Bridging the Gap Between Gyrokinetics and Gyrofluids: A Hermite-Laguerre Spectral Velocity Formulation of Gyrokinetics

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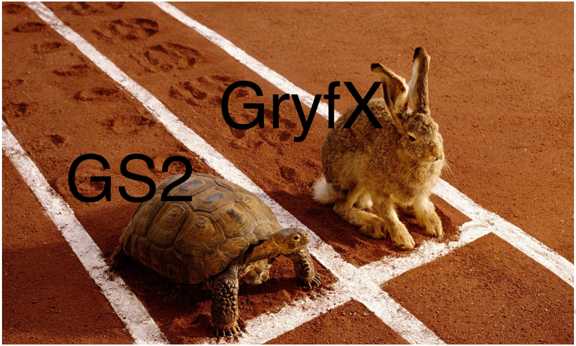
Motivation

Gyrokinetics vs Gyrofluids

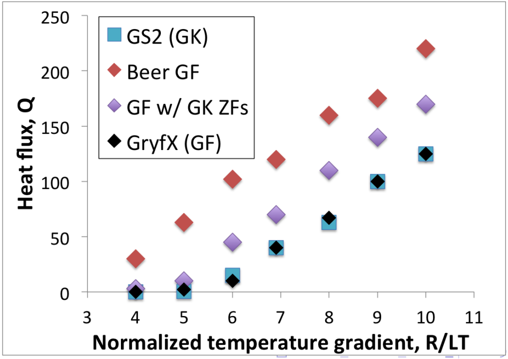
- Gyrofluid models are efficient, but have to work very hard (*read: most of* **my** *academic career thus far*) to get good fidelity
- Gyrokinetics is accurate, but have to wait a long time (*read: most of* **your** *academic careers*) to get results



Motivation: GryfX results

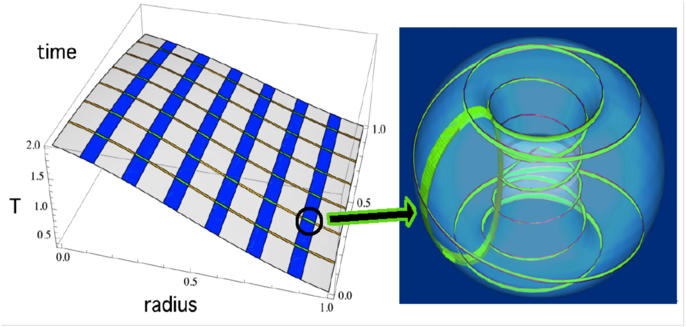
- Improvements to Beer 4+2 GF model: GryfX, a GPU-based gyrofluid code with gyrokinetic zonal flows
- Nonlinear GryfX simulations produce heat fluxes that agree with the gyrokinetic code GS2
    - Zonal flow improvements important at low R/L_T, produce desired Dimits shift at $R/L_T < 6$
    - Nonlinear phase mixing important at larger R/L_T

GryfX simulations:
$\sim$1 hour on 1 GPU + 16 CPU cores
GS2 simulations:
$\sim$1000 core hours



Motivation

- A goal: Use flux-tube turbulence calculations in multi-scale simulations like Trinity
- GS2 is too expensive, GryfX may not be accurate enough
- Want to be able to flexibly interpolate between GK and GF



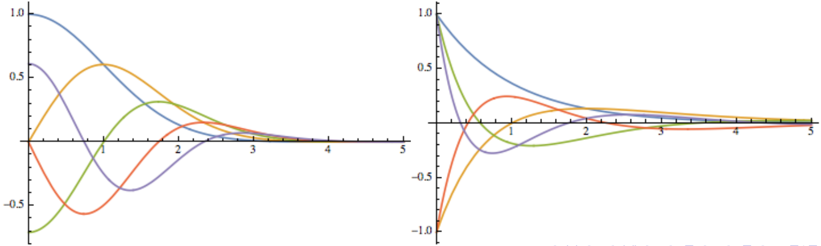
Hermite-Laguerre Formulation

- GK equation in $(v_\|, \mu)$ coordinates:

$$\frac{\partial g}{\partial t} + v_{ts} v_\| \hat{\mathbf{b}} \cdot \nabla g + (J_0 \mathbf{v}_E) \cdot \nabla (F_0 + g) + \mathbf{v}_d \cdot \nabla g + \frac{Z_s}{\tau_s} v_{ts} v_\| \hat{\mathbf{b}} \cdot \nabla (J_0 \Phi) F_0$$
$$- v_{ts} \mu \hat{\mathbf{b}} \cdot \nabla B \frac{\partial g}{\partial v_\|} - v_\|^2 (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot (J_0 \mathbf{v}_E) F_0 = C(h)$$

- Expand g in Maxwellian-weighted Hermite-Laguerre basis:

$$g = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{\ell!}} \mathrm{He}_\ell(v_\|) \mathrm{L}_m(\mu B) F_0 G_{\ell,m} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \phi^\ell \psi^m G_{\ell,m}$$



Hermite-Laguerre Formulation

- Moments defined by:

$$G_{\ell,m} = \int d^3v \, \frac{(-1)^m}{\sqrt{\ell!}} \mathrm{He}_\ell(v_\parallel) \mathrm{L}_m(\mu B) \; g = \int d^3v \; \phi_\ell \psi_m \; g$$

- Direct relation to gyrofluid moments:

$$\left(G_{0,0}, \;\; G_{1,0}, \;\; \sqrt{2} G_{2,0}, \;\; \sqrt{6} G_{3,0}, \;\; G_{0,1}, \;\; G_{1,1}\right) = \left(n, \;\; u_\parallel, \;\; T_\parallel, \;\; q_\parallel, \;\; T_\perp, \;\; q_\perp\right)$$

Hermite-Laguerre Formulation

- FLR accuracy is tied to Laguerre resolution:

$$J_0(\sqrt{2\mu Bb}) = \sum_{m=0}^{\infty} \psi_m \frac{1}{m!}\left(-\frac{b}{2}\right)^m e^{-b/2} \equiv \sum_{m=0}^{\infty} \psi_m \mathcal{J}_m(b)$$

$$\text{with} \quad \mathcal{J}_m \equiv \frac{1}{m!}\left(-\frac{b}{2}\right)^m e^{-b/2}$$

- Important for real space density (appears in quasineutrality)

$$\bar{n} = \int d^3v J_0 g = \sum_{m=0}^{\infty} \mathcal{J}_m G_{0,m}$$

Hermite-Laguerre Formulation

- General moment equation:

$$
\begin{aligned}
\frac{dG_{\ell,m}}{dt} &+ v_{ts}\nabla_{\parallel}\left(\sqrt{\ell+1}\,G_{\ell+1,m}+\sqrt{\ell}\,G_{\ell-1,m}\right)\\
&+ v_{ts}\Big[-\sqrt{\ell+1}\,(m+1)\,G_{\ell+1,m}-\sqrt{\ell+1}\,m\,G_{\ell+1,m-1}\\
&\quad+\sqrt{\ell}\,m\,G_{\ell-1,m}+\sqrt{\ell}\,(m+1)\,G_{\ell-1,m+1}\Big]\nabla_{\parallel}\ln B\\
&+ i\omega_d\Big[\sqrt{(\ell+1)(\ell+2)}\,G_{\ell+2,m}+(m+1)\,G_{\ell,m+1}\\
&\quad+2\,(\ell+m+1)\,G_{\ell,m}+\sqrt{\ell(\ell-1)}\,G_{\ell-2,m}+m\,G_{\ell,m-1}\Big]=C(H_{\ell,m})
\end{aligned}
$$

- Note: $\ell = 0, 1, 2$ equations have sources $\propto \Phi$ from gradients and parallel electric field
- Coupling to higher moments presents closure problem

Hermite-Laguerre Formulation

- Nonlinear term couples all Laguerre moments in convolution:

$$\frac{dG_{\ell,m}}{dt} \equiv \frac{\partial G_{\ell,m}}{\partial t} + \sum_{k=0}^{\infty} \sum_{n=|k-m|}^{k+m} C_{kmn} \left(\mathcal{J}_n \mathbf{v}_E \right) \cdot \nabla G_{\ell,k}$$

$$C_{kmn} = \int_0^{\infty} d\mu B \ \psi_k \psi^m \psi_n$$

- Manifestation of **nonlinear phase mixing**
- Can be seen in 4+2 GF equations, but only 2 Laguerre moments there

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \mathbf{v}_\Psi \cdot \nabla n + \left[\frac{1}{2} \hat{\nabla}_\perp^2 \mathbf{v}_\Psi \right] \cdot \nabla T_\perp + NLPM$$

Collision Operator

- Model collision operator

$$C(h) = \nu \Bigg\{ \left[\frac{\partial}{\partial v_\parallel}\left(\frac{\partial}{\partial v_\parallel} + v_\parallel\right) + 2\frac{\partial}{\partial \mu}\left(\frac{\mu}{B}\frac{\partial}{\partial \mu} + \mu\right) - k_\perp^2 \rho^2 \right] h$$

$$+ \left[J_0 \delta \bar{T} \left[(v_\parallel^2 - 1) + 2(\mu B - 1) \right] + \bar{\mathbf{u}} \cdot \mathbf{v} \right] F_0 \Bigg\}$$

$$= \nu \left[\frac{1}{v^2}\frac{\partial}{\partial \xi}\left((1-\xi^2)\frac{\partial}{\partial \xi} \right) + \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 \frac{\partial}{\partial v} + v^3 \right) - k_\perp^2 \rho^2 \right] h + \ldots$$

- $C(F_0) = 0$ ✓
- Conserves number, momentum, energy ✓
- Pitch angle scattering ✓
- H-theorem ✓
- Self adjoint ✓
- $\frac{dW}{dt} \leq 0$ ✓

Collision Operator

- Eigenfunctions are Hermite and Laguerre polynomials ✓

$$\begin{aligned}
\frac{dG_{0,m}}{dt} + \cdots = \quad & -\nu\,(b+2m)\,H_{0,m} + \nu\sqrt{b}\,(\mathcal{J}_m + \mathcal{J}_{m-1})\,\bar{u}_\perp \\
& + \nu\,\frac{2}{3}\,[m\mathcal{J}_{m-1} + 2m\mathcal{J}_m + (m+1)\mathcal{J}_{m+1}]\,\big(\bar{T}_\parallel + 2\,\bar{T}_\perp\big) \\
\vdots \quad & \\
\frac{dG_{\ell,m}}{dt} + \cdots = \quad & -\nu(b+\ell+2m)\;H_{\ell,m} \qquad\qquad (\ell > 2)
\end{aligned}$$

Free Energy

- Hermite-Laguerre projected free energy evolution:

$$\frac{\partial W_s}{\partial t} = \tau_s \int d^3\mathbf{r} \left[\sum_{\ell=0}^{L} \sum_{m=0}^{M} G_{\ell,m} \frac{\partial G_{\ell,m}}{\partial t} + \frac{Z_s}{\tau_s} \sum_{m=0}^{M} \mathcal{J}_m \Phi \frac{\partial G_{0,m}}{\partial t} \right]$$
$$= \tau_s \int d^3\mathbf{r} \left[\sum_{\ell=0}^{L} \sum_{m=0}^{M} H_{\ell,m} \frac{\partial G_{\ell,m}}{\partial t} \right]$$

- In the infinite moment limit (just gyrokinetics), free energy conserved in absence of driving and damping

$$\lim_{L,M\to\infty} \frac{dW}{dt} = \mathcal{D} - \mathcal{C},$$

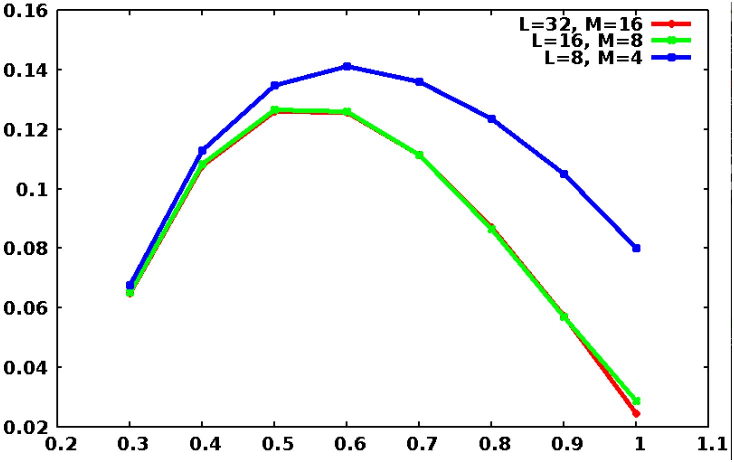
- Also conserved when closing moment series by truncation (i.e. unevolved moments $\to 0$)
- Must require $dW/dt \leq 0$ for any other closure choices

Closures

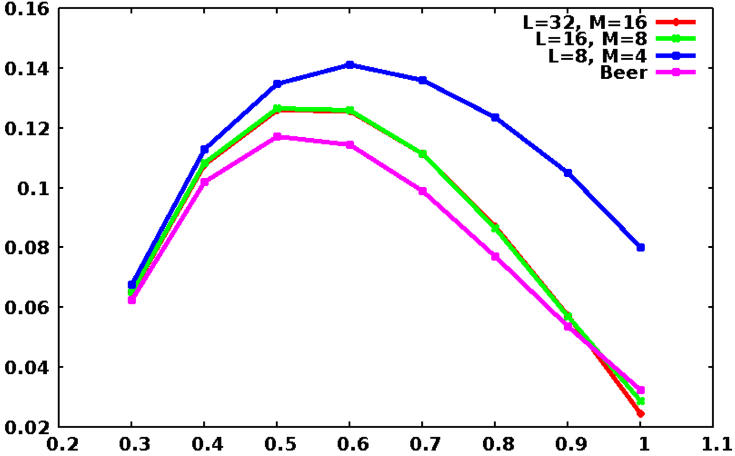
Options:

- Truncation (i.e. set unevolved moments to 0)
  - For given collisionality, there is a cutoff at some L and M
- Truncate + a hypercollisionality model to induce cutoff at lower resolution
  - $\sim -\nu_{hyper}\left[(\ell/L)^{p_\ell} + (m/M)^{p_m}\right]$
- Use generalized gyrofluid-like closures
  - Express unevolved moments in terms of lower, evolved moments by fitting to kinetic dispersion relation
  - Have derived a general closure (following Smith) in perpendicular direction (Laguerre) for phase mixing from ∇B drift
  - Deriving similar closure in parallel direction (Hermite) is complicated by the combination of phase mixing from parallel convection and curvature drift; still some ideas to try here

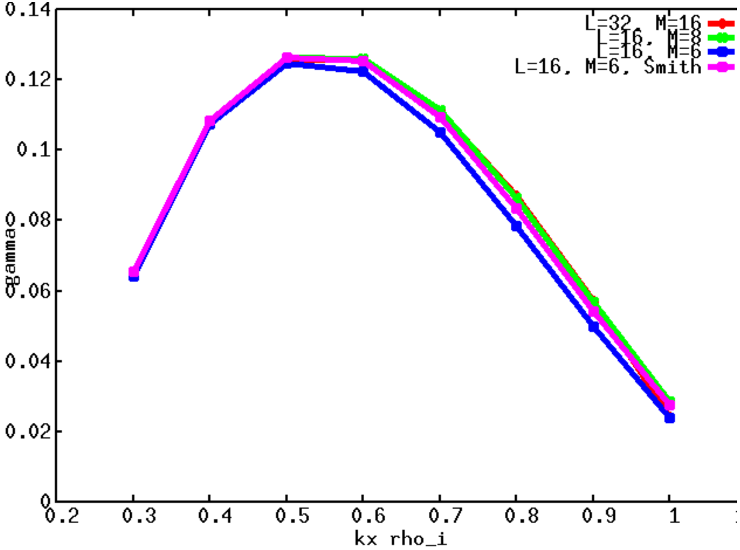
Linear growth rates: convergence



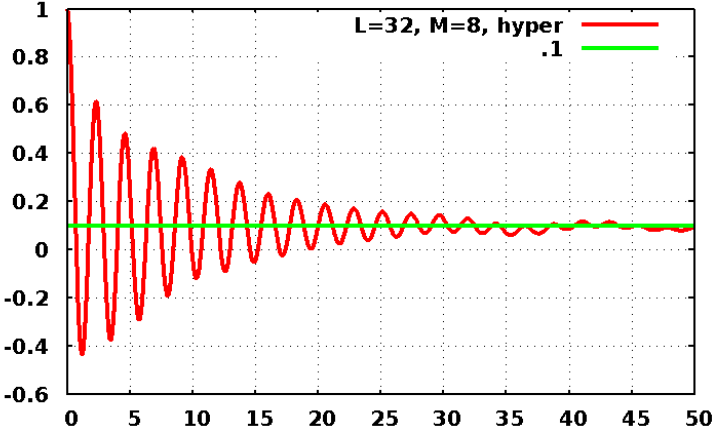
Linear growth rates: convergence



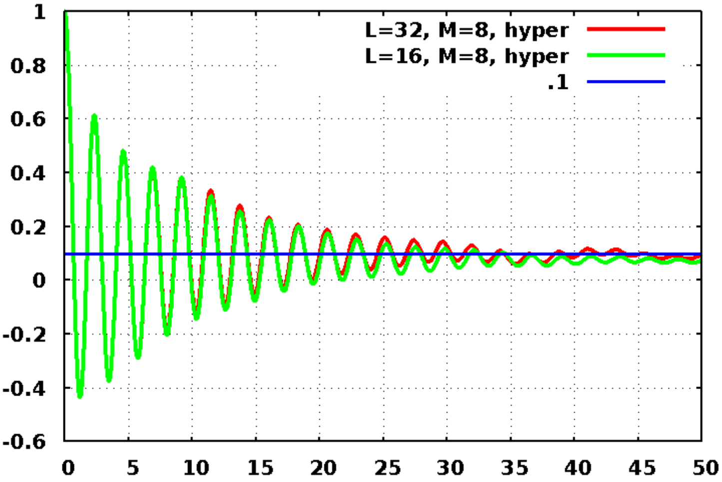
Linear growth rates: Smith perpendicular closure



Rosenbluth-Hinton zonal flow residual



Rosenbluth-Hinton zonal flow residual



Conclusions & Future Work

- Hermite-Laguerre formulation of gyrokinetics gives flexibility between gyrofluids and gyrokinetics
  - Opportunities for dynamic $v-$resolution refinement, adaptive closures, etc.
- New code (GryfX++) solves H-L system, is $\sim 3-4\times$ faster than GryfX, runs on single GPU (for now)
  - Will be able to afford better resolution (x and v) than most GK codes
- Can apply to any flux-tube geometry
  - Already ready to run in general tokamak and stellarator geometries
- Straightforward to extend to electromagnetic (e.g. w/ isothermal electrons)
  - But maybe some additional closure complications

Complete Equations

$$
\begin{aligned}
\frac{dG_{0,m}}{dt} &+ v_{ts}\nabla_{\parallel} G_{1,m} - v_{ts}\left[(m+1)G_{1,m} + mG_{1,m-1}\right]\nabla_{\parallel}\ln B \\
&+ i\omega_d\left[\sqrt{2}G_{2,m} + (m+1)G_{0,m+1} + 2(m+1)G_{0,m} + mG_{0,m-1}\right] \\
&+ i\omega_d\left[m\mathcal{J}_{m-1}\Phi + 2(m+1)\mathcal{J}_m\Phi + (m+1)\mathcal{J}_{m+1}\Phi\right] \\
&= i\omega_*\left[\frac{a}{L_{ns}}\mathcal{J}_m\Phi + \frac{a}{L_{Ts}}\left[m\mathcal{J}_{m-1}\Phi + 2m\mathcal{J}_m\Phi + (m+1)\mathcal{J}_{m+1}\Phi\right]\right] \\
&\quad - \nu\,(b+2m)\,H_{0,m} + \nu\sqrt{b}\,\left(\mathcal{J}_m + \mathcal{J}_{m-1}\right)\bar{u}_{\perp} \\
&\quad + \nu\,\frac{2}{3}\left[m\mathcal{J}_{m-1} + 2m\mathcal{J}_m + (m+1)\mathcal{J}_{m+1}\right]\left(\bar{T}_{\parallel} + 2\bar{T}_{\perp}\right)
\end{aligned}
$$

Complete Equations

$$
\begin{aligned}
\frac{dG_{1,m}}{dt} &+ v_{ts}\nabla_{\parallel}\left(\sqrt{2}\,G_{2,m} + G_{0,m}\right) + v_{ts}\frac{Z_s}{\tau_s}\nabla_{\parallel}\mathcal{J}_m\Phi \\
&+ v_{ts}\left[-\sqrt{2}\,(m+1)\,G_{2,m} - \sqrt{2}\,m\,G_{2,m-1} + m\,G_{0,m} + (m+1)\,G_{0,m+1}\right]\nabla_{\parallel}\ln B \\
&+ v_{ts}\frac{Z_s}{\tau_s}\left[m\,\mathcal{J}_m\Phi + (m+1)\,\mathcal{J}_{m+1}\Phi\right]\nabla_{\parallel}\ln B \\
&+ i\omega_d\left[\sqrt{6}\,G_{3,m} + (m+1)\,G_{1,m+1} + 2(m+2)\,G_{1,m} + m\,G_{1,m-1}\right] \\
&= -\nu\,(b+1+2m)H_{1,m} + \nu\,\mathcal{J}_m\bar{u}_{\parallel}
\end{aligned}
$$

Complete Equations

$$
\begin{aligned}
\frac{dG_{2,m}}{dt} &+ v_{ts}\nabla_{\parallel}\left(\sqrt{3}\,G_{3,m} + \sqrt{2}\,G_{1,m}\right) \\
&+ v_{ts}\left[-\sqrt{3}(m+1)\,G_{3,m} - \sqrt{3}\,m\,G_{3,m-1} + \sqrt{2}\,m\,G_{1,m} + \sqrt{2}\,(m+1)\,G_{1,m+1}\right]\nabla_{\parallel}\ln B \\
&+ i\omega_d\left[\sqrt{12}\,G_{4,m} + (m+1)\,G_{2,m+1} + 2(m+3)\,G_{2,m} + \sqrt{2}\,G_{0,m} + m\,G_{2,m-1}\right] \\
&+ \sqrt{2} i\omega_d\,\mathcal{J}_m\Phi \\
&= \frac{1}{\sqrt{2}} i\omega_*\,\frac{a}{L_{Ts}}\,\mathcal{J}_m\Phi - \nu\,(b+2+2m)\,H_{2,m} + \nu\,\frac{\sqrt{2}}{3}\,\mathcal{J}_m\left(\bar{T}_{\parallel} + 2\bar{T}_{\perp}\right)
\end{aligned}
$$

Complete Equations

For hydrogenic plasma with Boltzman electrons, the quasineutrality equation reduces to

$$\sum_{m=0}^{\infty} \mathcal{J}_m G_{0,m} = \tau^{-1} \left[\Phi - \langle\langle \Phi \rangle\rangle\right] + \left[1 - \Gamma_0(b)\right] \Phi.$$

Conservation terms

$$\bar{u}_{\|} = \int d^3 v J_0 v_{\|} h = \sum_{m=0}^{\infty} \mathcal{J}_m H_{1,m},$$

$$\bar{u}_{\perp} = \int d^3 v J_1 v_{\perp} h = \sqrt{b} \sum_{m=0}^{\infty} \left(\mathcal{J}_m + \mathcal{J}_{m-1}\right) H_{0,m},$$

$$\bar{T}_{\|} = \int d^3 v J_0 \left(v_{\|}^2 - 1\right) h = \sqrt{2} \sum_{m=0}^{\infty} \mathcal{J}_m H_{2,m},$$

$$\bar{T}_{\perp} = \int d^3 v J_0 \left(\mu B - 1\right) h = \sum_{m=0}^{\infty} \left[m\mathcal{J}_{m-1} + 2m\mathcal{J}_m + (m+1)\mathcal{J}_{m+1}\right] H_{0,m}$$

Nonlinear term convolution

$$\frac{dG_{\ell,m}}{dt} \equiv \frac{\partial G_{\ell,m}}{\partial t} + \sum_{k=0}^{\infty} \sum_{n=|k-m|}^{k+m} C_{kmn} \left(\mathcal{J}_n \mathbf{v}_E\right) \cdot \nabla G_{\ell,k}$$

The convolution arises from finite Larmor radius (FLR)-induced coupling. The convolution coefficients are given by (Watson)

$$C_{kmn} = \int_0^{\infty} d\mu B \; \psi_k \psi^m \psi_n = \sum_j \frac{(k+m-j)! \quad 2^{2j-k-m+n}}{(k-j)!(m-j)!(2j-k-m+n)!(k+m-n-j)!}$$