frac{Z_s e}{m_s}\nabla\varphi_0\right]$$

Drift velocities

Gyrokinetic physics: Gyration + streaming + drifts



Highly anisotropic, because particles stream freely along the magnetic field lines.

Plane perpendicular to magnetic field is special.

Self-consistent currents and fields.

E x **B** drift, flexing, stretching and tearing of field lines, included.

GK describes field perturbations larger *and smaller* than the gyration radii.

At next order, the system closes! (non-trivial)

$$f_s = F_{0s}(\psi(\boldsymbol{R}_s), \varepsilon_s) + F_{1s}(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma) + F_{2s}(\boldsymbol{r}, \boldsymbol{v}) - \frac{Z_s e}{T_s}\delta\varphi'(\boldsymbol{r}) F_{0s} + h_s(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \sigma) + \delta f_{2s}(\boldsymbol{r}, \boldsymbol{v}) + \cdots$$

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_\psi V' \langle n_s \rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \langle \Gamma_s \rangle_\psi = \left\langle S_s^{(n)} \right\rangle_\psi$$

$$\langle \Gamma_s \rangle_\psi = \left\langle \int d^3\boldsymbol{w} \left(\frac{\boldsymbol{w} \times \boldsymbol{b}}{\Omega_s} \cdot \nabla\psi \right) C[F_{0s}] \right\rangle_\psi + \left\langle \int d^3\boldsymbol{w} F_s^{(\mathrm{nc})} \boldsymbol{V}_{\mathrm{D}s} \cdot \nabla\psi \right\rangle_\psi - \langle n_s \rangle_\psi I(\psi) \frac{\langle \boldsymbol{E} \cdot \boldsymbol{B} \rangle_\psi}{\langle B^2 \rangle_\psi} + \left\langle \left\langle \int d^3\boldsymbol{w} \langle h_s \boldsymbol{V}_\chi \rangle_{\boldsymbol{r}} \cdot \nabla\psi \right\rangle_{\mathrm{turb}} \right\rangle_\psi$$

And similarly, one can derive equations for the momentum and temperature profiles

Where are the waves?

- We can look at the *linearized* (small amplitude) equations and find dispersion relations for the waves
- Let us do that briefly, without the complications of toroidal geometry

Gyrokinetic Equations

- The drift frequency $i\omega_*^T = n_0 c \partial F_0 / \partial \Psi$, where n_0 labels the α Fourier harmonic of the perturbation

- The perpendicular drifts (curvature, grad-B) are

$$\omega_d = \mathbf{k}_\perp \cdot \mathbf{B}_0 \times \left(m v_\parallel^2 \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} + \mu \nabla B_0 \right) / (m B_0 \Omega),$$

- Potentials for the fields appear as

$$\chi = J_0(\gamma) \left(\Phi - \frac{v_\parallel}{c} A_\parallel \right) + \frac{J_1(\gamma)}{\gamma} \frac{m v_\perp^2}{q} \frac{\delta B_\parallel}{B}; \qquad \gamma \equiv k_\perp v_\perp / \Omega$$

Gyrokinetic Field Equations

- Maxwell's equations, neglecting displacement current.

- Poisson's equation: $[\nabla \cdot \mathbf{E} = 4\pi\rho]$

$$\nabla_\perp^2 \Phi = 4\pi \sum_s \int d^3v\, q \left[q\Phi \frac{\partial F_0}{\partial \epsilon} + h \exp{(iL)} \right],$$

  where $L = (\mathbf{v} \times \hat{\mathbf{b}} \cdot k_\perp)/\Omega$ accounts for the gyrophase dependence.

- Preferred velocity space coordinates are (E, μ, ξ), so that

$$\int d^3v = \frac{B}{m^2} \int \frac{dE\, d\mu\, d\xi}{|v_\parallel|} \equiv \frac{1}{2\pi} \int d^2v\, d\xi$$

- Integrate over the gyrophase to find

$$\nabla_\perp^2 \Phi = 4\pi \sum_s \int d^2v\, q \left[q\Phi \frac{\partial F_0}{\partial \epsilon} + J_0(\gamma) h \right]$$

- Similarly, Ampere's law provides the two components of the perturbed magnetic field:

$$\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} \sum_s \int d^2v\, q v_\parallel J_0(\gamma) h \qquad \left[\hat{\mathbf{b}} \cdot \nabla \times \mathbf{B} = \frac{4\pi J_\parallel}{c} \right]$$

$$\frac{\delta B_\parallel}{B} = -\frac{4\pi}{B^2} \sum_s \int d^2v\, m v_\perp^2 \frac{J_1(\gamma)}{\gamma} h \qquad \left[4\pi \delta p_\perp + \frac{\delta B_\parallel}{B} = 0 \right]$$

(Fast waves -> pressure balance)

Linear Dispersion Relation

- Take $C = \omega_d = \omega_* = 0$ (collisionless, homogenous plasma, straight field lines)

- Linear GKE is:

$$-i\omega h + ik_\parallel v_\parallel h = -\frac{i\omega q}{T}\chi F_0$$

- For clarity, consider hydrogenic plasma, with $k_\perp \rho_e \ll 1$.

Linear Dispersion Relation

Shear Alfvèn Slow mode

$$\left[\left(\frac{k_\parallel v_A}{\omega}\right)^2 - \left(\frac{1-\mathcal{I}_0}{k_\perp^2\rho_i^2}\right)\left(1-\frac{1-\mathcal{I}_0}{S}\right)\right]\left[1-\frac{\beta_i}{2}C_2+\frac{\beta_i}{2}\frac{C_1^2}{S}\right]$$

$$=\frac{\beta_i}{2}\left(\frac{1}{k_\perp^2\rho_i^2}\right)\left[(1-\mathcal{I}_1)-(1-\mathcal{I}_0)\frac{C_1}{S}\right]^2$$

Finite Larmor radius coupling

$S = 1 + \mathcal{I}_{0i}\zeta_i Z(\zeta_i) + \frac{T_i}{T_e}[1+\zeta_e Z(\zeta_e)]$
$C_j = \sum_s \mathcal{I}_j \zeta Z(\zeta)\,(T_i/T_s)$

Long wavelength limits [small $b = (k_\perp \rho_i)^2$]:

$\mathcal{I}_0 \sim 1 - b$ $\mathcal{I}_1 \sim 1 - \frac{3}{2}b$ $\mathcal{I}_1 \sim 2 - 3b$

Long wavelength waves $(k_\perp \rho_i \ll 1, \beta \sim 1)$

Shear Alfvèn Slow mode

$$\left[\left(\frac{k_\parallel v_A}{\omega}\right)^2 - 1\right]\left[1 - \frac{\beta_i}{2}C_2 + \frac{\beta_i}{2}\frac{C_1^2}{S}\right]$$

$$= 0$$

Finite Larmor radius coupling

- Shear Alfvèn waves: $\omega^2 = k_\parallel^2 v_A^2$

- Slow wave: $1 - \frac{\beta_i}{2}C_2 + \frac{\beta_i}{2}\frac{C_1^2}{S} = 0$

The nonlinear version of this decoupling exists and is “Kinetic Reduced MagnetoHydroDynamics” or KRMHD

- **Kinetic** because the model includes phase mixing/Landau damping
- **Reduced** because the model describes small amplitude, spatially anisotropic fluctuations (long wavelengths along ***B*** and short wavelengths across)
- **MHD** because the Alfvenic fluctuations are the same as in reduced MHD.
- Just as in MHD, we represent the Alfvenic fluctuations in Elsasser variables
- The long wavelength $(k_\perp \rho \ll 1)$ kinetic fluctuations are described by “Vlasov”-like equations

Kinetic Reduced MHD

- Alfven waves decouple:

$$\left(\frac{\partial}{\partial t} \mp v_A \frac{\partial}{\partial z}\right) \omega^{\pm} = -\left[\zeta^{\mp}, \omega^{\pm}\right] - \left[\partial_i \zeta^{\mp}, \partial_i \zeta^{\pm}\right]$$

- Here, $\omega^{\pm} = \nabla_{\perp}^2 \zeta^{\pm}$ and $\zeta^{\pm} = \Phi \pm \Psi$. Physically, this is a complicated looking way to describe counter-propagating Alfven waves in reduced MHD. No compressional effects!

- Kinetic equations for the ions are simple *(i=1, 2)*

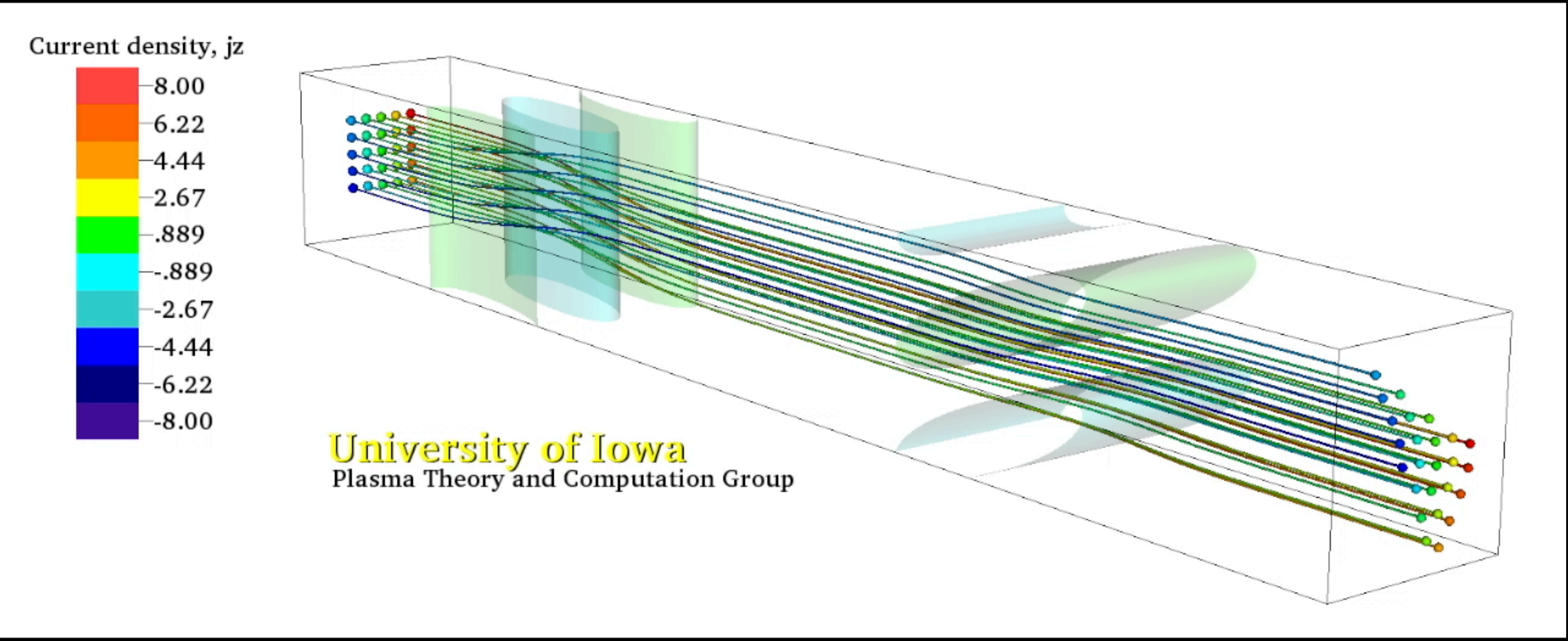
$$\frac{dg^{(i)}}{dt} + v_{\|} \nabla_{\|} g^{(i)} + v_{\|} F_0 \nabla_{\|} \phi^{(i)} = 0$$

- But the operators are the nonlinear ones

$$d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \quad \text{and} \quad \nabla_{\|} = v_A \partial/\partial z + \mathbf{b} \cdot \nabla$$

These waves interact! (unlike ordinary light)

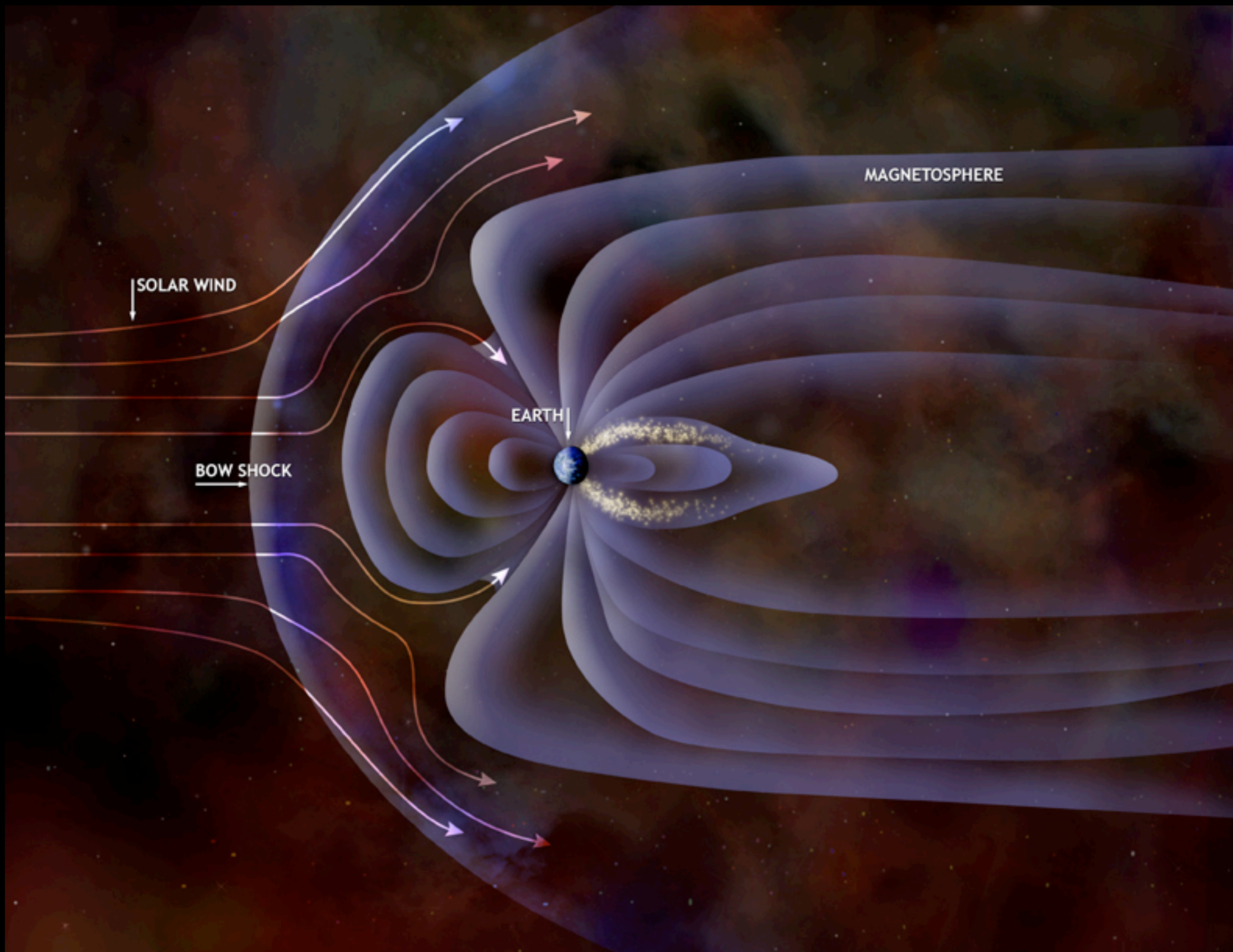
- Simulation of colliding Alfven waves using AstroGK



Mathematical "angles"

- Deriving the appropriate multi-scale reductions
- Identifying conserved quantities and symmetries
- Identifying exact, nonlinear solutions
- Developing discretizations and numerical algorithms
- Developing closures (How do unresolved scales affect solutions?)
- Optimizing codes (mostly by inventing new algorithms)
- Extracting meaning! —— "I" rather than "AI"

Applications of gyrokinetics: Nature



The solar wind is a pressure-driven, outward flow of plasma from the sun.

The pressure should drop as the plasma expands, and the flow should stagnate.

Why doesn't this happen? Perhaps turbulent heating -- in this case, gyrokinetic turbulent heating.

What we learn from solar wind may be applied to astrophysical systems.

The End