Ripple modifications to alpha transport in tokamaks & quasisymmetric stellarators

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QS stellarators & tokamaks are isomorphic (both have a drift kinetic canonical angular momentum constant of motion) can do tokamaks then use $n \Rightarrow (nM-mN)/(M-qN)$

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Overview

- Alpha & ripple background
- Need to solve α with ∇B drift, not $E \times B$
- Transit averaged kinetic equation
- Pitch angle scattering stronger than drag
- $\sqrt{\nu}$ regime diffusivity estimate
- superbanana plateau (sbp) estimate
- Alpha depletion a concern
- Boundary layer analysis
- Comments

Background

Magnetic ripple δ due to N (~ 16-20) toroidal field coils:

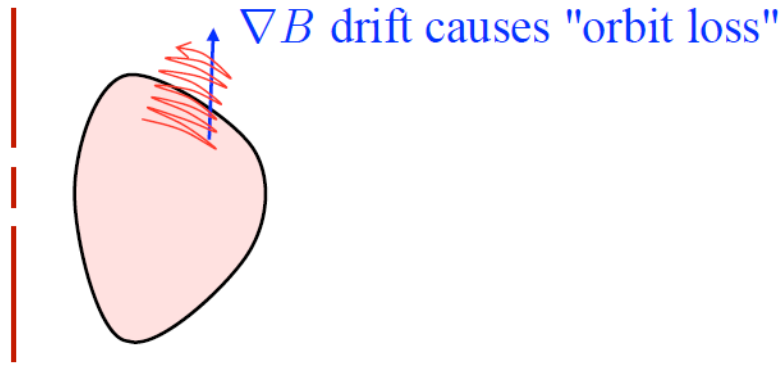
$$\delta = (B_{max} - B_{min})/(B_{max} + B_{min}) < 10^{-2}$$

Alphas must heat before being lost

Typically use $B = B_0[1-\varepsilon\cos\vartheta-\delta\cos(n\zeta)]$ for a tokamak

Extrema: $\varepsilon\sin\vartheta+qn\delta\sin(n\zeta) = 0$ for fixed $\alpha = \zeta - q\vartheta$

Assume $qn\delta << \varepsilon$: ripple only affects radial drift (no ripple trapping $\Rightarrow$)



Need tangential drift & streaming $\Rightarrow$ unperturbed

No $1/\nu$ regime for alphas

- D-T: T = 10 keV, $n_e = 10^{14}$ cm^{-3}, B = 5 T, R = 10 m, a = 3 m, and for 3.5 MeV alphas
  - alpha birth speed = $v_0 \approx 1.3\times10^9$ cm/sec
  - alpha gyroradius = $\rho_0 = v_0/\Omega_0 \approx 5.4$ cm
  - alpha slowing down time = $\tau_s \approx 0.63$ sec
  - tangential drift frequency = $\omega \sim \omega_{\alpha\nabla B}$
- $R/v_0\tau_s \sim 10^{-6} \Rightarrow$ can transit average
- $\rho_0/a \sim 10^{-2} \Rightarrow$ radial scales >> alpha gyroradius

∇B: $\omega_{\alpha\nabla B}R \sim v_0\rho_0/R \sim 7\times10^6 << v_i \approx 10^8 \Rightarrow \omega_{\alpha\nabla B}\tau_s \sim 4\times10^3$

- $\rho_0/a >> R/v_0\tau_s \Rightarrow$ Good/bad news $\Rightarrow \sqrt{\nu}$ & sbp regimes

expect narrow collisional boundary layers

Alpha collision operator & distribution function

Alpha slowing down tail distribution function satisfies

$$\frac{Ze}{Mc}\vec{v}\times\vec{B}\cdot\nabla_v f_s = C\{f_s\} + \frac{S\delta(v-v_0)}{4\pi v^2}$$

α birth & reaction rates related by $S = n_D n_T \langle\sigma v\rangle_{DT}$ &

$$f_s = f_s(\psi, v) = \frac{S(\psi)\tau_s(\psi)H(v_0 - v)}{4\pi[v^3 + v_c^3(\psi)]}$$

Alpha collision operator

$$C\{f\} = \frac{1}{\tau_s}\nabla_v\cdot[(\frac{v^3+v_c^3}{v^3})\vec{v}f + \frac{v_\lambda^3}{2v^3}(v^2\overleftrightarrow{I} - \vec{v}\vec{v})\cdot\nabla_v f]$$

v_c = critical speed for equal electron & ion drag & $v_\lambda \sim v_c$

The slowing down density for $v_0^3 >> v_c^3$ is

$$n_s = \int d^3v f_s \simeq S\tau_s \ell n(v_0/v_c)$$

Transit averaged kinetic equation

Using $f = f_s + h$ the transit averaged equation is

$$\overline{\vec{v}_d \cdot \nabla\psi}\frac{\partial f_s}{\partial\psi} + \overline{\vec{v}_d \cdot \nabla\alpha}\left.\frac{\partial \bar{h}_t}{\partial\alpha}\right|_\zeta = \overline{C\{\bar{h}_t\}}$$

where passing h vanishes ($\bar{h}_p = 0$), $\alpha = \zeta - q\vartheta$ &

$$\vec{B} = B\vec{b} = \nabla\alpha \times \nabla\psi = K(\psi,\vartheta,\zeta)\nabla\psi + G(\psi)\nabla\vartheta + I(\psi)\nabla\zeta$$

with $G/qI \sim rB_p/qRB_t \sim \varepsilon^2/q^2 << 1$ & $B_p \Rightarrow$ poloidal field

Tangential & radial drifts ($\rho_{p0} \simeq \rho_0 B_0/B_p$):

$$-\overline{\vec{v}_d \cdot \nabla\alpha} = \omega \sim (v_\perp^2/2\Omega)(\partial B/\partial\psi) \sim \rho_{p0} v_0/R^2 \Rightarrow \nabla B$$ but...

$$\frac{\overline{\vec{v}_d \cdot \nabla\psi}}{RB_p} = -\frac{B_0(\partial/\partial\alpha)(\oint_\alpha d\zeta\, v_\parallel / B)}{RB_p\Omega_0(\oint_\alpha d\zeta / v_\parallel B)} \sim \frac{qn\delta}{\varepsilon}\frac{\rho_0 v_0}{R}$$

Pitch angle scattering dominates

Trapped fraction $\varepsilon^{1/2}$

Tangential rotation = $\omega \Rightarrow$ boundary layer width w << $\varepsilon^{1/2}$

Pitch angle scatter time $\tau_p = (v_0^3/v_\lambda^3)\tau_s >> \tau_s$ = drag time, but

$$\overline{C\{\bar{h}\}} \sim \frac{v_\lambda^3}{\tau_s v_0^3}\frac{\partial^2 \bar{h}}{\partial \lambda^2} \sim \frac{\bar{h}}{\tau_p w^2} >> \frac{\bar{h}}{\tau_s} \Rightarrow w^2 << \frac{v_\lambda^3}{v_0^3} \sim \frac{\tau_s}{\tau_p} \sim 2\times 10^{-2}$$

Balance collisions by tangential drift $\Rightarrow$ narrow b. layer:

$$\bar{h}/w^2\tau_p \sim \overline{C\{\bar{h}\}} \sim \overline{\vec{v}_d \cdot \nabla\alpha}\,\partial\bar{h}/\partial\alpha \sim \omega n\bar{h} \sim \bar{h} n \rho_{p0} v_0/R^2$$

w ~ eff. trap. fract: $w \sim F \sim (1/n\omega\tau_p)^{1/2} \sim (rR/qn\rho_0 v_0 \tau_p)^{1/2} << \varepsilon^{1/2}$

Eff. pitch angle scatter time = $w^2\tau_p << \tau_s$ = slowing down

$$n\omega\tau_s >> 1$$

$\sqrt{\nu}$ regime diffusivity estimate

Eff. trapped fraction: $w \sim F \sim (1/n\omega\tau_p)^{1/2}$

Effective drift decorrelation time: $\tau = F^2\tau_p \sim 1/n\omega$

∇B ripple radial drift: $V \sim v_0\rho_0 qn\delta/r$

Tangential rotation: $\omega \sim \rho_{p0}v_0/R^2 \sim q\rho_0 v_0/rR$

Eff. radial ripple step: $\Delta \sim V\tau \sim R\delta$

Crude $\sqrt{\nu}$ regime alpha diffusivity:

$$D_{\sqrt{\nu}} \sim F\Delta^2/\tau = (R\delta)^2(n\omega/\tau_p)^{1/2} = (R\delta)^2\sqrt{\frac{qn\rho_0 v_0}{rR\tau_p}} \propto \frac{R\delta^2}{T^{3/4}}\sqrt{\frac{nn_e}{B_p}}$$

Galeev *et al.* (1969 thank Grad for "frank & comradely discussions" as in Vienna), Ho & Kulsrud (1987)

Superbanana plateau due to a resonance

Sbp occurs because there is a zero at $\kappa_0^2 \simeq 0.83$:

$$\omega \simeq -\frac{v^2[2E(\kappa)-K(\kappa)]}{2R^2\Omega_p K(\kappa)} \rightarrow \frac{v^2}{4R^2\Omega_p}\begin{cases} 2 & \kappa^2 \rightarrow 1 \\ \dfrac{\kappa-\kappa_0}{\kappa_0(1-\kappa_0^2)} & \kappa^2 \simeq \kappa_0^2 \end{cases}$$

with $\lambda = 1/(1-\varepsilon+2\varepsilon\kappa^2)$, $\lambda = 2\mu B_0/v^2$ & $\lambda-\lambda_0 \sim (\kappa-\kappa_0)\varepsilon$

Now $\bar{h}/w^2\tau_p \sim \overline{C\{\bar{h}\}} \sim \overline{\vec{v}_d \cdot \nabla\alpha}\,\partial\bar{h}/\partial\alpha \sim w\varepsilon^{-1}\omega n\bar{h} \sim wn\rho_{p0}v_0/rR$

Wider bound. lay: $w \sim (1/n\omega\tau_p)^{1/3}$ & Reduced rotation: $w\varepsilon^{-1}\omega$

Eff. trapped fraction of trapped fraction - normalize by $\varepsilon^{1/2}$

$$F \sim w/\varepsilon^{1/2} \sim \varepsilon^{-1/2}(1/n\omega\tau_p)^{1/3}$$

Superbanana plateau diffusivity estimate

Boundary layer width: $w \sim (1/n\omega\tau_p)^{1/3}$

Effective trapped fraction: $F \sim \varepsilon^{-1/2}(1/n\omega\tau_p)^{1/3} << 1$

Effective drift decorrelation time: $\tau = w^2\tau_p \sim \tau_p^{1/3}/(n\omega)^{2/3}$

Smaller tangential rotation = $w\varepsilon^{-1}\omega$ with $\omega \sim \rho_{p0}v_0/R^2$

∇B ripple radial drift: $V \sim v_0\rho_0 qn\delta/r \sim n\omega R\delta$

Eff. radial ripple step increase: $\Delta \sim V\tau \sim (n\omega\tau_p)^{1/3}R\delta$

Crude sbp regime alpha diffusivity:

$$D_{sbp} \sim F\Delta^2/\tau = (R\delta)^2 n\omega/\sqrt{\varepsilon} = \delta^2 n\rho_{p0}v_0/\sqrt{\varepsilon} \propto n\delta^2/B_p\sqrt{\varepsilon}$$

Large B_p & ε desirable

Comparing diffusivity estimates

Ratio

$$\frac{D_{sbp}}{D_{\sqrt{\nu}}} = \frac{(n\omega\tau_p)^{1/2}}{\sqrt{\varepsilon}} = \frac{(n\omega\tau_s v_0^3/v_\lambda^3)^{1/2}}{\sqrt{\varepsilon}} >> 1$$

so sbp dominates as long as $n\rho_{p0}/r >> R/v_0\tau_p$.

To avoid depleting slowing down tail

$$1 >> \frac{\tau_s D_{sbp}}{a^2} = \left(\frac{R\delta}{a}\right)^2 \frac{n\omega\tau_s}{\sqrt{\varepsilon}}$$

allows only small imperfections

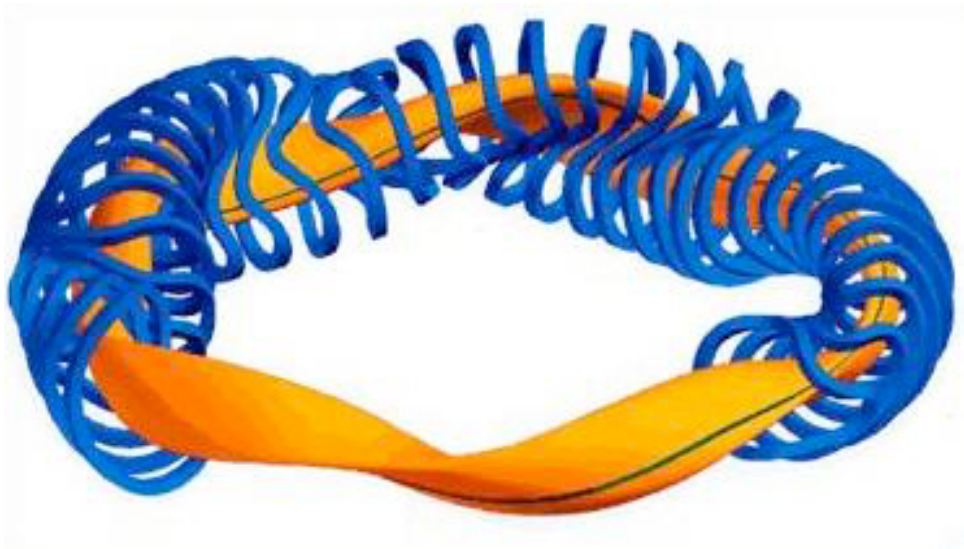
$$\delta << \frac{a\varepsilon^{1/4}}{R\sqrt{n\omega\tau_s}} \sim 10^{-3} - 10^{-4}.$$

Need a careful boundary layer analysis

Stellarator estimates

A quasisymmetric flux surface mod B closes after M toroidal turns and N toroidal turns: $B=B_0[1-\varepsilon\cos(M\vartheta-N\zeta)]$

N = 6 & M =1
(cheat alert: really W7-X so not QS, but close enough)
N = 0: quasiaxisymmetry
M = 0: quasipoloidal sym.
other N & M quasihelical



Quasisymmetric stellarators isomorphic with tokamaks
$B/B_0 = 1-\varepsilon\cos\vartheta-\delta\cos(n\zeta) \Rightarrow 1-\varepsilon\cos(M\vartheta-N\zeta)-\Sigma\delta\cos(m\vartheta-n\zeta)$
with Σ a sum over m and n and $(M-qN)\varepsilon >> (m-qn)\delta$

Just make replacement $n \Rightarrow |(nM-mN)/(M-qN)|$

Superbanana plateau kinetic equation

Trapped satisfy

$$\frac{v_0^3(2\kappa_0^2-1)\partial^2 f_t}{8\tau_p v^3\delta\kappa_0^2\,\partial\kappa^2}+\frac{v^2(\kappa-\kappa_0)}{4\Omega_p R^2\kappa_0(1-\kappa_0^2)}\frac{\partial f_t}{\partial\alpha}=-\frac{B_0 n\delta\lambda v^2}{2\Omega_0}\frac{\partial f_s}{\partial\psi}\cos(qn\pi)\sin(n\alpha)$$

Let $\chi\equiv(1-\kappa)/8$, $\chi_0\equiv(1-\kappa_0)/8$, & $f_t\equiv \mathrm{Im}[H(\chi)e^{in\alpha}]$ to find

$$\frac{\partial^2 H}{\partial\chi^2}-is^3(\chi-\chi_0)H=-\Upsilon.$$

Su & Oberman (1968) solved Airy form long ago to find

$$H=\frac{\Upsilon}{s^2}\int_0^\infty d\tau\, e^{-is(\kappa_0-\kappa)\tau/8-\tau^3/3}\underset{s(\chi-\chi_0)\to\infty}{\longrightarrow}-\frac{i\Upsilon}{s^3(\chi-\chi_0)}\underset{\kappa\to 1}{\longrightarrow}\frac{i\Upsilon}{s^3\chi_0}.$$

Use WKB to match to $\kappa\to 1$, $\sqrt{\nu}$ boundary layer $\Rightarrow$ add but $\sqrt{\nu}$ always small. Particle & energy diffusivity

$$D_{sbp}=\frac{\kappa_0^2(1-\kappa_0^2)\delta^2 n\rho_{p0}v_0\cos^2(qn\pi)}{2\sqrt{2\varepsilon}\,\ell n(v_0/v_c)}$$

Superbanana plateau coefficient small

Resonant trapped turn at $\kappa_0^2 \simeq 0.83$ giving $\kappa_0^2(1-\kappa_0^2) \simeq 0.14$, reducing coefficient by ≈ 1/20. Therefore, for a tokamak

$$1 >> \frac{\tau_s D_{sbp}}{a^2} = \left(\frac{R\delta}{a}\right)^2 \frac{n\omega\tau_s}{20\sqrt{\varepsilon}},$$

& the resonance is at $\vartheta_0 \simeq 3\pi/4$. Need

$$\delta << \frac{a\varepsilon^{1/4}\sqrt{20}}{R\sqrt{n\omega\tau_s}} \sim 10^{-3} \Rightarrow \text{usual need to keep } \delta \lesssim 10^{-4}.$$

Better if $a\varepsilon^{1/4}$ & n^{-1} ⇒ (M-qN)/(nM-mN) larger

$$\left|\frac{nM-mN}{M-qN}\right| \Rightarrow \begin{cases} n & N=0 & M\varepsilon >> (m-qn)\delta & \text{QAS} \\ q^{-1}m & M=0 & qN\varepsilon >> (m-qn)\delta & \text{QPS} \\ nM/qN & m/n << q & N\varepsilon >> n\delta & \text{QHS} \\ nM/m & m/n >> q & M\varepsilon >> m\delta & \text{QHS} \end{cases}.$$

Comments & warnings!

Superbanana plateau always dominates over $\sqrt{\nu}$ for alphas since $n\rho_{p0}/r >> R/v_0\tau_p$

Larger B, a, n_e & ε, at lower T better:

$$\delta << \frac{a\varepsilon^{1/4}\sqrt{20}}{R\sqrt{n\omega\tau_s}} \propto a\varepsilon^{1/4}\frac{B^{1/2}n_e^{1/2}}{T^{3/4}}$$

Highly idealized model of a stellarator (it is hard to be so close to QS) & want error fields with $\left|\frac{M-qN}{nM-mN}\right| >> 1$

Superbanana plateau is a transition to a superbanana regime with D linear in ν as $\kappa = \kappa(\psi,\lambda)$ (see Beidler & D'haeseleer 1995 PPCF) or some other regime