A scale separated framework for studying cross scale interactions in plasma turbulence

M.R.Hardman[1,2], M.Barnes[1,2], C.M.Roach[2]
[1] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford, UK*
[2] *CCFE, Culham Science Centre, Abingdon, Oxon, UK*

Introduction

Anomalous transport is driven by turbulence,

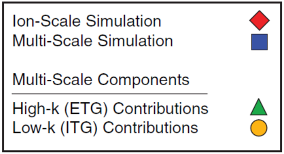
- IS : at scales where $k\rho_i \lesssim 1$
- ES : at scales where $1 \ll k\rho_e \lesssim 1$

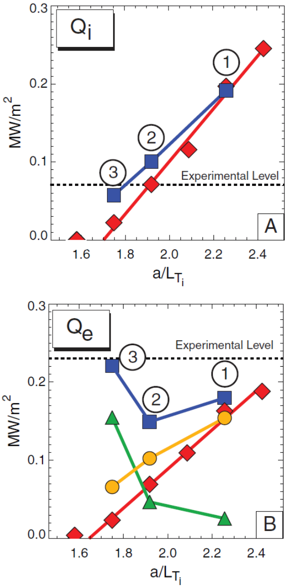
- do all scales matter?
- is cross scale coupling important?

- To answer these questions we take a scale separated approach

Introduction: do all scales matter?

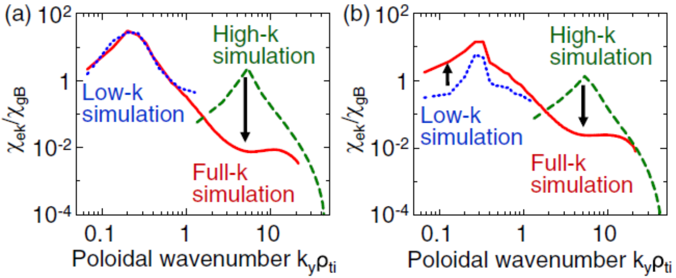
- simulation evidence where $Q_e \sim 10 Q_{egB} \sim (?) Q_{igB}$ e.g. Jenko and Dorland (2002)
- recent experimental evidence on NSTX Ren et al. (2017)
- Howard et al. (2016) Fig 3:





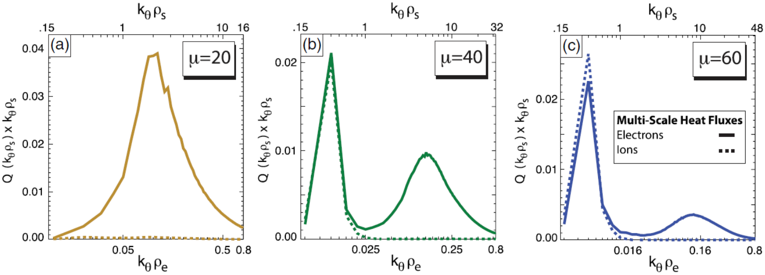
Introduction: is cross scale coupling important?

- Fig 2 from Maeyama et al. (2015):

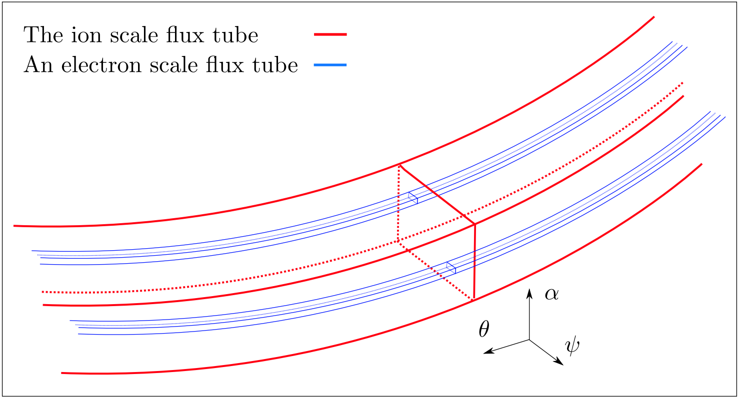


Introduction: can we reduce the mass ratio?

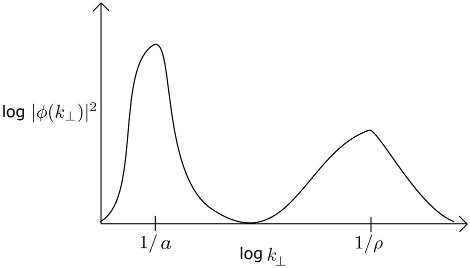
- Fig 5 from Howard et al. (2015):



Introduction: a scale separated approach



A Quick Reminder: Scale separation in turbulence



- scale separation: $\rho_* = \rho/a \to 0 \Rightarrow \quad f = F + \delta f$
- statistical periodicity: $\langle \delta f \rangle_{\text{turb}} = 0$
- gyro average: $\langle \cdot \rangle|_{\mathbf{R}}^{\text{gyro}}$
- orderings:

$$\delta f \sim \rho_* F$$

$$\nabla F \sim \nabla_\perp \delta f \sim \rho_*^{-1} \nabla_\| \delta f$$

$$\partial_t \delta f \sim (v_t/a)\delta f \sim \rho_* \Omega \delta f$$

$$\partial_t F \sim \rho_*^3 \Omega F$$

A Quick Reminder: The Gyrokinetic Equation

The gyrokinetic equation for $h = \delta f - (Ze\phi/T)F_0$:

$$\frac{\partial h}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial h}{\partial\theta} + (\mathbf{v}_M + \mathbf{v}_E)\cdot\nabla h + \mathbf{v}_E\cdot\nabla F_0 = \frac{ZeF_0}{T}\frac{\partial\varphi}{\partial t}, \tag{1}$$

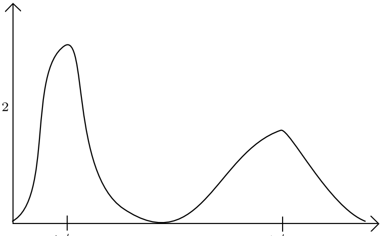
where,

$$\varphi = \langle\phi\rangle|_{\mathbf{R}}^{\text{gyro}}, \quad \mathbf{v}_E = \frac{c}{B}\mathbf{b}\wedge\nabla\varphi. \tag{2}$$

Closed by quasi-neutrality,

$$\sum_\alpha Z_\alpha e\Big(\int d^3\mathbf{v}|_{\mathbf{r}} h_\alpha\Big) = \sum_\alpha \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha}\phi(\mathbf{r}). \tag{3}$$

Separating Ion and Electron Scale Turbulence



- scale separation: $\rho_e/\rho_i \sim v_{ti}/v_{te} \sim \sqrt{m_e/m_i} \to 0$, $\Rightarrow \delta f = \overline{\delta f} + \tilde{\delta f}$
- electron scale statistical periodicity: $\left\langle \tilde{\delta f} \right\rangle^{\mathrm{ES}} = 0$
- orderings:

$$\nabla_\perp \overline{\delta f} \sim \rho_i^{-1} \overline{\delta f}, \quad \partial_t \overline{\delta f} \sim (v_{ti}/a) \overline{\delta f}$$

$$\nabla_\perp \tilde{\delta f} \sim \rho_e^{-1} \tilde{\delta f}, \quad \partial_t \tilde{\delta f} \sim (v_{te}/a) \tilde{\delta f}.$$

Separating Ion and Electron Scale Turbulence: The Coupled Equations

- ion scale equations, with new back reaction term:

$$\frac{\partial \overline{h}_i}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial \overline{h}_i}{\partial \theta} + (\mathbf{v}_{Mi} + \overline{\mathbf{v}}_{Ei})\cdot\nabla\overline{h}_i + \overline{\mathbf{v}}_{Ei}\cdot\nabla F_{0i} = \frac{Z_i e F_{0i}}{T_i}\frac{\partial \overline{\varphi}_i}{\partial t}, \quad (4)$$

$$\frac{\partial \overline{h}_e}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial \overline{h}_e}{\partial \theta} + (\mathbf{v}_{Me} + \overline{\mathbf{v}}_{Ee})\cdot\nabla\overline{h}_e + \overline{\mathbf{v}}_{Ee}\cdot\nabla F_{0e} + \nabla\cdot\left\langle \frac{c}{B}\tilde{h}_e\tilde{\mathbf{v}}_{Ee}\right\rangle^{ES} = -\frac{e F_{0e}}{T_e}\frac{\partial \overline{\varphi}_e}{\partial t}, \quad (5)$$

$$\int d^3\mathbf{v}|_{\mathbf{r}}(Z_i e\overline{h}_i - e\overline{h}_e) = (\frac{e^2 Z_i^2 n_i}{T_i} + \frac{e^2 n_e}{T_e})\overline{\phi}, \quad (6)$$

- electron scale equations, with the new advection and drive terms:

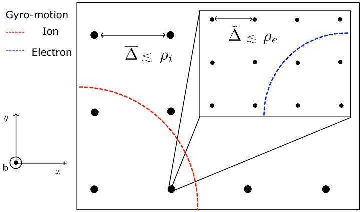
$$\frac{\partial \tilde{h}_e}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\theta\frac{\partial \tilde{h}_e}{\partial \theta} + (\mathbf{v}_{Me} + \tilde{\mathbf{v}}_{Ee} + \overline{\mathbf{v}}_{Ee})\cdot\nabla\tilde{h}_e + \tilde{\mathbf{v}}_{Ee}\cdot(\nabla\overline{h}_e + \nabla F_{0e}) = -\frac{e F_{0e}}{T_e}\frac{\partial \tilde{\varphi}_e}{\partial t}. \quad (7)$$

$$-\int d^3\mathbf{v}|_{\mathbf{r}} e\tilde{h}_e = (\frac{e^2 Z_i^2 n_i}{T_i} + \frac{e^2 n_e}{T_e})\tilde{\phi}, \quad (8)$$

Separating Ion and Electron Scale Turbulence: Difficult Points

- non-locality of the gyro average
- ions at electron scales

  ⇒ solved by considering allowed sizes of the fluctuations in dominant balance



- the parallel boundary condition

  ⇒ solved by choosing an electron scale boundary condition which ensures that the electron scale turbulence follows the field lines of the ion scale

Relative Size of the Fluctuations

The coupled equations allow the maximal ordering:

$$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\tilde{\phi}}{T} \sim \rho_{e*} \tag{9}$$

$$\frac{\overline{h}_i}{F_{0i}} \sim \frac{\overline{h}_e}{F_{0e}} \sim \frac{e\overline{\phi}}{T}, \quad \frac{\tilde{h}_e}{F_{0e}} \sim \frac{e\tilde{\phi}}{T}, \quad \frac{\tilde{h}_i}{F_{0i}} \sim (\frac{m_e}{m_i})^{1/4}\frac{e\tilde{\phi}}{T} \tag{10}$$

with subsidiary orderings:

$$\frac{e\overline{\phi}}{T} \lesssim \rho_{e*}, \quad \frac{e\tilde{\phi}}{T} \sim \rho_{e*} \tag{11}$$

$\Rightarrow$ ES sets saturation level and suppresses IS turbulence

$$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\tilde{\phi}}{T} \ll \rho_{e*} \tag{12}$$

$\Rightarrow$ IS sets saturation level and suppresses ES turbulence

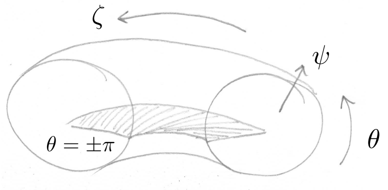
The Parallel Boundary Condition

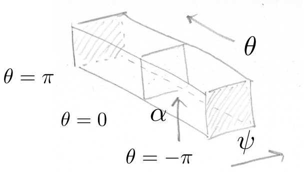
- ψ: radial, α: field line label,
  θ: poloidal angle, ζ: toroidal angle
- $\alpha(\zeta,\theta,\psi) = \alpha_0 + \zeta - q_0(\psi)\theta = \alpha_0 + \zeta - q_0\theta + q_0'(\psi-\psi_0)\theta$
- $\alpha(\zeta,\theta+2\pi,\psi) - \alpha(\zeta,\theta,\psi) = \underbrace{-2\pi q_0}_{\text{neglected}} -2\pi q_0'(\psi-\psi_0)$

$$A(\theta+2\pi,\alpha(\zeta,\theta+2\pi,\psi),\psi) = A(\theta,\alpha(\zeta,\theta,\psi),\psi) \tag{13}$$

Beer et al. (1995)

$\Rightarrow$ b.c. enforces statistical periodicity on a (ψ,ζ) plane
$\Rightarrow$ b.c. couples in α





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$$A(\theta + 2\pi, \alpha(\zeta, \theta + 2\pi, \psi), \psi) = A(\theta, \alpha(\zeta,\theta,\psi),\psi) \tag{13}$$

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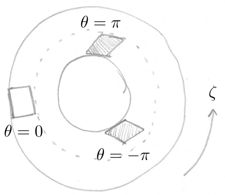
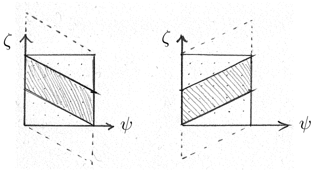




$\theta = \pi$ $\theta = -\pi$

The Parallel Boundary Condition

- consider

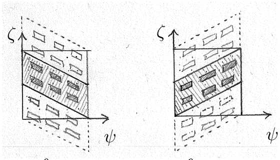
$$\bar{\mathbf{v}}_E \cdot \nabla \tilde{h} = \frac{c}{B} \nabla\alpha \wedge \nabla\psi \cdot \mathbf{b} \left(\frac{\partial\overline{\varphi}}{\partial\alpha}\frac{\partial\tilde{h}}{\partial\psi} - \frac{\partial\overline{\varphi}}{\partial\psi}\frac{\partial\tilde{h}}{\partial\alpha} \right) \tag{14}$$

- need parallel boundary condition for $\tilde{h}$ consistent with boundary condition on $\overline{\varphi}$
- $\bar{\mathbf{v}}_E \cdot \nabla \tilde{h}$ should be continuous along extended θ

$\Rightarrow$ correct b.c. :

$$\begin{aligned} &\tilde{A}(\theta + 2\pi, \alpha(\zeta, \theta + 2\pi, \tilde{\psi}), \tilde{\psi}; \alpha(\zeta, \theta + 2\pi, \overline{\psi}), \overline{\psi}) \\ &= \tilde{A}(\theta, \alpha(\zeta, \theta, \tilde{\psi}), \tilde{\psi}; \alpha(\zeta, \theta, \overline{\psi}), \overline{\psi}) \end{aligned} \tag{15}$$

$\Rightarrow$ b.c. enforces statistical periodicity on a (ψ, ζ) plane



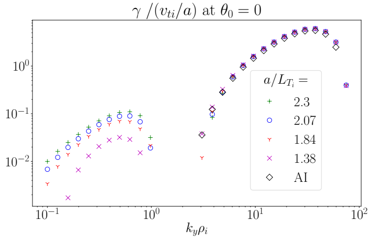
$\theta = \pi$ $\theta = -\pi$

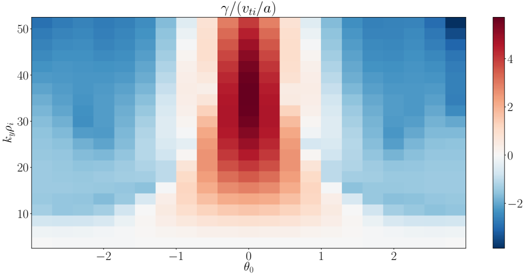
$\Rightarrow$ b.c. couples ES fluxtubes in α

Simulations: modification of ES linear physics: CBC

$$\frac{e\overline{\phi}}{T} \sim \rho_{i*}, \quad \frac{e\tilde{\phi}}{T} \ll \rho_{e*}$$

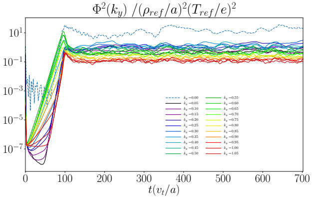
$\Rightarrow$ IS sets saturation level and suppresses ES turbulence





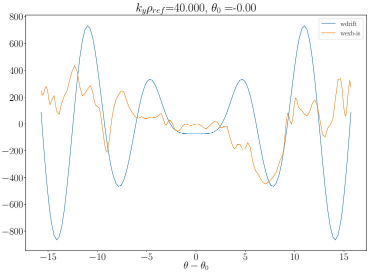
- Kinetic Ions and Electrons
- Adiabatic Ions, Kinetic Electron

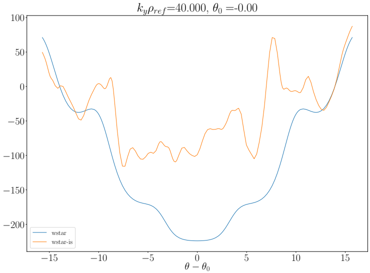
Simulations: modification of ES linear physics: CBC $a/L_{T_i} = 2.3$



- Run IS to saturation
- Form coefficients for ES linear calculation (*)

(*) For continuity require $k_x \rho_i \sim 10$ and/or $\|$ b.c. enforcing filter



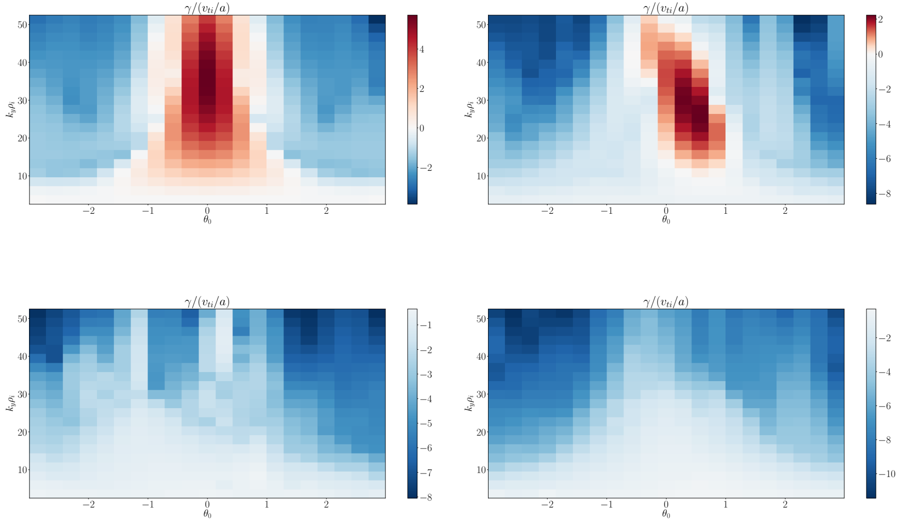


- $\bar{\mathbf{v}}_{Ee} \cdot \nabla \tilde{h}_e / \tilde{h}_e$
- $\mathbf{v}_{Me} \cdot \nabla \tilde{h}_e / \tilde{h}_e$

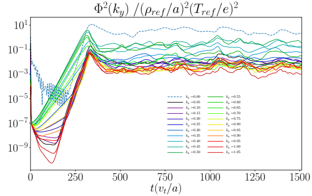
- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla \overline{h}_e / \tilde{\phi}$
- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla F_{0e} / \tilde{\phi}$

Simulations: modification of ES linear physics: CBC $a/L_{T_i} = 2.3$

Top left: No IS gradients. Rest: IS gradients from different IS (α, ψ) locations



Simulations: modification of ES linear physics: CBC $a/L_{T_i} = 1.38$



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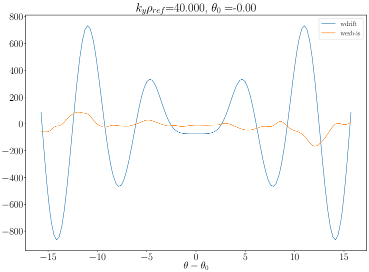

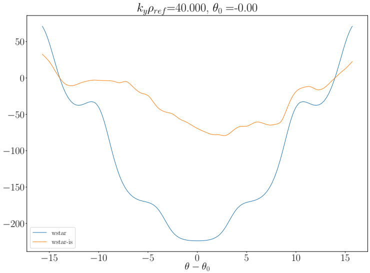

- $\bar{\mathbf{v}}_{Ee} \cdot \nabla\tilde{h}_e/\tilde{h}_e$
- $\mathbf{v}_{Me} \cdot \nabla\tilde{h}_e/\tilde{h}_e$
- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla\bar{h}_e/\tilde{\phi}$
- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla F_{0e}/\tilde{\phi}$

Simulations: modification of ES linear physics: CBC $a/L_{T_i} = 1.38$

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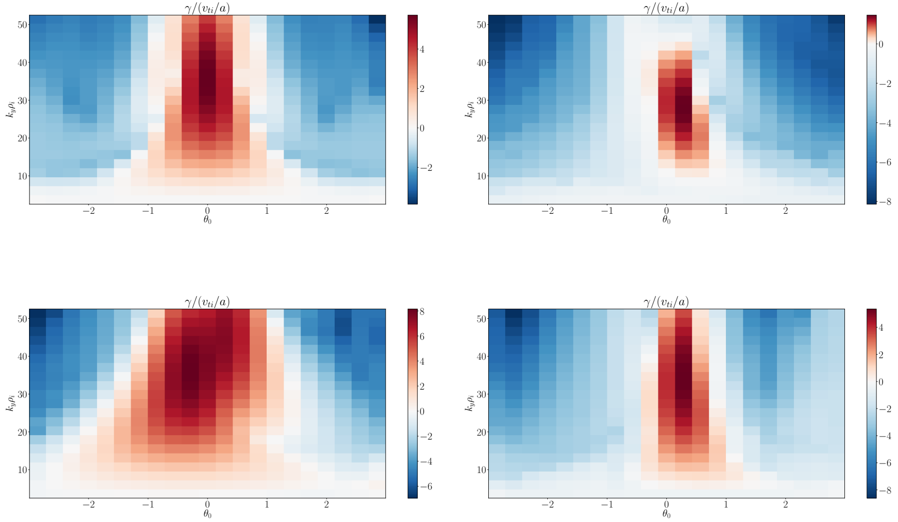

Summary

We have derived equations for the IS and ES turbulence:

- scale separated
- non-local (k) interaction terms - no shear!
- a parallel b.c. - introduces perpendicular coupling
- interesting subsidiary orderings

The ES terms have been implemented in `GS2`:

- Look at the effect on ES linear physics

Questions:

- Which structures enhance/retard ES instability and why?
- Which pieces of **v** space matter and why?
- Does parallel b.c. break scale separation? (α)
- Can we resolve an IS simulation with only $k_x \rho_i \sim 1$? (ψ)
- the effect of the back reaction on IS physics?
- the effect on ES nonlinear physics?
- timescale separation in a coupled IS-ES system?

Thanks to Felix Parra, Alex Schekochihin, Paul Dellar, Bill Dorland for discussion

Thank you for listening!

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Should We Expect Cross Scale Interaction?

Yes! Because:

- electron scale eddies have $\tilde{l}_\perp \sim \rho_e$
- ion scale eddies have $\bar{l}_\perp \sim \rho_i$
- ambient gradient argument $\Rightarrow \tilde{h}_e \sim \rho_e^* F_{0e}, \quad \overline{h}_e \sim \rho_i^* F_{0e}$
- $\Rightarrow \nabla \tilde{h}_e \sim \nabla \overline{h}_e \sim \nabla F_{0e}$

$\Rightarrow$ gradients of the distribution function are comparable at all scales
$\Rightarrow$ electron scale eddies can be driven by ion scale gradients

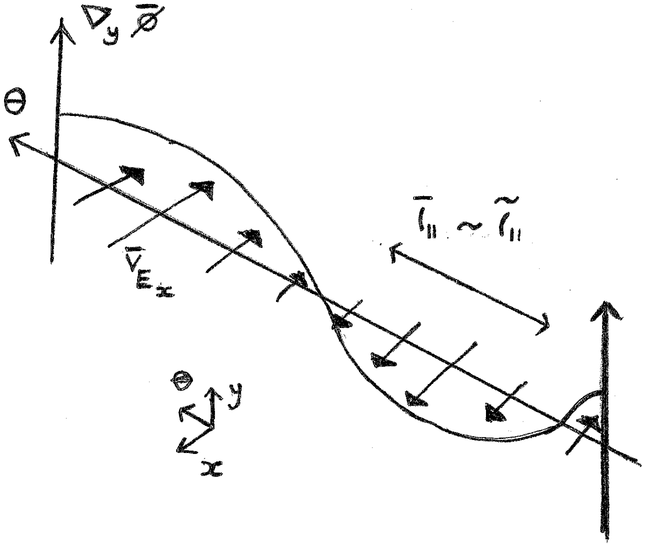
- applying the same argument to $\mathbf{E} = -\nabla\phi$
- $\Rightarrow \nabla\tilde{\phi} \sim \nabla\overline{\phi}$

$\Rightarrow$eddy E×B drifts $v_{\mathrm{E\times B}}$, are comparable at all scales

- applying the critical balance argument
- $v_{te}/\tilde{l}_\| \sim \tilde{\tau}_{nl}^{-1} \sim \tilde{v}_{\mathrm{E\times B}}/\tilde{l}_\perp$
- $v_{ti}/\bar{l}_\| \sim \overline{\tau}_{nl}^{-1} \sim \overline{v}_{\mathrm{E\times B}}/\bar{l}_\perp$
- $\tilde{l}_\| \sim \bar{l}_\|$

$\Rightarrow$ parallel correlation lengths are the same for ion scale and electron scale eddies
$\Rightarrow$ electron scale eddies are long enough to be differentially advected by $\overline{v}_{\mathrm{E\times B}}$

Visualising the Ion Scale E×B Velocity with θ



Separating Ion and Electron Scale Turbulence: Technicalities

- We introduce a fast spatial variable $\mathbf{r}_f$ and a slow spatial variable $\mathbf{r}_s$ and the fast and slow times t_f, t_s
- In the gyrokinetic equation we send,

$$\delta f(t,\mathbf{r}) \to \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f), \quad \nabla \to \nabla_s + \nabla_f, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}, \tag{16}$$

- then asymptotically expand in the mass ratio $(m_e/m_i)^{1/2}$
- remembering $\nabla_s \sim (m_e/m_i)^{1/2}\nabla_f$, and $\partial/\partial t_s \sim (m_e/m_i)^{1/2}\partial/\partial t_f$
- explicitly define the electron scale average,

$$\overline{\delta f}(t_s,\mathbf{r}_s) = \left\langle \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f) \right\rangle^{\mathrm{ES}} = \frac{1}{\tau_c A}\int_{t_s-\tau_c/2}^{t_s+\tau_c/2} dt_f \int_{A,\mathbf{r}_s} d^2\mathbf{r}_f \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f), \tag{17}$$

- We assume that,

$$\delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f) = \delta f(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f + n\Delta_{cx}\hat{\mathbf{x}} + m\Delta_{cy}\hat{\mathbf{y}}), \tag{18}$$

- This enforces $\left\langle \tilde{\delta f} \right\rangle^{\mathrm{ES}} = 0$.

Splitting the Quasi-Neutrality Relation

- We split the guiding centre into a slow $\mathbf{R}_s$ and a fast $\mathbf{R}_f$ part.
- $\mathbf{R} = \mathbf{r} - \rho(\mathbf{r})$, where $\rho(\mathbf{r})$ is the vector gyroradius
- Thus using the periodicity property equation (18) the electron scale average may be taken over guiding centre or real space coordinates.
- This observation allows us to note that the electron scale average commutes with the gyro average,

$$\left\langle \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\phi(\mathbf{r}_s,\mathbf{r}_f)\right\rangle^{\mathrm{ES}} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\left\langle \phi(\mathbf{r}_s,\mathbf{r}_f)\right\rangle^{\mathrm{ES}} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\overline{\phi}(\mathbf{r}_s), \tag{19}$$

The splitting of the quasi neutrality relation follows directly,

$$\sum_\alpha Z_\alpha e\left(\int d^3\mathbf{v}|_{\mathbf{r}}\overline{h}_\alpha(\mathbf{R}_s)\right) = \sum_\alpha \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha}\overline{\phi}(\mathbf{r}_s), \tag{20}$$

$$\sum_\alpha Z_\alpha e\left(\int d^3\mathbf{v}|_{\mathbf{r}}\tilde{h}_\alpha(\mathbf{R}_s,\mathbf{R}_f)\right) = \sum_\alpha \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha}\tilde{\phi}(\mathbf{r}_s,\mathbf{r}_f). \tag{21}$$

Addressing the Non-Locality of the Gyro Average

- Taking the gyro average at fixed guiding centre $\langle\cdot\rangle|_{\mathbf{R}}^{\text{gyro}}$, couples multiple $\mathbf{r}_s$ points.
- but we aim to find scale separated equations!
- Expanding both the slow and the fast spatial variable in Fourier series we note that,

$$\tilde{\varphi}(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) = \langle\tilde{\phi}(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f)\rangle|_{\mathbf{R}}^{\text{gyro}} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}}\tilde{\phi}(t_s, t_f, \mathbf{r}_s, \mathbf{r}_f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}} \sum_{\mathbf{k}_s,\mathbf{k}_f} \tilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{r}_s} e^{i\mathbf{k}_f\cdot\mathbf{r}_f} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma|_{\mathbf{R}} \sum_{\mathbf{k}_s,\mathbf{k}_f} \tilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{R}_s} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} e^{-i(\mathbf{k}_s+\mathbf{k}_f)\cdot\rho}$$

$$= \sum_{\mathbf{k}_s,\mathbf{k}_f} \tilde{\phi}_{\mathbf{k}_s,\mathbf{k}_f} e^{i\mathbf{k}_s\cdot\mathbf{R}_s} e^{i\mathbf{k}_f\cdot\mathbf{R}_f} J_0(|(\mathbf{k}_s+\mathbf{k}_f)|\rho), \tag{22}$$

for electrons:

- $|\mathbf{k}_f|\rho_e \sim 1$ and $|\mathbf{k}_s|\rho_e \sim (m_e/m_i)^{1/2}$
- we can expand the Bessel function to return to a local picture in the slow variable with $O(m_e/m_i)^{1/2}$ error.
- We will exploit this in scale separation.

for ions:

- $|\mathbf{k}_s|\rho_i \sim 1$ and $|\mathbf{k}_f|\rho_i \sim (m_e/m_i)^{-1/2}$.
- we are unable to expand the Bessel function
- we are unable to avoid the coupling of multiple $\mathbf{r}_s$ in the equations for ions at electron scale

Addressing the Non-Locality of the Gyro Average: continued

- assume we can neglect the ion contribution to electronscale quasi neutrality - shown later,

$$\tilde{\varphi}_e(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) = \sum_{\mathbf{k}_s, \mathbf{k}_f} \tilde{\phi}_{\mathbf{k}_s, \mathbf{k}_f} e^{i\mathbf{k}_s \cdot \mathbf{R}_s} e^{i\mathbf{k}_f \cdot \mathbf{R}_f} J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho)$$

$$= -\frac{T_e}{n_e e} \sum_{\mathbf{k}_s, \mathbf{k}_f} e^{i\mathbf{k}_s \cdot \mathbf{R}_s} e^{i\mathbf{k}_f \cdot \mathbf{R}_f} J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho) \int d^3\mathbf{v}\, \tilde{h}_{e,\mathbf{k}_s,\mathbf{k}_f} J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho) \tag{23}$$

- now we use that,

$$J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho_e) = J_0(|\mathbf{k}_f|\rho_e) + O(\mathbf{k}_s \cdot \mathbf{k}_f \rho_e^2 \frac{dJ_0(z)}{dz}|_{z=|\mathbf{k}_f|\rho_e}), \tag{24}$$

- exploit that $|\mathbf{k}_s|\rho_e \sim (m_e/m_i)^{1/2}$ to bring $\mathbf{R}_s$ under the velocity integral
- regard $\mathbf{R}_s$ as a fixed parameter in the integration, to find,

$$\tilde{\varphi}_e(t_s, t_f, \mathbf{R}_s, \mathbf{R}_f) =$$

$$-e(\sum_{\nu} \frac{Z_\nu^2 n_\nu e^2}{T})^{-1} \sum_{\mathbf{k}_f} e^{i\mathbf{k}_f \cdot \mathbf{R}_f} J_0(|(\mathbf{k}_s + \mathbf{k}_f)|\rho) \int d^3\mathbf{v}|_{\mathbf{R}_s} \tilde{h}_{e\mathbf{k}_f}(\mathbf{R}_s) J_0(|\mathbf{k}_f|\rho_e)(1 + O(m_e/m_i)^{1/2}) \tag{25}$$

- we can evaluate quasi-neutrality purely locally in the slow variable.

Splitting the Gyrokinetic Equation

- we apply the electronscale average to the gyrokinetic equation
- we neglect terms which are small by $(m_e/m_i)^{1/2}$

Ion scale equation:

$$\frac{\partial \overline{h}}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla\theta \frac{\partial \overline{h}}{\partial \theta} + (\mathbf{v}_M + \overline{\mathbf{v}}_E) \cdot \nabla_s \overline{h} + \nabla_s \cdot \left\langle \frac{c}{B} \tilde{h} \tilde{\mathbf{v}}_E \right\rangle^{\mathrm{ES}} + \overline{\mathbf{v}}_E \cdot \nabla F_0 = \frac{ZeF_0}{T} \frac{\partial \overline{\varphi}}{\partial t_s}. \quad (26)$$

- we subtract the ion scale equation from the full equation and neglect terms

Electron scale equation:

$$\frac{\partial \tilde{h}}{\partial t_f} + v_\parallel \mathbf{b} \cdot \nabla\theta \frac{\partial \tilde{h}}{\partial \theta} + (\mathbf{v}_M + \tilde{\mathbf{v}}_E + \overline{\mathbf{v}}_E) \cdot \nabla_f \tilde{h} + \tilde{\mathbf{v}}_E \cdot (\nabla_s \overline{h} + \nabla F_0) = \frac{ZeF_0}{T} \frac{\partial \tilde{\varphi}}{\partial t_f}, \quad (27)$$

where

$$\overline{\mathbf{v}}_E = \frac{c}{B} \mathbf{b} \wedge \nabla_s \overline{\varphi}, \quad \tilde{\mathbf{v}}_E = \frac{c}{B} \mathbf{b} \wedge \nabla_f \tilde{\varphi}. \quad (28)$$

Note that,

- there are two additional terms on the electron scale, $\tilde{\mathbf{v}}_E \cdot \nabla_f \tilde{h}$ and $\tilde{\mathbf{v}}_E \cdot \nabla_s \overline{h}$
- there is one new term at the ion scale, $\nabla_s \cdot \left\langle \frac{c}{B} \tilde{h} \tilde{\mathbf{v}}_E \right\rangle^{\mathrm{ES}}$
- $\overline{\mathbf{v}}_E$ cannot be removed with the boost or a solid body rotation because of the θ dependence of $\overline{\varphi}$

Scaling Work: the Relative Size of the Fluctuations

- if we assume the following scalings:

$$\overline{h}_i \sim \frac{e\overline{\phi}}{T} F_{0i}, \quad \tilde{h}_e \sim \frac{e\tilde{\phi}}{T} F_{0e}, \quad \tilde{h}_i \sim (\frac{m_e}{m_i})^{1/4} \frac{e\tilde{\phi}}{T} F_{0i},$$

$$\overline{h}_e \sim (\frac{m_e}{m_i})^{1/2} \frac{e\overline{\phi}}{T} F_{0e} \text{ -parallel gradient term}, \quad \overline{h}_e \sim \frac{e\overline{\phi}}{T} F_{0e} \text{ -}\theta \text{ constant piece.} \tag{29}$$

- Then we can show that:

$$\frac{e\tilde{\phi}}{T} \sim \rho_e^*, \quad \frac{e\overline{\phi}}{T} \sim \rho_i^* \tag{30}$$

Scaling Work: Neglecting Ions at Electron Scales

note that:

- $J_0(\mathbf{k}_f \rho_i) \sim (m_e/m_i)^{1/4}$
- so:

$$\int d^3\mathbf{v}|_\mathbf{r} \tilde{h}_i \sim (\frac{m_e}{m_i})^{1/4}(\frac{m_e}{m_i})^{1/4}\frac{en\tilde{\phi}}{T} \tag{31}$$

Ions at electron scales can be neglected to $O((m_e/m_i)^{1/2})$ in the electronscale equations!

note that:

- $\nabla_s \cdot \left\langle \frac{c}{B}\tilde{h}_i \tilde{\mathbf{v}}_{Ei} \right\rangle^{\mathrm{ES}} \sim O((m_e/m_i)^1 \overline{\mathbf{v}}_{Ei} \cdot \overline{h}_i)$

Ions at electron scales can be neglected to $O((m_e/m_i)^1)$ in the ion scale equations!

Scaling Work: which multiscale terms do we keep?

The only remaining multiscale terms are in electron species equations:

note that:

- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla_s \overline{h}_e \sim \overline{\mathbf{v}}_{Ee} \cdot \nabla_f \tilde{h}_e \sim \tilde{\mathbf{v}}_{Ee} \cdot \nabla_f \tilde{h}_e$
- ion scale gradients contribute at $O(1)$ to the electron scale
- ion scale shear can be neglected to $O((m_e/m_i)^{1/2})$ at the electron scale

- $\nabla_s \cdot \left\langle \frac{c}{B} \tilde{h}_e \tilde{\mathbf{v}}_{Ee} \right\rangle^{\mathrm{ES}} \sim O((m_e/m_i)^{1/2} \overline{\mathbf{v}}_{Ee} \cdot \overline{h}_e)$
- back reaction contributes at $O((m_e/m_i)^{1/2})$ to the electron equation at ion scales
- small but can be self consistently included

Scaling work: Heat Flux

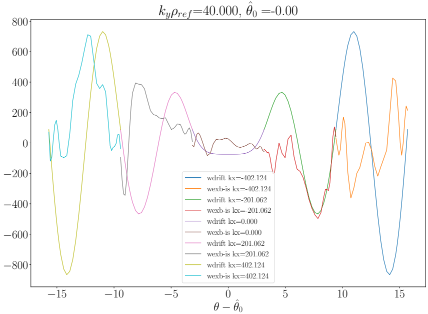
Substituting for the gyro Bohm scalings of the potential we find that,

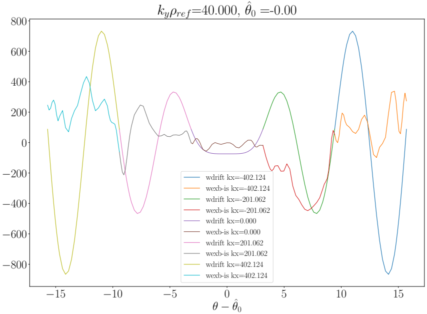
$$\frac{\tilde{Q}_i}{\overline{Q}_i} \sim (\frac{m_e}{m_i}), \quad \frac{\tilde{Q}_e}{\overline{Q}_e} \sim (\frac{m_e}{m_i})^{1/2}, \quad \overline{Q}_i \sim \overline{Q}_e. \tag{32}$$

Discontinuities and filtering: CBC $a/L_{T_i} = 2.3$

- $\bar{\mathbf{v}}_{Ee} \cdot \nabla \tilde{h}_e / \tilde{h}_e$
- $\mathbf{v}_{Me} \cdot \nabla \tilde{h}_e / \tilde{h}_e$

- filter in extended ballooning angle τ for each chain coupled by $\parallel$ b.c.
- $\exp\left[-D(\tau/\tau_{\rm max})^4\right]$





Discontinuities and filtering: CBC $a/L_{T_i} = 2.3$

- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla \overline{h}_e / \tilde{\phi}$
- $\tilde{\mathbf{v}}_{Ee} \cdot \nabla F_{0e} / \tilde{\phi}$

- filter in extended ballooning angle τ for each chain coupled by $\|$ b.c.
- $\exp\left[-D(\tau/\tau_{\max})^4\right]$