From Drift Waves to Zonal Flows: An Investigation of the Dimits Shift

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What is the Dimits Shift?

Drift waves are generated by the free energy available through linear instabilities such as gradients. These drift waves can themselves generated zonal flows, which in turn can 'quench' drift-wave turbulence.

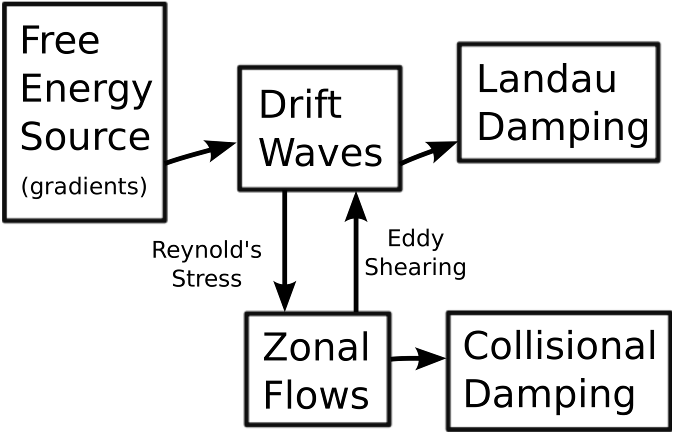

Figure: Energy diagram of a the typical drift wave/zonal flow interaction.

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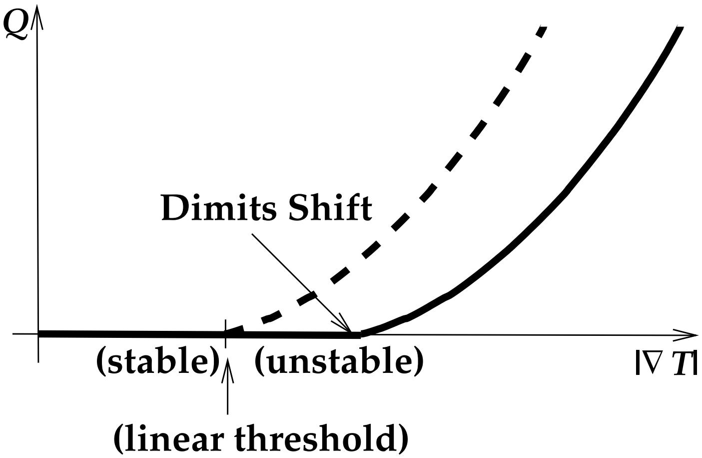

Figure: An illustration of the Dimits shift by the sudden appearance of turbulent heat flux Q with increasing temperature gradient ∇T.

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- This is caused by a shearing away of turbulent streamers by poloidal zonal flows (ZF).
- The zonal flows have their own instabilities.
- Andris Dimits and others noticed this shift in gyrokinetic simulations in the '90s.

Big Picture

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- We want to calculate the Dimits shift.
- We want to use a statistical framework to make our results as general as possible.
- We want to use the simplest models and closures possible, and then build up.

The Second Order Cumulant Expansion (CE2)

We want to use a simple statistical closure. As an example, let's take the 2D incompressible Navier-Stokes equations with stream function φ and vorticity $\zeta = \nabla^2 \varphi$,

$$\partial_t \zeta + \partial_y \varphi \partial_x \zeta - \partial_x \varphi \partial_y \zeta = \nu \nabla^2 \zeta.$$

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Decompose field into average $\overline{\varphi}(y) \equiv L_y^{-1}\int \mathrm{d}y\, \varphi(x,y)$ and fluctuation quantities $\varphi' = \varphi - \overline{\varphi}$. The resulting equations are

$$(\partial_t - \nu\partial_x^2)U + \partial_x\overline{u'v'} = 0,$$
$$(\partial_t - \nu\nabla^2)\zeta' + U\partial_y\zeta' + v'\partial_x^2 U = \cancel{\mathbf{v}'\cdot\nabla w' - \overline{\mathbf{v}'\cdot\nabla w'}},$$

where $\mathbf{v}' = \{v', u'\} = \nabla\varphi'$, and $U \equiv \partial_x\overline{\varphi}$. Neglecting the terms on the RHS result in the quasi-linear equations.

The Second Order Cumulant Expansion (CE2)

Then we can instead use the two-point covariance

$$\Phi(x, \overline{x}, y, t) = \frac{1}{L_y} \int_0^{L_y} \mathrm{d}\overline{y}\, \varphi'(x_1, y_1, t)\varphi'(x_2, y_2, t)$$

where we've defined sum $\overline{\mathbf{x}} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ and difference $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ coordinates.

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$$\begin{aligned}(\partial_t - \nu\partial_{\overline{x}}^2) U(\overline{x}) &= -\frac{1}{2}\partial_{\overline{x}}(\partial_x \partial_y \Phi)_{x=y=0}, \\ (\partial_t - 2\nu \nabla_+^2 \nabla_-^2) Z &= -(U_+ - U_-)\partial_y Z - (U_+'' \nabla_-^2 - U_-'' \nabla_+^2)\partial_y \Phi,\end{aligned}$$

where $U_\pm \equiv U(\overline{x} \pm x/2)$. These are the CE2 equations that evolve the Gaussian statistics of the original system.

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- Captures the essential physics of zonal flow/drift wave interactions.
- Great success in the context of planetary atmospheres and the zonostrophic/modulational instability.

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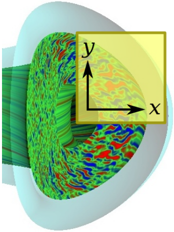

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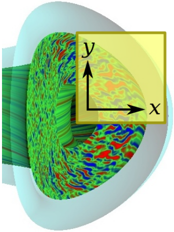

We use a two field model with ion density n_i and temperature T. We also only consider perturbations of the electrostatic potential φ, so that our drift velocity is the $\mathbf{E} \times \mathbf{B}$ velocity

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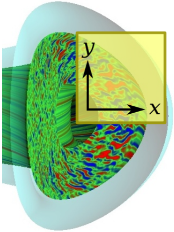

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The electrostatic potential can be related to the density via the Poisson equation.

We include a background temperature gradient $\nabla T = -L_T^{-1}\hat{\mathbf{x}}$ and the curvature drift term $\mathbf{v}_{\nabla B} = -2\epsilon\rho_*\hat{\mathbf{y}}$ where ϵ is the inverse aspect ratio and $\rho_* = \rho_\mathrm{s}/a$. This captures the toroidal ITG mode.

Two-field Model Equations

Consider the curl of $\mathbf{v}_{\mathrm{E}\times\mathrm{B}}$,

$$\nabla_\perp \times \mathbf{v}_{\mathrm{E}\times\mathbf{B}} = \nabla_\perp \times \left(\hat{\mathbf{z}} \times \nabla_\perp \varphi\right) = \nabla_\perp^2 \varphi.$$

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The equations we then consider are

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + \hat{\mathbf{z}} \times \nabla_\perp \varphi \cdot \nabla_\perp \zeta + 2\epsilon\rho_* \frac{\partial T}{\partial y} &= -\alpha\nu\zeta, \\
\frac{\partial T}{\partial t} + \hat{\mathbf{z}} \times \nabla_\perp \varphi \cdot \nabla_\perp T + \frac{1}{L_T}\frac{\partial \phi}{\partial y} \quad &= -\alpha\nu T.
\end{aligned}$$

Here,

$$\zeta \equiv \overline{\nabla}^2 \varphi = -n_{\mathrm{i}}$$

is the modified vorticity,

$$\overline{\nabla}^2 \equiv \nabla_\perp^2 - \alpha.$$

is the modified Laplacian and α is an operator such that it's zero acting on zonal modes and 1 otherwise.

Two-field Model Equations

We can linearize our equations by Fourier analysis with $\partial_{\mathbf{X}} = i\mathbf{k}$ and $\partial_t = \lambda$.

$$-\lambda(1+k^2)\phi + 2\epsilon\rho_* i k_y T = \alpha\nu(1+k^2)\phi,$$

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This leads to the dispersion relation

$$(\lambda+\nu)^2 = \omega_T \omega_{\mathrm{d}}.$$

The condition for linear stability is

$$\nu^2 > \omega_T \omega_{\mathrm{d}}.$$

One-field Toy Model

We can also construct a one-field model that mimics the growth rates of the unstable branch,

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \gamma \frac{\partial^2 \varphi}{\partial y^2} - \alpha\mu\zeta + \alpha\nu\nabla_\perp^2\zeta.$$

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Stability requirement

$$\gamma < \left(\sqrt{\mu} + \sqrt{\nu}\right)^2.$$

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and the CE2 statistical closure

$$\begin{aligned}
\partial_t Z =& -(U_+ - U_-)\partial_y Z + \left(U_+'' \overline{\nabla}_-^2 - U_-'' \overline{\nabla}_+^2\right)\Phi \\
&+ \gamma \partial_y^2 (\overline{\nabla}_+^2 + \overline{\nabla}_-^2)\Phi - 2\left(\mu + \nu \nabla_+^2 \nabla_-^2\right) Z, \\
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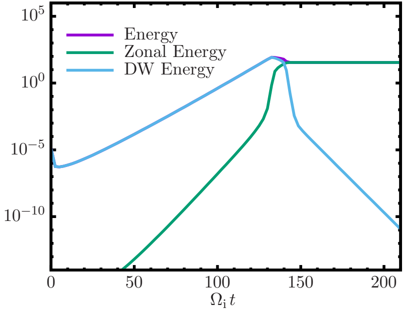
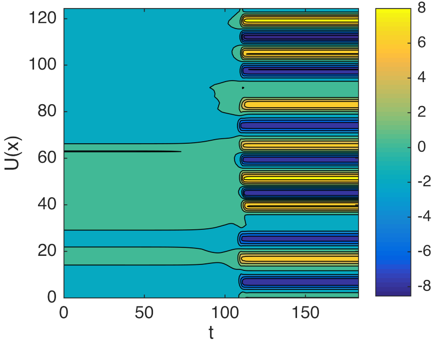
QUESTION: Do these simplified models exhibit a Dimits shift?

Numerical Results

Now I'll show some movies for various Direct Numerical Simulations (DNS) of some of the systems in question.





Analytics: Four-mode Truncation (4MT)

The Fourier transform of the one-field model equation gives

$$\partial_t \phi_{\mathbf{k}} = \gamma_{\mathbf{k}} \phi_{\mathbf{k}} - \frac{1}{\alpha_{\mathbf{k}} + k^2} \sum_{\mathbf{k}_1, \mathbf{k}_2} k_{1,x} k_{2,y} (\alpha_2 - \alpha_1 + k_2^2 - k_1^2) \times \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2).$$

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Considering only the modes $\mathbf{p} = (0, p_y)$, $\mathbf{q} = (q_x, 0)$, and sidebands $\mathbf{r}_\pm = (\pm q_x, p_y)$,

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Considering only the modes $\mathbf{p} = (0, p_y)$, $\mathbf{q} = (q_x, 0)$, and sidebands $\mathbf{r}_\pm = (\pm q_x, p_y)$, we have the four mode truncation (4MT)

$$\partial_t \phi_{\mathbf{p}} = \gamma_{\mathbf{p}} \phi_{\mathbf{p}} + \frac{q_x p_y^3}{1 + p_y^2} \left(\phi_{\mathbf{q}} \phi_{\mathbf{r}-} - \phi_{\mathbf{q}}^* \phi_{\mathbf{r}+} \right),$$

$$\partial_t \phi_{\mathbf{r}+} = \gamma_{\mathbf{r}} \phi_{\mathbf{r}+} + \frac{q_x p_y}{1 + r^2} (p_y^2 - q_x^2) \phi_{\mathbf{p}} \phi_{\mathbf{q}},$$

$$\partial_t \phi_{\mathbf{r}-}^* = \gamma_{\mathbf{r}} \phi_{\mathbf{r}-}^* - \frac{q_x p_y}{1 + r^2} (p_y^2 - q_x^2) \phi_{\mathbf{p}}^* \phi_{\mathbf{q}},$$

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\end{aligned}$$

This can be solved for $\phi_{\mathbf{q}}$,

$$\phi_{\mathbf{q}}'' - (\gamma_{\mathbf{p}} + \gamma_{\mathbf{r}})\phi_{\mathbf{q}}' - 2 q_x p_y \beta \phi_0^2 e^{2\gamma_{\mathbf{p}} t} \phi_{\mathbf{q}} = 0,$$

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\partial_t \phi_{\mathbf{q}} &= q_x p_y \big(\phi_{\mathbf{r}_+} - \phi_{\mathbf{r}_-}^*\big) e^{\gamma_{\mathbf{p}} t}\phi_0.
\end{aligned}$$

This can be solved for $\phi_{\mathbf{q}}$,

$$\phi_{\mathbf{q}}'' - (\gamma_{\mathbf{p}} + \gamma_{\mathbf{r}})\phi_{\mathbf{q}}' - 2 q_x p_y \beta \phi_0^2 e^{2\gamma_{\mathbf{p}} t}\phi_{\mathbf{q}} = 0,$$

with asymptotic behaviour

$$\phi_{\mathbf{q}} \sim \exp\{\gamma_{\mathbf{p}}^{-1}\sqrt{2 q_x p_y \beta \phi_0^2} e^{\gamma_{\mathbf{p}} t} + \gamma_{\mathbf{r}} t/2\}.$$

Analytics: Four-mode Truncation (4MT)

Let us just consider the linear growth phase with an exponentially growing drift wave mode $\phi_{\mathbf{p}} = e^{\gamma_{\mathbf{p}} t}\phi_0$. WLOG, we let $\phi_0^* = \phi_0$. Defining $\beta \equiv q_x p_y (p_y^2 - q_x^2)/(1 + r^2)$, the remaining equations become

$$\partial_t \phi_{\mathbf{r}-}^* = \gamma_{\mathbf{r}} \phi_{\mathbf{r}-}^* - \beta \phi_{\mathbf{q}} e^{\gamma_{\mathbf{p}} t}\phi_0,$$
$$\partial_t \phi_{\mathbf{r}+} = \gamma_{\mathbf{r}} \phi_{\mathbf{r}+} + \beta \phi_{\mathbf{q}} e^{\gamma_{\mathbf{p}} t}\phi_0,$$
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Compare with standard modulational instability growth rate,

$$\lambda = \pm\sqrt{2 q_x p_y \beta \phi_0^2 - (\omega_{\mathbf{p}} - \omega_{\mathbf{r}})^2}.$$

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Solution: One must carefully do a conditional ensemble average of the initial state.

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- These models have been reformulated under the CE2 statistical framework.
- Numerical work has been performed, demonstrating rich behaviour in the models.
- Preliminary analytical work has already revealed important difference from the standard modulational stability.

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