Turbulence in magnetized pair plasmas

Nuno Loureiro

Plasma Science and Fusion Center, MIT

MIT Plasma Science & Fusion Center

Magnetized pair plasmas

Fluid equations for a low beta, non-relativistic pair plasma

Start from two-fluid equations for electrons and positrons:

$$n^{\pm}m\left(\frac{\partial \mathbf{v}^{\pm}}{\partial t}+\mathbf{v}^{\pm}\cdot\nabla\mathbf{v}^{\pm}-\mu^{\pm}\nabla^2\mathbf{v}^{\pm}\right)=\pm n^{\pm}e\left(\mathbf{E}+\frac{\mathbf{v}^{\pm}\times\mathbf{B}}{c}\right)-\nabla p^{\pm}\mp\mathcal{R}$$

Sum and subtract:

$$nm\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\frac{1}{n^2e^2}\mathbf{j}\cdot\nabla\mathbf{j}-\mu\nabla^2\mathbf{v}\right)=\frac{\mathbf{j}\times\mathbf{B}}{c}-\nabla p$$

v is center-of-mass velocity

$$\frac{m}{ne^2}\left(\frac{\partial \mathbf{j}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{j}+\mathbf{j}\cdot\nabla\mathbf{v}-\mu\nabla^2\mathbf{j}\right)=\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}-\eta\mathbf{j}.$$

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Impose reduced-MHD-like ordering: $\mathbf{B} = B_0\hat{z} + \mathbf{B}_\perp, \text{with } B_\perp/B_0 \sim \epsilon$

Introduce potentials: $\mathbf{v}_\perp = \hat{z} \times \nabla_\perp\phi;$ $\quad \frac{\mathbf{B}_\perp}{\sqrt{4\pi\rho}} = \hat{z} \times \nabla_\perp\psi,$

$$\frac{\partial}{\partial t}\nabla_\perp^2\phi + \{\phi, \nabla_\perp^2\phi\} = \{\psi, \nabla_\perp^2\psi\} + V_A\frac{\partial}{\partial z}\nabla_\perp^2\psi + \mu\nabla_\perp^2\phi,$$

momentum equation

$$\frac{\partial}{\partial t}\left(1 - d_e^2\nabla_\perp^2\right)\psi + \left\{\phi, (1 - d_e^2\nabla_\perp^2)\psi\right\} = V_A\frac{\partial\phi}{\partial z} + \eta\nabla_\perp^2\psi - \mu d_e^2\nabla_\perp^4\psi$$

Ohm’s law

These reduce to the familiar RMHD equations when electron inertia is neglected (unsurprisingly).

Magnetized pair plasmas

Invariants and waves

These equations have two *exact* invariants at *all* scales:

$$\mathcal{E} = \frac{1}{2}\int dV \left\{ (\nabla_\perp \psi)^2 + d_e^2 \left(\nabla_\perp^2 \psi\right)^2 + (\nabla_\perp \phi)^2 \right\}$$

energy

$$\mathcal{H}^C = \int dV \left\{ \nabla_\perp^2 \phi \left(1 - d_e^2 \nabla_\perp^2\right) \psi \right\}$$

(generalized) cross helicity

Only one wave is supported:

$$\omega_l = \pm \frac{k_z V_A}{\sqrt{1 + k_\perp^2 d_e^2}}$$

Alfvén wave modified at kinetic scales by electron inertia

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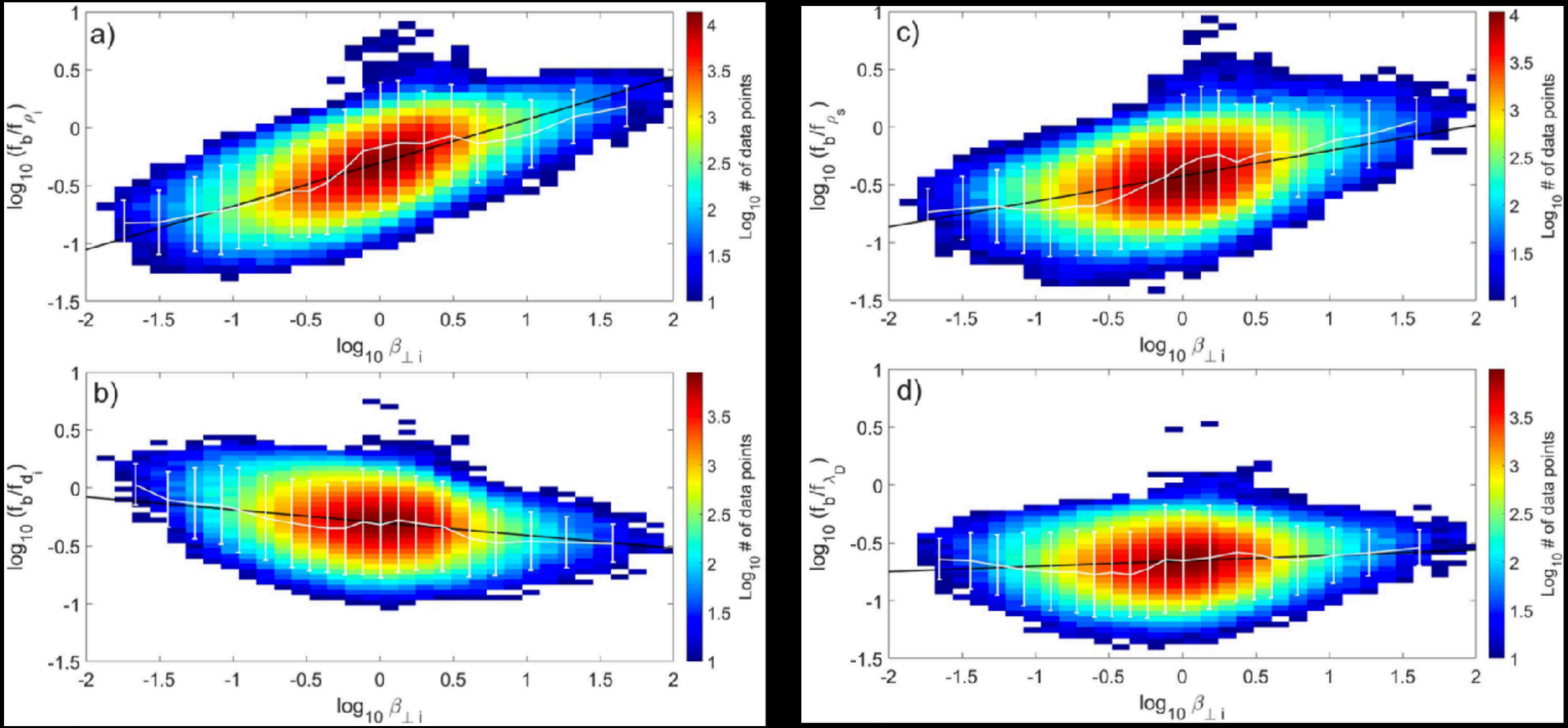
Turbulence at MHD scales

At **MHD scales** ($kd_e << 1$), the equations are the same as for ion-electron plasmas. So turbulence will be the same:

1. expect $k^{-3/2}$ up until the reconnection scale,
2. followed by a transition to a k^{-3} (or -8/3) due to reconnection (?)



Vech *et al.*, ApJ 2018

Let us work out the spectrum expected from a Kolmogorov-like energy cascade at **kinetic scales** ($kd_e >> 1$).

$$\mathcal{E} = \frac{1}{2}\int dV \left\{(\nabla_\perp\psi)^2 + d_e^2\left(\nabla_\perp^2\psi\right)^2 + (\nabla_\perp\phi)^2\right\}$$

$$\Rightarrow \frac{1}{2}\int dV \left\{d_e^2(\nabla_\perp^2\psi)^2 + (\nabla_\perp\phi)^2\right\}$$

Expect equipartition between these two terms (parallel and perpendicular kinetic energies) at these scales.

$$k_\perp^2 \phi_\lambda^2 / \tau_\lambda \sim \varepsilon$$

$$\tau_\lambda = 1/\omega_{nl} \sim 1/(k_\perp^2 \phi_\lambda)$$

This results in:

$$\phi_\lambda \sim \varepsilon^{1/3} k_\perp^{-4/3} \quad \Rightarrow \quad E_\phi(k_\perp) dk_\perp \sim \varepsilon^{2/3} k_\perp^{-11/3} dk_\perp$$

and same scaling for magnetic energy, by equipartition

Finally, *declare* that the fluctuations are *critically balanced* at these scales: $\omega_l \sim \omega_{nl}$

$$k_{\|} \sim \varepsilon^{1/3} d_e V_A^{-1} k_\perp^{5/3}$$

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Dealing with the cross helicity

Recall cross-helicity: $$\mathcal{H}^C = \int dV \left\{\nabla_\perp^2 \phi \left(1 - d_e^2 \nabla_\perp^2\right) \psi\right\}$$ It is *not positive definite*.

Estimate the flux of cross helicity as $$\left(k_\perp^2 \phi_\lambda\right) \left(d_e^2 k_\perp^2 \psi_\lambda\right) R_\lambda / \tau_\lambda \sim \varepsilon^c$$

where R is a dimensionless cancelation factor at scale lambda. From the energy invariant:

$$k_\perp^2 d_e^2 \psi_\lambda^2 \sim \phi_\lambda^2$$ so the cross helicity flux becomes $$k_\perp d_e (k_\perp^2 \phi_\lambda^2) R_\lambda / \tau_\lambda \sim \varepsilon^c$$

But energy flux (constant) is $$k_\perp^2 \phi_\lambda^2 / \tau_\lambda \sim \varepsilon$$ Thus: $$R_\lambda \propto 1/(k_\perp d_e).$$

Conclusions

Presented a tentative theory of turbulence in low beta, strongly magnetized, non-relativistic pair plasmas:

- same as MHD at fluid scales
- Energy cascade implies $k_\perp^{-11/3}$ spectrum prediction at kinetic scales.
- Critical balance then gives $k_{||} \sim k_\perp^{5/3}$
- Notice that reconnection is not allowed at $k_\perp d_e \gg 1$ (flux is no longer frozen) so no need to worry about that.

Loureiro & Boldyrev, arXiv:1805.09224

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