Modelling Coupled Ion and Electron Scale Turbulence in Magnetic Confinement Fusion Plasmas

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Anomalous transport is driven by turbulence,
- at scales where $k\rho_i \lesssim 1$ - ion scale
- at scales where $k\rho_e \sim 1$ - electron scale

We want to answer the following questions:
- do all scales matter?
- is cross scale coupling important?

To answer these questions we take a scale separated approach
Introduction: do all scales matter?

- Simulation evidence where \( Q_e \sim Q_{igB} \sim \sqrt{m_i/m_e} Q_{egB} \) e.g. Jenko and Dorland (2002)
- Recent experimental evidence on NSTX Ren et al. (2017)
- Howard et al. (2016) Fig 3:
Introduction: is cross scale coupling important?

- Fig 2 from Maeyama et al. (2015):
Introduction: can we reduce the mass ratio?

Fig 5 from Howard et al. (2015):
Introduction: a scale separated approach

The ion scale flux tube
An electron scale flux tube
A Quick Reminder: The Gyrokinetic Equation

The gyrokinetic equation:

\[
\frac{\partial h}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial h}{\partial \theta} + (\mathbf{v}_M + \mathbf{v}_E) \cdot \nabla h + \mathbf{v}_E \cdot \nabla F_0 = \frac{ZeF_0}{T} \frac{\partial \phi}{\partial t},
\]

(1)

where,

\[
\mathbf{v}_E = \frac{c}{B} \mathbf{b} \wedge \nabla \phi.
\]

(2)

Closed by quasi-neutrality,

\[
\sum_{\alpha} Z_\alpha e \left( \int d^3v |r h_\alpha \right) = \sum_{\alpha} \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha} \phi(r).
\]

(3)

Ingredients:

- a kinetic equation for \(f\)
- scale separation: \(\rho_* = \rho/a \rightarrow 0\), \(f = F + \delta f\)
- statistical periodicity: \(\langle \delta f \rangle_{\text{turb}} = 0\)
- orderings: \(\delta f \sim \rho_* F\), \(\nabla F \sim \nabla_\perp \delta f \sim \rho_*^{-1} \nabla_\parallel \delta f\)
Separating Ion and Electron Scale Turbulence

Using the ingredients:

- scale separation: \( \sqrt{m_e/m_i} \to 0 \), an electron scale average, \( \langle \cdot \rangle \)
- scale separation: \( \langle \delta f \rangle = \bar{\delta f} \), \( \delta f = \bar{\delta f} + \tilde{\delta f} \)
- electron scale statistical periodicity: \( \langle \tilde{\delta f} \rangle = 0 \)
- orderings:

\[
\nabla_\perp \bar{\delta f} \sim \rho_i^{-1} \bar{\delta f}, \quad \frac{\partial \bar{\delta f}}{\partial t} \sim \frac{v_{ti}}{a} \bar{\delta f}, \quad \nabla_\perp \tilde{\delta f} \sim \rho_e^{-1} \tilde{\delta f}, \quad \frac{\partial \tilde{\delta f}}{\partial t} \sim \frac{v_{te}}{a} \tilde{\delta f}.
\]

we can derive the coupled equations!
The Coupled Equations

- ion scale equations, with new back reaction term:

\[
\frac{\partial \tilde{h}_i}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_i}{\partial \theta} + (\mathbf{v}_M + \mathbf{v}_E) \cdot \nabla \tilde{h}_i + \mathbf{v}_E \cdot \nabla F_{0i} = \frac{Z_i e F_{0i}}{T_i} \frac{\partial \bar{\varphi}_i}{\partial t},
\]

(4)

\[
\frac{\partial \tilde{h}_e}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} + (\mathbf{v}_M + \mathbf{v}_E) \cdot \nabla \tilde{h}_e + \mathbf{v}_E \cdot \nabla F_{0e} + \nabla \cdot \left( \frac{c}{B} \tilde{h}_e \bar{\mathbf{v}}_E \right) = - \frac{e F_{0e}}{T_e} \frac{\partial \bar{\varphi}_e}{\partial t},
\]

(5)

\[
\int d^3 \mathbf{v} |_{r} (Z_i e \tilde{h}_i - e \tilde{h}_e) = \left( \frac{e^2 Z_i^2 n_i}{T_i} + \frac{e^2 n_e}{T_e} \right) \bar{\phi},
\]

(6)

- electron scale equations, with the new advection and drive terms:

\[
\frac{\partial \tilde{h}_e}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}_e}{\partial \theta} + (\mathbf{v}_M + \mathbf{v}_E + \mathbf{v}_E) \cdot \nabla \tilde{h}_e + \mathbf{v}_E \cdot (\nabla \bar{h}_e + \nabla F_{0e}) = - \frac{e F_{0e}}{T_e} \frac{\partial \bar{\varphi}_e}{\partial t}.
\]

(7)

\[
- \int d^3 \mathbf{v} |_{r} e \tilde{h}_e = \left( \frac{e^2 Z_i^2 n_i}{T_i} + \frac{e^2 n_e}{T_e} \right) \bar{\phi},
\]

(8)
The Coupled Equations: Sticky Points

Deriving parallisable coupled equations requires dealing with:

- non-locality of the gyro average
- the relative size of fluctuations - gyro Boehm scaling
- ions at electron scales
- the parallel boundary condition (*)

(*) Not yet resolved!
Visualising the Ion Scale $E \times B$ Velocity with $\theta$
multigs2 runs N+1 instances of gs2 and handles communication of gradients and fluxes between them

\[ \nabla \bar{h}_e \quad \nabla E_e \]

\[ \nabla \cdot \left( \vec{v}_{Ee} \bar{h}_e \right) \]
Electron Scale Simulations: Modification of the Linear Growth Rate
Electron Scale Simulations: Modification of the Linear Growth Rate

\[ \Phi/(\rho_{\text{ref}}/a)(T_{\text{ref}}/e) \text{ at } t = 163.25625/(a/v_{\text{ref}}) \]
Electron Scale Simulations: Modification of the Linear Growth Rate

\[ \gamma \left( \frac{a}{v_{ti}} \right) \]

\[ x \text{ index} \]

\[ 0 \quad 20 \quad 40 \quad 60 \quad 80 \quad 100 \quad 120 \]
Electron Scale Simulations: Modification of the Linear Growth Rate

\[ \frac{d(\delta T)}{dx} \]

\[ \theta \]

\[ \text{ix=1} = 22 \]
\[ \text{=22} \]
\[ \text{=43} \]
\[ \text{=64} \]
\[ \text{=85} \]
\[ \text{=106} \]
Electron Scale Simulations

\[ \Phi^2 / (\rho_i/a)^2 (T_i/e)^2 \]

![Graph showing the relationship between \( t (v_{ti}/a) \) and \( \Phi^2 / (\rho_i/a)^2 (T_i/e)^2 \) for different values of \( a/L_{Te} \).]
Electron Scale Simulations

\[ \frac{\Phi^2}{(\rho_i/a)^2(T_i/e)^2} \]

![Graph showing electron scale simulations with different curves and annotations](image)

- Blue curve: \( a/L_{Te} = 3.28 \)
- Orange curve: \( a/L_{Te} = 2.6 \)
- Green curve: \( a/L_{Te} = 1.92 \)
Electron Scale Simulations

\[ \Phi^2(k_y) \]

\[ t(v_t/a) \]

\[ 10^{-7} \]

\[ 10^{-6} \]

\[ 10^{-5} \]

\[ 10^{-4} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ k_y = 0.0 \]

\[ k_y = 2.5 \]

\[ k_y = 5.0 \]

\[ k_y = 7.5 \]

\[ k_y = 10.0 \]

\[ k_y = 12.5 \]

\[ k_y = 15.0 \]

\[ k_y = 17.5 \]

\[ k_y = 20.0 \]

\[ k_y = 22.5 \]

\[ k_y = 25.0 \]

\[ k_y = 27.5 \]

\[ k_y = 30.0 \]

\[ k_y = 32.5 \]

\[ k_y = 35.0 \]

\[ k_y = 37.5 \]

\[ k_y = 40.0 \]

\[ k_y = 42.5 \]

\[ k_y = 45.0 \]

\[ k_y = 47.5 \]

\[ k_y = 50.0 \]

\[ k_y = 52.5 \]
Summary

Conclusions:
- we have derived coupled equations for the ion and electron scale turbulence
- the electron scale terms have been implemented in gs2
- we have begun a search for a suitable proof of concept case for simulation

Future Work:
- understanding the electron scale equation through simulation
- including the back reaction in simulations


Should We Expect Cross Scale Interaction?

Yes! Because:

- electron scale eddies have $\tilde{l}_\perp \sim \rho_e$
- ion scale eddies have $\bar{l}_\perp \sim \rho_i$
- ambient gradient argument $\Rightarrow \tilde{h}_e \sim \rho_e^* F_{0e}, \quad \bar{h}_e \sim \rho_i^* F_{0e}$
- $\Rightarrow \nabla \tilde{h}_e \sim \nabla \bar{h}_e \sim \nabla F_{0e}$

$\Rightarrow$ gradients of the distribution function are comparable at all scales
$\Rightarrow$ electron scale eddies can be driven by ion scale gradients

- applying the same argument to $E = -\nabla \phi$
- $\Rightarrow \nabla \tilde{\phi} \sim \nabla \bar{\phi}$

$\Rightarrow$ eddy $E \times B$ drifts $v_{E \times B}$, are comparable at all scales

- applying the critical balance argument
- $v_{te} / \tilde{l}_\parallel \sim \tilde{l}^{-1} \sim \tilde{v}_{E \times B} / \tilde{l}_\perp$
- $v_{ti} / \bar{l}_\parallel \sim \bar{l}^{-1} \sim \bar{v}_{E \times B} / \bar{l}_\perp$
- $\tilde{l}_\parallel \sim \bar{l}_\parallel$

$\Rightarrow$ parallel correlation lengths are the same for ion scale and electron scale eddies
$\Rightarrow$ electron scale eddies are long enough to be differentially advected by $\bar{v}_{E \times B}$
Separating Ion and Electron Scale Turbulence: Technicalities

- We introduce a fast spatial variable \( r_f \) and a slow spatial variable \( r_s \) and the fast and slow times \( t_f, t_s \).
- In the gyrokinetic equation we send,

\[
\delta f(t, r) \rightarrow \delta f(t_s, t_f, r_s, r_f), \quad \nabla \rightarrow \nabla_s + \nabla_f, \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f},
\]

(9)

- then asymptotically expand in the mass ratio \((m_e/m_i)^{1/2}\)
- remembering \( \nabla_s \sim (m_e/m_i)^{1/2} \nabla_f \), and \( \partial/\partial t_s \sim (m_e/m_i)^{1/2} \partial/\partial t_f \)
- explicitly define the electron scale average,

\[
\overline{\delta f}(t_s, r_s) = \left\langle \delta f(t_s, t_f, r_s, r_f) \right\rangle = \frac{1}{\tau_c A} \int_{t_s - \tau_c/2}^{t_s + \tau_c/2} dt_f \int_A d^2 r_f \delta f(t_s, t_f, r_s, r_f),
\]

(10)

- We assume that,

\[
\delta f(t_s, t_f, r_s, r_f) = \delta f(t_s, t_f, r_s, r_f + n \Delta_{cx} \hat{x} + m \Delta_{cy} \hat{y}),
\]

(11)

- This enforces \( \left\langle \delta f \right\rangle = 0 \).
Splitting the Quasi-Neutrality Relation

- We split the guiding centre into a slow $R_s$ and a fast $R_f$ part.
- $R = r - \rho(r)$, where $\rho(r)$ is the vector gyroradius.
- Thus using the periodicity property equation (11) the electron scale average may be taken over guiding centre or real space coordinates.
- This observation allows us to note that the electron scale average commutes with the gyro average,

$$\left\langle \frac{1}{2\pi} \int_0^{2\pi} d\gamma |_R \phi(r_s, r_f) \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\gamma |_R \left\langle \phi(r_s, r_f) \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\gamma |_R \tilde{\phi}(r_s), \right\rangle$$

(12)

The splitting of the quasi neutrality relation follows directly,

$$\sum_\alpha Z_\alpha e(\int d^3v |_r \tilde{h}_\alpha(R_s)) = \sum_\alpha \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha} \phi(r_s),$$

(13)

$$\sum_\alpha Z_\alpha e(\int d^3v |_r \tilde{h}_\alpha(R_s, R_f)) = \sum_\alpha \frac{Z_\alpha^2 e^2 n_\alpha}{T_\alpha} \tilde{\phi}(r_s, r_f).$$

(14)
Addressing the Non-Locality of the Gyro Average

- Taking the gyro average at fixed guiding centre \( \langle \cdot \rangle_{\text{gyro}}^R \), couples multiple \( r_s \) points.
- but we aim to find parallelisable equations!
- Expanding both the slow and the fast spatial variable in Fourier series we note that,

\[
\tilde{\varphi}(t_s, t_f, R_s, R_f) = \langle \tilde{\varphi}(t_s, t_f, r_s, r_f) \rangle_{\text{gyro}}^R = \frac{1}{2\pi} \int_0^{2\pi} d\gamma |R| \tilde{\varphi}(t_s, t_f, r_s, r_f)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} d\gamma |R| \sum_{k_s, k_f} \tilde{\phi}_{k_s, k_f} e^{i k_s \cdot R_s} e^{i k_f \cdot R_f} e^{-i (k_s + k_f) \cdot \rho} = \sum_{k_s, k_f} \tilde{\phi}_{k_s, k_f} e^{i k_s \cdot R_s} e^{i k_f \cdot R_f} J_0(||(k_s + k_f)|| \rho),
\]

(15)

for electrons:
- \(|k_f| \rho_e \sim 1 \) and \(|k_s| \rho_e \sim (m_e/m_i)^{1/2}\)
- we can expand the Bessel function to return to a local picture in the slow variable with \( O(m_e/m_i)^{1/2} \) error.
- We will exploit this in parallelisation.

for ions:
- \(|k_s| \rho_i \sim 1 \) and \(|k_f| \rho_i \sim (m_e/m_i)^{-1/2}\).
- we are unable to expand the Bessel function
- we are unable to avoid the coupling of multiple \( r_s \) in the equations for ions at electron scale
assume we can neglect the ion contribution to electronscale quasi neutrality,

\[ \tilde{\varphi}_e(t_s, t_f, R_s, R_f) = \sum_{k_s, k_f} \tilde{\varphi}_{k_s, k_f} e^{i k_s \cdot R_s} e^{i k_f \cdot R_f} J_0(||(k_s + k_f)||\rho) \]

\[ = -\frac{T_e}{n_e e} \sum_{k_s, k_f} e^{i k_s \cdot R_s} e^{i k_f \cdot R_f} \int d^3v \tilde{h}_{e, k_s, k_f} J_0^2(||(k_s + k_f)||\rho) \]  

\[ = -\frac{T_e}{n_e e} \sum_{k_s, k_f} e^{i k_s \cdot R_s} e^{i k_f \cdot R_f} \int d^3v \tilde{h}_{e, k_s, k_f} J_0^2(||(k_s + k_f)||\rho) \]  (16)

now we use that,

\[ J_0(||(k_s + k_f)||\rho) = J_0(||k_f||\rho) + O(k_s \cdot k_f \rho e^2 \frac{dJ_0(z)}{dz} |z = ||k_f||\rho e) \]  (17)

exploit that \( ||k_s||\rho e \sim (m_e/m_i)^{1/2} \) to bring \( R_s \) under the velocity integral

regard \( R_s \) as a fixed parameter in the integration, to find,

\[ \tilde{\varphi}_e(t_s, t_f, R_s, R_f) = -\frac{T_e}{n_e e} \sum_{k_f} e^{i k_f \cdot R_f} \int d^3v |R_s\tilde{h}_{e k_f}(R_s) J_0^2(||k_f||\rho) (1 + O(m_e/m_i)^{1/2}) \]  (18)

we can evaluate quasi-neutrality purely locally in the slow variable.
Splitting the Gyrokinetic Equation

- we apply the electronscale average to the gyrokinetic equation
- we neglect terms which are small by \((m_e/m_i)^{1/2}\)

Ion scale equation:

\[
\frac{\partial \bar{h}}{\partial t_s} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \bar{h}}{\partial \theta} + (v_M + \bar{v}_E) \cdot \nabla_s \bar{h} + \nabla_s \cdot \left\langle \frac{c}{B} \bar{h} \bar{v}_E \right\rangle + \bar{v}_E \cdot \nabla F_0 = \frac{ZeF_0}{T} \frac{\partial \bar{\varphi}}{\partial t_s}. \tag{19}
\]

- we subtract the ion scale equation from the full equation and neglect terms

Electron scale equation:

\[
\frac{\partial \tilde{h}}{\partial t_f} + v_\parallel \mathbf{b} \cdot \nabla \theta \frac{\partial \tilde{h}}{\partial \theta} + (v_M + \tilde{v}_E + \bar{v}_E) \cdot \nabla_f \tilde{h} + \tilde{v}_E \cdot (\nabla_s \tilde{h} + \nabla F_0) = \frac{ZeF_0}{T} \frac{\partial \tilde{\varphi}}{\partial t_f}, \tag{20}
\]

where

\[
\tilde{v}_E = \frac{c}{B} \mathbf{b} \wedge \nabla_s \bar{\varphi}, \quad \tilde{v}_E = \frac{c}{B} \mathbf{b} \wedge \nabla_f \tilde{\varphi}. \tag{21}
\]

Note that,

- there are two additional terms on the electron scale, \(\tilde{v}_E \cdot \nabla_f \tilde{h}\) and \(\tilde{v}_E \cdot \nabla_s \tilde{h}\)
- there is one new term at the ion scale, \(\nabla_s \cdot \left\langle \frac{c}{B} \bar{h} \bar{v}_E \right\rangle\)
- \(\bar{v}_E\) cannot be removed with the boost or a solid body rotation because of the \(\theta\) dependence of \(\bar{\varphi}\)
Scaling Work: the Relative Size of the Fluctuations

- if we assume the following scalings:

\[
\bar{h}_i \sim \frac{e\Phi}{T} F_{0i}, \quad \bar{h}_e \sim \frac{e\Phi}{T} F_{0e}, \quad \bar{h}_i \sim \left(\frac{m_e}{m_i}\right)^{1/4} \frac{e\Phi}{T} F_{0i},
\]

\[
\bar{h}_e \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{e\Phi}{T} F_{0e} \text{ - parallel gradient term,} \quad \bar{h}_e \sim \frac{e\Phi}{T} F_{0e} \text{ - } \theta \text{ constant piece. (22)}
\]

- Then we can show that:

\[
\frac{e\phi}{T} \sim \rho_e^*, \quad \frac{e\phi}{T} \sim \rho_i^*
\]  

(23)
note that:

- \( J_0(k_f \rho_i) \sim (m_e/m_i)^{1/4} \)
- so:

\[
\int d^3v|\tilde{h}_i| \sim (\frac{m_e}{m_i})^{1/4}(\frac{m_e}{m_i})^{1/4}\frac{en\tilde{\phi}}{T} \tag{24}
\]

Ions at electron scales can be neglected to \( O((m_e/m_i)^{1/2}) \) in the electronscale equations!

note that:

- \( \nabla_s \cdot \left\langle \frac{c}{B} \tilde{h}_i \tilde{v}_{Ei} \right\rangle \sim O((m_e/m_i)^{3/4}\tilde{v}_{Ei} \cdot \tilde{h}_i) \)

Ions at electron scales can be neglected to \( O((m_e/m_i)^{3/4}) \) in the ion scale equations!
Scaling Work: which multiscale terms do we keep?

The only remaining multiscale terms are in electron species equations:

note that:

- \( \tilde{v}_{EE} \cdot \nabla_s \tilde{h}_e \sim \tilde{v}_{EE} \cdot \nabla_f \tilde{h}_e \sim \tilde{v}_{EE} \cdot \nabla_f \tilde{h}_e \)
- ion scale gradients contribute at \( O(1) \) to the electron scale
- ion scale shear can be neglected to \( O((m_e/m_i)^{1/2}) \) at the electron scale

\[
\nabla_s \cdot \left\langle \frac{c}{B} \tilde{h}_e \tilde{v}_{EE} \right\rangle \sim O((m_e/m_i)^{1/2} \tilde{v}_{EE} \cdot \tilde{h}_e)
\]

- back reaction contributes at \( O((m_e/m_i)^{1/2}) \) to the electron equation at ion scales
- small but can be self consistently included
Substituting for the gyro Bohm scalings of the potential we find that,

$$\frac{\tilde{Q}_i}{\tilde{Q}_i} \sim \left( \frac{m_e}{m_i} \right), \quad \frac{\tilde{Q}_e}{\tilde{Q}_e} \sim \left( \frac{m_e}{m_i} \right)^{1/2}, \quad \overline{Q}_i \sim \overline{Q}_e.$$ (25)
The Parallel Boundary Condition

using the field line label $\alpha = \xi - q(\psi)\theta \simeq \xi - q_0\theta - q'_0(\psi - \psi_0)$ the fluctuations take the form:

$$A(\theta, \alpha, \psi) = \sum_{n_0,\theta_0} A_{n_0,\theta_0}(\theta)e^{in_0((\alpha-\alpha_0)+q_0\theta_0+q'_0(\psi-\psi_0))} \tag{26}$$

the parallel boundary condition in these variables is, Beer et al. (1995),

$$A_{n_0,\theta_0+2\pi N}(\theta + 2\pi N) = A_{n_0,\theta_0}(\theta) \tag{27}$$

If we have parametric ionscale coordinate ($\bar{\alpha}, \bar{\psi}$) dependence then this boundary condition should become:

$$A_{n_0,\theta_0+2\pi N}(\theta + 2\pi N, \bar{\alpha}(\theta + 2\pi N), \bar{\psi}) = A_{n_0,\theta_0}(\theta, \bar{\alpha}(\theta), \bar{\psi}), \tag{28}$$

where $\alpha(\theta + 2\pi N) - \alpha(\theta) \simeq -q_02\pi N - q'_02\pi N(\psi - \psi_0)$

which would couple the electronscale flux tubes together!