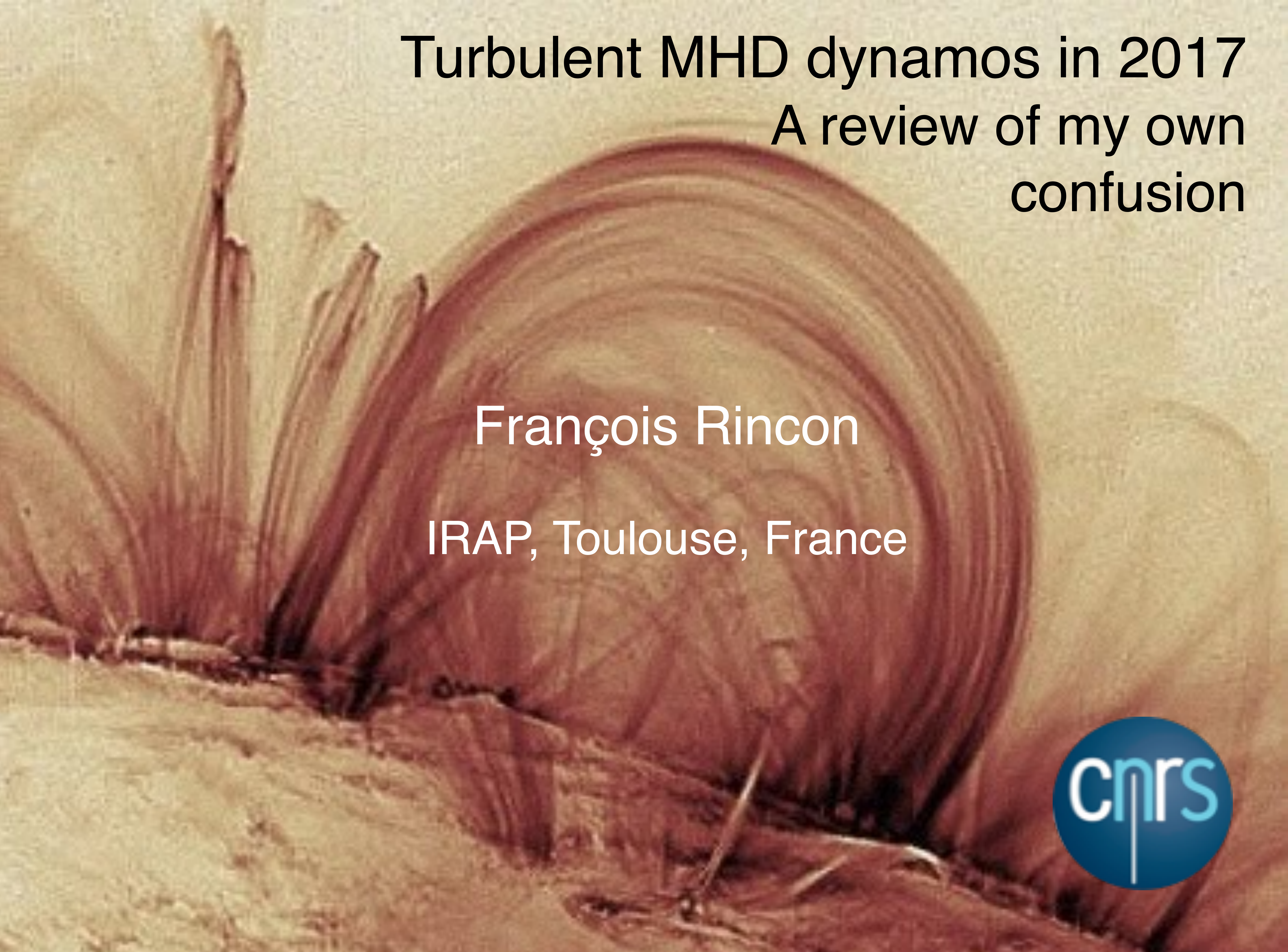
Turbulent MHD dynamos in 2017
A review of my own confusion



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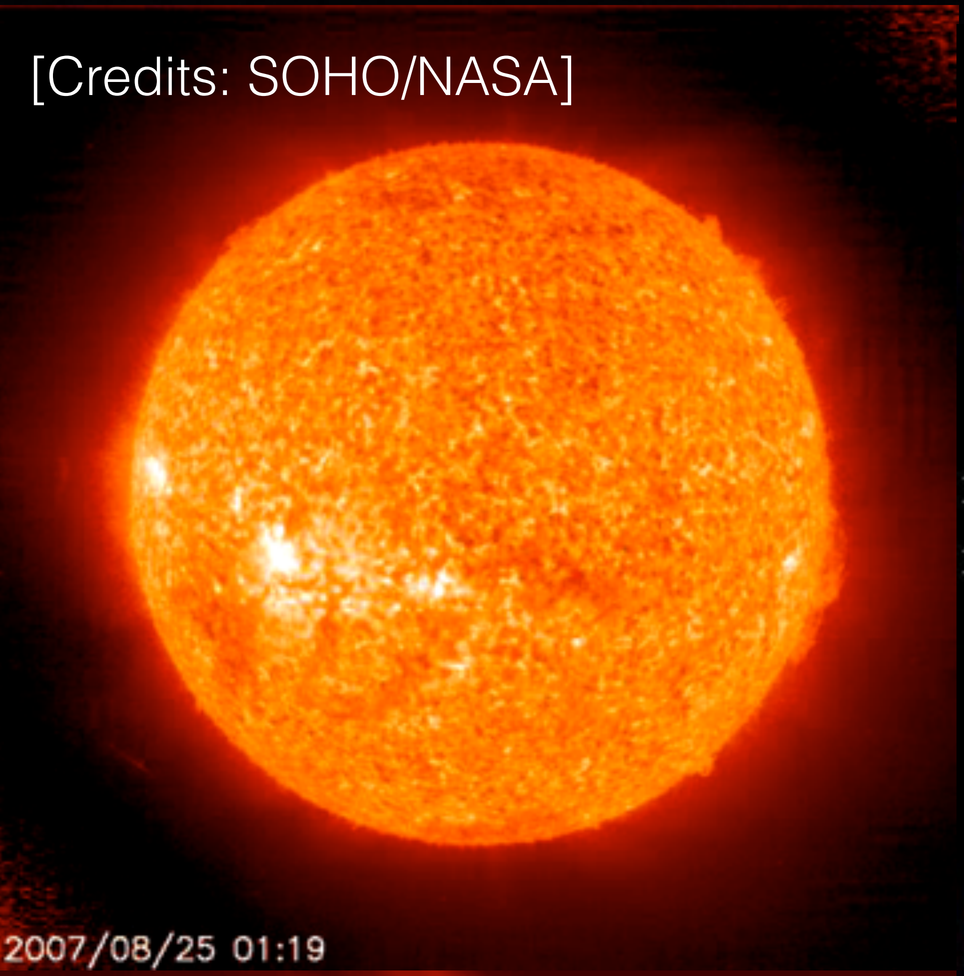
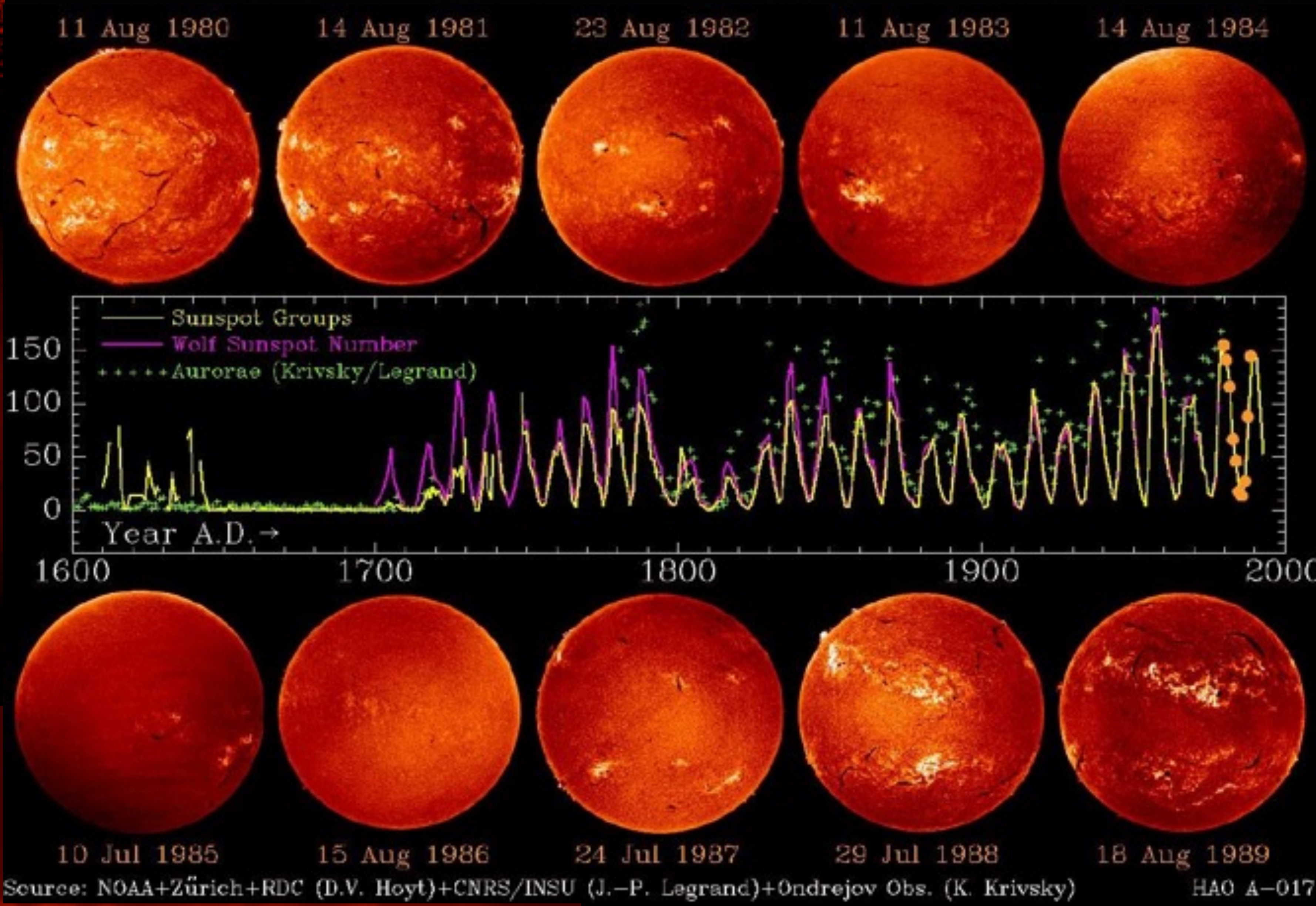
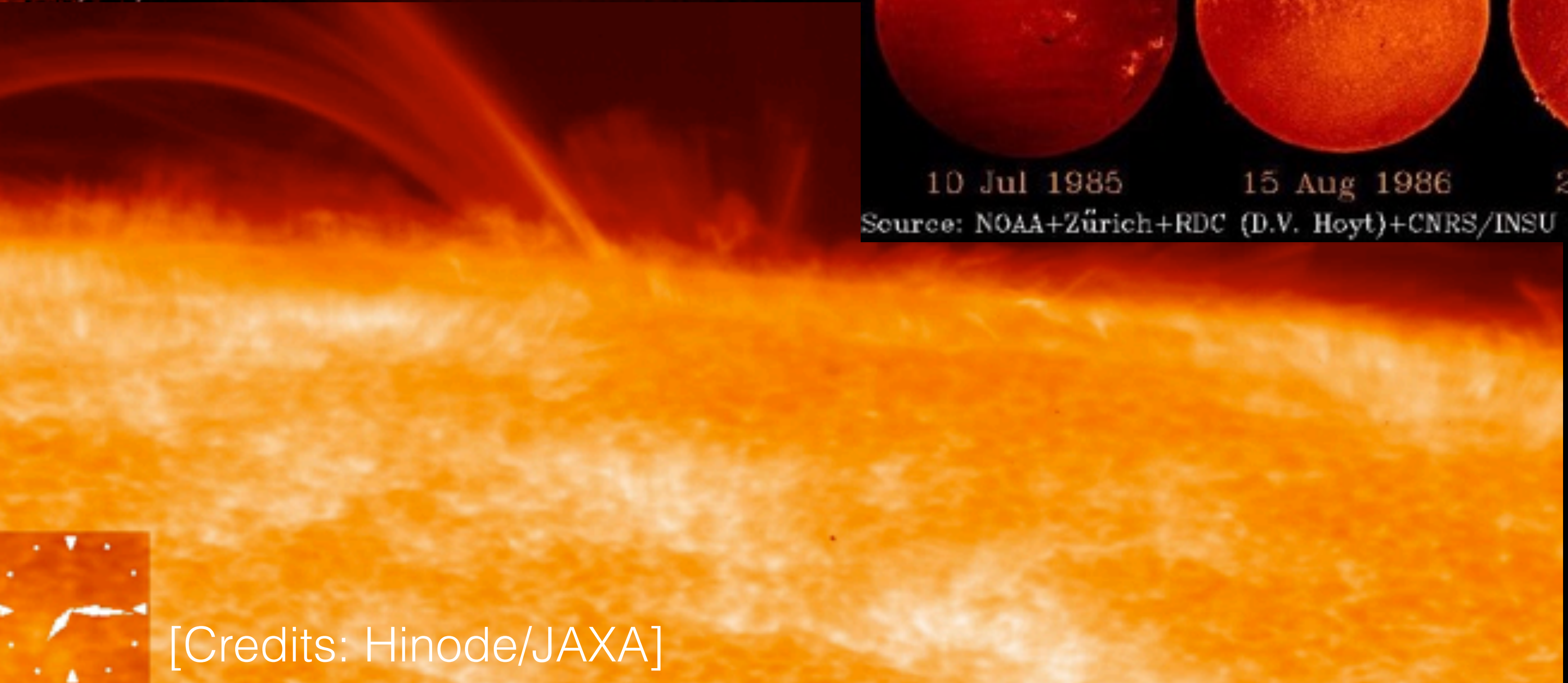
Introduction

Solar magnetism





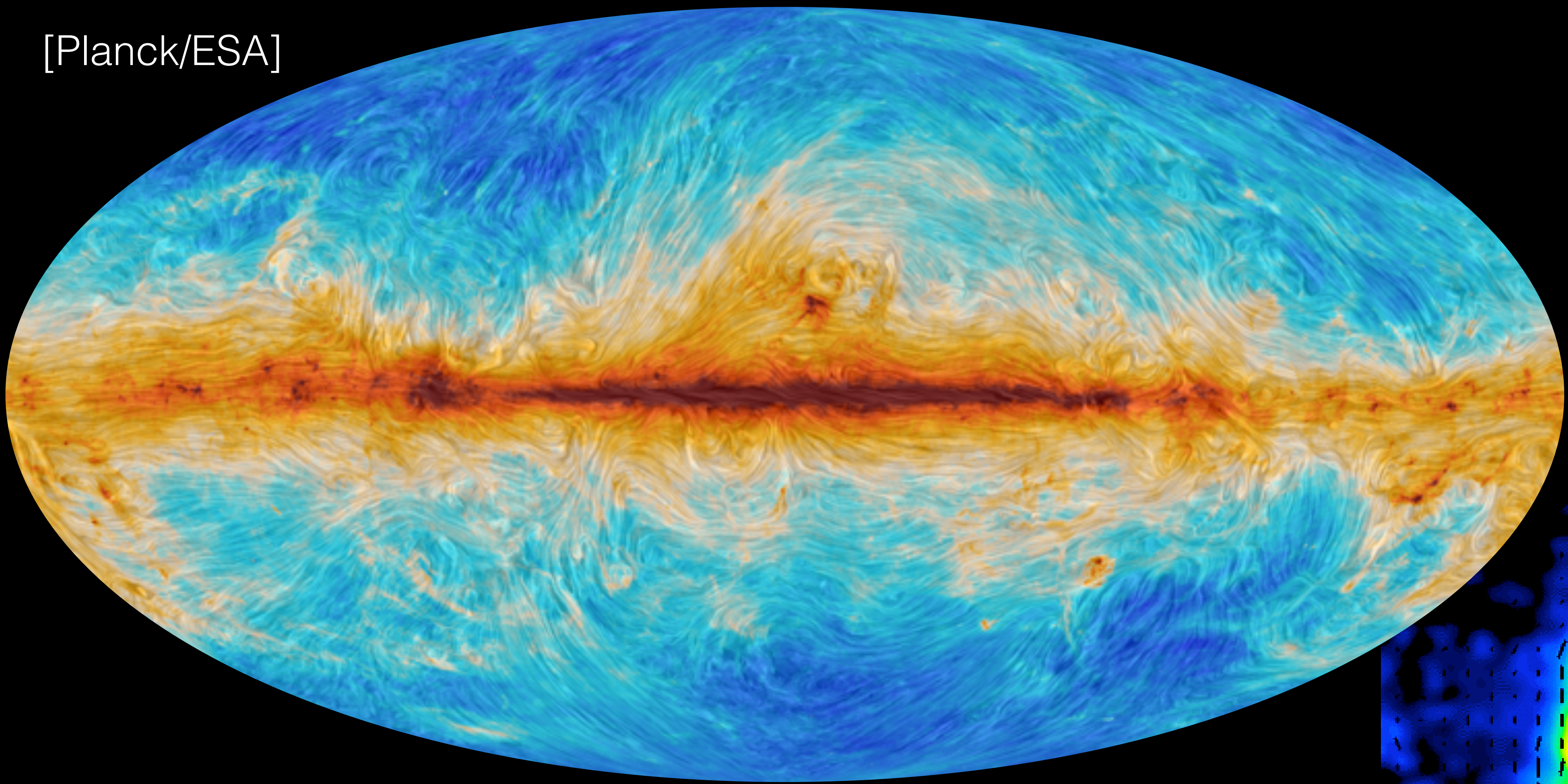
Global solar cycle dynamics



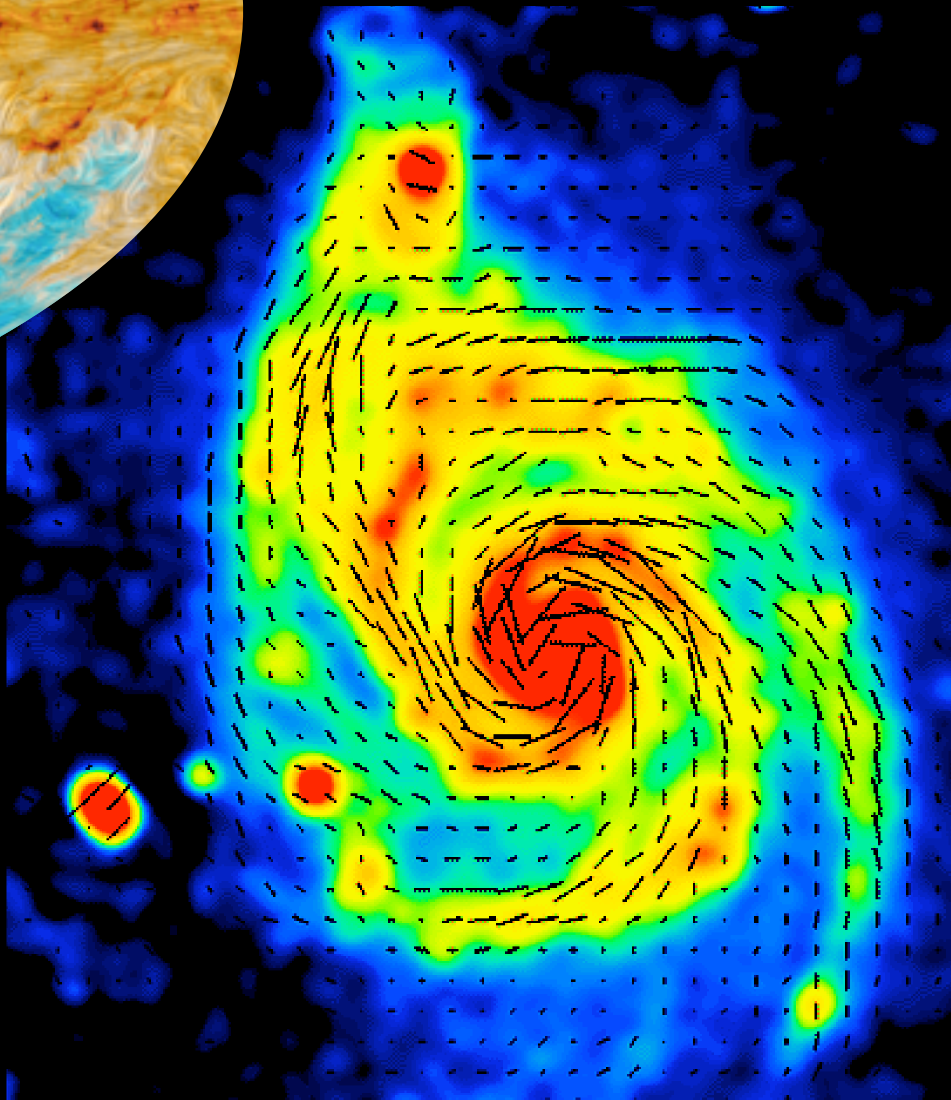
Small-scale surface dynamics

Galactic magnetism

[Planck/ESA]



Galactic magnetic field



M51 magnetic field

[Beck et al. VLA/Effelsberg]

Takeaway phenomenological points

- Many astrophysical objects have global, ordered fields
  - Differential rotation, global symmetries and geometry important
  - Coherent structures and MHD instabilities may also be very important
  - Motivation for the development of "large-scale" dynamo theories
- Lots of "small-scale", random fields also discovered from the 70s
  - These come hand in hand with global magnetism
  - Simultaneous development of "small-scale dynamo" theory
- Astrophysical magnetism is in a nonlinear, saturated state
  - Linear theory not the whole story (or using it requires non-trivial justification)
  - Multiple scale interactions expected to be important

Simplest MHD system for dynamo theory

- Incompressible, resistive, viscous MHD
  - Captures a great deal of the dynamo problem

Magnetic tension

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \mathbf{B} \cdot \nabla \mathbf{B} + \nu \Delta \mathbf{u} + \mathbf{f}(\mathbf{x}, t)$$

Induction

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B}$$

$$P = p + \frac{B^2}{2}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

p and $\mathbf{B}$ rescaled by ρ and $(4\pi\rho)^{1/2}$

- Often paired with simple periodic boundary conditions
  - Problematic in some cases

Scales and dimensionless numbers

- System/integral scale ℓ_0, U_0
- Fluid system with two dissipation channels
  - Dimensionless numbers:

$$\mathrm{Re} = \frac{\ell_0 U_0}{\nu} \qquad \mathrm{Rm} = \frac{\ell_0 U_0}{\eta} \qquad \mathrm{Pm} = \frac{\nu}{\eta}$$

  - Kolmogorov viscous scale $\ell_\nu \sim \mathrm{Re}^{-3/4}\, \ell_0$, $u_\nu \sim \mathrm{Re}^{-1/4}\, U_0$
  - Magnetic resistive scale ℓ_η (Pm-dependent)
- Another important dimensionless quantity
  - Eddy turnover time $\tau_{NL} \sim \ell_u/u$
  - Flow/eddy correlation time τ_c

$$\mathrm{St} = \frac{\tau_c}{\tau_{NL}}$$

Strouhal/Kubo number

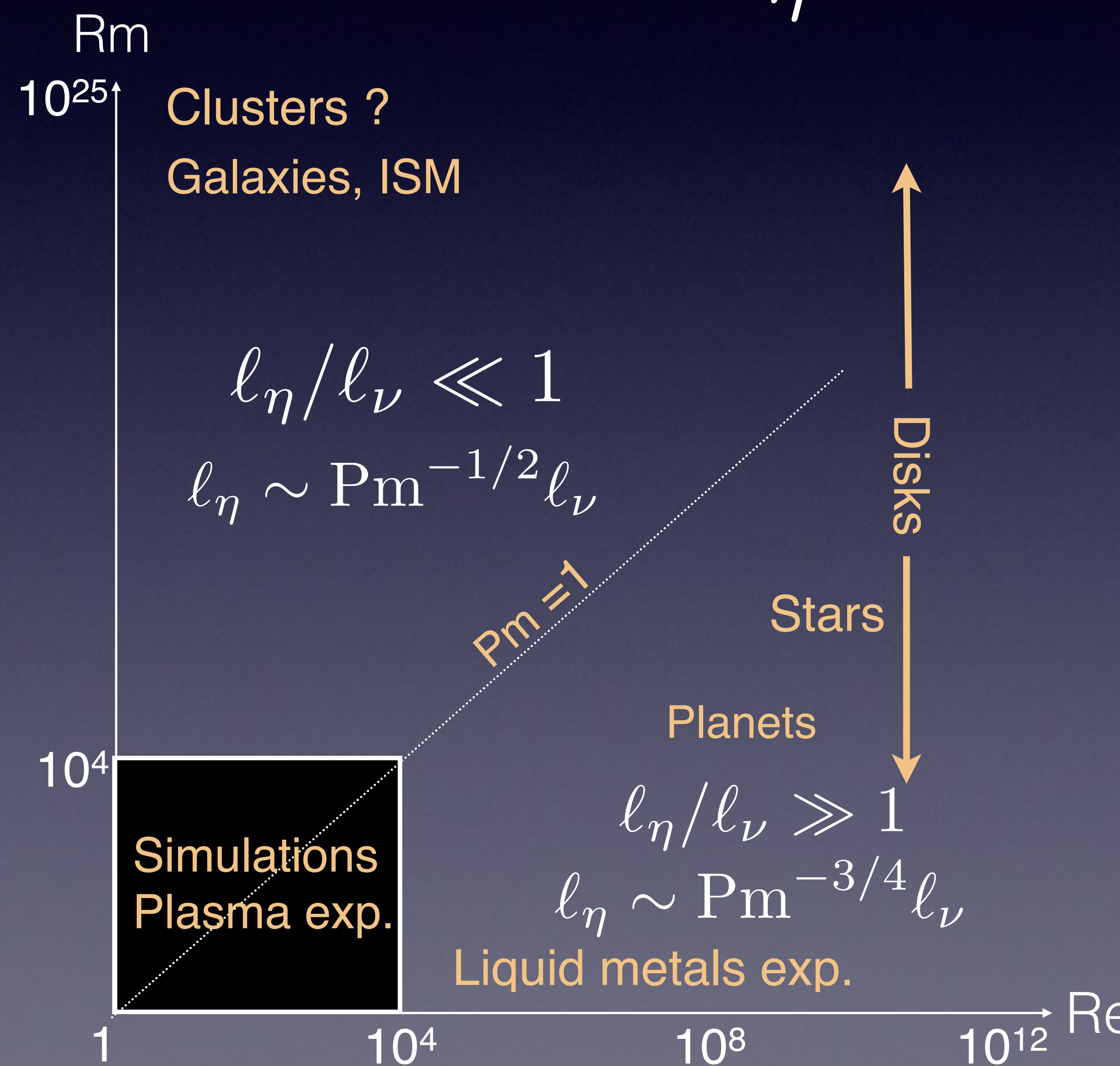
The magnetic Prandtl number landscape

- Wide range of Pm in nature
  - Liquid metals have Pm << 1
  - Computers have Pm ~ O(1)
- For a collisional hydrogen plasma [Te=Ti in K, *n* in S.I.]

$$\mathrm{Pm} = 2.5\times 10^{3}\frac{T^4}{n\ln\Lambda^2}$$

- Pm<1 and Pm>1 seemingly very different situations
  - Naively, Pm>1 makes life easier to magnetic fields

$$\mathrm{Pm} = \frac{\nu}{\eta}$$



Small-scale dynamos

Numerical evidence

- Homogeneous, isotropic, non-helical, incompressible, 3D turbulent flow of conducting fluid is a small-scale dynamo

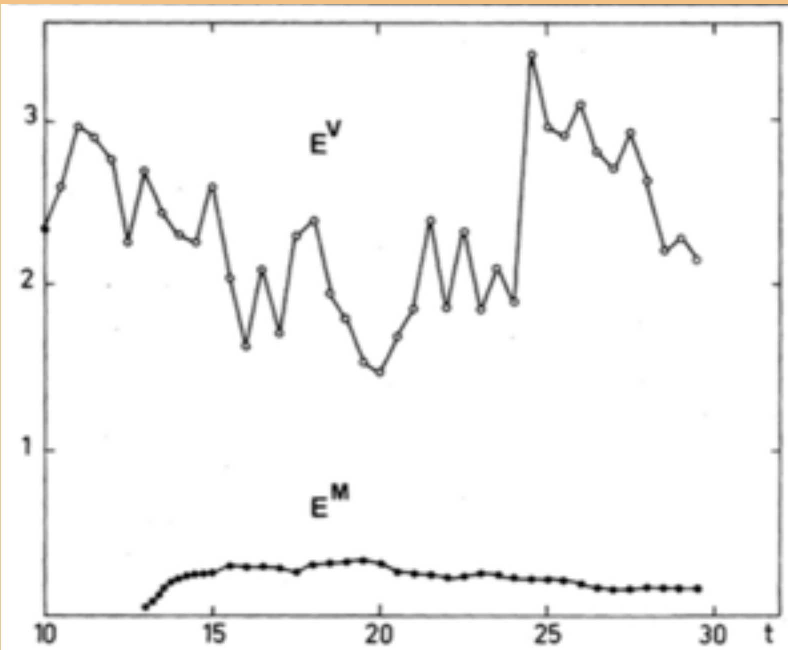


FIG. 1. Turbulent dynamo with nonhelical driving. Temporal variation of kinetic (E^V) and magnetic (E^M) energy. Reynolds numbers are $R^V = R^M \approx 100$. The time unit is the eddy-turnover time l_0/v_0.

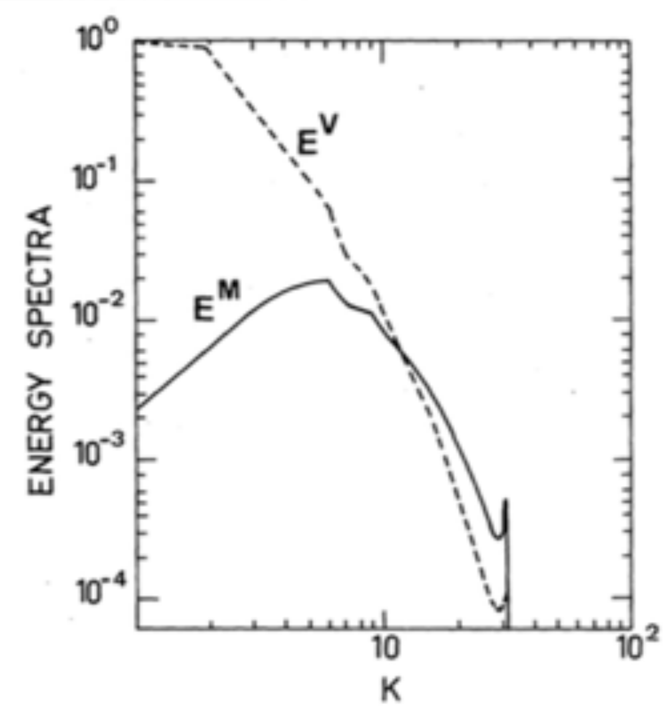


FIG. 2. Kinetic (E^V) and magnetic (E^M) energy spectra at $t = 27$. Nonhelical dynamo with $R^V = R^M \approx 100$.

64x64x64 spectral DNS simulations at Pm=1

[Meneguzzi, Frisch, Pouquet, PRL, 1981]

Zel'dovich phenomenology

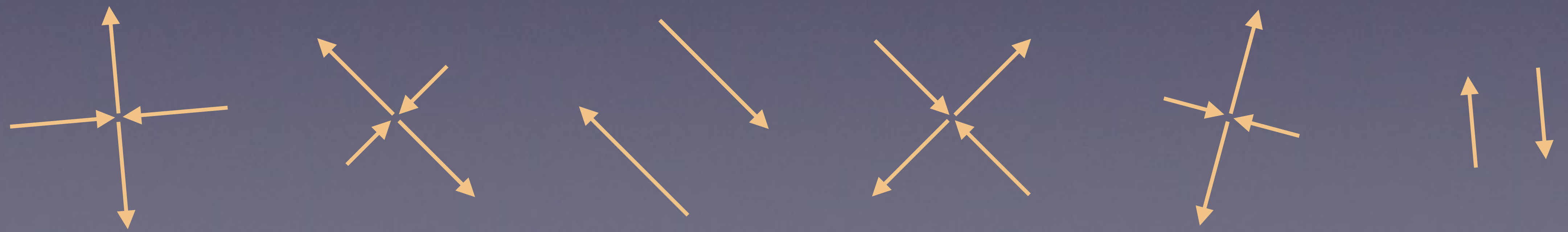
[Zel'dovich et al., JFM 144, 1 (1984)]

- Consider incompressible, kinematic dynamo problem

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B} \qquad \nabla \cdot \mathbf{B} = 0$$

- Assume that $\mathbf{B}(0, \mathbf{r}) = \mathbf{B}_0(\mathbf{r})$
    - has finite total, energy, no singularity
    - $\lim_{r \to \infty} \mathbf{B}_0(\mathbf{r}) = 0$

- Take simplest possible model of time-evolving "smooth" velocity field
    - Random linear shear: $\mathbf{u} = \mathsf{C}\mathbf{r}$ $\qquad \mathrm{Tr}\,\mathsf{C} = 0$ [incompressible]



[think of this as being 3D]

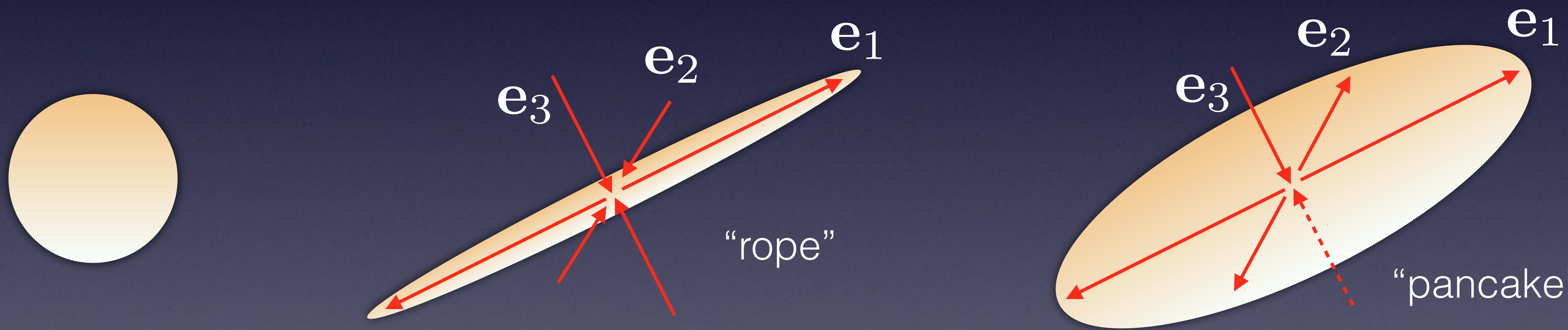
Stretching and squeezing

- Evolution of vector connecting 2 fluid particles: $\dfrac{d\delta r_i}{dt} = \mathsf{C}_{\mathsf{ik}}\delta r_k$
- Consider constant $\mathsf{C} = \mathrm{diag}(c_1, c_2, c_3)$

$$c_1 + c_2 + c_3 = 0$$

  - Exponential stretching along first axis

$$c_1 > 0 > c_2 > c_3 \qquad\qquad c_1 > c_2 > 0 > c_3$$



- In ideal MHD, we thus expect $B^2 \sim \exp(2c_1 t)$
  - However, perpendicular squeezing implies that even a tiny magnetic diffusion matters…is growth still possible in that case ?

Magnetic field evolution

- Decompose $\mathbf{B}(t,\mathbf{r}) = \int \mathbf{b}(t,\mathbf{k}_0) \exp\left(i\mathbf{k}(t)\cdot\mathbf{r}\right) d^3\mathbf{k}_0$

$$\frac{d\mathbf{b}}{dt} = \mathsf{C}\mathbf{b} - \eta \mathsf{k}^2\mathbf{b} \qquad \frac{d\mathbf{k}}{dt} = -\mathsf{C}^{\mathsf{T}}\mathbf{k} \qquad \mathbf{k}\cdot\mathbf{b} = 0$$

- Diffusive part of evolution ~ $\exp\left(-\eta\int_0^t k^2(s)ds\right)$
  - super-exponential decay of most Fourier modes because

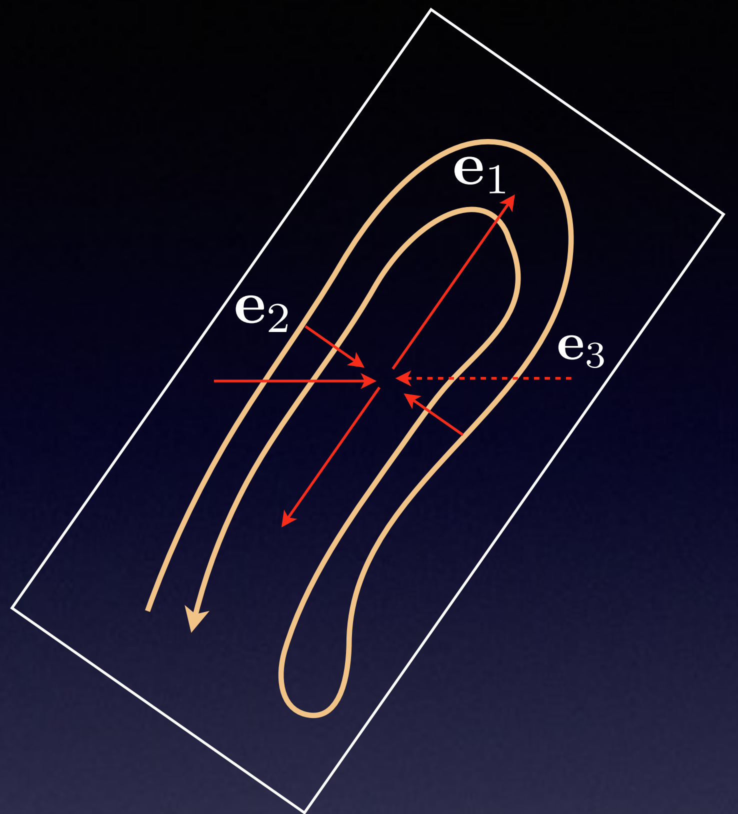
$$k_3 \sim k_{03}\exp(|c_3|t)$$

  - survivors live in an exponentially narrow cone of modes such that

$$\eta\int_0^t k^2(s)ds = O(1)$$

    - rope case: $k_{02} \sim \exp(-|c_2|t) \qquad k_{03} \sim \exp(-|c_3|t)$

Magnetic field evolution (ropes)



- Surviving modes at time t have an initial field
  - $b_1(0, \mathbf{k}_0) \sim b_2(0, \mathbf{k}_0) k_{02}/k_{01} \sim \exp(-|c_2|t)$
  - This field is stretched along the first axis, so

$$\mathbf{b}(t, \mathbf{k}_0) \sim \exp\left(c_1 t\right) \exp\left(-|c_2|t\right)$$

- Now, estimate the magnetic field in physical space

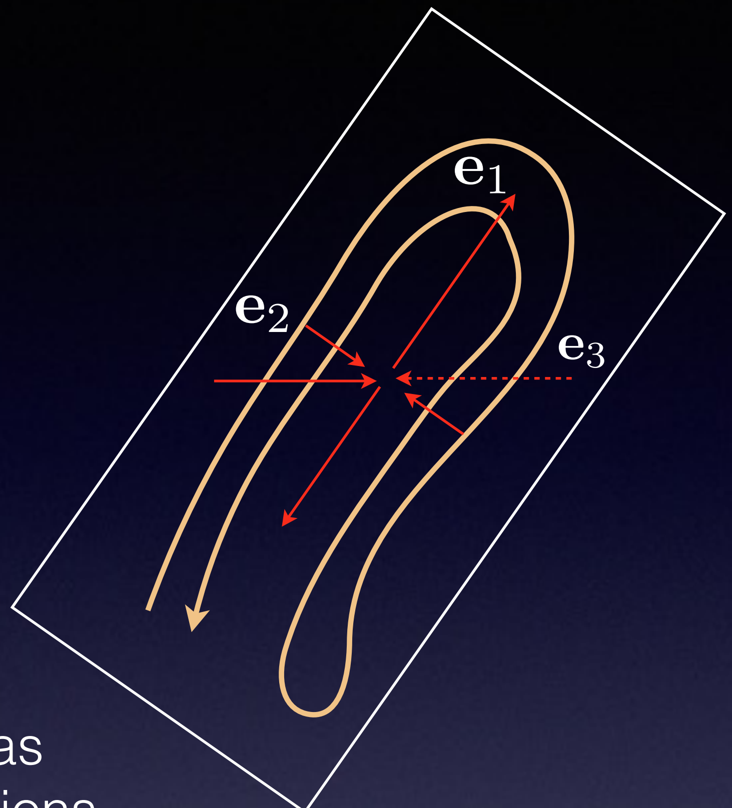
$$\mathbf{B}(t, \mathbf{r}) \sim \int \mathbf{B}_k d^3\mathbf{k}_0 \sim \exp(-|c_2|t)$$

$$\sim \exp\left[(c_1 - |c_2|)t\right] \qquad \sim \exp\left[(-|c_2| - |c_3|)t\right]$$

Magnetic field stretches into an asymptotically-decaying rope

Magnetic energy evolution (ropes)

- What about magnetic energy ?



$$E_\mathrm{m} = \int \mathbf{B}^2(t,\mathbf{r})d^3\mathbf{r}$$

$$B^2 \sim \exp\left(-2|c_2|t\right)$$

$$\text{Volume} \sim \exp(c_1 t)$$

Important: no shrinking along axis 2 and 3 as diffusion sets a minimum scale in these directions

$$E_\mathrm{m} \sim \exp\left[(c_1 - 2|c_2|)t\right] \sim \exp\left[(|c_3| - |c_2|)t\right] \text{(3D)}$$

Total magnetic energy grows ! (in 3D)

Volume occupied by the magnetic field grows faster than field decays pointwise

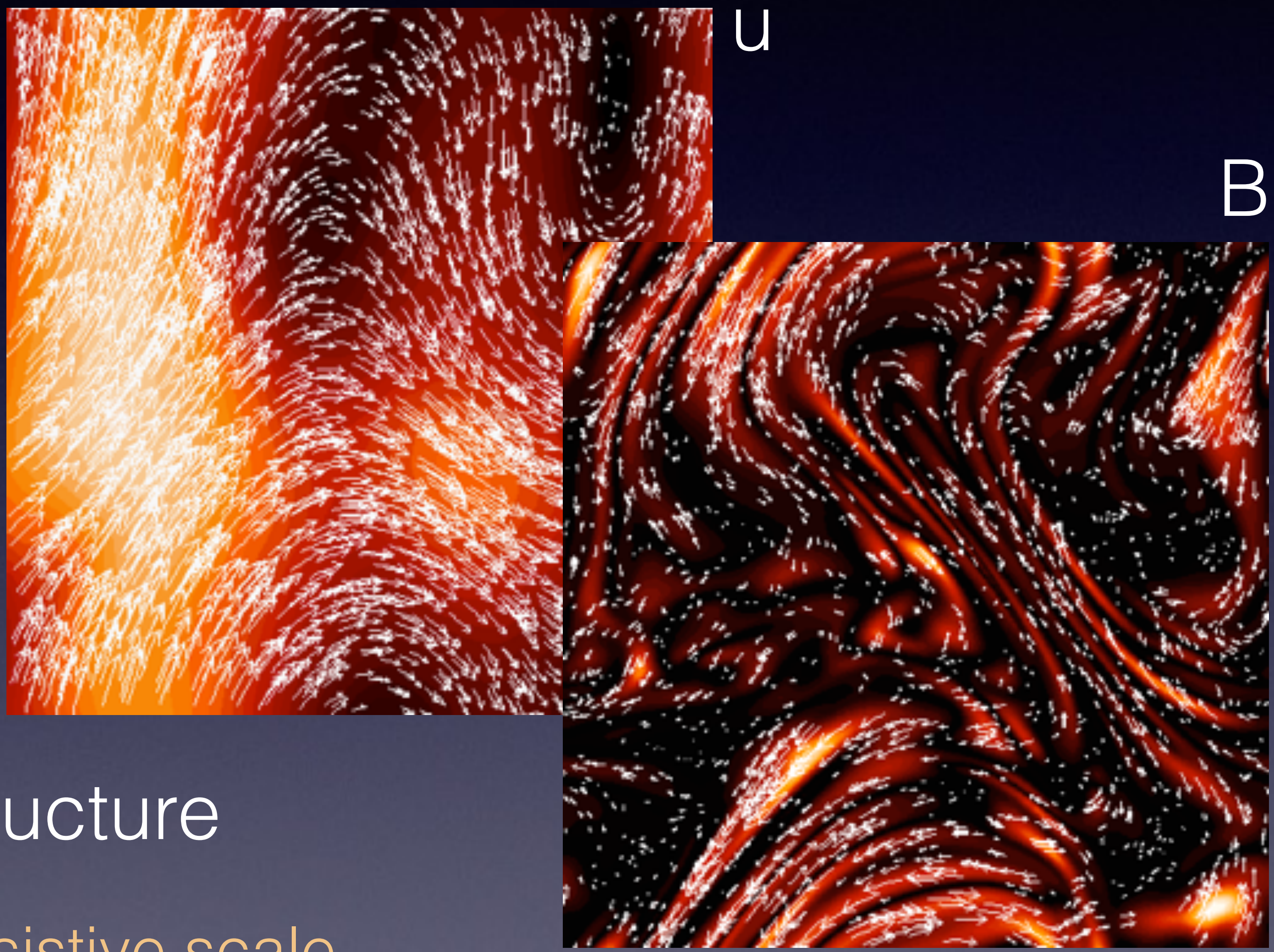
- Similar conclusions apply in the pancake case, but $E_\mathrm{m} \sim \exp\left[(c_1 - c_2)t\right]$

Small-scale dynamo fields at Pm ≥ 1

- Pm=Rm=1250, Re=1 [from Schekochihin et al., ApJ 2004]

u

B



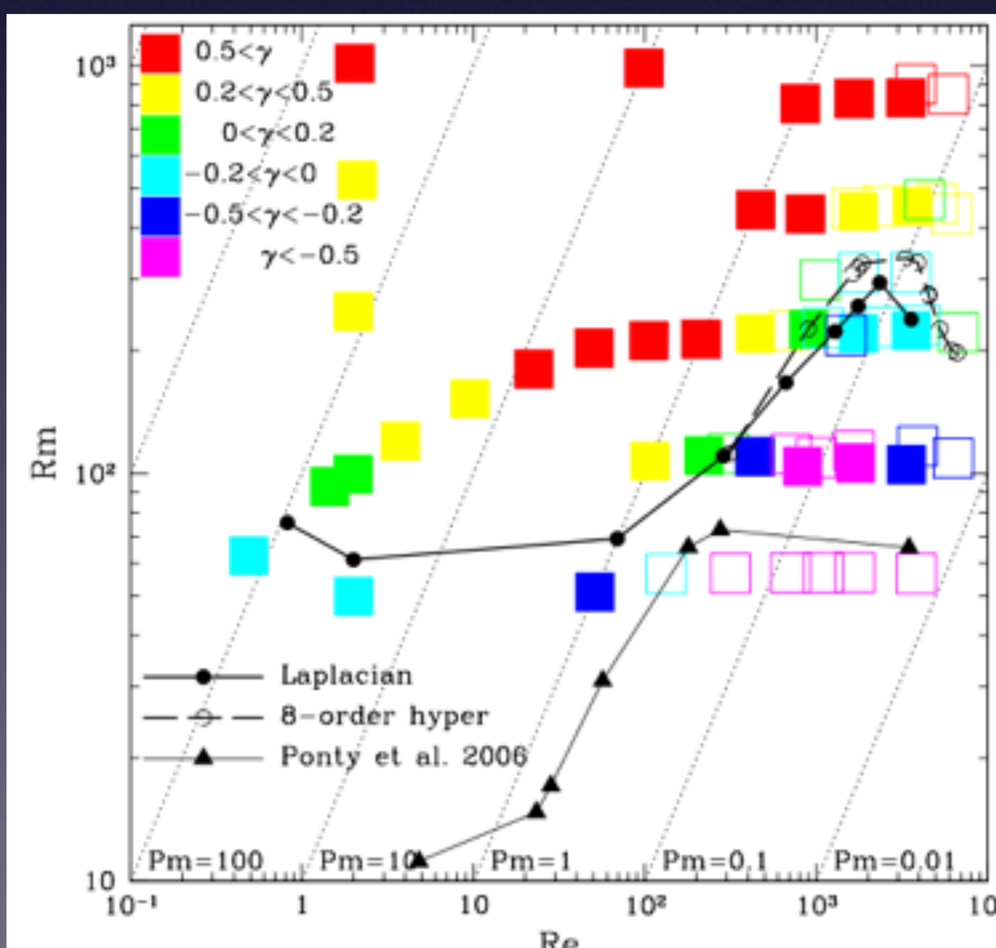
$\ell_\eta \sim \ell_\nu \mathrm{Pm}^{-1/2}$

- Folded field structure
  - Reversals at resistive scale
  - Folds coherent over flow scale
  - Field strength and curvature anticorrelated

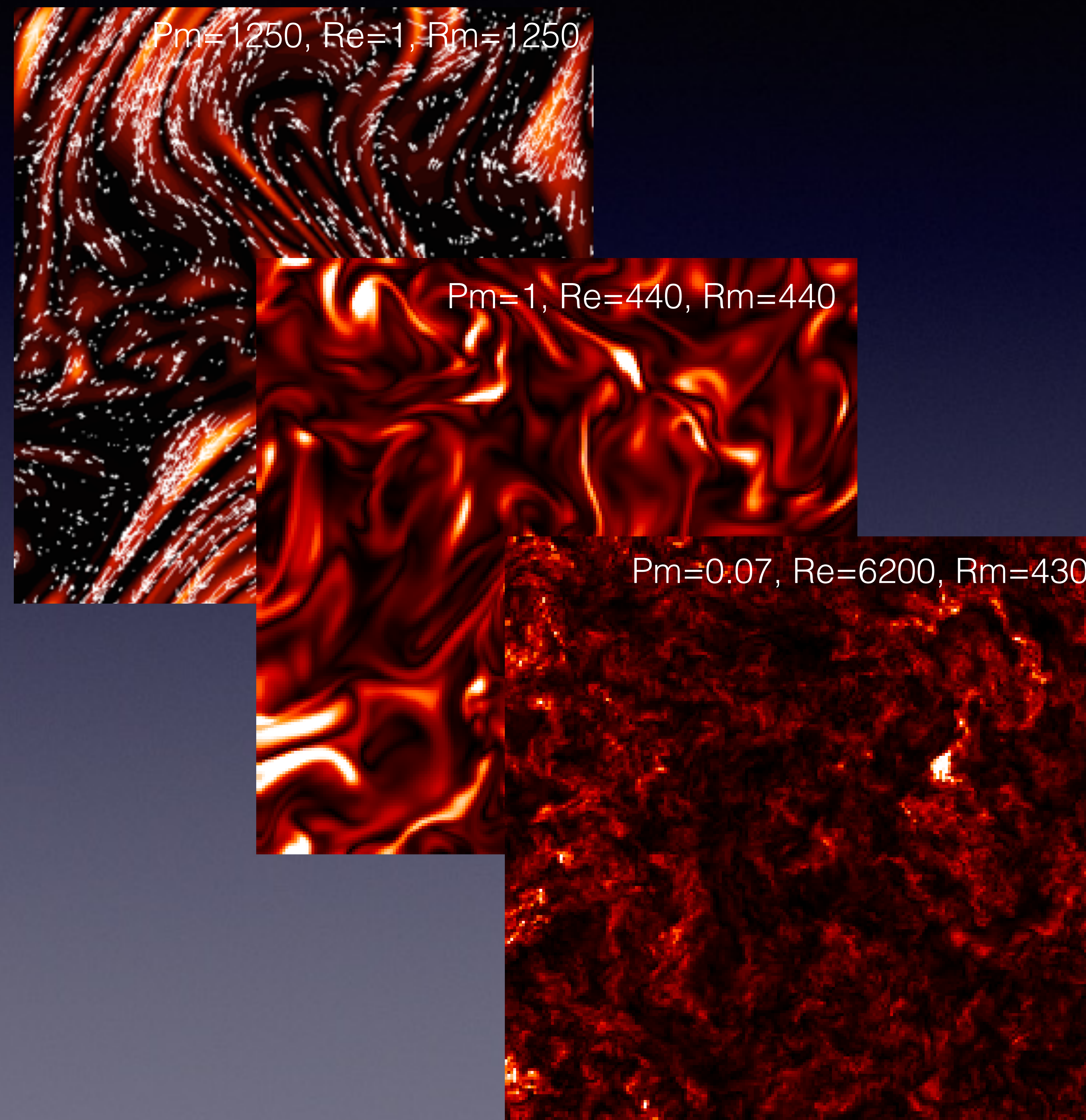
Critical Rm ~ 60

Small-scale dynamo at low Pm

- Yes, but much harder
  - Critical Rm~200
  - More complicated than Zel'dovich picture



[Iskakov et al., PRL 2007]



Kazantsev-Kraichnan model

- Consider again the following kinematic dynamo problem:

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B} \qquad \nabla \cdot \mathbf{B} = 0$$

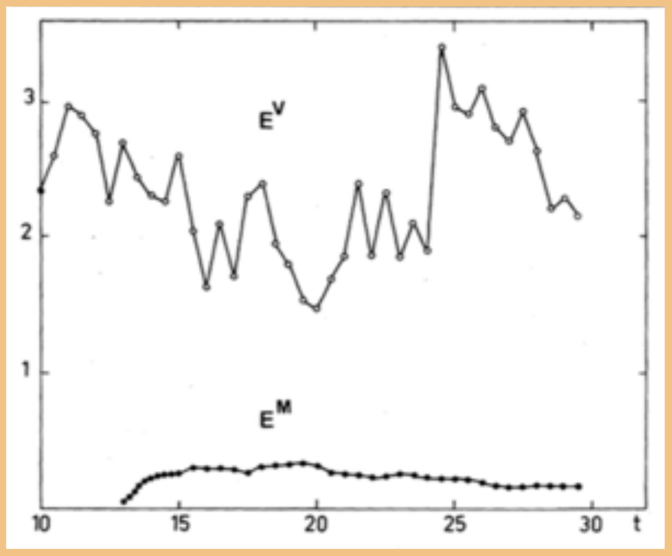
- This problem can be solved analytically if u is
    - a random Gaussian process with no memory (zero-correlation time)
    - The so-called Kraichnan ensemble $\left\langle u^i(\mathbf{x},t) u^j(\mathbf{x}',t') \right\rangle = \kappa^{ij}(\mathbf{r})\delta(t-t')$
- Obviously, not your usual turbulent flow, but still…
    - Very useful to understand the properties of small-scale dynamo modes
- Originally solved by Kazantsev [JETP, 1968]

[and further explored by Zel'dovich, Ruzmaikin, Sokoloff, Vainshtein, Kitchatinov, Vergassola, Vincenzi, Subramanian, Boldyrev, Schekochihin etc.]

Saturation of small-scale dynamo

- As B gets large-enough, Lorentz force saturates dynamo
    - What is “large-enough “?
    - How does it work ?

[Meneguzzi et al., PRL 1981]



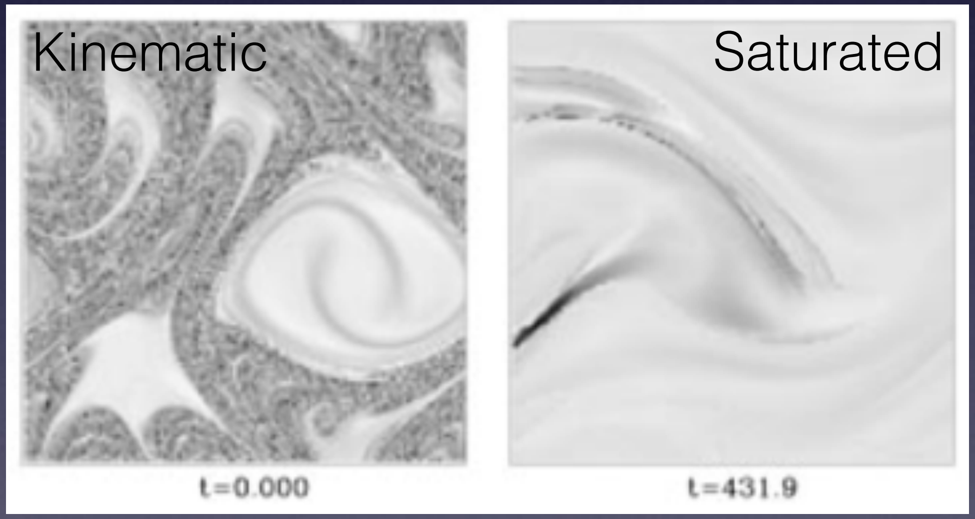
- Historical ideas
    - Batchelor argument [PRSL,1950]:
        - magnetic field is similar to hydrodynamic vorticity
        - should peak at viscous scale, hence saturation for $B^2 \sim \delta u_\nu^2$

$$\left\langle B^2 \right\rangle \sim \mathrm{Re}^{-1/2} \left\langle u^2 \right\rangle$$ Sub-equipartition unless Re=1

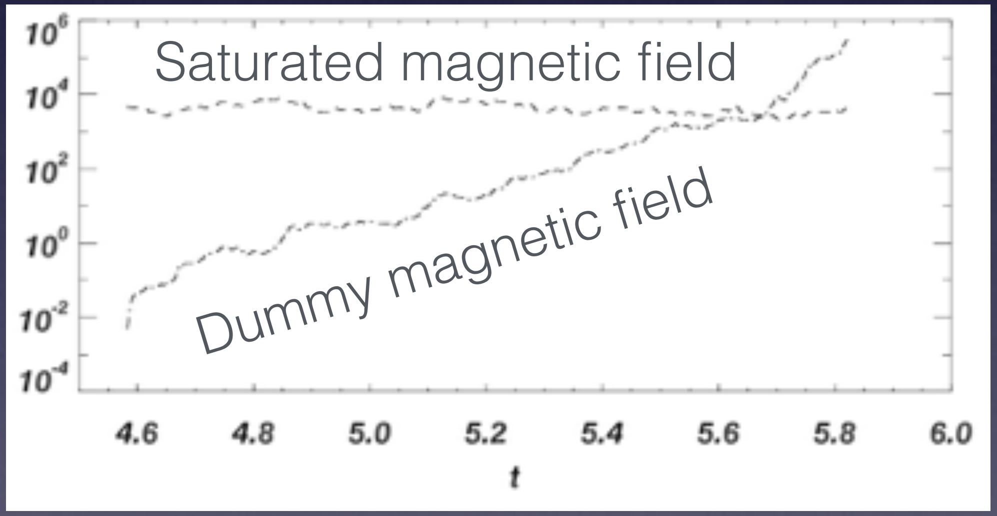
- Schlüter-Biermann argument [Z. Naturforsch.,1950]:
    - equipartition at all scales $\left\langle B^2 \right\rangle \sim \left\langle u^2 \right\rangle$

Saturation phenomenology

- Geometric structure and orientation of the field matters
  - Magnetic tension $\mathbf{B}\cdot\nabla\mathbf{B}$ encodes magnetic curvature
  - Reduction of stretching Lyapunov exponents
  - A field realization can only saturate itself



[Cattaneo et al., PRL 1996]



[Cattaneo & Tobias, JFM 2009]

- Saturation at low Pm
  - Pretty much *Terra incognita* (no published simulation)

Large Pm phenomenology

- Plausible (but not definitive) scenario from simulations
  [Schekochihin et al., ApJ 2002, 2004]
  - Lorentz force first suppresses stretching at viscous scales

$$\mathbf{B}\cdot\nabla\mathbf{B} \sim \mathbf{u}\cdot\nabla\mathbf{u} \sim \delta u_\nu^2/\ell_\nu$$

$$\sim B^2/\ell_\nu \quad \text{(folded structure)} \longrightarrow \left\langle B^2 \right\rangle \sim \mathrm{Re}^{-1/2} \left\langle u^2 \right\rangle$$

  - From there, slower, larger-scale eddies take over stretching
    - B keeps growing and acts on increasingly more energetic eddies…
      - Secular growth regime: $\left\langle B^2 \right\rangle \sim \varepsilon t$
  - Final state: $\left\langle B^2 \right\rangle \sim \left\langle u^2 \right\rangle$ after "suppression" of full inertial range
    - "Isotropic MHD turbulence", folded structure is preserved
- P[B] not log-normal anymore (likely exponential)

Large-scale dynamos

Numerical evidence

- Small-scale helical turbulence can generate large-scale field
    - Critical Rm is O(1), lower than that of the small-scale dynamo

[Brandenburg, ApJ 2001]

[Meneguzzi et al., PRL 1981 — again !]

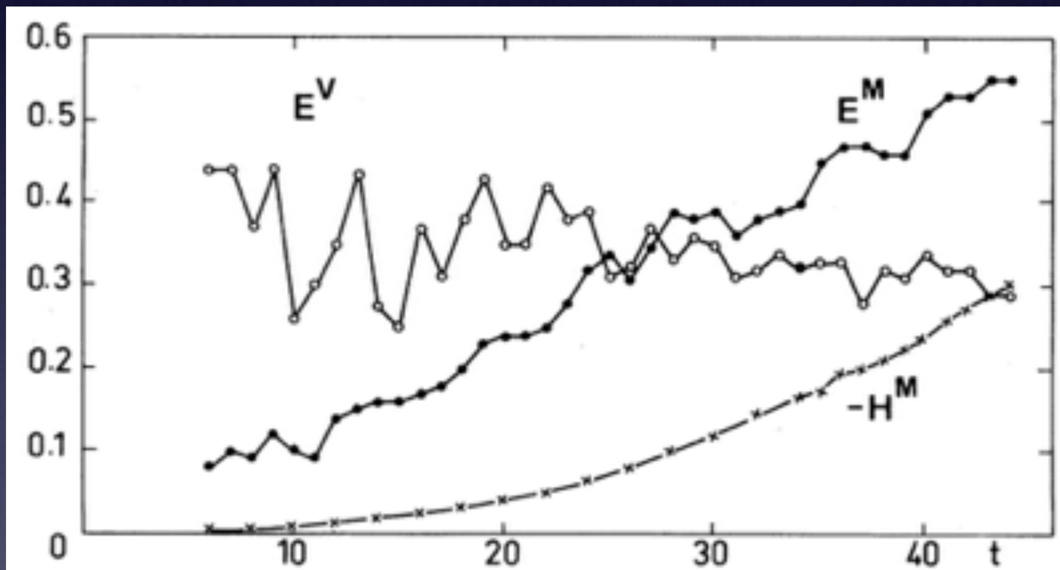
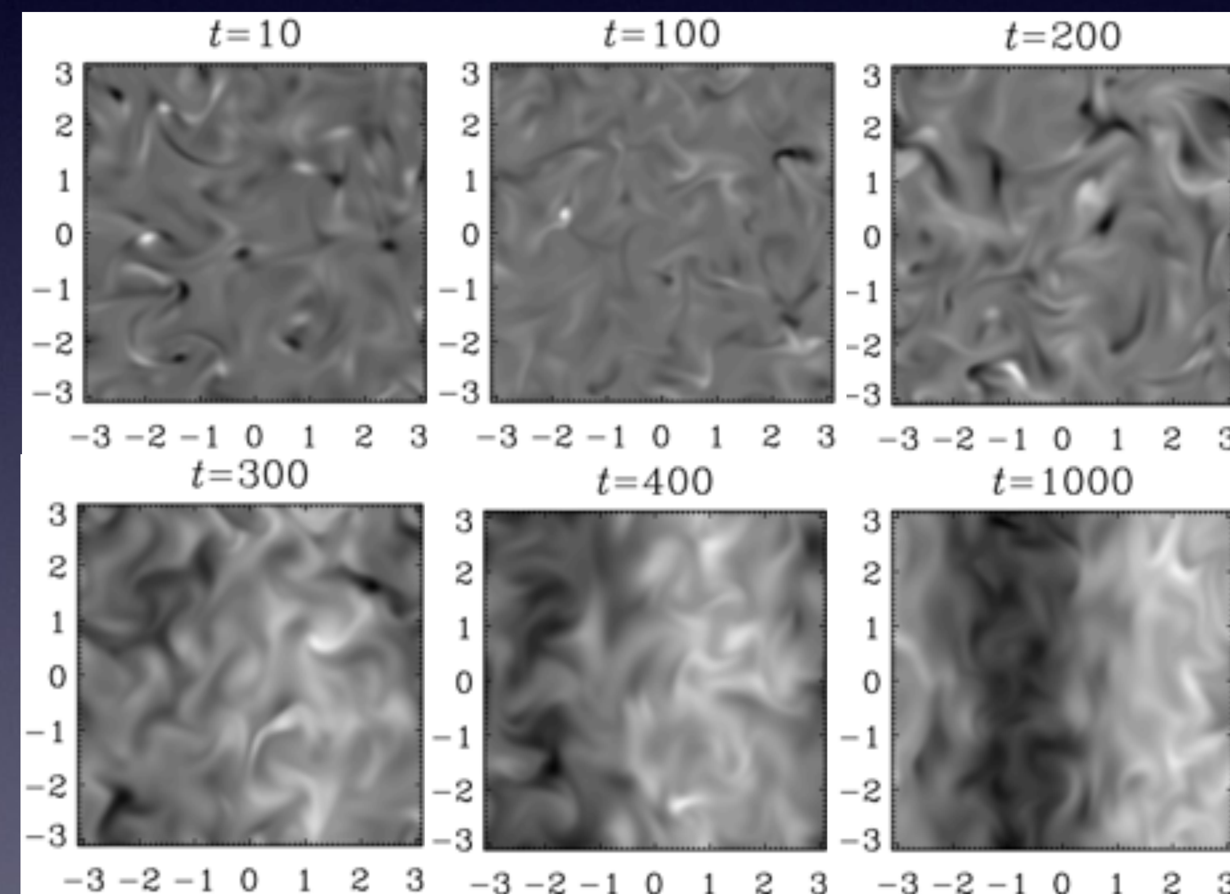


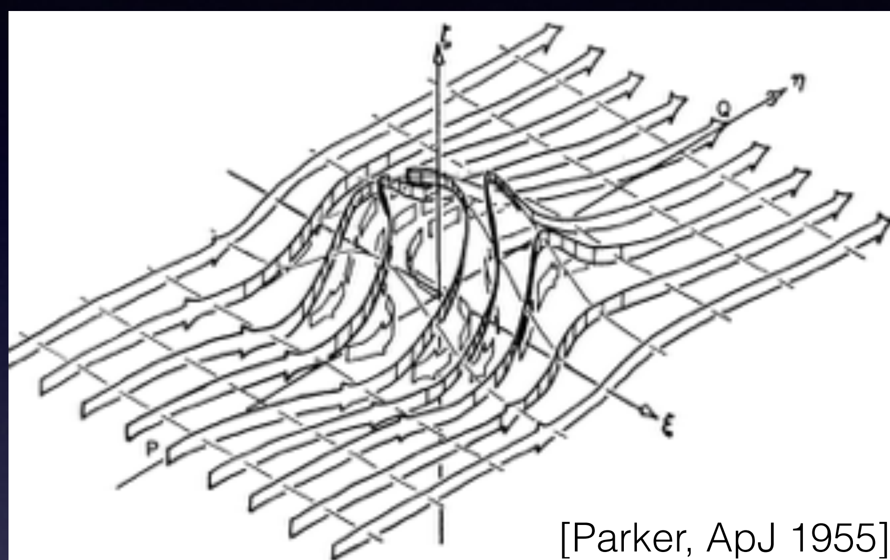
FIG. 4. Helical dynamo with driving at intermediate scales ($k = 5$). Temporal variation of kinetic energy (E^V), magnetic energy (E^M), and magnetic helicity ($-H^M$).



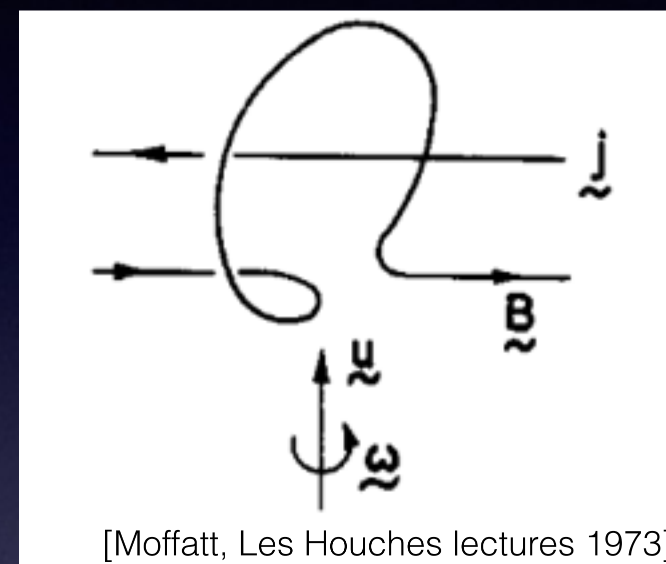
- Helicity seemingly key for large-scale dynamos (but see later)

Parker’s mechanism

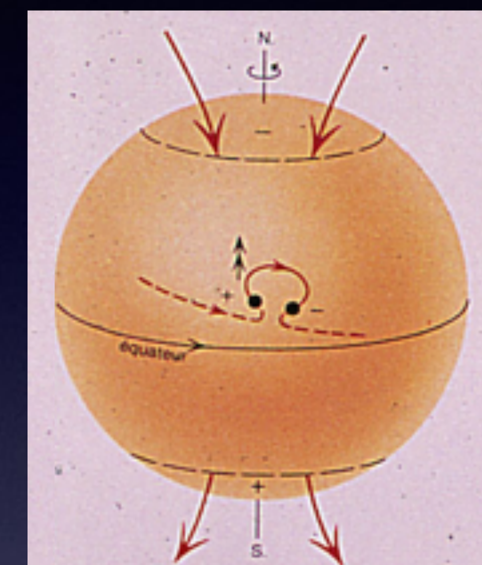
- Effect of a localized cyclonic swirl on a straight magnetic field



[Parker, ApJ 1955]



[Moffatt, Les Houches lectures 1973]



  - In polar geometry, this mechanism can produce axisymmetric poloidal field out of axisymmetric toroidal field — and the converse
    - Kinetic helicity in the swirl is essential

- This “alpha effect” can mediate statistical dynamo action
  - Ensemble of turbulent helical swirls should have a net effect of this kind
  - Cowling’s theorem does not apply as each swirl is localized (“non-axisymmetric”)

Mean-field approach

- Incompressible, kinematic problem with uniform diffusivity

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\Delta\mathbf{B}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

- Split fields into large-scale $(\ell > \ell_0)$ and fluctuating part $(\ell < \ell_0)$

$$\mathbf{B} = \overline{\mathbf{B}} + \tilde{\mathbf{B}} \qquad \mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}}$$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{B}} = \overline{\mathbf{B}}\cdot\nabla\overline{\mathbf{u}} + \nabla\times\left(\overline{\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}}\right) + \eta\Delta\overline{\mathbf{B}}$$

- To determine the evolution of $\overline{\mathbf{B}}$ we need to know $\overline{\mathcal{E}} = \overline{\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}}$
  - We cannot just sweep fluctuations under the rug: closure problem

[Any good review covers this, see references]

Mean-field approach

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times \left[(\tilde{\mathbf{u}} \times \overline{\mathbf{B}}) + \left(\overline{\mathbf{u}} \times \tilde{\mathbf{B}}\right) + \underline{\left(\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}\right) - \left(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}}\right)} \right] + \eta \Delta \tilde{\mathbf{B}}$$

Tangling/shearing of mean field

Tricky bit — closure problem !
[also known as the "pain in the neck" term]

- Assume linear relation between $\tilde{\mathbf{B}}$ and $\overline{\mathbf{B}}$
  - Expand $(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}})_i = \alpha_{ij} \overline{\mathbf{B}}_j + \beta_{ijk} \nabla_k \overline{\mathbf{B}}_j + \cdots$
  - Simplest pseudo-isotropic case: $\alpha_{ij} = \alpha \delta_{ij}, \;\; \beta_{ijk} = \beta \epsilon_{ijk}$

[Warning: hard to justify if there is small-scale dynamo !]

- For $\overline{u} = 0$, we obtain a closed "α^2" dynamo equation $(\eta \ll \beta)$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\alpha \overline{\mathbf{B}}\right) + \beta \Delta \overline{\mathbf{B}}$$

alpha effect   beta effect ("turbulent" diffusion)

  - Exponentially growing solutions with real eigenvalues $\gamma = |\alpha| k - \beta k^2$
  - Max growth rate $\gamma_{\max} = \alpha^2/(4\beta)$ at scale $\ell_{\max} = 2\beta/\alpha \gg \ell_0$

Calculation of mean-field coefficients

- We only know how to calculate α and β perturbatively for
  - small correlation times (low Strouhal number $\tau_c/\tau_{\rm NL}$, random waves)
  - low magnetic Reynolds number $\mathrm{Rm} \sim \tau_\eta/\tau_{\rm NL} \ll 1$

$$(\overline{\mathbf{u}} \doteq 0)$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times \left[(\tilde{\mathbf{u}} \times \overline{\mathbf{B}}) + \left(\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}\right) - \left(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}}\right) \right] + \eta \Delta \tilde{\mathbf{B}}$$

$O(\tilde{B}_{\rm rms}/\tau_c)$ $\quad O(\overline{B}/\tau_{\rm NL})$ $\quad \overline{\qquad} \; O(\tilde{B}_{\rm rms}/\tau_{\rm NL})$ $\quad O(\tilde{B}_{\rm rms}/\tau_\eta)$

$\tau_{\rm NL} = \ell_u/u_{\rm rms}$ tricky "pain in the neck" term G $\tau_\eta = \ell_u^2/\eta$

- In both cases we can justify neglecting the tricky term
  - First Order Smoothing Approximation (FOSA, SOCA, Born, quasilinear…)

[Steenbeck et al., Astr. Nach. 1966; see H. K. Moffatt's textbook, CUP 1978;
Brandenburg & Subramanian, Phys. Rep. 2005]

Calculation of mean-field coefficients

- Let's see how the calculation for $\tau_c/\tau_{\rm NL} \ll 1$
  - Neglecting the tricky term and assuming small resistivity,

$$\begin{aligned}\overline{\tilde{\mathbf{u}}(t) \times \tilde{\mathbf{B}}(t)} &= \overline{\tilde{\mathbf{u}}(t) \times \int_0^t \nabla \times \left[\tilde{\mathbf{u}}(t') \times \overline{\mathbf{B}}(t')\right] dt'} \\ &= \int_0^t \left[\hat{\alpha}(t-t')\overline{\mathbf{B}(t')} - \hat{\beta}(t-t')\nabla \times \overline{\mathbf{B}}\right] dt' \qquad \text{(isotropic case)}\end{aligned}$$

$$\hat{\alpha} = \frac{1}{3}\overline{\tilde{\mathbf{u}}(t)\cdot\tilde{\boldsymbol{\omega}}(t')} \qquad \hat{\beta} = \frac{1}{3}\overline{\tilde{\mathbf{u}}(t)\cdot\tilde{\mathbf{u}}(t')} \qquad \tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\mathbf{u}}$$

  - For slowly varying $\overline{\mathbf{B}}$ and short-correlated velocities, this simplifies as

$$\overline{\tilde{\mathbf{u}}(t) \times \tilde{\mathbf{B}}(t)} = \alpha\overline{\mathbf{B}} - \beta\nabla \times \overline{\mathbf{B}}$$

$$\alpha \simeq -\frac{1}{3}\tau_c\overline{(\tilde{\mathbf{u}}\cdot\tilde{\boldsymbol{\omega}})} \qquad \beta \simeq \frac{1}{3}\tau_c\overline{\tilde{\mathbf{u}}^2}$$

  - The role of kinetic helicity is explicit
- At low Rm, we have the similar result $\alpha \simeq -\frac{1}{3}\tau_\eta\overline{(\tilde{\mathbf{u}}\cdot\tilde{\boldsymbol{\omega}})}$

Dynamical regime of large-scale dynamos

- When B gets "large enough", the Lorentz force back-reacts
  - Big questions: what happens then, and what is "large-enough" ?
    [Brandenburg & Subramanian, Phys. Rep. 2005, and refs. therein: Proctor, 2003; Diamond et al. 2005]
- Equipartition argument: saturation when $\overline{\mathbf{B}}^2 \sim 4\pi\overline{\rho\,\tilde{\mathbf{u}}^2} \equiv B_{\mathrm{eq}}^2$, but
  - $\overline{\mathbf{B}}$ and $\tilde{\mathbf{u}}$ have very different scales
  - Large-scale dynamos alone produce plenty of small-scale field
- Equipartition of small-scale fields: $\overline{\tilde{\mathbf{b}}^2} \sim B_{\mathrm{eq}}^2$, with $\overline{\tilde{b}^2} \sim \mathrm{Rm}^p\overline{B}^2$
  - Not very astro-friendly: $\overline{\mathbf{B}}^2 \sim B_{\mathrm{eq}}^2/\mathrm{Rm}^p \ll B_{\mathrm{eq}}^2$ for *p=O(1)*
  - Possibility of "catastrophic" alpha quenching

$$\alpha(\overline{\mathbf{B}}) = \frac{\alpha_0}{1+\mathrm{Rm}^q\,(\overline{\mathbf{B}}^2/B_{\mathrm{eq}}^2)} \qquad\qquad q = O(1)$$

Quenching issue

- Physical origin of quenching debated:
  - Magnetized fluid has "memory": possible drastic reduction of statistical effects compared to random walk estimates [see review by Diamond et al., 2005]
  - Magnetic helicity conservation argument:
    - in "closed" systems, large-scale field can only reach equipartition on slow, large-scale resistive timescales [e.g. Brandenburg, ApJ 2001]
- Possible way out of problem is to evacuate magnetic helicity [Blackman & Field, ApJ 2000; see discussion by Brandenburg, Space Sci. Rev. (2009)]

$$\frac{d}{dt}\langle \mathbf{A}\cdot\mathbf{B}\rangle_V = -2\eta\,\langle\,(\nabla\times\mathbf{B})\cdot\mathbf{B}\rangle_V - \langle\nabla\cdot\mathbf{F}_{\mathcal{H}_m}\rangle$$

  - Requires open boundary conditions (periodic b.c. not ok)
  - Requires internal fluxes of helicity [Kleeorin et al., Vishniac-Cho etc.]

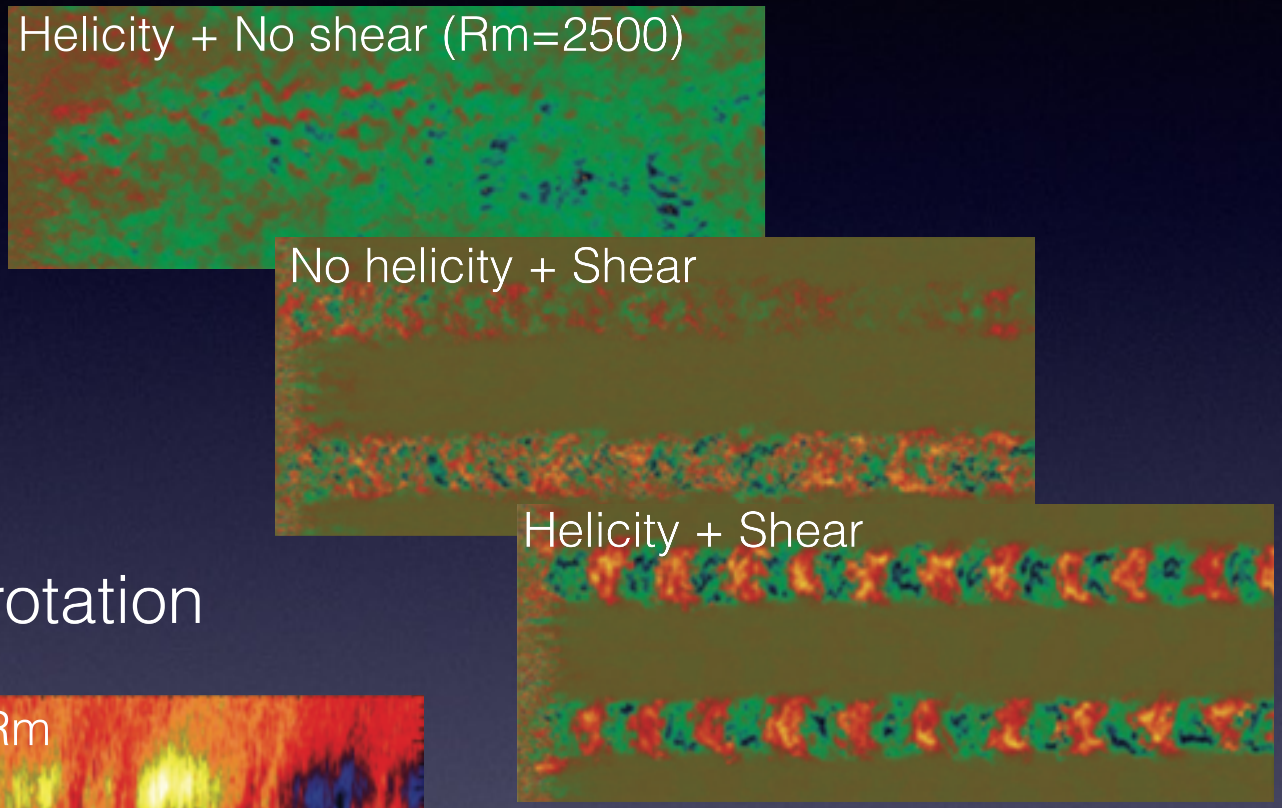
Remarks

- Historically, mean-field models have been at the core of modelling of
  - solar and stellar dynamos — “alpha” provided by cyclonic convection
  - galactic dynamos — “alpha” provided by supernova explosions
- But classical mean-field theory faces strong limitations
  - Astro turbulence typically has $\tau_c/\tau_{\mathrm{NL}} \sim 1$ and $\mathrm{Rm} \gg 1$
  - “Co-existence” with fast, small-scale dynamo for $\mathrm{Rm} \gg 1$
    - pain in the neck term exponentially growing…then what ?
    - linear relation between $\tilde{\mathbf{b}}$ and $\overline{\mathbf{B}}$ doubtful
  - Quenching problem
- Large-scale dynamos are “real” — independently of our limited theories
  - We have to think harder ! (and ask good questions to computers)

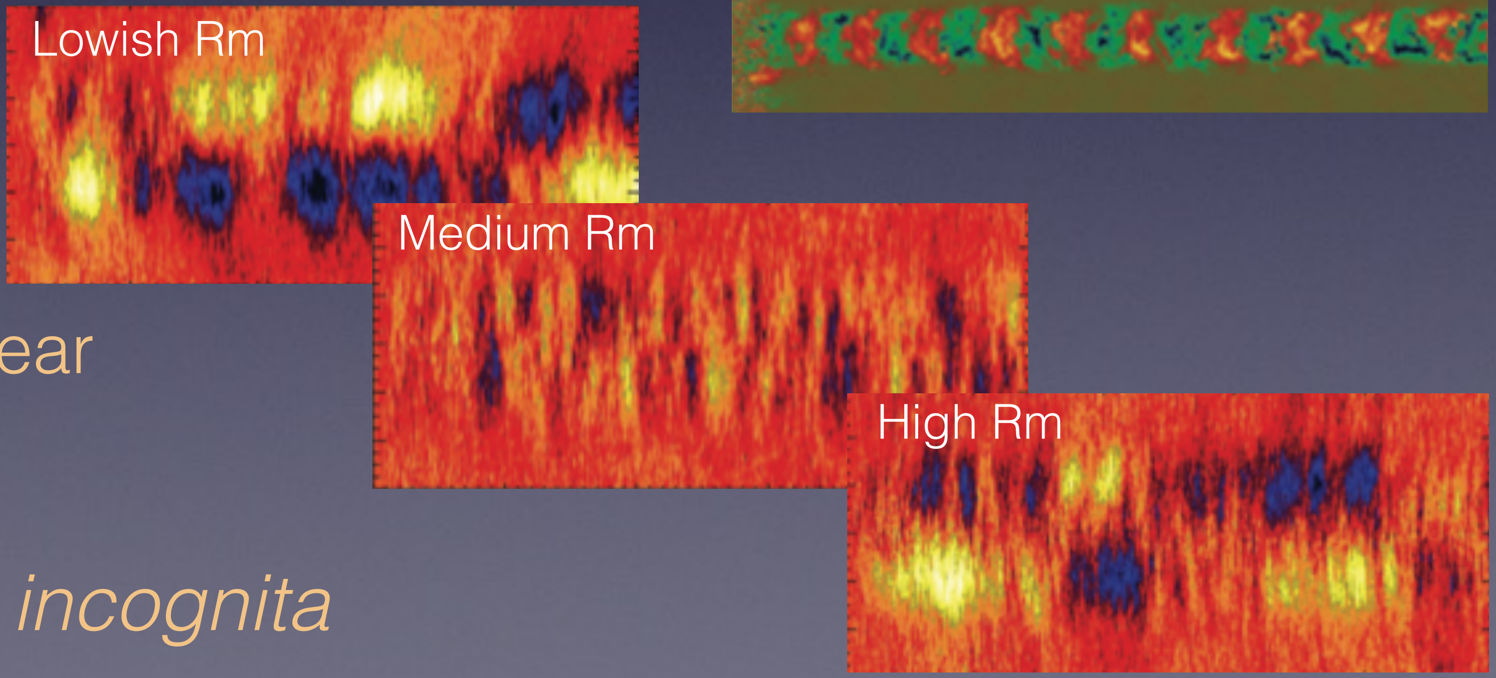
Large-scale meets small-scale and instabilities

Order out of chaos ?

- Large-scale dynamos at largish Rm now observed numerically
  - Galloway-Proctor flow + Shear
    [Tobias & Cattaneo, Nature 2013]
    - "Suppression" principle: shear turns off small-scale dynamo ?



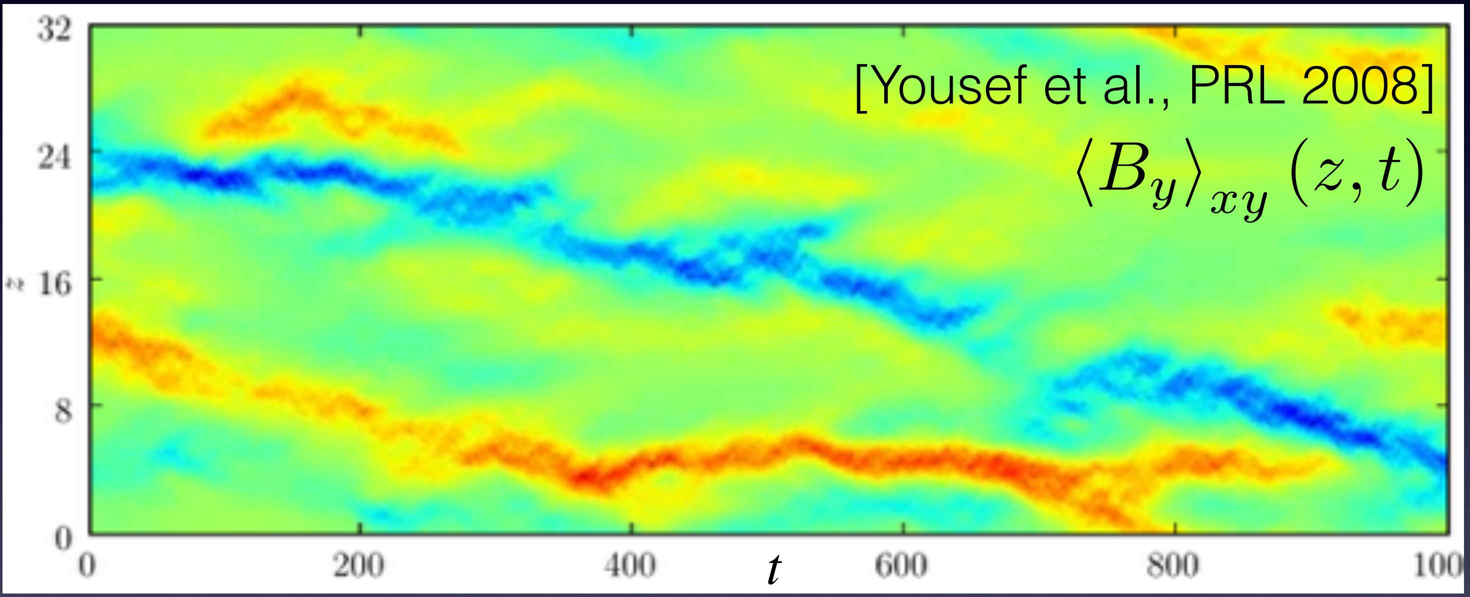
  - Turbulent convection + differential rotation
    [Hotta et al., Science 2016]
    - Small-scale dynamo reduces turb. diffusion ?



  - Asymptotic behaviour unclear
- Dynamical theory still *terra incognita*

Other (lack of) twists

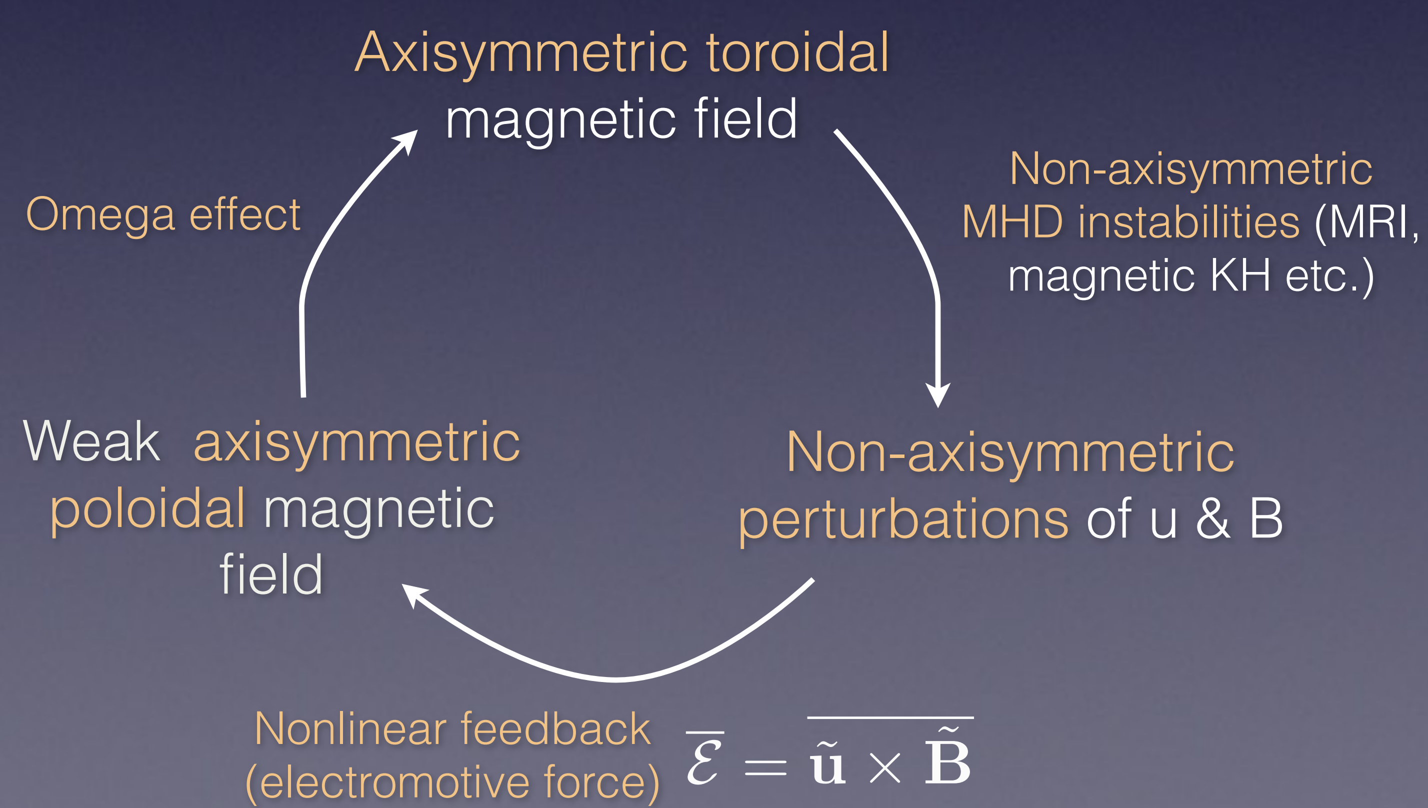
- Large-scale dynamo action is possible without net helicity
  - The shear dynamo: $\mathbf{u} = Sx\mathbf{e}_y +$ non-helical small-scale turbulence



  - Mean-field description in terms of “WxJ” effect [Kleeorin & Rogachevskii]
  - “Incoherent” alpha effect [Silant’ev 2007, Proctor 2007, Brandenburg 2008], etc.
- Recent developments [Squire & Battacharjee, PRL 2015]
  - Saturated small-scale dynamo in a shear flow can lead to large-scale dynamo

Instability-driven dynamos

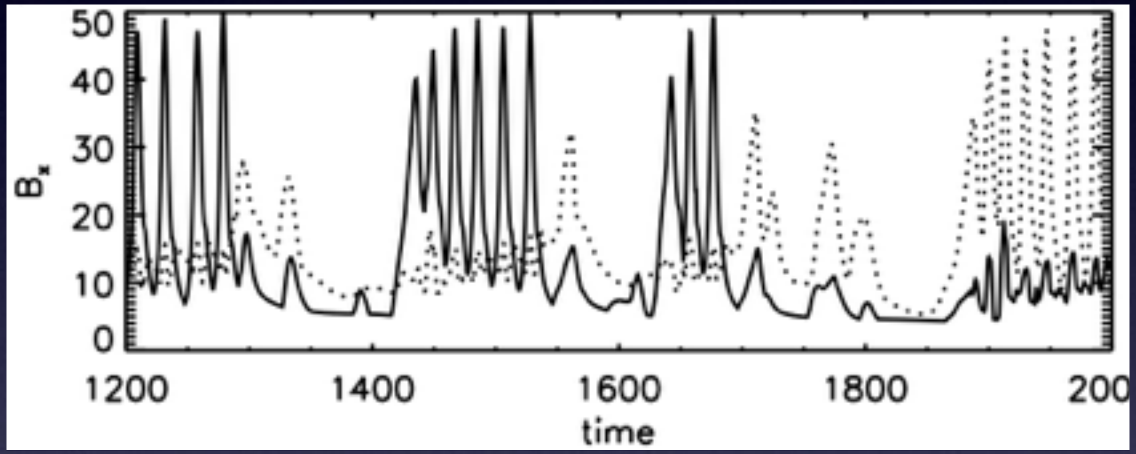
- Many astrophysical systems
  - host differential rotation: i.e. there is a background shear flow
  - are prone to non-axisymmetric MHD instabilities
- This can lead to specific nonlinear forms of dynamo action
  - Analogous to self-sustaining nonlinear process in hydro shear flows



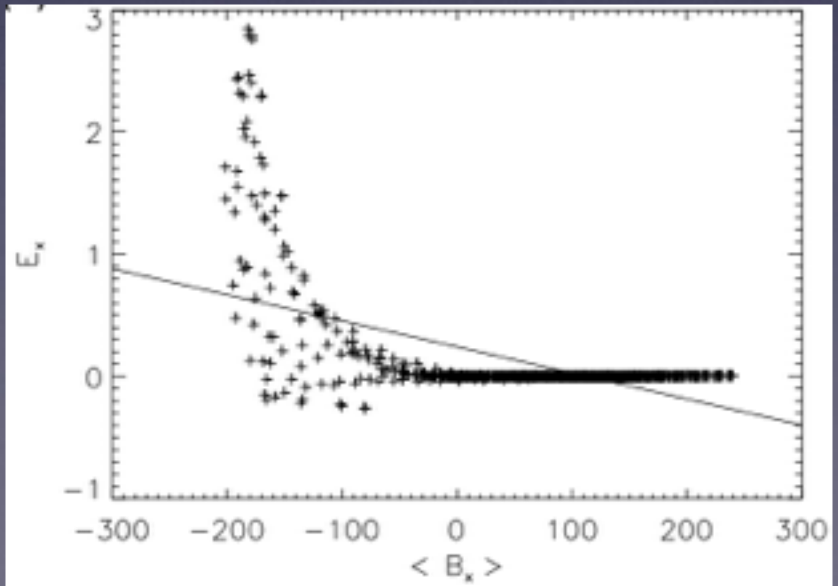
[Rincon et al., PRL 2007;
Astron. Nachr. 2008;
Riols et al., JFM 2013]

"Solar-like" magnetic buoyancy dynamo

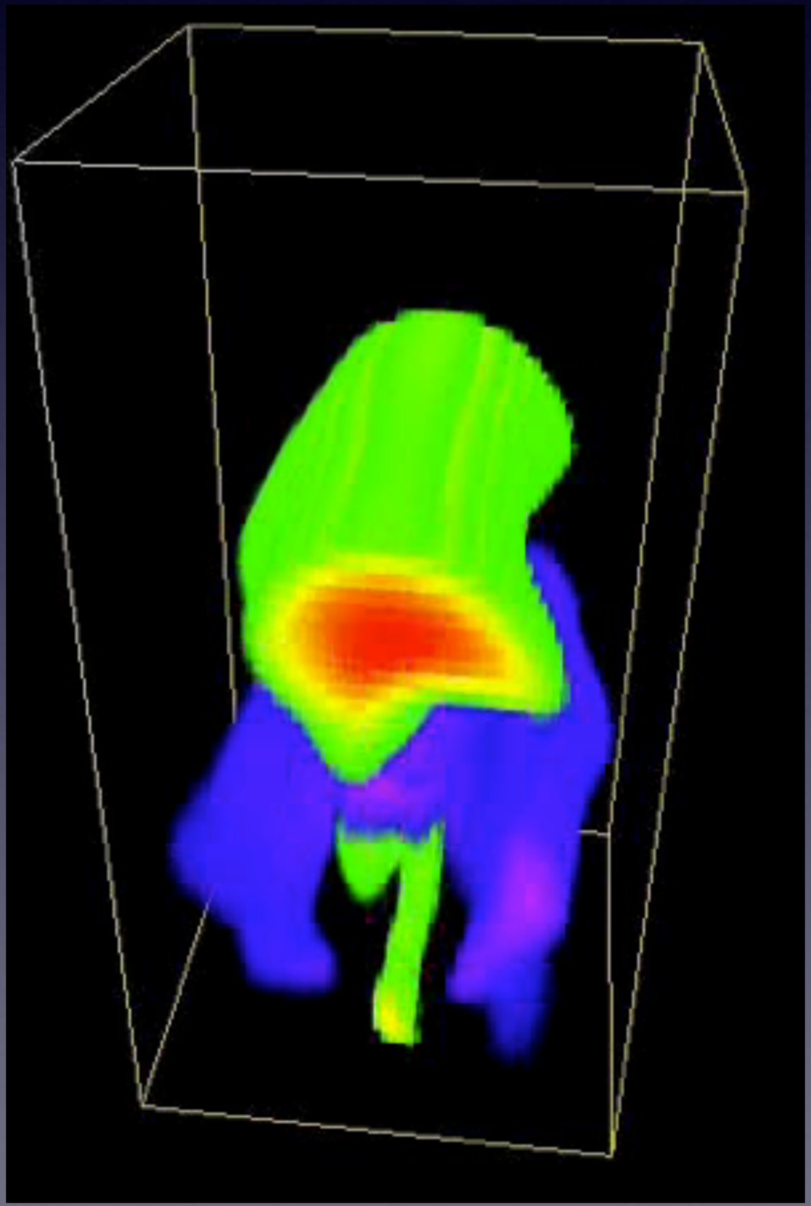
- Shear + Magnetic buoyancy + Kelvin-Helmholtz
  - Coherent, strongly chaotic dynamo action



  - Strongly nonlinear EMF / field relationship

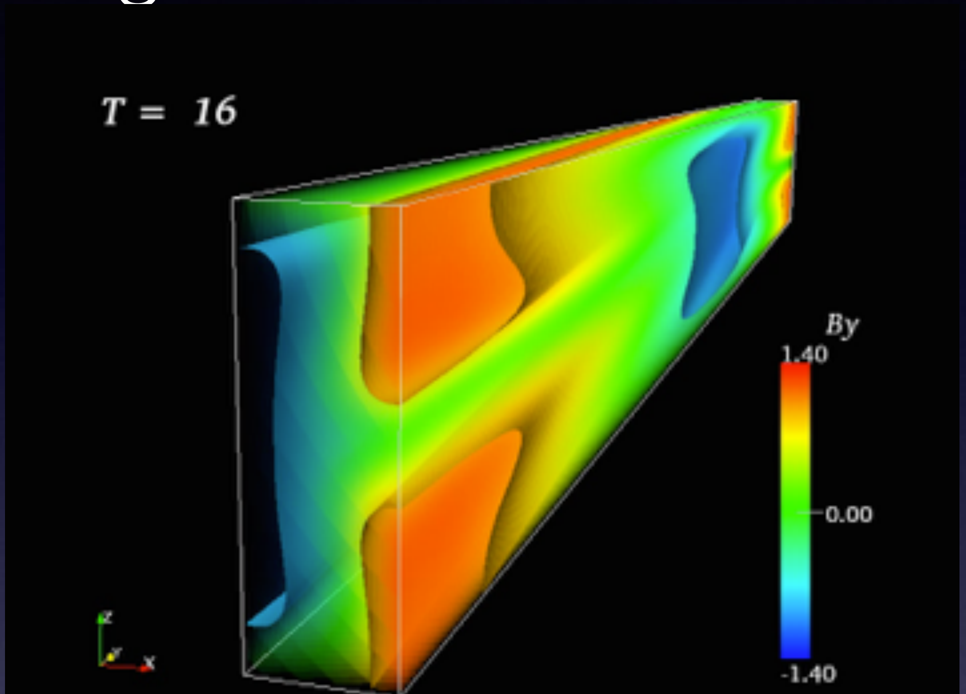


[Cline et al., ApJ 2003]



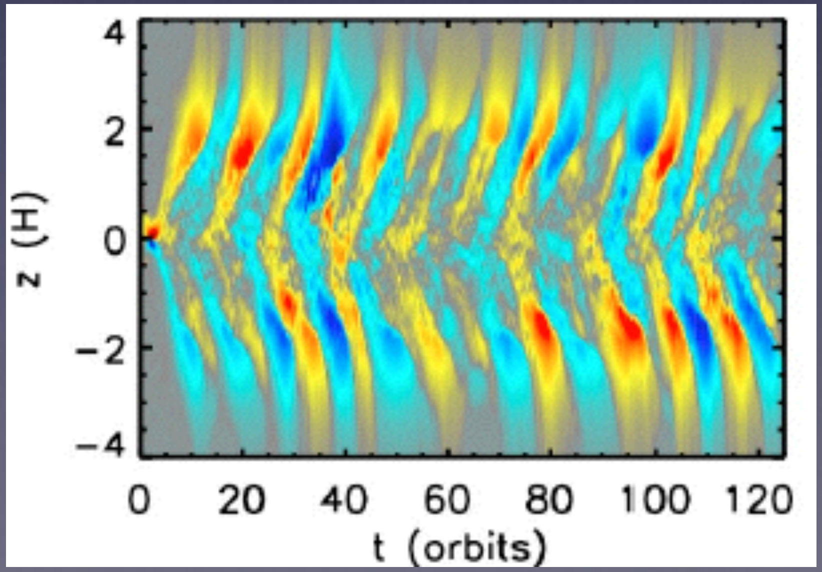
Accretion disk dynamo

- Keplerian shear flow turbulence is thought to be MRI-driven
  - Possible even in the absence of net magnetic flux [Hawley et al., ApJ 1996]

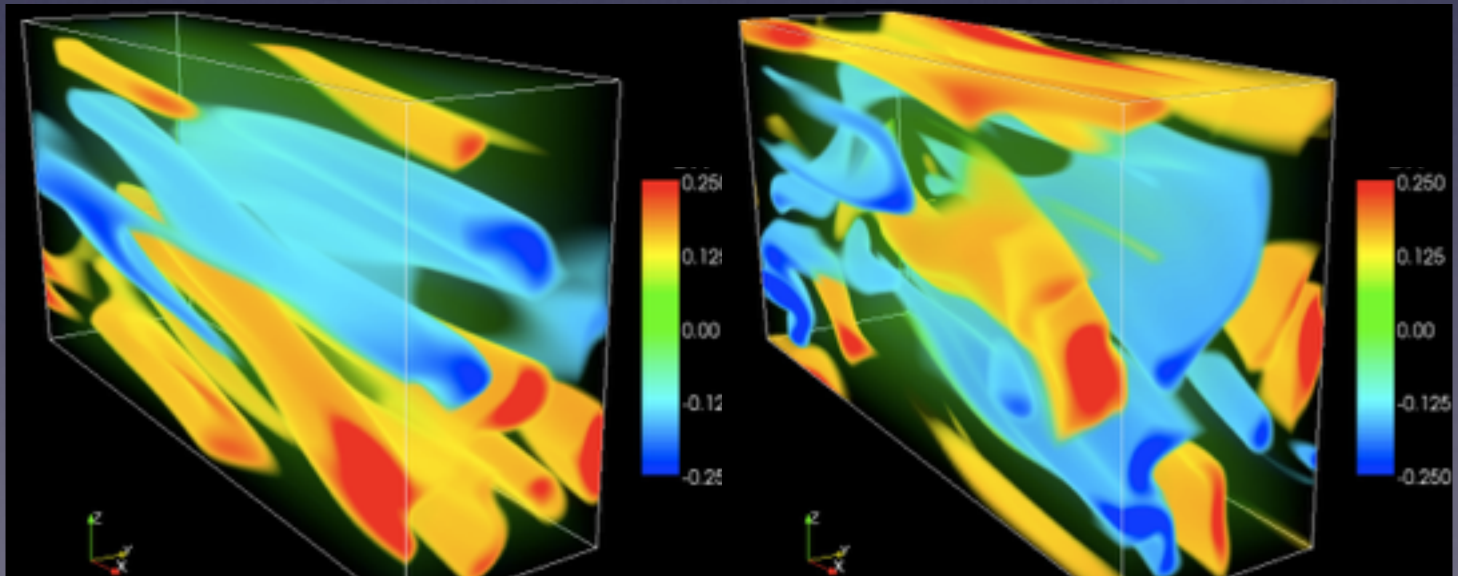


[Herault et al., PRE 2011]

- Characterised by dynamical reversals of large-scale field
  - Non-axisymmetric MRI of toroidal field critical (magnetic buoyancy)



[Davis et al., ApJ 2010]

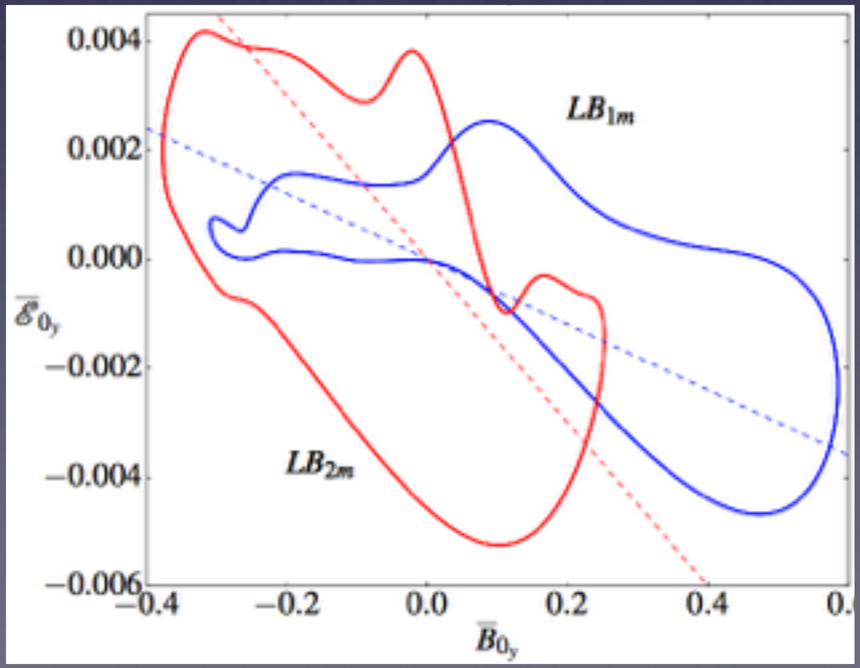


[Lesur & Ogilvie, A&A 2008]

From subcritical to statistical

- Such dynamos are subcritical / essentially nonlinear
  - "Egg and chicken" problem
    - Non-axisymmetric instability growth requires large-scale field
    - Large-scale field sustainement rests on non-axisymmetric instability
  - Non-axisymmetric $\tilde{\mathbf{u}}, \tilde{\mathbf{B}}$ jointly excited by instability: Lorentz force essential

[Riols et al., A&A 2017]



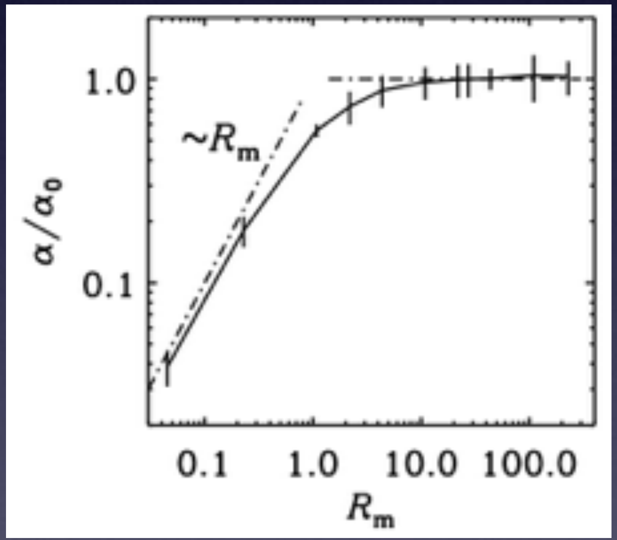
- Implications
  - No kinematic stage, homoclinic bifurcations
  - Nonlinear EMF/field relationship
- Statistical theory relevant but difficult
  - Mean-field approach controversial

Different ideas

“Test field”-like methods

- Pragmatic strategies have been devised for “astrophysical applications”
  - postulate generalised mean-field form for $\overline{\boldsymbol{\mathcal{E}}}(\overline{\mathbf{B}})$ (convolution integrals)
  - Measure effective transport coefficients in local simulations
  - Use the results in simpler 2D mean-field models

[Sur et al., MNRAS 2008, Brandenburg, Space Sci. Rev. 2009]



- Such procedures
  - produce converged values of transport coefficients
  - reproduce exact results in perturbative kinematic limits

- TFM-based modelling may be useful, but:
  - no rigorous justification as to why it should be accurate/appropriate (Rm>>1 !)
  - dynamical, tensorial convolution relations $\overline{\boldsymbol{\mathcal{E}}}(\overline{\mathbf{B}})$ can fit complex dynamics, but could well be degenerate with more physically-grounded nonlinear models
  - it can obfuscate the underlying physics, e.g. when MHD instabilities are involved

Kazantsev approaches

- Fokker-Planck equation for the pdf for basic Kazantsev

$$\frac{\partial}{\partial t} P\left[\mathbf{B}\right] = \frac{\kappa_2}{2} T^{ij}_{k\ell} B^k \frac{\partial}{\partial B^i} B^\ell \frac{\partial}{\partial B^j} P\left[\mathbf{B}\right]$$

Strain correlator [3D, incompressible]

$$P\left[\mathbf{B}\right] = P\left[B\right] G[\hat{\mathbf{b}}]$$

$$T^{ij}_{k\ell} = -\frac{1}{\kappa_2} \frac{\partial^2 \kappa^{ij}(\mathbf{r})}{\partial r^k \partial r^\ell} = \delta^{ij} \delta_{k\ell} - \frac{1}{4} \left(\delta^i_k \delta^j_\ell + \delta^i_\ell \delta^j_k \right)$$

- Amplitude pdf: $\frac{\partial}{\partial t} P\left[B\right] = \frac{\kappa_2}{4} \frac{1}{B^2} \frac{\partial}{\partial B} B^4 \frac{\partial}{\partial B} P\left[B\right]$

$$P[B](t) = \frac{1}{\sqrt{\pi \kappa_2 t}} \int_0^\infty \frac{dB'}{B'} P_0\left[B'\right] \exp\left(- \frac{\left[\ln(B/B') + (3/4)\kappa_2 t\right]^2}{\kappa_2 t} \right)$$

Log-normal

$$\langle B \rangle = B_0 \exp\left[2\kappa_2 t\right]$$

- Orientation pdf: $G[\hat{\mathbf{b}}] = 1 + \overline{B}^i \hat{\mathbf{b}}^i \exp\left[-2\kappa_2 t\right]$ w. $\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\alpha \overline{\mathbf{B}}\right) + \beta \Delta \overline{\mathbf{B}}$
- Overall vector mean-field follows: $\langle B^i \rangle = \overline{B}^i B_0 / d$

Boldyrev's large Pm extension of Kazantsev

- Add "viscously" saturated component to velocity field

$$u^i_k = -\frac{1}{\nu}\left(B^i B^k - \frac{1}{3}\delta^{ik} B^2\right) + \tilde{u}^i_k$$

Kazantsev velocity field

- Extra-term in the amplitude pdf equation

$$\frac{\partial}{\partial t} P[B] = \frac{\kappa_2}{4}\frac{1}{B^2}\frac{\partial}{\partial B} B^4 \frac{\partial}{\partial B} P[B] + \frac{2}{3\nu}\frac{1}{B^2}\frac{\partial}{\partial B} B^5 P[B]$$

  - Amplitude pdf is now a steady Gaussian

- Isotropization not compensated by growth of amplitude
  - Saturation of mean-field as soon as small-scale field saturates
  - Kazantsev approach to alpha quenching

Further ideas on nonlinear theory

- Relaxation model [Schekochihin et al., ApJ 2002]

$$\frac{d\mathbf{B}}{dt} = \mathbf{B}\cdot\nabla\tilde{\mathbf{u}} - \tau_r^{-1}(B)\mathbf{B}$$

  - Subtle dependence of saturated pdf on choice of B in $\tau_r^{-1} \sim \frac{B^2}{\nu}$

- Local anisotropization of velocity field in magnetic folds [Schekochihin et al., PRL 2004]

$$\begin{aligned}\kappa^{ij}(\mathbf{k}) &= \kappa^{(\mathrm{i})}(k,|\mu|)\left(\delta^{ij} - \hat{k}_i\hat{k}_j\right) \\ &+ \kappa^{a}(k,|\mu|)\left(\hat{b}^i\hat{b}^j + \mu^2\hat{k}_i\hat{k}_j - \mu\hat{b}^i\hat{k}_j - \mu\hat{k}_i\hat{b}^j\right)\end{aligned}$$

$$\hat{\mathbf{b}} = \mathbf{B}/B \quad \hat{\mathbf{k}} = \mathbf{k}/k$$

$$\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{b}}$$

  - As yet unexplored in the context of large-scale dynamo growth/saturation

- Variational calculation of non-perturbative instantons

Conclusions

Tomorrow’s fundamental theory challenges

- Turbulent large and small-scale dynamos
  - Unified, self-consistent nonlinear multiscale statistical dynamo theory
    - Requires physically justified closures
  - Description of asymptotic regimes
    - Re, Rm >> 1, Pm << 1, strong rotation etc.

- Interactions of different physical and geometrical effects
  - MHD instabilities combined to shear (magnetic buoyancy, MRI etc.)
  - Coherent structures (vortices, zonal flows, tangent cylinders etc.)
  - Plasma effects (batteries, pressure anisotropies, multi-fluid etc.)
  - Reconnexion