Cross-scale interactions and the Dimits shift in reduced models of drift-wave turbulence

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Outline

- Part 1: Cross-scale interactions (CSI)
  - Quasilinear Hasegawa–Mima model
  - Quasistatic equation of state, basic scalings (new)
  - Zonal-flow merging as the cause of the CSI (new)
  - Comparison of the quasilinear and nonlinear models (new)

- Part 2: Stability of zonal flows and nonlinear suppression of DW turbulence
  - Tertiary instability in the Hasegawa–Mima model
  - Tertiary instability in the Terry–Horton model (new)
  - Dimits shift in the Terry–Horton model (new)
  - Dimits shift in the Hasegawa–Wakatani model (in progress)

(new) = updates since June

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The main results are yet to be published, but see the background...


Gyrokinetic simulations show that low-\( k \) ITG turbulence can suppress high-\( k \) ETG turbulence. For example, see Maeyama et al., PRL (2015):

FIG. 2 (color online). Poloidal wave number spectrum of the time-averaged electron energy diffusivity \( \chi_{ek} \) for (a) electrostatic (\( \beta = 0.04\% \)) and (b) electromagnetic (\( \beta = 2.0\% \)) cases. The solid (red), dotted (blue), and dashed (green) lines plot \( \chi_{ek} \) as obtained from the full-\( k \), low-\( k \) (\( k_y \rho_{ti} < 1.3 \)), and high-\( k \) (\( k_y \rho_{ti} > 1.3 \)) simulations, respectively.

Our goal is to explain this effect within the simplest meaningful model.
As a starting point, consider the Hasegawa–Mima model.

Basic physics of DW turbulence is often studied within the Hasegawa–Mima model:

$$\partial_t w + \{\varphi, w\} + \beta \partial_x \varphi = 0, \quad w = (\nabla^2 - \hat{a}) \varphi, \quad \beta \sim \partial_y N$$

Electrons respond adiabatically to drift waves ($k_\parallel \neq 0$) and do not respond to zonal flows (ZF), which are spontaneously-generated banded shear flows with $k_\parallel = 0$.

$$w_{dw} = (\nabla^2 - 1) \varphi_{dw}, \quad w_{zf} = \nabla^2 \varphi_{zf}$$
We further reduce the model using the quasilinear approximation.

- The quasilinear approximation is sufficiently accurate to capture basic effects.

\[
\text{average: } \frac{\partial_t U}{\partial y} \tilde{v}_x \tilde{v}_y = 0, \quad \tilde{v} = \mathbf{Z} \times \nabla \tilde{\varphi}, \quad \tilde{w} = (\nabla^2 - 1) \tilde{\varphi}
\]

\[
\text{fluctuations: } \frac{\partial_t \tilde{w}}{U \partial_x \tilde{w}} + [\beta - (\partial_y^2 U)] \partial_x \tilde{\varphi} = \tilde{v} \cdot \nabla \tilde{w} - \tilde{v} \cdot \nabla \tilde{w}
\]

neglected (QL model)

- The equation for \( \tilde{w} \) can be expressed as a Schrödinger equation for “driftons”:

\[
 i \frac{\partial_t \tilde{w}}{\partial x} = \hat{H} \tilde{w} + \hat{N} \mathcal{L}, \quad \hat{H} = \hat{k}_x \hat{U} - \hat{k}_x (\beta - \hat{U}'')(1 + \hat{k}_1^2)^{-1}, \quad \hat{k} = -i \nabla
\]

Ruiz et al. (2016); Zhou et al. (2019)
The quasilinear HM model captures cross-scale interactions.

\[ \frac{k_{x2}}{k_{x1}} = 5 \]

\[ \frac{k_{x2}}{k_{x1}} = 15 \]

\[ Z_{dw} = \frac{1}{2} \int d^2x \, \bar{w}^2, \quad Z_{zf} = \frac{1}{2} \int dy \, (U')^2, \quad E_{dw} = -\frac{1}{2} \int d^2x \, \bar{w} \bar{\phi}, \quad E_{zf} = \frac{1}{2} \int dy \, U^2 \]
The main claim

Simulations show that ZFs exhibit substantial merging in multi-scale turbulence.

Claim: this merging is the cause of the high-$k$ turbulence demise and a generic property of multi-scale turbulence.

- To show this, some theory will be needed:
  - general wave-kinetic theory,
  - topology of the drifton phase-space,
  - approximate closure for $U$,
  - conditions for ZF merging.

- We will also argue that the physics beyond the quasilinear approximation is not very different.
A statistical theory is constructed by analogy with that in QM.

- The Wigner function \( \tilde{W}(t, y, k) \) is defined as
  \[
  \tilde{W}(t, y, k) = \int d^2 s e^{-i k \cdot s} \langle \tilde{w}(t, x + s/2) \tilde{w}(t, x - s/2) \rangle
  \]
  (i.e., the spectrum of the two-point correlator, or “quasiprobability distribution”) satisfies

  \[
  \frac{\partial \tilde{W}}{\partial t} = \{ \mathcal{H}_H, \tilde{W} \} + [\mathcal{H}_A, \tilde{W}], \quad \frac{\partial U}{\partial t} = \frac{\partial}{\partial y} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{1 + k_\perp^2} * k_x k_y \tilde{W} * \frac{1}{1 + k_\perp^2}
  \]

  \[
  \mathcal{H}_H = k_x U - \frac{\beta k_x}{1 + k_\perp^2} + \frac{1}{2} \left[ \{ U'', \frac{k_x}{1 + k_\perp^2} \} \right], \quad \mathcal{H}_A = \frac{1}{2} \left\{ \{ U'', \frac{k_x}{1 + k_\perp^2} \} \right\}
  \]

- Geometrical-optics limit: improved wave kinetic equation (iWKE) with new terms:

  \[
  \hat{\mathcal{L}} = O(\partial_x \partial_k) \sim (k L_{zf})^{-1} < 1, \quad A e^{i \hat{\mathcal{L}}/2} B = 1 + i/2 \{ A, B \} + \ldots
  \]

  \[
  \frac{\partial \tilde{W}}{\partial t} = \{ \mathcal{H}_H, \tilde{W} \} + 2 \mathcal{H}_A \tilde{W}, \quad \frac{\partial U}{\partial t} = \frac{\partial}{\partial y} \int \frac{d^2 k}{(2\pi)^2} \frac{k_x k_y \tilde{W}}{(1 + k_\perp^2)^2}
  \]

  \[
  \mathcal{H}_H \approx k_x U - k_x (\beta - U'')/(1 + k_\perp^2), \quad \mathcal{H}_A \approx -U''' k_x k_y/(1 + k_\perp^2)^2
  \]

  \[
  \hat{\mathcal{L}} = \{, \} \quad \{ A, B \} = 2 A \sin(\hat{\mathcal{L}}/2) B \quad [ A, B ] = 2 A \cos(\hat{\mathcal{L}}/2) B \quad A \ast B = A e^{i \hat{\mathcal{L}}/2} B
  \]

Ruiz et al. (2016); Parker (2016); cf. Smolyakov and Diamond (1999); Krommes and Kim (2000) . . .
Quasistatic approximation in the limit $U'' \ll \beta$

- Let us rewrite the iWKE in the following form using the group velocity $v_g$:

$$
\frac{\partial W}{\partial t} + \frac{\partial}{\partial y} (Wv_g) = \frac{\partial}{\partial k_y} \left( W \frac{\partial \mathcal{H}}{\partial y} \right) - \frac{U'''}{\beta - U''} Wv_g, \quad v_g = \frac{2k_x k_y}{(1 + k^2_\perp)^2} (\beta - U'')
$$

- By integrating the iWKE over $k$, one obtains an equation for the drifton density. The term on the right can be neglected compared to $\partial_y J$ when $U'' \ll \beta$.

$$
\partial_t N + \partial_y J = -JU'''/(\beta - U''), \quad N = \int W \, d^2k, \quad J = \int W v_g \, d^2k
$$

- In this “quasistatic” limit, $U$ becomes a local function of $N$ (“equation of state”):

$$
\frac{\partial U}{\partial t} = \frac{\partial}{\partial y} \left[ \frac{J}{2(\beta - U'')} \right] \approx \frac{\partial_y J}{2\beta} \approx -\frac{\partial_t N}{2\beta}
$$

$$
U \approx -\frac{N}{2\beta} + \text{const}
$$

For quasimonochromatic turbulence as a special case, we discussed this in Zhou et al. (2019).
Even at small $U$, the ZF stability depends on the ZF wavenumber $q$.

- The iWKE is only marginally applicable to ZF formation but can explain it qualitatively.

\[ \mathcal{H} = \frac{k_x (-\beta + U'')}{1 + k_x^2 + k_y^2} + k_x U, \quad U \approx -\frac{N}{2\beta} + \text{const}^* \]

\[ k_y^2 \ll 1 + k_x^2, \quad q^2 \approx -U''/U, \quad k_x = \text{const} \]

\[ \mathcal{H} \approx \frac{k_x}{\beta} \left( \frac{k_y^2}{2m} + V \right) + \text{const}, \quad \frac{1}{m} \approx \frac{2\beta^2}{(1 + k_x^2)^2}, \quad V \approx \left( \frac{q^2}{1 + k_x^2} - 1 \right) \frac{N}{2} \]

- If $q^2 < 1 + k_x^2$, driftons reside near minima of $V$, so the system is stable. 

- If $q^2 > 1 + k_x^2$, driftons reside near maxima of $V$. The system can lower the energy by bifurcating to a lower-$q$ state, so it is unstable to ZF merging.

* Here, we assume $U'' \lesssim \beta$. Unlike in single-scale turbulence, this does not rule out $q \gtrsim k_x$. 

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ZF\text{s have many regimes but not all of them are realized naturally.}

- In single-scale turbulence, $q$ corresponds to the maximum of $\gamma_{\text{secondary}}$. Then, there are passing and/or trapped trajectories, and many driftons survive.

\[ q \sim \min \left\{ k_x^2 \sqrt{N/\beta}, \sqrt{1 + k_x^2} \right\}, \quad U \lesssim U_{c1} \]

- Low-$k$ waves cause ZFs to merge down to $q^2 \sim 1 + k_x^2$. High-$k$ waves become runaways and dissipate. Low-$k$ waves remain passing because $U_{c1} = U_{c1}(k_x)$. 

\[ \text{Rayleigh–Kuo threshold} \]

\[ \text{TI-stable (no stationary modes)} \]

\[ \text{TI-unstable ($\gamma_{\text{TI}} > 0$)} \]

\[ U_{c2} = \frac{\beta}{q^2} \]

\[ U_{c1} = \frac{\beta}{2 (1 + k_x^2) - q^2} \]

\[ \text{weak ZFs} \]

\[ \text{merging} \]

\[ P, T, R \]

Zhu et al. (2019)
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Zhu et al. (2019)
• The scale separation persists in the poloidal-$k$ space.
• In the radial-$k$ space, the scales are determined by the ZFs, which start at the electron scale and then merge down to the ion scale.
• During the ZF merging, most high-$k$ driftons become runaways and dissipate.

Quasilinear simulations for two different ratios $k_{x2}/k_{x1}$
Nonlinear HM simulations demonstrate similar behavior.

![Graphs showing energy and enstrophy evolution](image)

- Drifton collisions serve as an additional channel though which high-$k$ driftons become runaways. Other than that, the effect remains the same.

Summary: the demise of high-$k$ turbulence via cross-scale interactions is robustly explained as a phase-space effect in the Hasegawa–Mima model.
In the Hasegawa–Mima model, the drifton Hamiltonian is pseudo-Hermitian, resulting in an instability of the Kelvin–Helmholtz type. (But is it relevant?)

\[ i \partial_t \tilde{w} = \hat{\mathcal{H}} \tilde{w}, \quad \hat{\mathcal{H}} = k_x \hat{U} - k_x (\beta - \hat{U}'') \left( 1 + k_x^2 + \hat{k}_y^2 \right)^{-1} \]

\[ \gamma_{TI} = |k_x U_0| \left( 1 - \frac{1}{q^2} \right) \sqrt{1 - \frac{\beta^2}{U_0^2 q^4}} \]
The dissipative TI is different. Consider the Terry–Horton model...

\[ \partial_t w + \{ \varphi, w \} = \beta \partial_y \varphi - \hat{D} w \]
\[ w = (\nabla^2 - \hat{\alpha} + i\delta) \varphi \]
\[ \hat{\delta} = \delta(\hat{k}_y) , \quad \hat{D} = 1 - \kappa \nabla^2 \]

- The “Hasegawa–Mima” TI mode becomes localized + other localized modes appear.

\[ \hat{H} = k_y \hat{U} + k_y (\beta + \hat{U}'' \left[ 1 + k_x^2 + k_y^2 - i\delta(\hat{k}_y) \right]^{-1} - i\hat{D} \]

For this modified Terry–Horton model, see St-Onge (2017). The function \( \delta(k_y) \) can be anything.
The largest growth rates belong to the lowest-order modes. Those are localized in \((x, k_x)\), so the drifiton Hamiltonian can be approximated with its Taylor expansion:

\[
\partial_t W = \{\{\mathcal{H}_H, W\}\} + [\mathcal{H}_A, W] \quad \Rightarrow \text{truncate } \mathcal{H} \quad \Rightarrow \quad \hat{\mathcal{H}} \approx c_0 + c_1 \tilde{x}^2 + c_2 \tilde{k}_x^2
\]

This yields an equation of a quantum harmonic oscillator with complex coefficients:

\[
\left(-\vartheta^2 \frac{d^2}{dx^2} + x^2\right) \tilde{w} = \varepsilon \tilde{w} \quad \Rightarrow \quad \tilde{w}_n \sim H_n \left(\frac{x}{\sqrt{\vartheta}}\right) e^{-x^2/2\vartheta}, \quad \varepsilon_n = (2n + 1) \vartheta
\]

\[
\vartheta \equiv -i\sqrt{2(1 + \beta/U_0''/U_0''')} \quad \varepsilon \equiv \frac{2}{k_y U_0''} \left[ \omega_{\text{TI}} - k_y U_0 + iD_0 - \frac{k_y(\beta + U_0'')}{1 + k_y^2 - i\delta} \right]
\]
The growth rate is obtained explicitly and agrees with simulations.

\[ \gamma_{TI} = -D_0 + \text{Im} \left[ \frac{k_y(\beta + U_0'') - i k_y U_0'' \sqrt{1 + \beta/U_0''/2}}{1 + k_y^2 - i \delta} \right] \]

Upper: analytic formula. Lower: numerical simulations + an alternative theory with a fitting parameter (not shown).
\[ \gamma_{TI} = -D_0 + \text{Im} \left[ \frac{k_y (\beta + U_0'') - i k_y U_0'' \sqrt{1 + \beta/U_0''}}{1 + k_y^2 - i \delta} \right] \equiv \gamma_{\text{primary}} + \Delta \gamma(U_0'') \]

- The tertiary instability can be viewed as the primary instability modified by ZFs.
  - If \( \gamma_{TI} < 0 \), turbulence is suppressed; ZFs survive, assuming \( \hat{D} \) acts only on DWs.
  - If \( \gamma_{TI} > 0 \), the system ends up in a turbulent state.
  - Due to \( \Delta \gamma \), the transition to the turbulent state occurs at plasma parameters different from those without ZFs. This is called the **Dimits shift**.

The figures are taken from Dimits et al. (2000) and St-Onge (2017).
Our explicit formula for the Dimits shift agrees with simulations.

- We calculate the values of $\beta$ that correspond to $\gamma_{\text{primary}}^{(\text{linear})} = 0$ and $\gamma_{\text{TI}} = 0$ using $U''_0 \sim q^2 U_{c1}$. The difference between these values is the Dimits shift (shaded).

- Compared with a related calculation by St-Onge (2017), our model is a better fit at both large and small $\delta$. For example, it has no spurious cutoff at $\delta = 2$.

\[ \beta_c \approx \frac{D[(1 + k_y^2)^2 + \delta^2]/k_y}{\delta - (1 + k_y^2)\sqrt{U''_0/2\beta}}, \quad \frac{U''_0}{\beta} \sim \frac{q^2}{k_y^2 + 1} \]

St-Onge (2017) used four-mode truncation (a stretch, also not intuitive) and did not calculate the Dimits shift \textit{per se}. 21/27
Modified Hasegawa–Wakatani system (work in progress)

- Next step: consider the Hasegawa–Wakatani model, with \( w = \nabla^2 \varphi - n \):
  \[
  \partial_t w + \{ \varphi, w \} = \beta \partial_y \varphi - \hat{D} w, \quad \partial_t n + \{ \varphi, n \} = \alpha(\tilde{\varphi} - \tilde{n}) - \beta \partial_y \varphi - \hat{D} n
  \]

- The linear equation for tertiary modes is somewhat similar to that in the Terry–Horton model:
  \[
  \omega \tilde{w} = \hat{H} \tilde{w}
  \]
  \[
  \hat{H} = k_y U + k_y (\beta + U'' - N') \tilde{K}^{-2} - iD
  \]

\[
\tilde{K}^2 = \tilde{k}_x^2 + \tilde{k}_y^2 + \frac{k_y (\beta - N') + i\alpha}{\omega + iD - k_y U + i\alpha}
\]

- The trapped and passing modes are still present.
- However, the dependence of the Hamiltonian on \( N' \) makes analytic predictions more difficult.
Collisionless ZFs: three different regimes are found numerically.

(a) Dimits regime, turbulence is suppressed; (ii) ZF-dominated regime, turbulence is localized, special structures; (iii) turbulence-dominated regime.
Collisional zonal flows

- When zonal flows are collisional, the complete suppression of turbulence is impossible. Instead, predator–prey oscillations are observed.
- A parabolic expansion of $\hat{H}$ leads to reasonable results, but more work is needed.
Summary

- Studying drift-wave turbulence in phase space requires:
  - looking beyond geometrical optics: do not neglect $\lambda/L$ and $U''/\beta$,
  - deriving WME/WKE from first principles: hand-waving leads to errors.

- Cross-scale interactions within the Hasegawa–Mima model:
  - Zonal flows tend to merge when there are driftons with $k_{\text{poloidal}} \lesssim q$.
  - Zonal-flow merging gradually reduces the DW radial scale.
  - High-$k_{\text{poloidal}}$ DWs are efficiently dissipated during this process.

- Tertiary instabilities and the Dimits shift:
  - Dissipation localizes the tertiary modes near the ZF-velocity extrema.
  - The growth rate of these modes can be made negative by $U'' \Rightarrow$ Dimits shift.
  - An analytic theory is developed within the Terry–Horton model.
  - The Hasegawa–Wakatani system is more subtle. (in progress)
[Dimits et al. (2000)]

[Hammett et al. (1993)]

[Howard et al. (2016)]

[Kim and Diamond (2002)]

[Krommes and Kim (2000)]

[Kuo (1949)]

[Maeyama et al. (2015)]

[Maeyama et al. (2017)]

[Numata et al. (2007)]

[Parker (2016)]

[Ruiz et al. (2019)]

[Ruiz et al. (2016)]

[Smolyakov and Diamond (1999)]
[St-Onge (2017)]

[Zhou *et al.* (2019)]

[Zhu *et al.* (2018)a]

[Zhu *et al.* (2018)b]

[Zhu *et al.* (2019)]

[Zhu *et al.* (2018)c]