UNIVERSITY OF OXFORD

Whisper waves: an unlikely addition to the microscale zoo

Sub-electron-Larmor-scale dynamics in a high-β plasma

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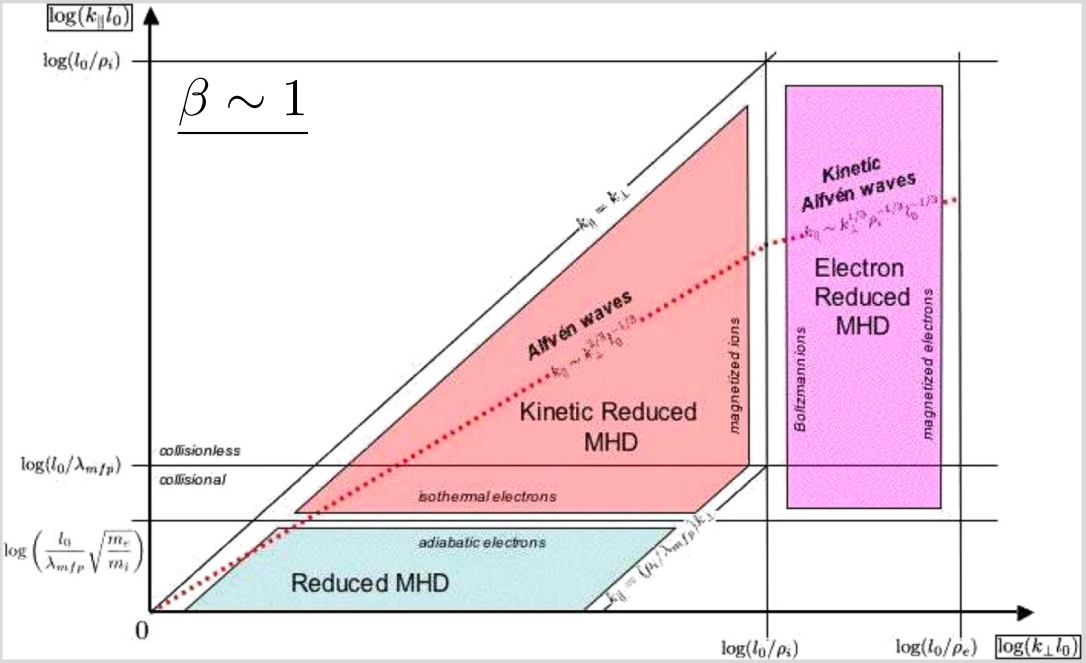
12th Plasma Kinetics Working Meeting, Wolfgang-Pauli Institute

Motivation

- Linear, low-frequency (gyrokinetic) modes comprehensively understood in $\beta \sim 1$ plasma
  - At sub-electron-Larmor scales, no propagating modes (Boldyrev 2013)

- Much is known about turbulent cascades in astrophysical plasma (Schekochihin 2009)
  - At sub-electron Larmor scales, find electron-entropy cascade – 'last cascade'



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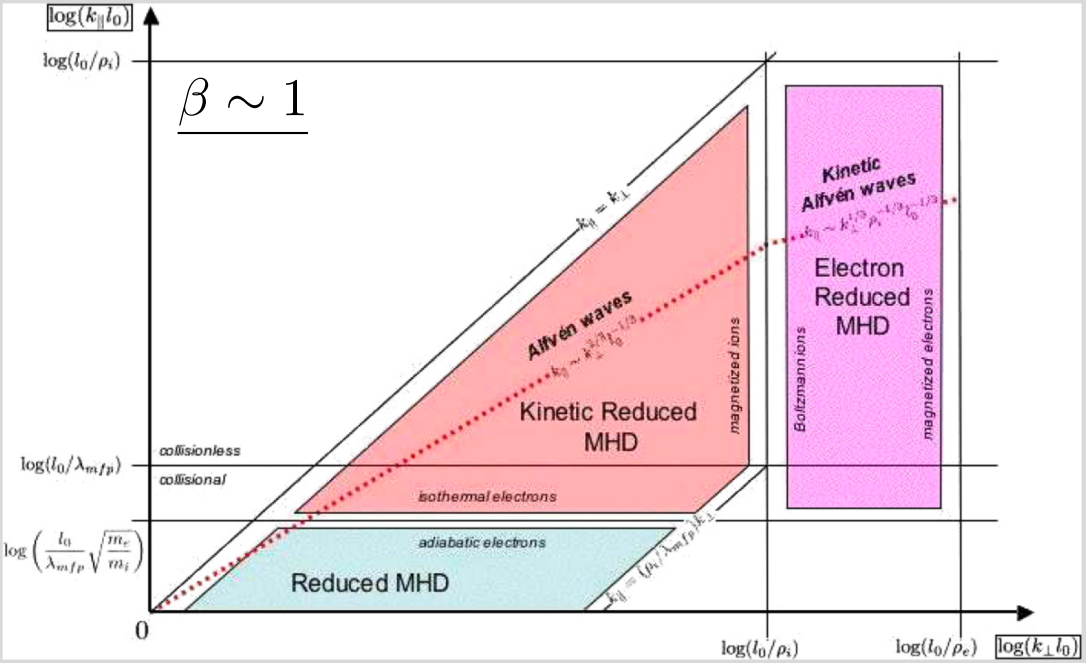

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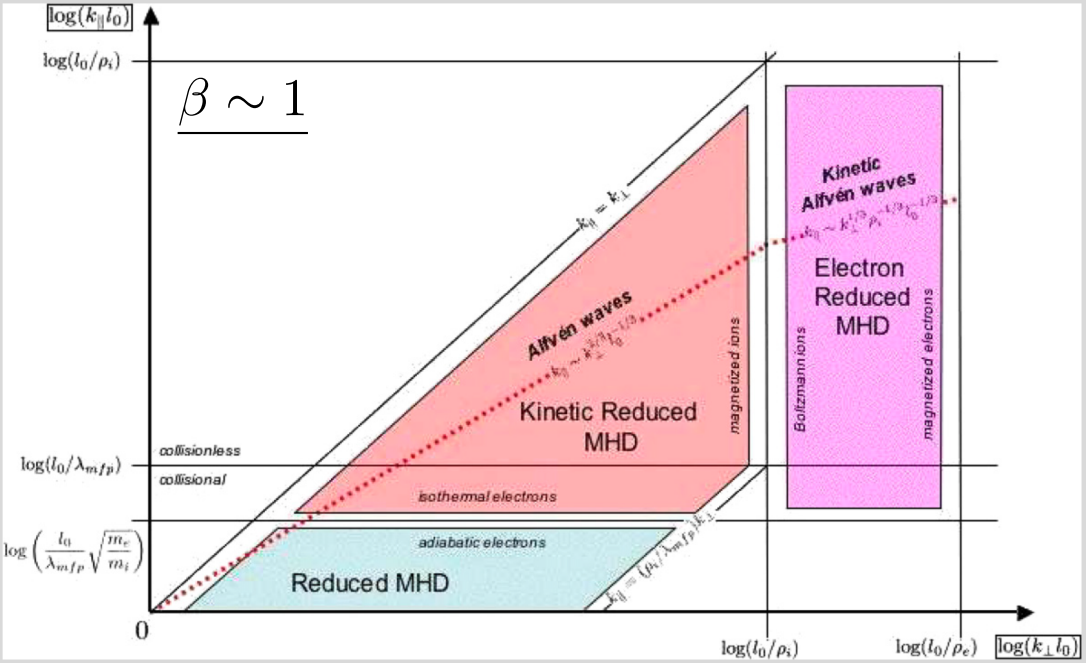

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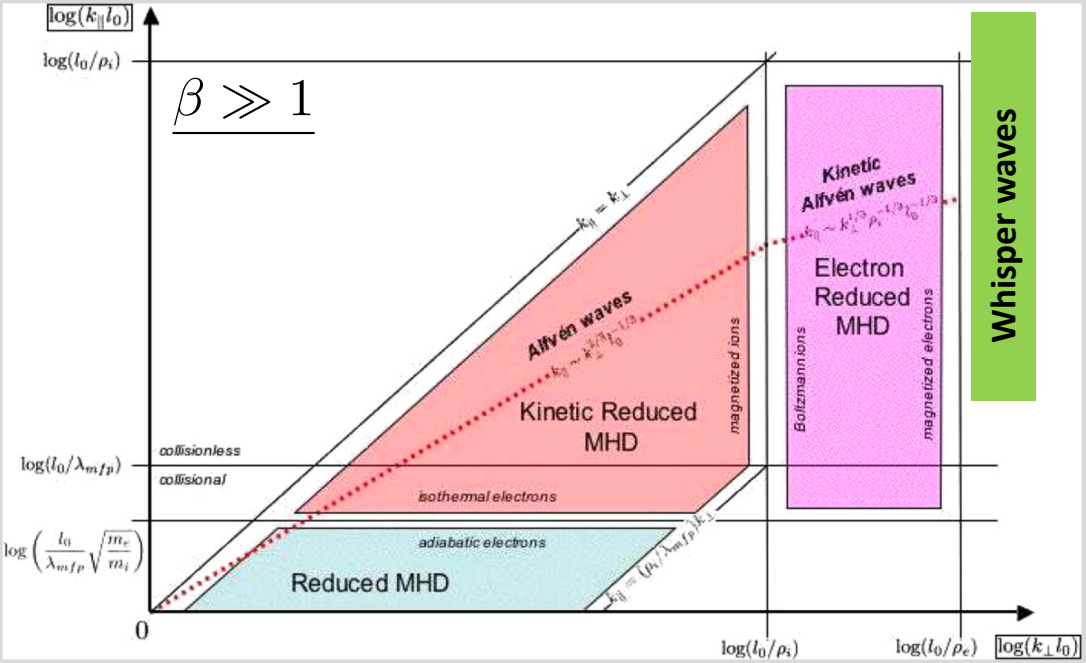

- Is the same true for $\beta \gg 1$?
  - Surprisingly not! For β sufficiently large, there exist propagating waves with $k_\perp \rho_e \gg 1$ (***whisper waves***).
  - Life can be electromagnetic beyond the electron-Larmor-scale in a high-β plasma

Talk overview

1. Linear theory of whisper waves
2. Nonlinear whisper-wave equations
3. Should anyone care about whisper waves?
4. Whispered wishes…

Whisper waves: orderings

- Assume that $\beta_e^{1/5} \gg 1$, where $\beta_e \equiv 8\pi n_{e0} T_{e0}/B_0^2$
- Frequency ω and perpendicular wavenumber $k_\perp$ of the whisper wave will then satisfy

$$\frac{\omega}{k_\parallel v_{\rm the}} \sim \frac{1}{k_\perp \rho_e} \sim \beta_e^{-1/5} \ll 1$$

Notes on frequency/wavenumber ordering

1. $\omega \ll \Omega_e$, so ***gyrokinetic equation for electron species valid***
2. Ion behaviour: for a hydrogen plasma with $T_{i0} = T_{e0}$, $\omega/k_\parallel v_{{\rm th}i} \sim \beta_e^{-1/5}(m_p/m_e)^{1/2}$
   - If $\beta_e \ll 10^8$, ***ions can be treated as static***

- Ordering for electromagnetic field components:

$$\phi \sim \frac{1}{k_\perp \rho_e}\frac{\omega}{k_\parallel v_{\rm the}}\frac{v_{\rm the} A_\parallel}{c} \sim \beta_e^{-2/5}\frac{v_{\rm the} A_\parallel}{c}\,,$$

$$\delta B_\parallel \sim \frac{eB_0}{T_{e0}}\frac{v_{\rm the} A_\parallel}{c} \sim \frac{1}{k_\perp \rho_e}\delta B_\perp \sim \beta_e^{-1/5}\delta B_\perp\,.$$

- ***Energetically dominant EM field component is*** $\delta B_\perp$

Full governing equations

- Gyrokinetic equation for electron distribution function:

$$\frac{\partial h_e}{\partial t} + v_\parallel \frac{\partial h_e}{\partial z} + \frac{c}{B_0}\left\{\langle\chi\rangle_{\boldsymbol{R}_e}, h_e\right\} = -\frac{eF_{e0}}{T_{e0}}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_e}}{\partial t} + \left(\frac{\partial h_e}{\partial t}\right)_c$$

- Maxwell's equations, with only electron source terms:

$$\frac{e^2\phi}{T_{e0}} n_{e0} = -e\int \mathrm{d}^3\boldsymbol{v}\,\langle h_e\rangle_{\boldsymbol{r}}\,,$$

$$\nabla_\perp^2 A_\parallel = \frac{4\pi e}{c}\int \mathrm{d}^3\boldsymbol{v}\, v_\parallel \langle h_e\rangle_{\boldsymbol{r}}\,,$$

$$\nabla_\perp^2 \delta B_\parallel = \frac{4\pi e}{c}\hat{\boldsymbol{z}}\cdot\left[\nabla_\perp \times \int \mathrm{d}^3\boldsymbol{v}\,\langle \boldsymbol{v}_\perp h_e\rangle_{\boldsymbol{r}}\right]$$

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- *Useful to write Maxwell's equations in Fourier-space representation:*

$$\frac{e\phi}{T_{e0}} = -\frac{1}{n_{e0}}\sum_{\boldsymbol{k}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right)\int \mathrm{d}^3\boldsymbol{v}\, J_0(a_e) h_{e\boldsymbol{k}}\left(t, v_\parallel, v_\perp\right) \ ,$$

$$\frac{c}{4\pi e n_{e0}}\nabla_\perp^2 A_\parallel = \frac{1}{n_{e0}}\sum_{\boldsymbol{k}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right)\int \mathrm{d}^3\boldsymbol{v}\, v_\parallel J_0(a_e) h_{e\boldsymbol{k}}\left(t, v_\parallel, v_\perp\right) \ ,$$

$$\frac{\delta B_\parallel}{B_0} = -\frac{\beta_e}{2}\frac{1}{n_{e0}}\sum_{\boldsymbol{k}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right)\int \mathrm{d}^3\boldsymbol{v}\, \frac{2v_\perp^2}{v_{\mathrm{the}}^2}\frac{J_1(a_e)}{a_e} h_{e\boldsymbol{k}}\left(t, v_\parallel, v_\perp\right) \ ,$$

because the ring-averaged gyrokinetic potential is then related to the Fourier-transformed EM fields via

$$\langle\chi\rangle_{\boldsymbol{R}_e,\boldsymbol{k}} = J_0(a_e)\left(\phi_{\boldsymbol{k}} - \frac{v_\parallel A_{\parallel\boldsymbol{k}}}{c}\right) + \frac{T_{e0}}{e}\frac{2v_\perp^2}{v_{\mathrm{the}}^2}\frac{J_1(a_e)}{a_e}\frac{\delta B_{\parallel\boldsymbol{k}}}{B_0}$$

Linearisation

- Could in principle apply whisper-wave orderings consistently to full governing equations…
  - …but won't do so yet! Easier to derive damping rate etc. by linearising first, then expanding linearised equations in small parameter $\epsilon \equiv \beta_e^{-1/5} \ll 1$

- Consider infinitesimal perturbation in the usual way (drop subscripts):

$$h(t, \boldsymbol{R}, v_\parallel, v_\perp) = h_{\boldsymbol{k},\omega}\left(v_\parallel, v_\perp\right) \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R} - \mathrm{i}\omega t\right),$$
$$\phi(t, \boldsymbol{r}) = \phi_{\boldsymbol{k},\omega} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t\right),$$
$$A_\parallel(t, \boldsymbol{r}) = A_{\parallel\boldsymbol{k},\omega} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t\right),$$
$$\delta B_\parallel(t, \boldsymbol{r}) = \delta B_{\parallel\boldsymbol{k},\omega} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t\right).$$

- Rearranging linearised gyrokinetic equation (neglecting collisions) gives distribution function in terms of Fourier-transformed EM fields:

$$h_{\boldsymbol{k},\omega} = -\frac{eF_0}{T_0}\frac{\omega}{\omega - k_\parallel v_\parallel}\left[J_0(a)\left(\phi_{\boldsymbol{k},\omega} - \frac{v_\parallel A_{\parallel\boldsymbol{k},\omega}}{c}\right) + \frac{T_0}{e}\frac{2v_\perp^2}{v_{\mathrm{th}}^2}\frac{J_1(a)}{a}\frac{\delta B_{\parallel\boldsymbol{k},\omega}}{B_0}\right]$$

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- Substitute result into Maxwell's equations (see Howes 2006), deduce algebraic equations:

$$\phi_{\boldsymbol{k},\omega} = -\Gamma_0(\alpha)\xi Z(\xi)\phi_{\boldsymbol{k},\omega} + [1+\xi Z(\xi)]\,\Gamma_0(\alpha)\frac{\omega A_{\parallel\boldsymbol{k},\omega}}{k_\parallel c} + \Gamma_1(\alpha)\xi Z(\xi)\frac{T_0}{e}\frac{\delta B_{\parallel\boldsymbol{k},\omega}}{B_0}\,,$$

$$\frac{\omega A_{\parallel\boldsymbol{k},\omega}}{k_\parallel c} = -\frac{4\pi e^2 n_0\omega^2}{T_0 k_\parallel^2 k_\perp^2 c^2}\,[1+\xi Z(\xi)]\left[\Gamma_0(\alpha)\left(\phi_{\boldsymbol{k},\omega} - \frac{\omega A_{\parallel\boldsymbol{k},\omega}}{k_\parallel c}\right) - \Gamma_1(\alpha)\frac{T_0}{e}\frac{\delta B_{\parallel\boldsymbol{k},\omega}}{B_0}\right]\,,$$

$$\frac{\delta B_{\parallel\boldsymbol{k},\omega}}{B_0} = \frac{4\pi e n_0}{B_0^2}\Gamma_1(\alpha)\left[-\xi Z(\xi)\phi_{\boldsymbol{k},\omega} + [1+\xi Z(\xi)]\,\frac{\omega A_{\parallel\boldsymbol{k},\omega}}{k_\parallel c} + 2\xi Z(\xi)\frac{T_0}{e}\frac{\delta B_{\parallel\boldsymbol{k},\omega}}{B_0}\right]\,,$$

where $\Gamma_0(\alpha) \equiv I_0(\alpha)\exp(-\alpha)$ and $\Gamma_1(\alpha) \equiv [I_0(\alpha) - I_1(\alpha)]\exp(-\alpha)$.

- In general determine frequencies by looking for solutions with vanishing determinant

Expansion procedure

- Now expand in $\epsilon = \beta^{-1/5} \sim \alpha^{-1/2} \sim \xi \ll 1$, and assume whisper-wave orderings:

$$\begin{aligned}
\omega &= \omega^{(0)} + \epsilon\omega^{(1)} + \dots\,, \\
\xi &= \xi^{(0)} + \epsilon\xi^{(1)} + \dots\,, \\
\phi_{\boldsymbol{k},\omega} &= \phi^{(0)}_{\boldsymbol{k},\omega} + \epsilon\phi^{(1)}_{\boldsymbol{k},\omega} + \dots\,, \\
A_{\|\boldsymbol{k},\omega} &= A^{(0)}_{\|\boldsymbol{k},\omega} + \epsilon A^{(1)}_{\|\boldsymbol{k},\omega} + \dots\,, \\
\delta B_{\|\boldsymbol{k},\omega} &= \delta B^{(0)}_{\|\boldsymbol{k},\omega} + \epsilon\delta B^{(1)}_{\|\boldsymbol{k},\omega} + \dots\,,
\end{aligned}$$

$$+$$

$$\begin{aligned}
\phi^{(0)}_{\boldsymbol{k},\omega} &\sim \epsilon^2 A^{(0)}_{\|\boldsymbol{k},\omega}\,, \\
\delta B^{(0)}_{\|\boldsymbol{k},\omega} &\sim \epsilon k_\perp A^{(0)}_{\|\boldsymbol{k},\omega}\,.
\end{aligned}$$

- Use asymptotic identities for special functions:

$$\Gamma_0(\alpha) = \frac{1}{\sqrt{2\pi\alpha}}\left[1 + \frac{1}{8\alpha} + O\left(\frac{1}{\alpha^2}\right)\right]\,,$$

$$\Gamma_1(\alpha) = \frac{1}{\sqrt{8\pi\alpha^3}}\left[1 + O\left(\frac{1}{\alpha}\right)\right]\,,$$

$$Z(\xi) \approx \mathrm{i}\sqrt{\pi}\left[1 + O(\xi)\right]\,.$$

Dispersion relation

- *Quasi-neutrality:* $\phi^{(0)}_{\boldsymbol{k},\omega} = \frac{1}{\sqrt{\pi}} \frac{\xi^{(0)}}{k_\perp \rho} \frac{v_{\rm th} A^{(0)}_{\parallel \boldsymbol{k},\omega}}{c}$ ➢ ***No electron density perturbation***
- *Perpendicular Ampere's law:* $0 = \frac{e}{2T_0} \frac{\beta \xi^{(0)}}{\sqrt{\pi} k_\perp^3 \rho^3} \left[\frac{v_{\rm th} A^{(0)}_{\parallel \boldsymbol{k},\omega}}{c} + 2{\rm i}\sqrt{\pi} \frac{T_0}{e} \frac{\delta B^{(0)}_{\parallel \boldsymbol{k},\omega}}{B_0} \right]$
  - ➢ $\frac{\delta B^{(0)}_{\parallel \boldsymbol{k},\omega}}{B_0} = \frac{\rm i}{2\sqrt{\pi}} \frac{e}{T_0} \frac{v_{\rm th} A^{(0)}_{\parallel \boldsymbol{k},\omega}}{c}$ ➢ ***Resonant/non-resonant perpendicular pressure balance***
- *Parallel Ampere's law:* $\frac{A^{(0)}_{\parallel \boldsymbol{k},\omega}}{k_\parallel c} = \frac{4\pi e^2 n_0 (\omega^{(0)})^2}{T_0 k_\parallel^2 k_\perp^2 c^2} \frac{1}{\sqrt{\pi} k_\perp \rho} \frac{A^{(0)}_{\parallel \boldsymbol{k},\omega}}{k_\parallel c}$.
  - ➢ $\omega^{(0)} = \pm \frac{\pi^{1/4}}{\sqrt{2}} \frac{k_\parallel v_{\rm th} (k_\perp \rho)^{3/2}}{\beta^{1/2}}$ ➢ ***Whisper-wave dispersion relation***

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- Alternative: `warbler' wave, after the willow warbler's song (descends in pitch…)
- Suggestions welcome…!

Damping of whisper waves

- *Damping rate:* need to go to next order in expansion of parallel Ampere's law, find

$$\gamma^{(1)} = -k_{\parallel} v_{\mathrm{th}} \left[\frac{\pi}{4} \frac{k_{\perp}^3 \rho^3}{\beta} + \frac{1}{4\sqrt{\pi} k_{\perp}^2 \rho^2} \right] .$$

- Presence of two terms result of two physical damping mechanisms operating on the whisper waves: *Landau damping* associated with the parallel electric field inherent in the wave, and *linear coupling to Barnes-damped modes*.
  - The efficiency of these mechanisms as $k_{\perp}\rho$ increases is counterposed.
  - Damping is minimised when $k_{\perp}\rho \approx 0.65\beta^{1/5}$, with $\gamma/k_{\parallel}v_{\mathrm{th}} \approx 0.55\beta^{-2/5}$
    - Whisper-wave frequency at this wavenumber satisfies $\omega/k_{\parallel}v_{\mathrm{th}} \approx 0.50\beta^{-1/5}$.
  - For $k_{\perp}\rho$ sufficiently small or large, damping rate of the whisper waves becomes comparable to their frequency – no longer waves!
    - Require $\beta^{1/7} \lesssim k_{\perp}\rho \lesssim \beta^{1/3}$ for whisper waves to propagate at all…

Phase-space structure of whisper waves

- For order-unity velocities, whisper-wave distribution function takes the form

$$h_{\boldsymbol{k},\omega} \approx \frac{eF_0}{T_0}\sqrt{\frac{2\Omega}{\pi k_\perp v_\perp}}\cos\left(\frac{k_\perp v_\perp}{\Omega} - \frac{\pi}{4}\right)\frac{\omega A_{\parallel \boldsymbol{k},\omega}}{k_\parallel c}$$

  - *Whisper waves have perpendicular phase-space structure…!*

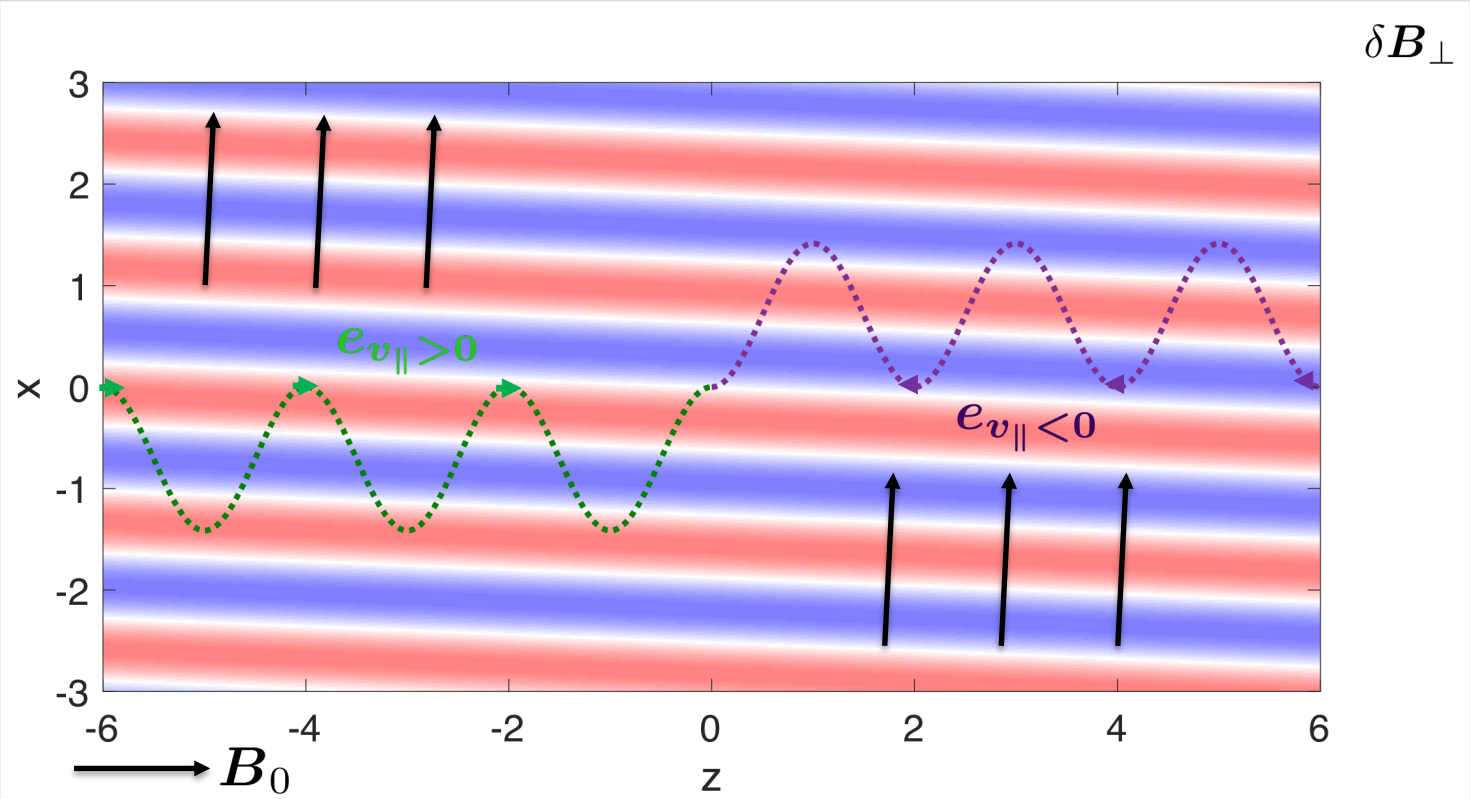
- Corollary: all particles are not equal… Whisper waves are kinetic!
  - Example: parallel current proportional to

$$\int \mathrm{d}^3\boldsymbol{v}\, v_\parallel J_0(a_e) h_{\boldsymbol{k},\omega}(v_\parallel, v_\perp) \propto \int_0^\infty \mathrm{d}v_\perp\, F_0(v)\cos^2\left(\frac{k_\perp v_\perp}{\Omega} - \frac{\pi}{4}\right)$$

  - *Particles with* $v_\perp = v_{\mathrm{th}}\frac{\pi}{k_\perp \rho}\left(n + \frac{1}{4}\right)$ *do not contribute significantly to current*

Physical mechanism of whisper waves

- For propagating wave-motion to be possible, need $\delta \boldsymbol{B}_\perp$ in anti-phase with $\delta \boldsymbol{E}_\parallel$, and $\pi/2$ phase difference with parallel current perturbation
  - **Q:** How does $\delta \boldsymbol{E}_\parallel$ generate current perturbation?
  - **A:** *Interaction with gyro-orbiting electrons with* $\delta \boldsymbol{E}_\parallel$ *at edge of their orbits*



Deriving self-consistent whisper-wave equations I

- Derive closed 'whisper-wave equations' by applying whisper-wave orderings directly to the full gyrokinetic equation
  - First, convenient to decompose the ring-centre distribution function into even and odd components:

$$h^+(t,\boldsymbol{R},v_\parallel,v_\perp) \equiv \frac{h(t,\boldsymbol{R},v_\parallel,v_\perp)+h(t,\boldsymbol{R},-v_\parallel,v_\perp)}{2}\,,$$

$$h^-(t,\boldsymbol{R},v_\parallel,v_\perp) \equiv \frac{h(t,\boldsymbol{R},v_\parallel,v_\perp)-h(t,\boldsymbol{R},-v_\parallel,v_\perp)}{2}\,.$$

  - Then defining modified gyrokinetic potential $\bar{\chi} = \chi + v_\parallel A_\parallel/c$, find

$$\frac{\partial h^-}{\partial t} + v_\parallel \frac{\partial h^+}{\partial z} + \frac{c}{B_0}\left\{\langle\bar{\chi}\rangle_{\boldsymbol{R}}, h^-\right\} - \frac{v_\parallel}{B_0}\left\{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^+\right\} = \frac{eF_0}{cT_0} v_\parallel \frac{\partial\langle A_\parallel\rangle_{\boldsymbol{R}}}{\partial t}\,,$$

$$\frac{\partial h^+}{\partial t} + v_\parallel \frac{\partial h^-}{\partial z} + \frac{c}{B_0}\left\{\langle\bar{\chi}\rangle_{\boldsymbol{R}}, h^+\right\} - \frac{v_\parallel}{B_0}\left\{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^-\right\} = -\frac{eF_0}{T_0}\frac{\partial\langle\bar{\chi}\rangle_{\boldsymbol{R}}}{\partial t}\,,$$

$$\nabla_\perp^2 A_\parallel = \frac{4\pi e}{c}\int \mathrm{d}^3\boldsymbol{v}\, v_\parallel \langle h^-\rangle_{\boldsymbol{r}}\,, \qquad \nabla_\perp^2 \delta B_\parallel = \frac{4\pi e}{c}\hat{\boldsymbol{z}}\cdot\left[\nabla_\perp \times \int \mathrm{d}^3\boldsymbol{v}\,\langle \boldsymbol{v}_\perp h^+\rangle_{\boldsymbol{r}}\right]\,.$$

Deriving self-consistent whisper-wave equations II

- Now use linear analysis to show that for $v \sim v_{\mathrm{th}}$,

$$h^+ \sim \frac{e}{T_0}\frac{n_0}{v_{\mathrm{th}}^3}\frac{\omega\langle A_\parallel\rangle_{\boldsymbol{R}}}{k_\parallel c}\,, \quad h^- \sim \epsilon h^+\,, \quad \langle\bar{\chi}\rangle_{\boldsymbol{R}} \approx -\frac{\langle \boldsymbol{v}_\perp\cdot\boldsymbol{A}_\perp\rangle_{\boldsymbol{R}}}{c} \sim \epsilon\frac{\langle v_\parallel A_\parallel\rangle_{\boldsymbol{R}}}{c}\,.$$

  - Electrostatic potential determined passively by quasi-neutrality
  - Parallel magnetic field determined by the vanishing of the perpendicular current:

$$\hat{\boldsymbol{z}}\cdot\left[\nabla_\perp\times\int \mathrm{d}^3\boldsymbol{v}\,\langle\boldsymbol{v}_\perp h^+\rangle_{\boldsymbol{r}}\right] \approx 0\,.$$

- Final equations: parallel Ampere, and

$$\frac{eF_0}{cT_0}\frac{\partial\langle A_\parallel\rangle_{\boldsymbol{R}}}{\partial t} - \frac{\partial h^+}{\partial z} \approx -\frac{1}{B_0}\left\{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^+\right\}\,,$$

$$\frac{\partial h^+}{\partial t} + v_\parallel\frac{\partial h^-}{\partial z} + \frac{eF_0}{T_0}\frac{\partial\langle\bar{\chi}\rangle_{\boldsymbol{R}}}{\partial t} \approx \frac{v_\parallel}{B_0}\left\{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^-\right\} - \frac{c}{B_0}\left\{\langle\bar{\chi}\rangle_{\boldsymbol{R}}, h^+\right\}\,.$$

- Time-evolution of h^- is slow, so can be neglected

Nonlinear whisper-wave interactions are kinetic

- Informative to take zeroth moments of governing equations:

$$\frac{e}{cT_0}\frac{\partial}{\partial t}\left[\int \mathrm{d}^3\boldsymbol{v}\, F_0 \langle\langle A_\parallel\rangle_{\boldsymbol{R}}\rangle_{\boldsymbol{r}}\right] - \frac{\partial}{\partial z}\left[\int \mathrm{d}^3\boldsymbol{v}\, \langle h^+\rangle_{\boldsymbol{r}}\right] = -\frac{1}{B_0}\int \mathrm{d}^3\boldsymbol{v}\, \langle\{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^+\}\rangle_{\boldsymbol{r}}\,,$$

$$\frac{\partial}{\partial t}\left[\int \mathrm{d}^3\boldsymbol{v}\, \langle h^+\rangle_{\boldsymbol{r}}\right] + \frac{c}{4\pi e}\frac{\partial}{\partial z}\nabla_\perp^2 A_\parallel = \frac{1}{B_0}\int \mathrm{d}^3\boldsymbol{v}\, \langle v_\parallel \{\langle A_\parallel\rangle_{\boldsymbol{R}}, h^-\} - c\{\langle\bar{\chi}\rangle_{\boldsymbol{R}}, h^+\}\rangle_{\boldsymbol{r}}\,.$$

- Linearized equations are fluid-like, and recover whisper-wave dispersion relation:

$$\frac{1}{n_0}\frac{\partial^2}{\partial t^2}\left[\int \mathrm{d}^3\boldsymbol{v}\, F_0 \langle\langle A_\parallel\rangle_{\boldsymbol{R}}\rangle_{\boldsymbol{r}}\right] + \frac{d_e^2}{2}\frac{\partial^2}{\partial z^2}\nabla_\perp^2 A_\parallel = 0\,.$$

- Nonlinear equations are not fluid-like; most obvious in Fourier-space representation:

$$\frac{e}{cT_0}\frac{1}{\sqrt{\pi}k_\perp\rho}\frac{\partial A_{\parallel\boldsymbol{k}}}{\partial t} - ik_\parallel \int \mathrm{d}^3\boldsymbol{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) h_{\boldsymbol{k}}^+$$
$$= \frac{1}{B_0}\sum_{\boldsymbol{k}'} \hat{\boldsymbol{z}}\cdot\left(\boldsymbol{k}_\perp \times \boldsymbol{k}'_\perp\right) A_{\parallel\boldsymbol{k}-\boldsymbol{k}'} \int \mathrm{d}^3\boldsymbol{v}\, J_0\left(\frac{[k_\perp - k'_\perp]\, v_\perp}{\Omega}\right) J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) h_{\boldsymbol{k}'}^+\,.$$

Nonlinear cascades of whisper waves I

- Equations have conserved generalised energy:

$$W = \int \mathrm{d}^3\boldsymbol{r} \left[\frac{\delta \boldsymbol{B}^2}{8\pi} + \int \mathrm{d}^3\boldsymbol{v}\, \frac{T_0 \langle (h^+)^2 \rangle_{\boldsymbol{r}}}{2F_0} \right] .$$

- Scaling arguments for whisper-wave cascade: suppose some energy flux is delivered to sub-electron-Larmor scales. Then, assuming constant flux across perpendicular scales λ ,

$$\frac{v_{\mathrm{the}}^8 h_{e\lambda}^2}{n_{e0}^2 \tau_{h\lambda}} \sim \varepsilon_h$$

- Balancing nonlinear term of gyrokinetic equation with the field-ring interaction term gives

$$\tau_{h\lambda}^{-1} \sim \rho_e^{-1/2} \lambda^{-1/2} v_{\mathrm{the}} \frac{\delta B_{\perp\lambda}}{B_0} \leftrightarrow h_{e\lambda} \sim \frac{n_{e0}}{v_{\mathrm{the}}^3} \frac{1}{\beta_e^{1/2}} \frac{\delta B_{\perp\lambda}}{B_0} .$$

- Critical balance ($\tau_{h\lambda} \sim \ell_{\parallel}/v_{\mathrm{the}}$) then gives desired scalings (for $\ell_0 \equiv v_{\mathrm{the}}^3/\varepsilon_h$):

$$h_{e\lambda} \sim \frac{n_{e0}}{v_{\mathrm{the}}^3} \beta^{-1/6} \lambda^{1/6} \rho_e^{1/6} \ell_0^{-1/3} , \quad \frac{\delta B_{\perp\lambda}}{B_0} \sim \beta^{1/3} \lambda^{1/6} \rho_e^{1/6} \ell_0^{-1/3} , \quad \ell_{\parallel} \sim \beta^{-1/3} \lambda^{1/3} \rho_e^{1/3} \ell_0^{1/3} .$$

Nonlinear cascades of whisper waves II

- Scalings:

$$h_{e\lambda} \sim \frac{n_{e0}}{v_{\rm the}^3}\beta^{-1/6}\lambda^{1/6}\rho_e^{1/6}\ell_0^{-1/3}\,, \quad \frac{\delta B_{\perp\lambda}}{B_0} \sim \beta^{1/3}\lambda^{1/6}\rho_e^{1/6}\ell_0^{-1/3}\,, \quad \ell_\parallel \sim \beta^{-1/3}\lambda^{1/3}\rho_e^{1/3}\ell_0^{1/3}\,.$$

- In terms of spectral scalings, this becomes

Whisper-wave cascade

$$E_{\delta B_\perp}(k_\perp) \propto k_\perp^{-4/3}\,,$$
$$E_{\delta B_\parallel}(k_\perp) \propto k_\perp^{-10/3}\,,$$
$$E_\phi(k_\perp) \propto k_\perp^{-7/3}.$$

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- In terms of spectral scalings, this becomes

Whisper-wave cascade	Electron-entropy cascade
$E_{\delta B_\perp}(k_\perp) \propto k_\perp^{-4/3}\,,$	$E_{\delta B_\perp}(k_\perp) \propto k_\perp^{-16/3}\,,$
$E_{\delta B_\parallel}(k_\perp) \propto k_\perp^{-10/3}\,,$	$E_{\delta B_\parallel}(k_\perp) \propto k_\perp^{-16/3}\,,$
$E_\phi(k_\perp) \propto k_\perp^{-7/3}.$	$E_\phi(k_\perp) \propto k_\perp^{-10/3}.$

- If correct, one would measure much shallower spectra below the electron Larmor scale (or at least a knee!).
  - Like with KAWs/ion entropy cascade, not clear how efficient energy injection is…
- Relationship to velocity-space structure is the usual: $\dfrac{\delta v_\perp}{v_{\rm the}} \sim \dfrac{1}{k_\perp \rho_e}$
- Transition scale to electron-entropy cascade at $k_\perp \rho_e \sim \beta_e^{1/3}$

Does any of this matter?

- Expansion parameter $\epsilon \equiv \beta_e^{-1/5} \ll 1$ is not very small in most astrophysical plasmas of interest…
  - Schekochihin law of small numbers $(1/3 \ll 1)$ implies $\beta_e \gtrsim 250$
- Yet even if asymptotic theory not entirely accurate, still the case that electromagnetic interactions extend beyond electron Larmor scale at moderately high β_e
  - E.g. for $\beta_e = 100$ find that pure electrostatic cascade only emerges for $k_\perp \rho_e \gtrsim 5 \gg 1$

- Perhaps more relevantly, whisper waves can be very important in plasma with negative electron pressure anisotropy – they become unstable (*whisper-wave instability):*
  - Describable by pressure-anisotropic gyrokinetics
  - Assuming electron pressure anisotropy satisfies $|\Delta_e| \sim \beta_e^{-3/5}$, can show that only changes to governing equations is additional term in parallel Ampere's law:

$$\nabla_\perp^2 A_\parallel = \frac{4\pi e}{c} \int \mathrm{d}^3\boldsymbol{v}\, v_\parallel \langle h^- \rangle_{\boldsymbol{r}} + \frac{\Delta_e}{d_e^2} A_\parallel$$

  - Thus, have identical eigenmodes to stable whisper waves, and dispersion relation

$$\omega = \pm \frac{\pi^{1/4}}{\sqrt{2}} k_\parallel v_{\mathrm{th}} \left[k_\perp \rho_e \left(k_\perp^2 d_e^2 + \Delta_e \right) \right]^{1/2} .$$

Sub-electron-Larmor scale microinstabilities

- Whisper-wave instability is special example of sub-electron Larmor scale microinstabilities which only exist in plasma with a background magnetic field
  - Smaller and smaller scale microinstabilities triggered as β increases (for fixed $|\Delta_e|$)
  - Example: microinstabilities of the electron Chapman-Enskog distribution function, with

$$\Delta_e = -0.02\sqrt{m_e/m_i}$$

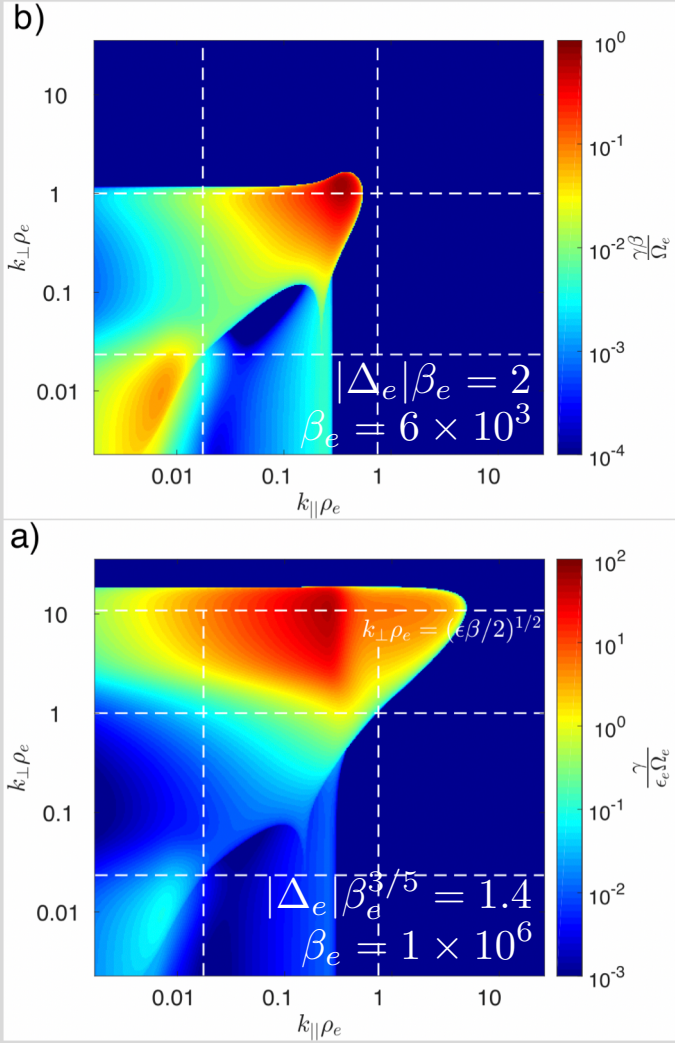
- Peak growth rates for such microinstabilities often much faster than microinstabilities at comparable scales in unmagnetised plasma (e.g. Weibel/transverse instability)

$$\gamma_{\mathrm{T}} \approx \frac{2}{3\sqrt{3\pi}}(|\Delta_e|\beta)^{1/2}|\Delta_e|\Omega_e$$

$$\gamma_{\mathrm{WW}} \approx \frac{\pi^{1/4}}{\sqrt{2\log 1/|\Delta_e|}}\left(|\Delta_e|\beta\right)^{1/4}|\Delta_e|^{1/2}\Omega_e$$

- Whisper-wave instability dominates for

$$\beta_e^{-5/7} \lesssim |\Delta_e| \lesssim \beta_e^{-1/3}$$



Whispered wishes…

- Remain number of open questions about whisper waves/sub-electron-Larmor scale dynamics
  - How are results affected by finite but large β_e?
  - Can one construct more quantitative theories of nonlinear interactions between whisper waves?
  - How do whisper waves interact with large-scale motions?
  - How does the whisper-wave instability saturate?
    - Does it affect the thermal conductivity?

- To help answer these questions, numerics would be helpful – AstroGK/modified Pegasus simulations?
  - Working on the former – very preliminary results support existence of mode

- Could experimental evidence for whisper waves be found?
  - Solar-wind data not appropriate, as β_e too low
  - Laser-plasma experiments more promising – very high plasma β_e, and can access relevant regime at high-energy facilities

Conclusions

1. There can exist propagating waves (*whisper waves*) describable by gyrokinetics with perpendicular length scales below the electron-Larmor scale in a high-β plasma

2. These waves are electromagnetic rather than electrostatic (dominated by $\delta B_\perp$), which implies that the turbulent-cascade model appropriate for $\beta \sim 1$ plasma at sub-electron-Larmor scales don't apply to high-β plasma

3. Whisper waves have interesting properties:
   I. They have significant perpendicular phase-space structure
   II. Their nonlinear interactions are fundamentally kinetic (i.e. cannot be described by a few moments)
   III. For sufficiently large electron-pressure anisotropies ($\beta_e^{-5/7} \lesssim |\Delta_e| \lesssim \beta_e^{-1/3}$), whisper-waves are destabilised, and are the most rapidly growing of all unstable modes (including unmagnetised instabilities)

4. Results testable both in simulations (gyrokinetic or PIC) and potentially laser-plasma experiments

Any questions?