WPI, Vienna 5 Aug 2016

5D Phase Mixing in GK Plasma

on the Example of Electron Sub-Larmor Turbulence

Alex Schekochihin (Oxford)

with

Steve Cowley (Oxford),
Bill Dorland (U of Maryland),
Tomo Tatsuno (UEC Tokyo),
Gabriel Plunk (IPP, Greifswald)

[ApJS 182, 310 (2009), section 7.12]

WPI, Vienna 5 Aug 2016

5D Phase Mixing in GK Plasma

on the Example of Electron Sub-Larmor Turbulence

Alex Schekochihin (Oxford)

with

Steve Cowley (Oxford),
Bill Dorland (U of Maryland),
Tomo Tatsuno (UEC Tokyo),
Gabriel Plunk (IPP, Greifswald)

[ApJS 182, 310 (2009), section 7.12]

Sequel: 4D Phase Mixing in a Drift-Kinetic Plasma

with J. Parker (RAL), E. Highcock (Chalmers), P. Dellar (Oxford),
G. Hammett (Princeton), W. Dorland, A. Kanekar (Maryland),
R. Meyrand (Berkeley), N. Loureiro (MIT), L. Stipani, F. Califano (Pisa)

[JPP 82, 905820212 (2016); PoP 23, 070703 (2016)]

…this was last year's talk

WPI, Vienna 5 Aug 2016

5D Phase Mixing in GK Plasma

on the Example of Electron Sub-Larmor Turbulence

Alex Schekochihin (Oxford)

with

Steve Cowley (Oxford),
Bill Dorland (U of Maryland),
Tomo Tatsuno (UEC Tokyo),
Gabriel Plunk (IPP, Greifswald)

[ApJS 182, 310 (2009), section 7.12]

Sequel: 4D Phase Mixing in a Drift-Kinetic Plasma

with J. Parker (RAL), E. Highcock (Chalmers), P. Dellar (Oxford),
G. Hammett (Princeton), W. Dorland, A. Kanekar (Maryland),
R. Meyrand (Berkeley), N. Loureiro (MIT), L. Stipani, F. Califano (Pisa)

[JPP 82, 905820212 (2016); PoP 23, 070703 (2016)]

Prequel: 2D Phase Mixing in a Vlasov-Poisson Plasma

with Toby Adkins (Oxford)

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$ electron Larmor rings are >> spatial scale of e-m fluctuations

$\omega \ll \Omega_e$ but electron Larmor period << time scale of e-m fluctuations

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\mathrm{the}}$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\mathbf{the}}$

$$f_e = \underset{\substack{\uparrow \\ \text{equilibrium} \\ \text{Maxwellian} \\ \text{(yes, I know…)}}}{F_0} + \underset{\substack{\uparrow \\ \text{Boltzmann} \\ \text{response} \\ \varphi = e\phi/T_e}}{\varphi}(t, \mathbf{r})F_0 + h(t, \underset{\substack{\uparrow \\ \text{gyrocentre} \\ \mathbf{R} = \mathbf{r} - \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}}}{\mathbf{R}}, v_\perp, v_\parallel) \leftarrow \text{distribution of rings}$$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\| \ll k_\perp$, because $\omega \sim k_\| v_{\rm the}$

$$f_e = F_0 + \varphi(t,\mathbf{r})F_0 + h(t,\mathbf{R},v_\perp,v_\|) \leftarrow$$

distribution of rings

equilibrium Maxwellian (yes, I know…)

Boltzmann response $\varphi = e\phi/T_e$

gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

energy injection (from larger scales)

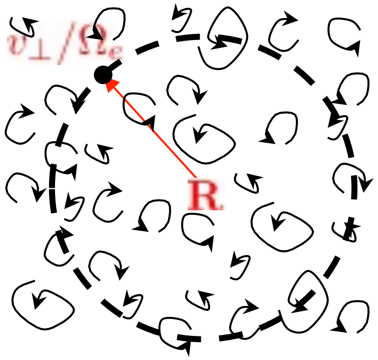
$$\boxed{\frac{\partial h}{\partial t} + v_\| \nabla_\| h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi}$$

parallel particle streaming (more of it later!)

gyroaveraged **E**x**B** drift velocitty $\mathbf{u}_\perp = \dfrac{\rho_e v_{\rm the}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$

gyroaveraged wave-ring interaction $-\left\langle \dfrac{d\varepsilon}{dt} \dfrac{\partial f_e}{\partial \varepsilon} \right\rangle_\mathbf{R}$

collisions



$$\langle \varphi \rangle_\mathbf{R} = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \, \varphi\left(\mathbf{R} + \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e} \right)$$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\rm the}$

$$f_e = F_0 + \varphi(t,\mathbf{r})F_0 + h(t,\mathbf{R},v_\perp,v_\parallel) \leftarrow$$

distribution of rings

equilibrium Maxwellian (yes, I know…)

Boltzmann response $\varphi = e\phi/T_e$

gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

energy injection (from larger scales)

$$\boxed{\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi}$$

parallel particle streaming (more of it later!)

gyroaveraged **E**x**B** drift velocitty

$$\mathbf{u}_\perp = \frac{\rho_e v_{\rm the}}{2}\hat{\mathbf{b}} \times \nabla_\perp \varphi$$

gyroaveraged wave-ring interaction

$$-\left\langle \frac{d\varepsilon}{dt}\frac{\partial f_e}{\partial \varepsilon} \right\rangle_{\mathbf{R}}$$

collisions

$$\langle \varphi \rangle_{\mathbf{R}} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta\, \varphi\left(\mathbf{R} + \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}\right) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)\varphi_{\mathbf{k}}$$

Gyroaveraging is a Bessel operator, so, at $k_\perp \rho_e \gg 1$, $\langle \varphi \rangle_{\mathbf{R}} = \hat{J}_0 \varphi \sim \dfrac{\varphi}{\sqrt{k_\perp \rho_e}}$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\| \ll k_\perp$, because $\omega \sim k_\| v_{\rm the}$

$$f_e = \underset{\uparrow}{F_0} + \underset{\uparrow}{\varphi}(t,\mathbf{r})F_0 + h(t,\underset{\uparrow}{\mathbf{R}},v_\perp,v_\|) \leftarrow$$

distribution of rings

equilibrium Maxwellian (yes, I know…)

Boltzmann response $\varphi = e\phi/T_e$

gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

$$\frac{\partial h}{\partial t} + v_\| \nabla_\| h + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}}} \cdot \nabla_\perp h = -\underset{\uparrow}{\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t}} F_0 + C[h] + \chi$$

To calculate φ, use quasineutrality:

$$\frac{\delta n_e}{n_e} = \varphi + \frac{1}{n_e}\int d^3\mathbf{v}\,\langle h \rangle_{\mathbf{r}} \overset{\downarrow}{=} \frac{\delta n_i}{n_i}$$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\rm the}$

$$f_e = F_0 + \varphi(t, \mathbf{r}) F_0 + h(t, \mathbf{R}, v_\perp, v_\parallel) \leftarrow$$

distribution of rings

↑ equilibrium Maxwellian (yes, I know…)

↑ Boltzmann response $\varphi = e\phi/T_e$

↑ gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

↑ ↑

To calculate φ, use quasineutrality:

↓

$$\frac{\delta n_e}{n_e} = \varphi + \frac{1}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \frac{\delta n_i}{n_i} = -\frac{e\phi}{T_i} = -\frac{T_e}{T_i} \varphi$$

↑

Ions have Boltzmann response because everything else averages out over their (huge!) Larmor orbits

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\mathrm{the}}$

$$f_e = \underset{\uparrow}{F_0} + \underset{\uparrow}{\varphi}(t,\mathbf{r})F_0 + h(t,\underset{\uparrow}{\mathbf{R}},v_\perp,v_\parallel) \leftarrow \text{distribution of rings}$$

equilibrium Maxwellian (yes, I know…)

Boltzmann response $\varphi = e\phi/T_e$

gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}}} \cdot \nabla_\perp h = -\underset{\uparrow}{\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t}} F_0 + C[h] + \chi$$

To calculate φ, use quasineutrality: ↓

$$\frac{\delta n_e}{n_e} = \varphi + \frac{1}{n_e}\int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \frac{\delta n_i}{n_i} = -\frac{e\phi}{T_i} = -\frac{T_e}{T_i}\varphi$$

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e}\int d^3\mathbf{v}\, J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1 + T_e/T_i}$$

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$

$\omega \ll \Omega_e$

this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\mathrm{the}}$

$$f_e = \underset{\substack{\uparrow\\ \text{equilibrium}\\ \text{Maxwellian}\\ \text{(yes, I know…)}}}{F_0} + \underset{\substack{\uparrow\\ \text{Boltzmann}\\ \text{response}\\ \varphi = e\phi/T_e}}{\varphi}(t,\mathbf{r})F_0 + \underset{\substack{\uparrow\\ \text{gyrocentre}}}{h}(t,\mathbf{R},v_\perp,v_\parallel) \leftarrow \text{distribution of rings}$$

$$\mathbf{R} = \mathbf{r} - \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$$

$$\boxed{\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi}$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}}$$

Closed system

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1 + T_e/T_i}$$

Magnetic Fluctuations @ Sub-Larmor Scales

Our equations are electrostatic. Is this a good approximation?

Closed system

$$\boxed{\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|} h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi}$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}}$$

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1 + T_e/T_i}$$

Magnetic Fluctuations @ Sub-Larmor Scales

Our equations are electrostatic. Is this a good approximation? – YES:

Parallel Ampere's law: $\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} j_\parallel = \frac{4\pi e}{c}\int d^3\mathbf{v}\, v_\parallel \langle h\rangle_{\mathbf{r}}$

$$\boxed{\frac{\delta\mathbf{B}_{\perp\mathbf{k}}}{B_0} = -\frac{\hat{\mathbf{b}}\times i\mathbf{k}_\perp A_{\parallel\mathbf{k}}}{B_0} = \underbrace{\frac{\beta_e}{k_\perp\rho_e}}_{\text{small factor!}}\hat{\mathbf{b}}\times i\mathbf{k}_\perp \frac{1}{n_e}\int d^3\mathbf{v}\,\frac{v_\parallel}{v_{the}}\,J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}} \ll \varphi}$$

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1+T_e/T_i}$$

Magnetic Fluctuations @ Sub-Larmor Scales

Our equations are electrostatic. Is this a good approximation? – YES:

Parallel Ampere's law: $\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} j_\parallel = \frac{4\pi e}{c} \int d^3\mathbf{v}\, v_\parallel \langle h \rangle_{\mathbf{r}}$

$$\boxed{\frac{\delta \mathbf{B}_{\perp\mathbf{k}}}{B_0} = -\frac{\hat{\mathbf{b}} \times i\mathbf{k}_\perp A_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp \rho_e} \hat{\mathbf{b}} \times i\mathbf{k}_\perp \frac{1}{n_e} \int d^3\mathbf{v}\, \frac{v_\parallel}{v_{\mathrm{th}e}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}} \ll \varphi}$$

small factor!

Perpendicular Ampere's law:

$$\nabla_\perp^2 \delta B_\parallel = -\frac{4\pi}{c} \hat{\mathbf{b}} \cdot (\nabla_\perp \times \mathbf{j}_\perp) = \frac{4\pi e}{c} \hat{\mathbf{b}} \cdot \left(\nabla_\perp \times \int d^3\mathbf{v}\, \langle \mathbf{v}_\perp h \rangle_{\mathbf{r}}\right)$$

$$\boxed{\frac{\delta B_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp \rho_e} \frac{1}{n_e} \int d^3\mathbf{v}\, \frac{v_\perp}{v_{\mathrm{th}e}} J_1\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}} \ll \varphi}$$

small factor!

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1 + T_e/T_i}$$

Magnetic Fluctuations @ Sub-Larmor Scales

Our equations are electrostatic. Is this a good approximation? – YES:

Parallel Ampere's law: $\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} j_\parallel = \frac{4\pi e}{c}\int d^3\mathbf{v}\, v_\parallel \langle h\rangle_\mathbf{r}$

$$\boxed{\frac{\delta\mathbf{B}_{\perp\mathbf{k}}}{B_0} = -\frac{\hat{\mathbf{b}}\times i\mathbf{k}_\perp A_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp\rho_e}\hat{\mathbf{b}}\times i\mathbf{k}_\perp \frac{1}{n_e}\int d^3\mathbf{v}\,\frac{v_\parallel}{v_{the}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k} \ll \varphi}$$

small factor!

Perpendicular Ampere's law:

$$\nabla_\perp^2 \delta B_\parallel = -\frac{4\pi}{c}\hat{\mathbf{b}}\cdot(\nabla_\perp\times\mathbf{j}_\perp) = \frac{4\pi e}{c}\hat{\mathbf{b}}\cdot\left(\nabla_\perp\times\int d^3\mathbf{v}\,\langle\mathbf{v}_\perp h\rangle_\mathbf{r}\right)$$

$$\boxed{\frac{\delta B_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp\rho_e}\frac{1}{n_e}\int d^3\mathbf{v}\,\frac{v_\perp}{v_{the}} J_1\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k} \ll \varphi}$$

small factor!

Key point: magnetic spectra are slaved to the spectra of density and of φ:

$$\frac{\delta B}{B_0} \sim \frac{\beta_e}{k_\perp\rho_e}\varphi$$

Plan: Theory ⇒ Observables

1. Solve this system for h and φ :

$$\boxed{\begin{gathered}\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|}h + \langle \mathbf{u}_{\perp}\rangle_{\mathbf{R}} \cdot \nabla_{\perp} h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi \\ \mathbf{u}_{\perp} = \frac{\rho_e v_{\mathrm{the}}}{2}\,\hat{\mathbf{b}} \times \nabla_{\perp}\varphi \\ \langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_{\perp}v_{\perp}}{\Omega_e}\right)\varphi_{\mathbf{k}} \\ \varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e}\int d^3\mathbf{v}\, J_0\left(\frac{k_{\perp}v_{\perp}}{\Omega_e}\right) h_{\mathbf{k}}\end{gathered}}$$

…and get spectra $E_{\varphi}(k_{\perp}) \propto k_{\perp}^{-\mu}, \quad E_h(k_{\perp}) \propto k_{\perp}^{-\nu}$

2. Infer density spectra: $E_n(k_{\perp}) \propto k_{\perp}^{-\mu}$ because $\dfrac{\delta n_e}{n_e} = \dfrac{\varphi}{\alpha} = -\left(1 + \dfrac{T_e}{T_i}\right)\varphi$

magnetic-field spectra: $E_B(k_{\perp}) \propto k_{\perp}^{-\mu-2}$ because $\dfrac{\delta B}{B} \sim \dfrac{\beta_e}{k_{\perp}\rho_e}\varphi$

electric-field spectra: $E_E(k_{\perp}) \propto k_{\perp}^{-\mu+2}$ because $\mathbf{E}_{\perp} = -\nabla_{\perp}\phi \propto k_{\perp}\varphi$

Free Energy

1. Solve this system for h and φ :

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_\mathrm{the}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$$

$$\langle \varphi \rangle_\mathbf{R} = \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_\mathbf{k}$$

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_\mathbf{r} = \alpha \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k}$$

Rather than "solving," we can resort to Kolmogorov-ology: scalings will be set assuming constant flux of some conserved quantity, viz., free energy:

$$\frac{d}{dt}\underset{\uparrow}{\left[\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h^2}{2F_0} + \int d^3\mathbf{r}\, \frac{\varphi^2}{2\alpha}\right]} = \underset{\uparrow}{\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h\chi}{F_0}} + \underset{\uparrow}{\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{hC[h]}{F_0}}$$

free energy

injection
$\equiv \varepsilon$

collisional dissipation
(negative definite!)

[AAS et al. 2008, PPCF 50, 24024]

Free Energy

1. Solve this system for h and φ :

$$\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|}h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{th}e}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}}$$

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}$$

Rather than "solving," we can resort to Kolmogorov-ology: scalings will be set assuming constant flux of some conserved quantity, viz., free energy:

$$\frac{d}{dt}\left[\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h^2}{2F_0} + \int d^3\mathbf{r}\, \frac{\varphi^2}{2\alpha}\right] = \frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h\chi}{F_0} + \frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{hC[h]}{F_0}$$

↑ free energy

↑ injection $\equiv \varepsilon$

↑ collisional dissipation (negative definite!)

NB: free energy has to get to small scales in velocity space, to dissipate.

Free Energy

In general, the free energy in δf kinetics is

$$\mathcal{F} = -\sum_s T_s \delta S = -\sum_s T_s\, \delta\left[\iint d^3\mathbf{v}\, d^3\mathbf{r}\, f_s \ln f_s\right] = \sum_s \iint d^3\mathbf{v}\, d^3\mathbf{r}\, \frac{T_s \delta f_s^2}{2F_{0s}}$$

$$= n_e T_e \left[\frac{1}{n_e}\iint d^3\mathbf{v}\, d^3\mathbf{R}\, \frac{h^2}{2F_0} + \int d^3\mathbf{r}\, \frac{\varphi^2}{2\alpha}\right]$$ in our case

This has a long history:

Kruskal & Oberman 1958
Bernstein 1958
Fowler 1963, 68
Krommes & Hu 1994
Krommes 1999
Sugama et al. 1996
Hallatschek 2004
Howes et al. 2006
Schekochihin et al. 2007-09
Scott 2010
Banon, Jenko et al. 2011-14
Plunk et al 2012
Abel et al. 2013
Kunz et al. 2015…

Rather than "solving," we can resort to Kolmogorov-ology: scalings will be set assuming constant flux of some conserved quantity, viz., free energy:

$$\frac{d}{dt}\left[\frac{1}{n_e}\iint d^3\mathbf{v}\, d^3\mathbf{R}\, \frac{h^2}{2F_0} + \int d^3\mathbf{r}\, \frac{\varphi^2}{2\alpha}\right] = \frac{1}{n_e}\iint d^3\mathbf{v}\, d^3\mathbf{R}\, \frac{h\chi}{F_0} + \frac{1}{n_e}\iint d^3\mathbf{v}\, d^3\mathbf{R}\, \frac{hC[h]}{F_0}$$

↑ free energy

↑ injection $\equiv \varepsilon$

↑ collisional dissipation (negative definite!)

So our conserved quantity is (minus) entropy!

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\frac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \frac{h}{F_0}$ at each scale $k_\perp^{-1}$

Rather than “solving,” we can resort to Kolmogorov-ology: scalings will be set assuming constant flux of some conserved quantity, viz., free energy:

$$\boxed{\frac{d}{dt}\left[\frac{1}{n_e}\iint d^3\mathbf{v}\,d^3\mathbf{R}\,\underbrace{\frac{h^2}{2F_0}}_{\text{free energy}} + \int d^3\mathbf{r}\,\frac{\varphi^2}{2\alpha}\right] = \underbrace{\frac{1}{n_e}\iint d^3\mathbf{v}\,d^3\mathbf{R}\,\frac{h\chi}{F_0}}_{\substack{\text{injection}\\ \equiv\,\varepsilon}} + \underbrace{\frac{1}{n_e}\iint d^3\mathbf{v}\,d^3\mathbf{R}\,\frac{hC[h]}{F_0}}_{\substack{\text{collisional dissipation}\\ \text{(negative definite!)}}}}$$

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\dfrac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \dfrac{h}{F_0}$ at each scale $k_\perp^{-1}$

cascade time

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}}$$

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\frac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \frac{h}{F_0}$ at each scale $k_\perp^{-1}$

↑

cascade time

↓

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi \sim \rho_e^2 \Omega_e k_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}} \sim \hat{J}_0 \varphi \sim \frac{\varphi}{\sqrt{k_\perp \rho_e}}$$

Cascade time: $\boxed{\tau^{-1} \sim k_\perp \langle u_\perp \rangle_{\mathbf{R}} \sim \Omega_e (k_\perp \rho_e)^2 \hat{J}_0 \varphi \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi}$

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\frac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \frac{h}{F_0}$ at each scale $k_\perp^{-1}$

cascade time

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi \sim \rho_e^2 \Omega_e k_\perp \varphi$$

$$\langle \varphi \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_{\mathbf{k}} \sim \hat{J}_0 \varphi \sim \frac{\varphi}{\sqrt{k_\perp \rho_e}}$$

Cascade time: $\boxed{\tau^{-1} \sim k_\perp \langle u_\perp \rangle_{\mathbf{R}} \sim \Omega_e (k_\perp \rho_e)^2 \hat{J}_0 \varphi \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi}$

NB: $\tau^{-1} \ll \Omega_e$ provided $\varphi \ll \frac{1}{(k_\perp \rho_e)^{3/2}}$ (we'll check this later)

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\frac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \frac{h}{F_0}$ at each scale $k_\perp^{-1}$

Cascade time: $\boxed{\tau^{-1} \sim k_\perp \langle u_\perp \rangle_{\mathbf{R}} \sim \Omega_e (k_\perp \rho_e)^2 \hat{J}_0 \varphi \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi}$

NB: $\tau^{-1} \ll \Omega_e$ provided $\varphi \ll \frac{1}{(k_\perp \rho_e)^{3/2}}$ (we'll check this later)

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

Cascade time: $\boxed{\tau^{-1} \sim k_\perp\langle u_\perp\rangle_{\mathbf{R}} \sim \Omega_e(k_\perp\rho_e)^2\hat{J}_0\varphi \sim \Omega_e(k_\perp\rho_e)^{3/2}\varphi}$

NB: $\tau^{-1} \ll \Omega_e$ provided $\varphi \ll \frac{1}{(k_\perp\rho_e)^{3/2}}$ (we'll check this later)

[AAS et al. 2008, PPCF 50, 24024]

Gyroaveraged Response

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_\mathbf{r} = \alpha\sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_\mathbf{k}(v_\perp)$$

[AAS et al. 2008, PPCF 50, 24024]

Gyroaveraged Response

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{\mathrm{th}e}} \sim \frac{1}{k_\perp\rho_e}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_{\mathbf{k}}(v_\perp)$$

$$\approx\left(\frac{2\Omega_e}{\pi k_\perp v_\perp}\right)^{1/2}\cos\left(\frac{k_\perp v_\perp}{\Omega_e}-\frac{\pi}{4}\right)$$

oscillatory integral, sign changes with period

$$\frac{\Delta v_\perp}{v_{\mathrm{th}e}} = \frac{2\pi}{k_\perp\rho_e}$$

[AAS et al. 2008, PPCF 50, 24024]

Gyroaveraged Response

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

…and we now need a relationship between φ and $\hat{h}$:

we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{\mathrm{th}e}} \sim \frac{1}{k_\perp\rho_e}$

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}(v_\perp)$$

$$\approx \left(\frac{2\Omega_e}{\pi k_\perp v_\perp}\right)^{1/2}\cos\left(\frac{k_\perp v_\perp}{\Omega_e}-\frac{\pi}{4}\right)$$

oscillatory integral, sign changes with period

$$\frac{\Delta v_\perp}{v_{\mathrm{th}e}} = \frac{2\pi}{k_\perp\rho_e}$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\,\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

from J_0 (gyroaverging)

integral accumulates as a random walk,

$$N \sim \frac{v_{\mathrm{th}e}}{\delta v_\perp} \sim k_\perp\rho_e$$

[AAS et al. 2008, PPCF 50, 24024]

Nonlinear Phase Mixing

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

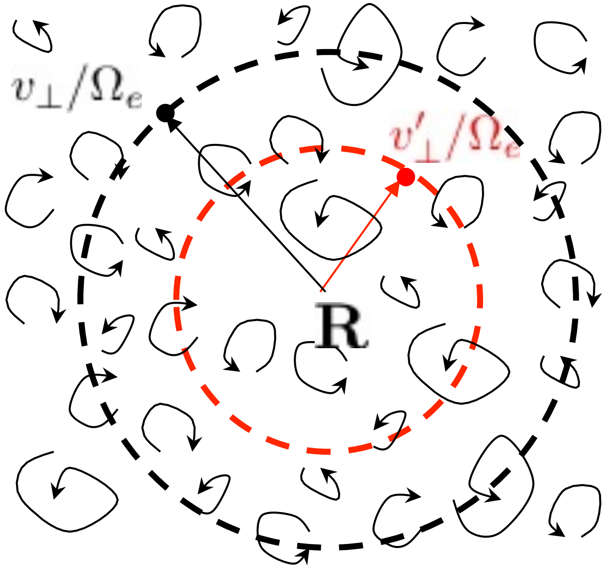
we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp\rho_e}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_{\mathbf{k}}(v_\perp)$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

$$\frac{\partial h}{\partial t} + v_\parallel\nabla_\parallel h + \langle\mathbf{u}_\perp\rangle_{\mathbf{R}}\cdot\nabla_\perp h = -\frac{\partial\langle\varphi\rangle_{\mathbf{R}}}{\partial t}F_0 + C[h] + \chi$$



Two values of gyroveraged $\mathbf{E}\times\mathbf{B}$ velocity $\langle\mathbf{u}_\perp\rangle_{\mathbf{R}}(v_\perp)$ and $\langle\mathbf{u}_\perp\rangle_{\mathbf{R}}(v'_\perp)$ come from spatially decorrelated fluctuations if

$$\left|\frac{v_\perp}{\Omega_e} - \frac{v'_\perp}{\Omega_e}\right| \gtrsim \frac{1}{k_\perp} \quad\Rightarrow\quad \boxed{\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp\rho_e}}$$

coherence scale in velocity space, q.e.d.

[AAS et al. 2008, PPCF 50, 24024]

Nonlinear Phase Mixing

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

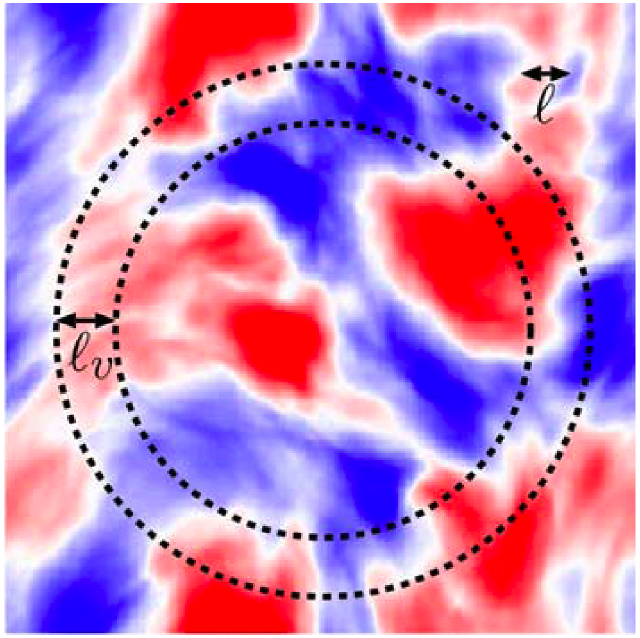
we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{\rm the}} \sim \frac{1}{k_\perp\rho_e}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_\mathbf{r} = \alpha\sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_\mathbf{k}(v_\perp)$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

$$\frac{\partial h}{\partial t} + v_\parallel\nabla_\parallel h + \langle\mathbf{u}_\perp\rangle_\mathbf{R}\cdot\nabla_\perp h = -\frac{\partial\langle\varphi\rangle_\mathbf{R}}{\partial t}F_0 + C[h] + \chi$$



Two values of gyroveraged $\mathbf{E}\times\mathbf{B}$ velocity $\langle\mathbf{u}_\perp\rangle_\mathbf{R}(v_\perp)$ and $\langle\mathbf{u}_\perp\rangle_\mathbf{R}(v'_\perp)$ come from spatially decorrelated fluctuations if

$$\left|\frac{v_\perp}{\Omega_e} - \frac{v'_\perp}{\Omega_e}\right| \gtrsim \frac{1}{k_\perp} \quad\Rightarrow\quad \boxed{\frac{\delta v_\perp}{v_{\rm the}} \sim \frac{1}{k_\perp\rho_e}}$$

coherence scale in velocity space, q.e.d.

[Tatsuno et al. 2009, PRL 103, 015003]

Entropy Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

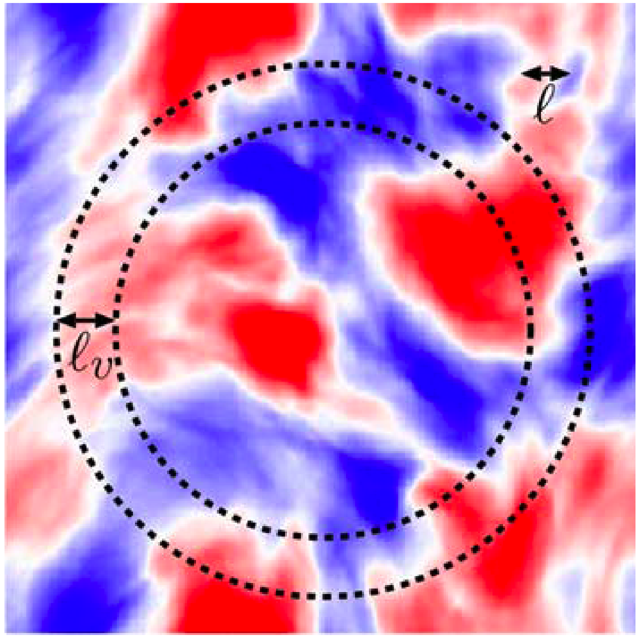
we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp \rho_e}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_{\mathbf{k}}(v_\perp)$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

Thus, we have a phase-space cascade ("entropy cascade"), simultaneous in position and velocity.



$\boxed{\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp\rho_e}}$ coherence scale in velocity space.

[Tatsuno et al. 2009, PRL 103, 015003]

Entropy Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

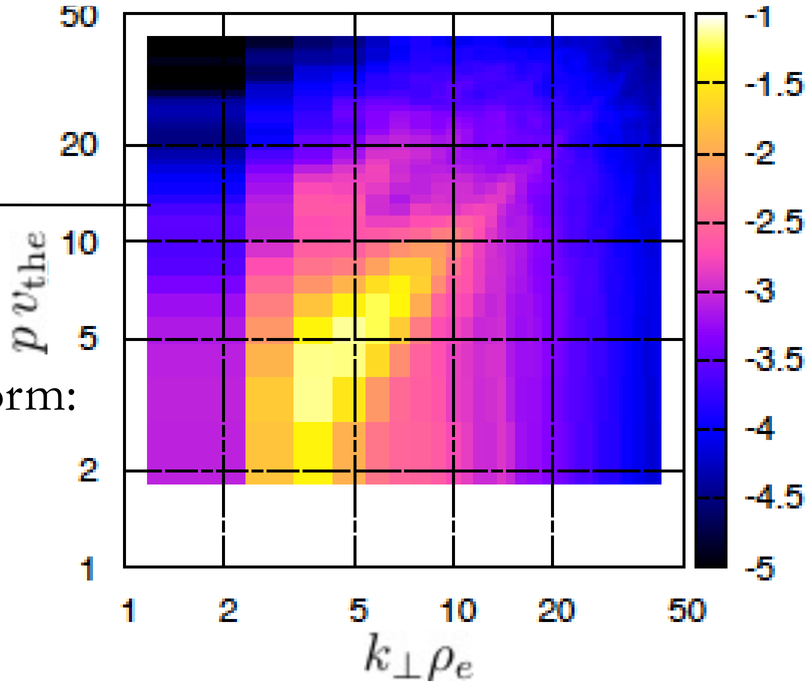
…and we now need a relationship between φ and $\hat{h}$:

we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{\mathrm{the}}} \sim \frac{1}{k_\perp\rho_e}$

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_{\mathbf{k}}(v_\perp)$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

Thus, we have a phase-space cascade ("entropy cascade"), simultaneous in position and velocity.



Spectral representation in terms of Hankel transform:

$$\tilde{h}_{\mathbf{k}}(p) = 2\pi\int dv_\perp v_\perp J_0(pv_\perp)h_{\mathbf{k}}(v_\perp)$$

Phase-space spectrum: $E_h(k_\perp, p) = p\,\overline{\left|\tilde{h}_{\mathbf{k}}(p)\right|^2}$

[Plunk et al. 2010, JFM, 664, 407]

$\boxed{p\,v_{\mathrm{the}} \sim k_\perp\rho_e}$ coherence scale in velocity space.

[Tatsuno et al. 2009, PRL 103, 015003]

Entropy Cascade

Constant flux of free energy: $\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}$ $\Rightarrow$ $\hat{h}^3 \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-1/2}$

$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

Entropy Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$ $\Rightarrow$ $\hat{h}^3 \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-1/2}$

$$\boxed{\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}}$$

$$\boxed{\hat{h} \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-1/6}} \Rightarrow E_h \propto k_\perp^{-4/3}$$

$$\boxed{\varphi \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-7/6}} \Rightarrow E_\varphi \propto k_\perp^{-10/3}$$

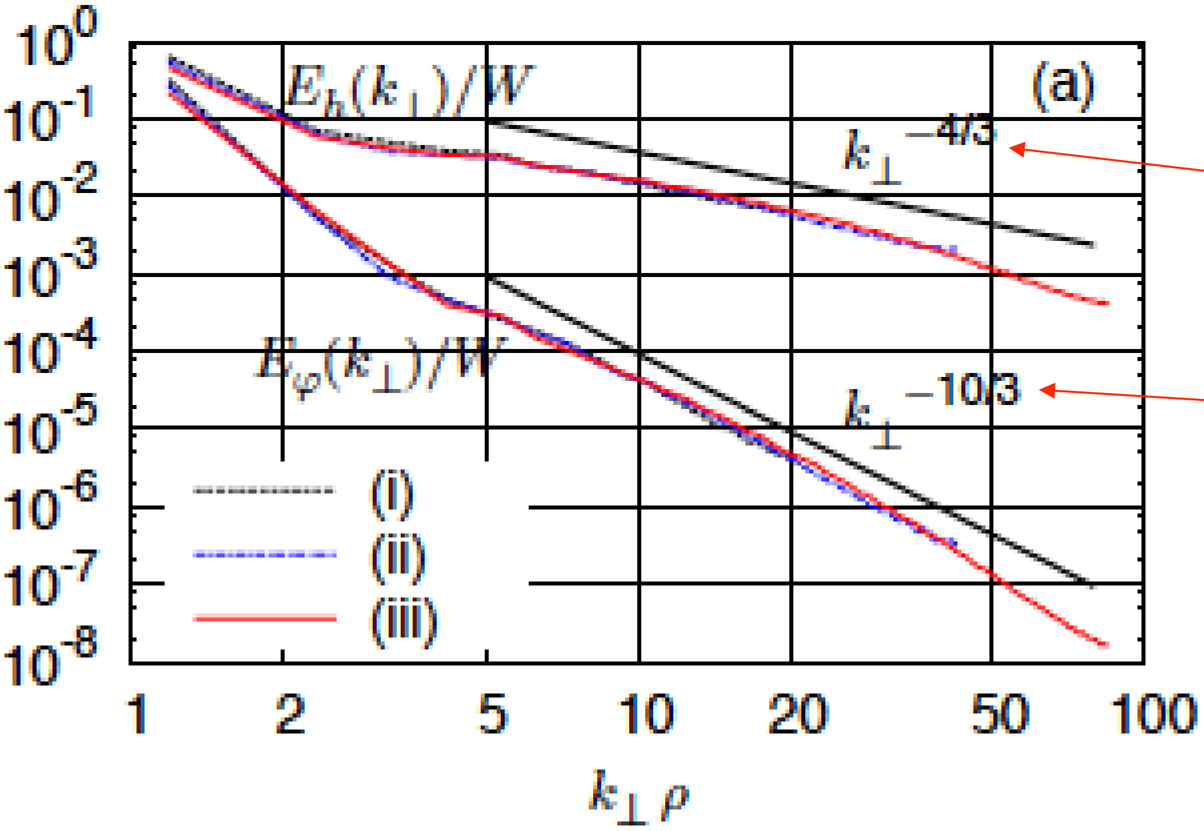
$$E \sim k_\perp\phi \Rightarrow E_E \propto k_\perp^{-4/3}$$

$$\boxed{\frac{\delta B}{B_0} \sim \frac{\beta_e}{k_\perp\rho_e}\varphi \sim \beta_e\left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-13/6}}$$

$$\Rightarrow E_B \propto k_\perp^{-16/3}$$

Theory vs. Simulations

GK SIMULATIONS by T. Tatsuno (2D, electrostatic, decaying):



[Tatsuno et al. 2009, PRL 103, 015003]

THEORY:

$E_h \propto k_\perp^{-4/3}$

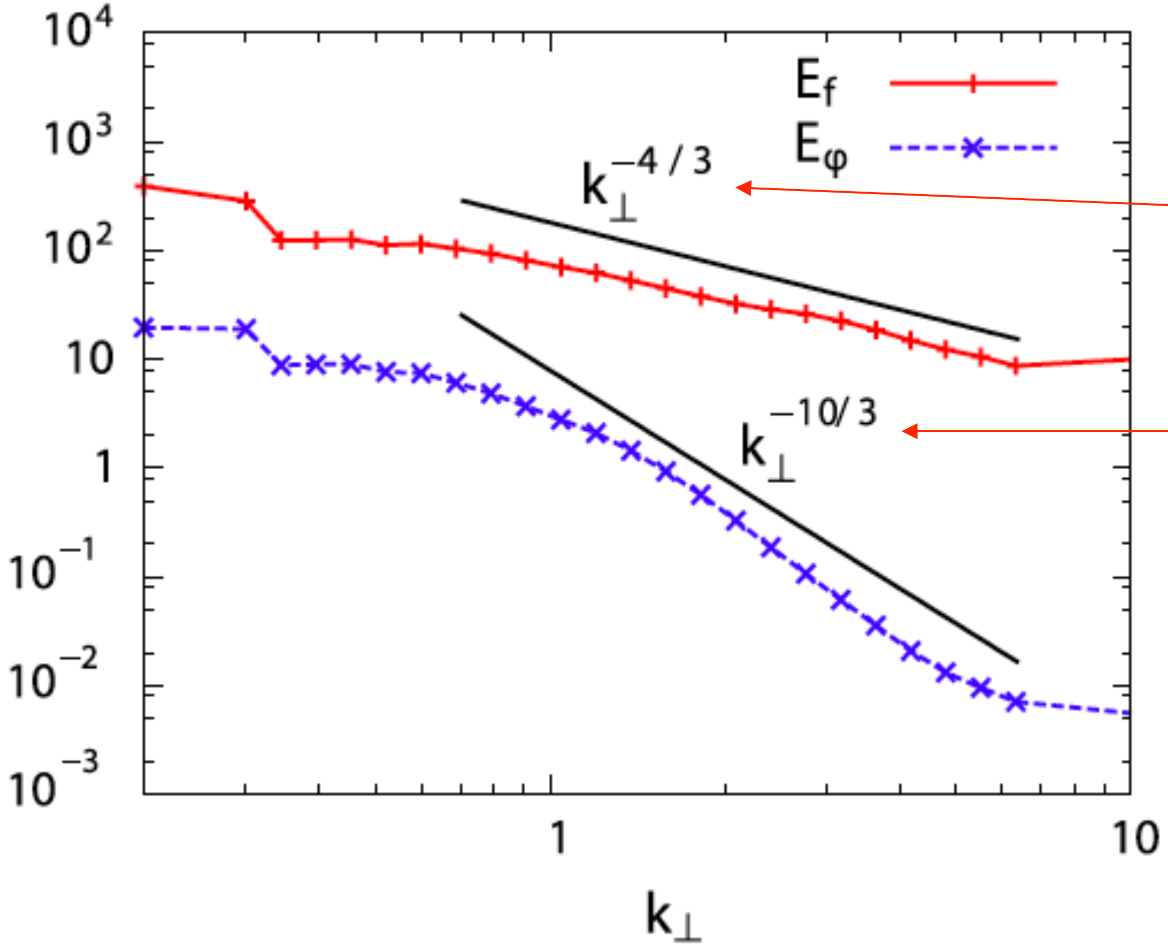
$E_\varphi \propto k_\perp^{-10/3}$

$E_E \propto k_\perp^{-4/3}$

$E_B \propto k_\perp^{-16/3}$

Theory vs. Simulations

GK SIMULATIONS (3D electrostatic, ITG):



[Banon Navarro et al. 2011, PRL 106, 055001]

THEORY:

$E_h \propto k_\perp^{-4/3}$

$E_\varphi \propto k_\perp^{-10/3}$

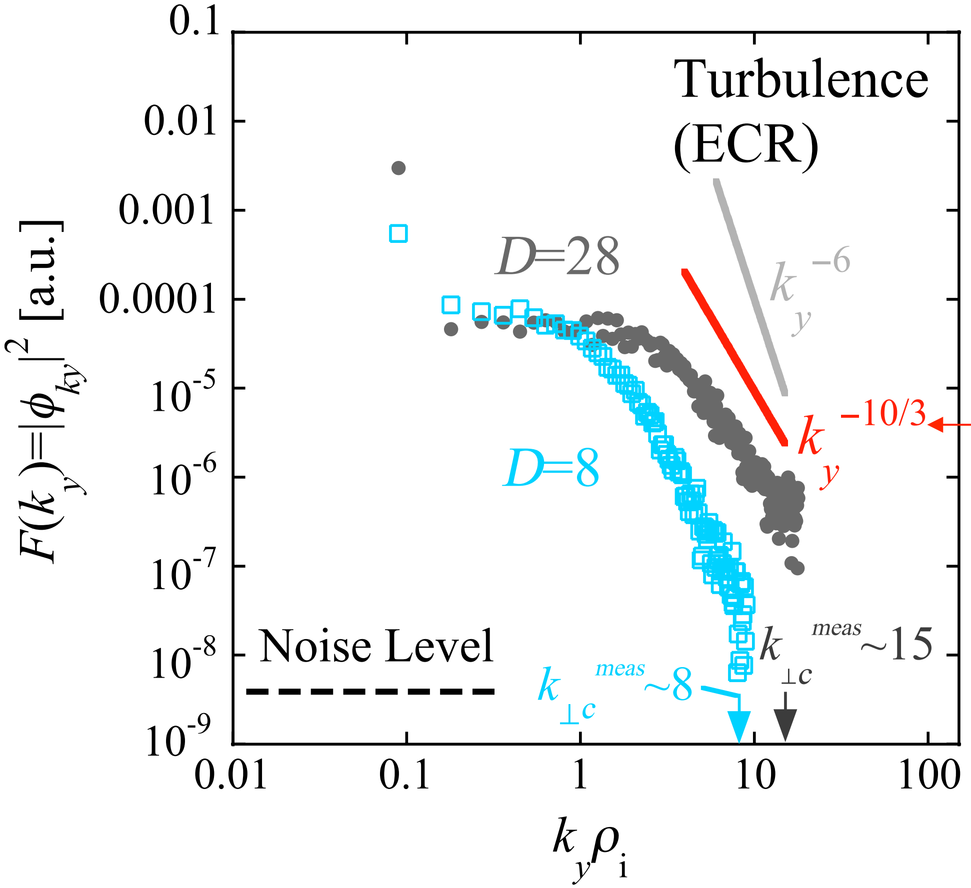
$E_E \propto k_\perp^{-4/3}$

$E_B \propto k_\perp^{-16/3}$

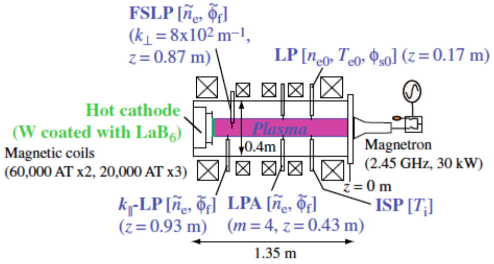
This was done for ion entropy cascade, but in the electrostatic limit, the theory and results are exactly the same [AAS et al. 2008, PPCF 50, 24024]

Theory vs. Experiment!

LABORATORY EXPERIMENT:



[Kawamori (2013), PRL 110, 195001]



THEORY:

$E_h \propto k_\perp^{-4/3}$

$E_\varphi \propto k_\perp^{-10/3}$

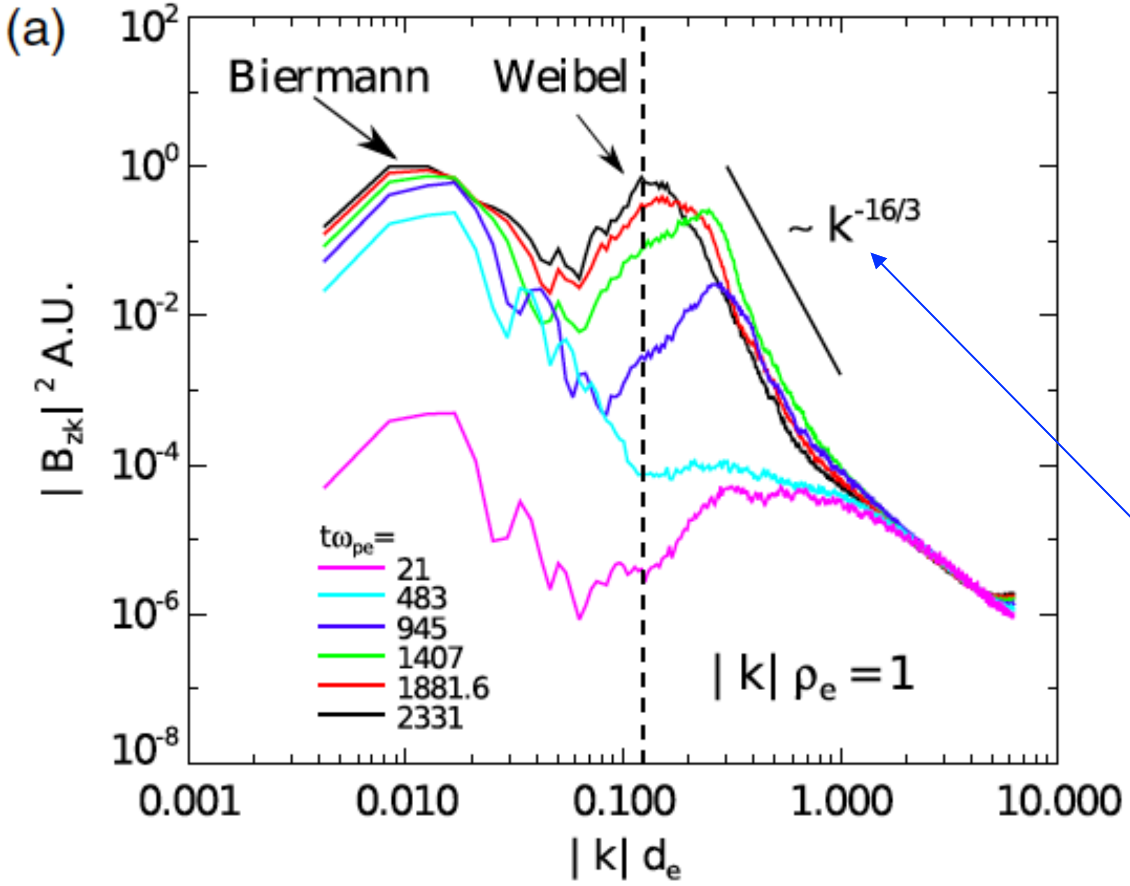
$E_E \propto k_\perp^{-4/3}$

$E_B \propto k_\perp^{-16/3}$

This was done for ion entropy cascade, but in the electrostatic limit, the theory and results are exactly the same [AAS et al. 2008, PPCF 50, 24024]

Theory vs. Simulations

PIC SIMULATIONS (3D, self-generated m. field):



[Schoeffler et al. (2014), PRL 112, 175001]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

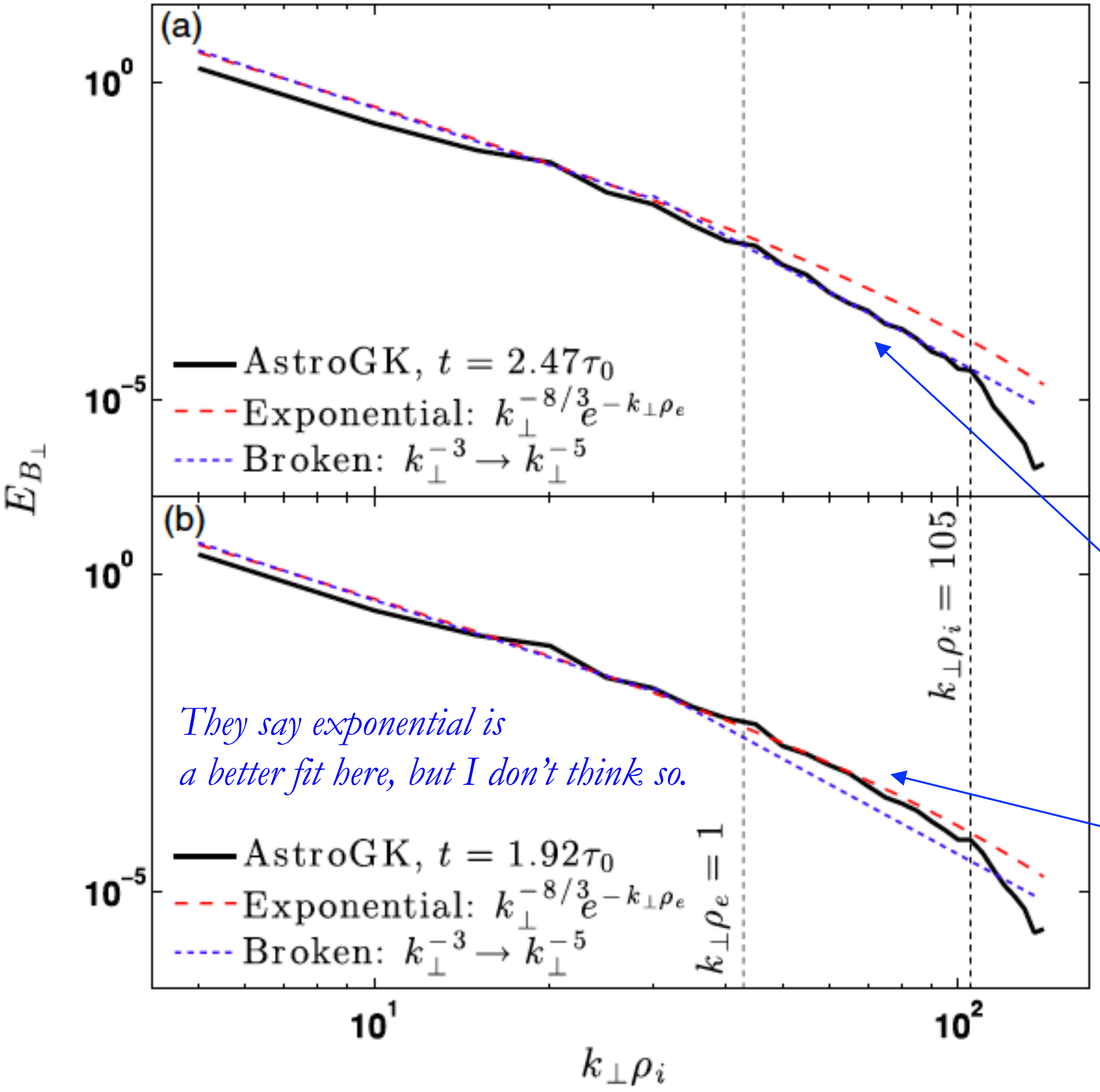
$$E_\varphi \propto k_\perp^{-10/3}$$

$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

Theory vs. Simulations

GK SIMULATIONS by J. TenBarge (3D, forced):



They say exponential is a better fit here, but I don't think so.

[TenBarge et al. (2013), ApJ 774, 139]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

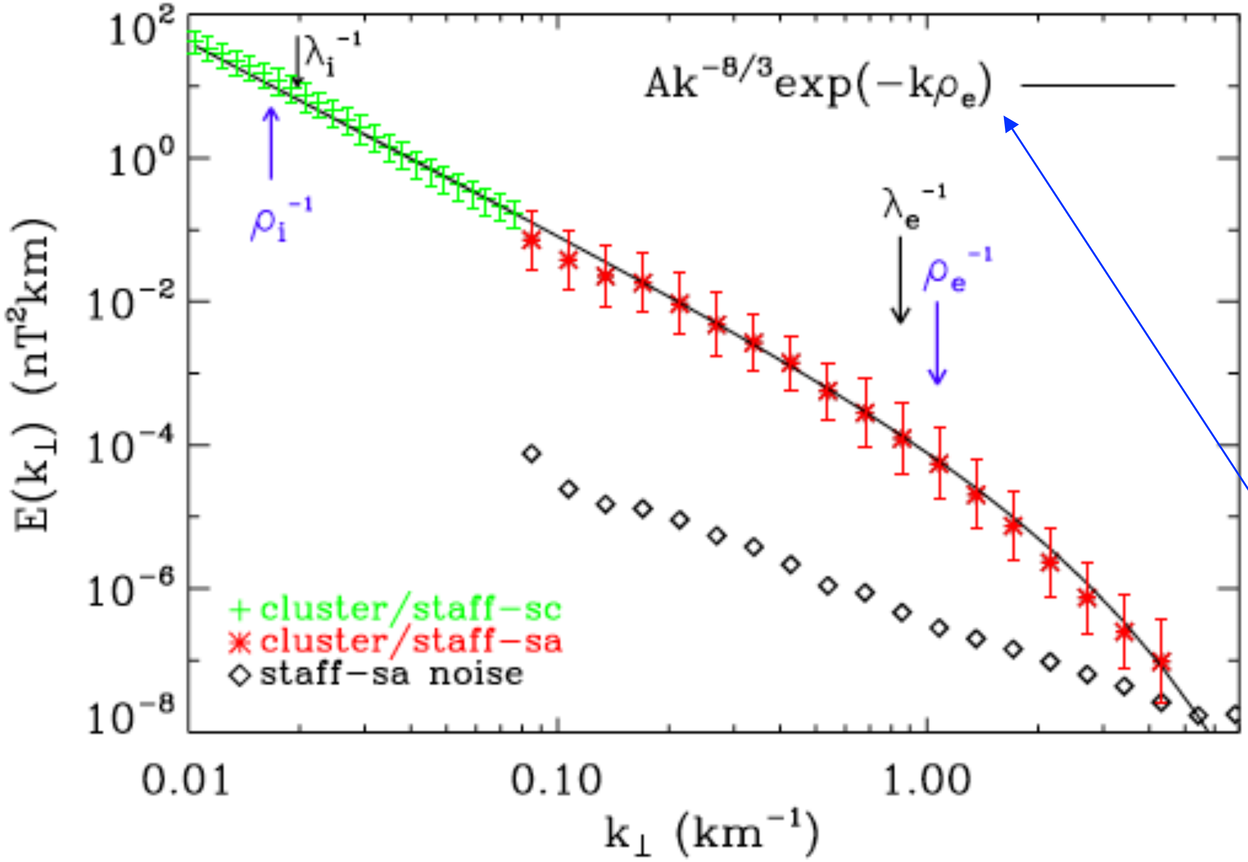
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

SOLAR WIND OBSERVATIONS (Cluster):



[Alexandrova et al. (2012), ApJ 760, 121]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

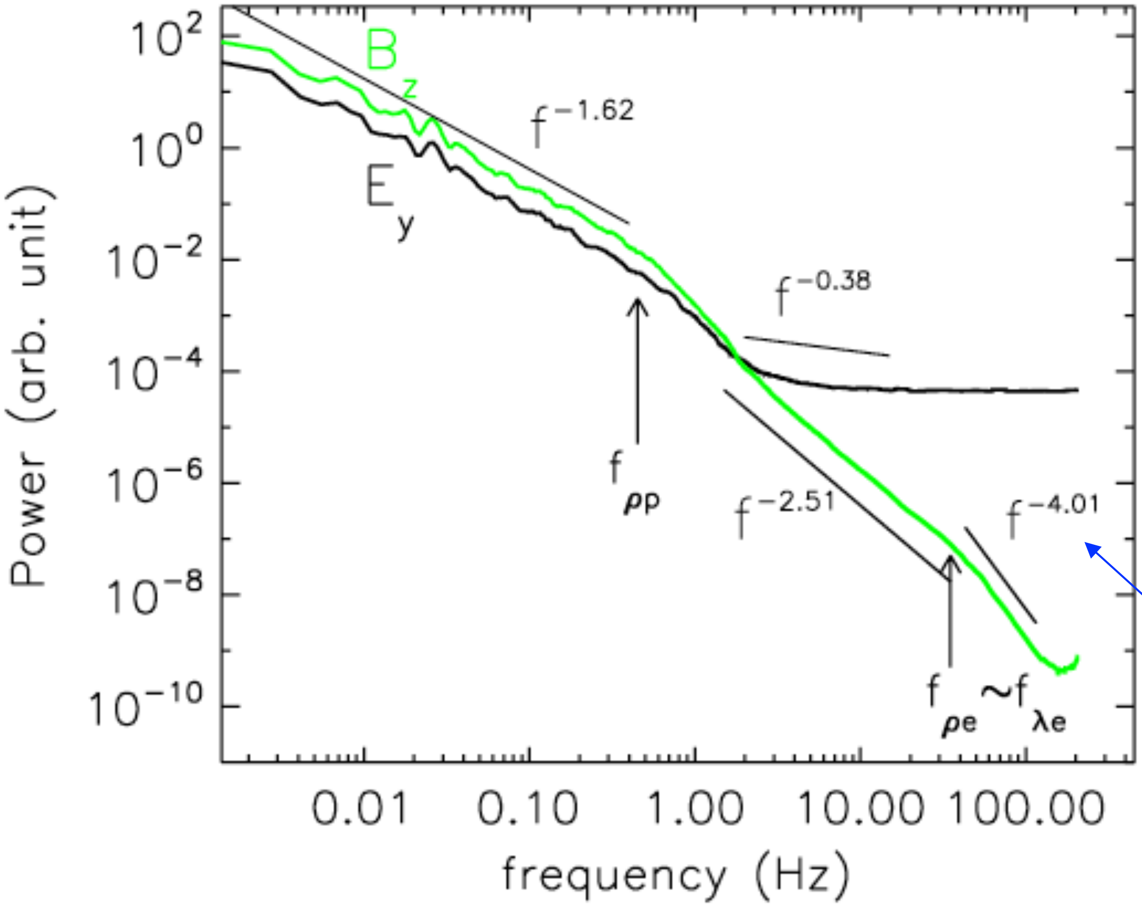
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

SOLAR WIND OBSERVATIONS (Cluster):



[Sahraoui et al. (2009), PRL 102, 231102]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

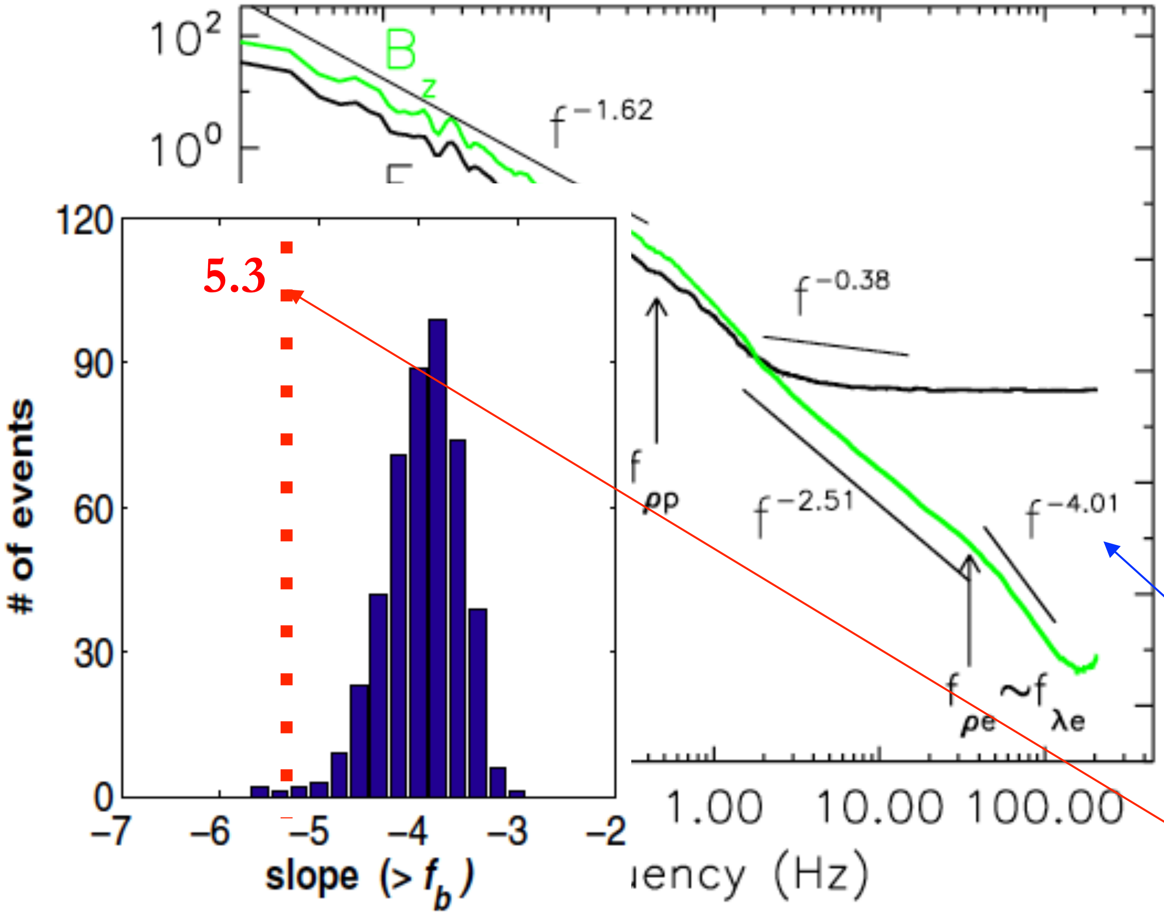
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

SOLAR WIND OBSERVATIONS (Cluster):



[Sahraoui et al. (2013), ApJ 777, 15]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

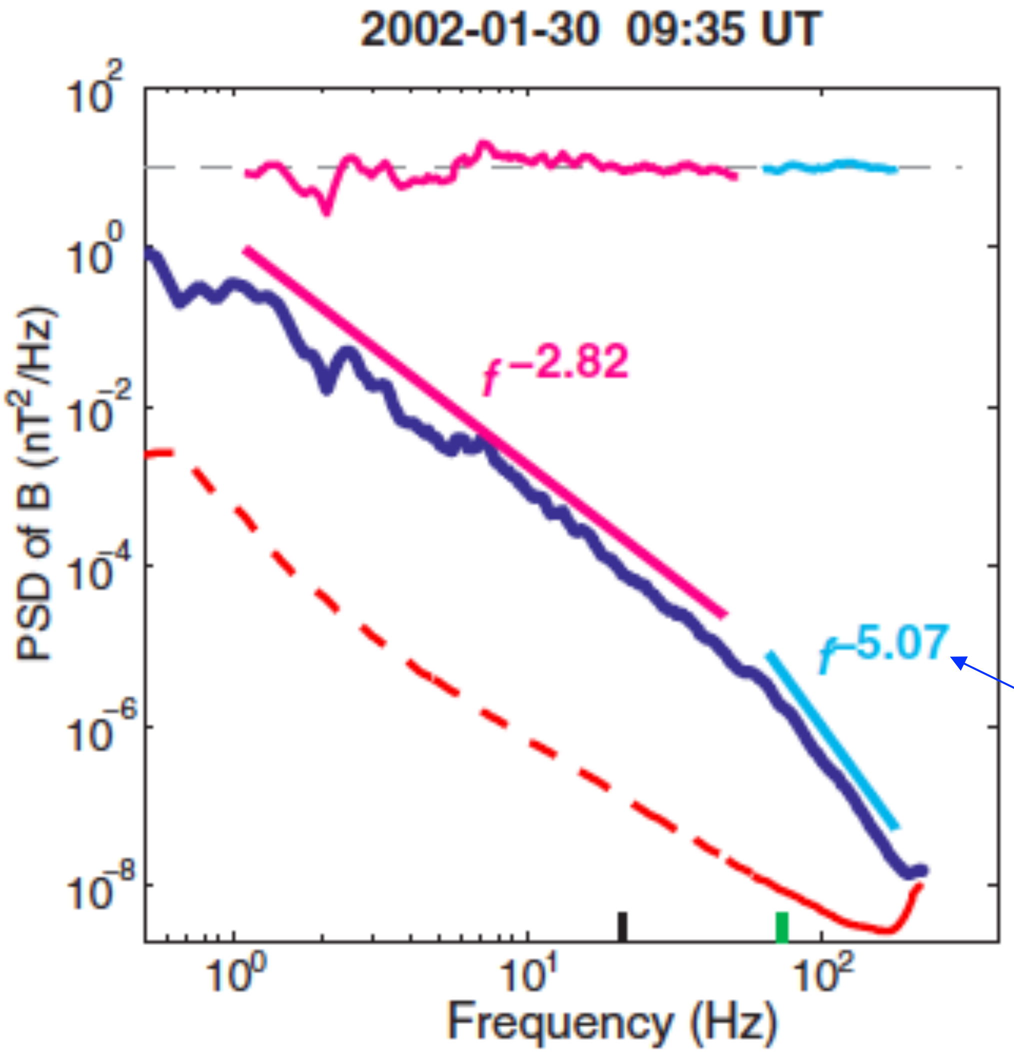
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

MAGNETOSHEATH OBSERVATIONS (Cluster):



[Huang, Sahraoui et al. (2014), ApJ 789, L28]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

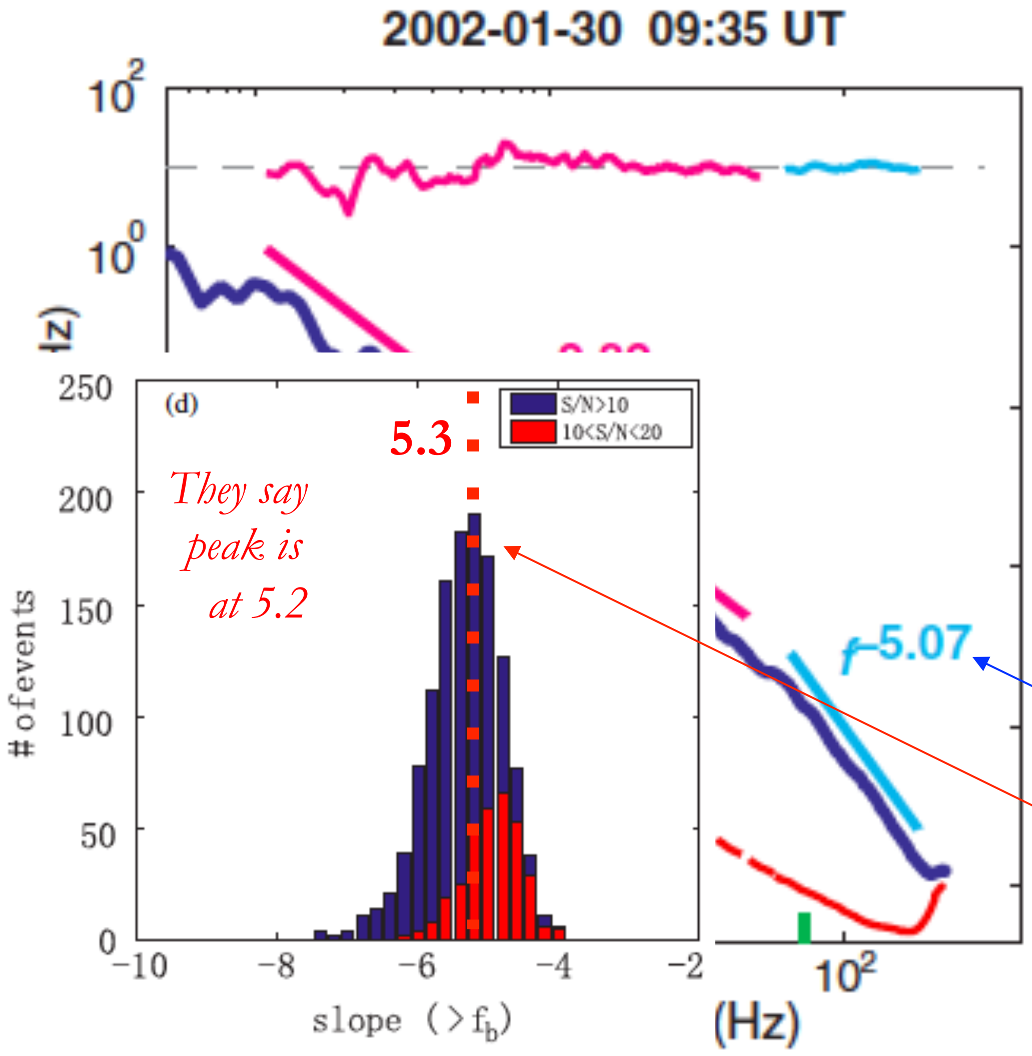
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

MAGNETOSHEATH OBSERVATIONS (Cluster):



5.3

They say peak is at 5.2

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

[Huang, Sahraoui et al. (2014), ApJ 789, L28]

“Kolmogorov” Scale

Where does the electron entropy cascade cut off?

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + \underset{\uparrow}{C[h]} + \chi$$

nonlinear advection

collisional dissipation

$$\tau^{-1} \sim k_\perp \langle u_\perp \rangle_\mathbf{R} \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi$$

$$\sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$$

$$C \sim \nu_e v_{the}^2 \frac{\partial^2}{\partial v_\perp^2} \sim \nu_e \left(\frac{\delta v_\perp}{v_{the}}\right)^{-2}$$

$$\sim \nu_e (k_\perp \rho_e)^2$$

because

$$\varphi \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{-7/6}$$

[AAS et al. 2008, PPCF 50, 24024]

“Kolmogorov” Scale

Where does the electron entropy cascade cut off?

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \underset{\substack{\uparrow \\ \text{nonlinear advection}}}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + \underset{\substack{\uparrow \\ \text{collisional dissipation}}}{C[h]} + \chi$$

$$\tau^{-1} \sim k_\perp \langle u_\perp \rangle_{\mathbf{R}} \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi \qquad C \sim \nu_e v_{\mathrm{the}}^2 \frac{\partial^2}{\partial v_\perp^2} \sim \nu_e \left(\frac{\delta v_\perp}{v_{\mathrm{the}}} \right)^{-2}$$

$$\boxed{\Omega_e \left(\frac{\varepsilon}{\Omega_e} \right)^{1/3} (k_\perp \rho_e)^{1/3} \quad \sim \quad \nu_e (k_\perp \rho_e)^2}$$

$$\Downarrow$$

Collisional cutoff: $\displaystyle \frac{1}{k_{\perp c} \rho_e} \underset{\uparrow}{\sim} \frac{\delta v_{\perp c}}{v_{\mathrm{the}}} \sim \underset{\uparrow}{(\nu_e \tau_{\rho_e})^{3/5}} \equiv \underset{\uparrow}{\mathrm{Do}^{-3/5}}$

NB: spatial and velocity
resolution are linked!

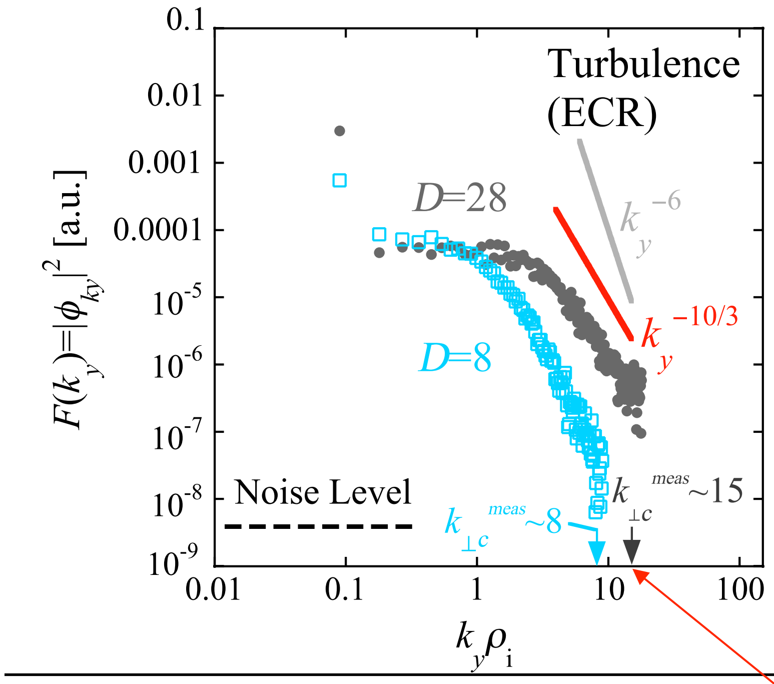
nonlinear time
at Larmor scale

“Dorland number”

$$\tau_{\rho_e}^{-1} \sim \Omega_e \varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e} \frac{\delta B_{\rho_e}}{B_0}$$

[AAS et al. 2008, PPCF 50, 24024]

“Kolmogorov” Scale



This appears to have been checked in a laboratory experiment (for ions)

[Kawamori (2013), PRL 110, 195001]

Collisional cutoff:

$$\frac{1}{k_{\perp c}\rho_e} \sim \frac{\delta v_{\perp c}}{v_{the}} \sim (\nu_e\tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5}$$

NB: spatial and velocity resolution are linked!

nonlinear time at Larmor scale

“Dorland number”

$$\tau_{\rho_e}^{-1} \sim \Omega_e\varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}$$

[AAS et al. 2008, PPCF 50, 24024]

Validity of Low-Frequency Limit

$$\tau^{-1} \sim \tau_{\rho_e}^{-1}(k_\perp\rho_e)^{1/3} \ll \Omega_e \quad \Leftrightarrow \quad k_\perp\rho_e \ll (\Omega_e\tau_{\rho_e})^3 \sim \varphi_{\rho_e}^{-3} \sim \left(\frac{1}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}\right)^{-3}$$

Collisional cutoff:

$$\boxed{\frac{1}{k_{\perp c}\rho_e} \underset{\uparrow}{\sim} \frac{\delta v_{\perp c}}{v_{\mathrm{th}e}} \sim (\nu_e\underset{\uparrow}{\tau_{\rho_e}})^{3/5} \equiv \underset{\uparrow}{\mathrm{Do}}^{-3/5}}$$

NB: spatial and velocity resolution are linked!

nonlinear time at Larmor scale

$$\tau_{\rho_e}^{-1} \sim \Omega_e\varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}.$$

"Dorland number"

[AAS et al. 2008, PPCF 50, 24024]

Validity of Low-Frequency Limit

$$\tau^{-1} \sim \tau_{\rho_e}^{-1}(k_\perp\rho_e)^{1/3} \ll \Omega_e \quad \Leftrightarrow \quad k_\perp\rho_e \ll (\Omega_e\tau_{\rho_e})^3 \sim \varphi_{\rho_e}^{-3} \sim \left(\frac{1}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}\right)^{-3}$$

Thus, the entropy cascade stays within low-frequency limit if $\varphi_{\rho_e} \ll \mathrm{Do}^{-1/5}$, or

$$\boxed{\varphi_{\rho_e} \ll \left(\frac{\nu_e}{\Omega_e}\right)^{1/6}}$$

can't be
too difficult!

Otherwise all sorts of high-frequency physics will kick in…

Collisional cutoff: $$\boxed{\frac{1}{k_{\perp c}\rho_e} \sim \frac{\delta v_{\perp c}}{v_{\mathrm{th}e}} \sim (\nu_e\tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5}}$$

NB: spatial and velocity
resolution are linked!

nonlinear time
at Larmor scale

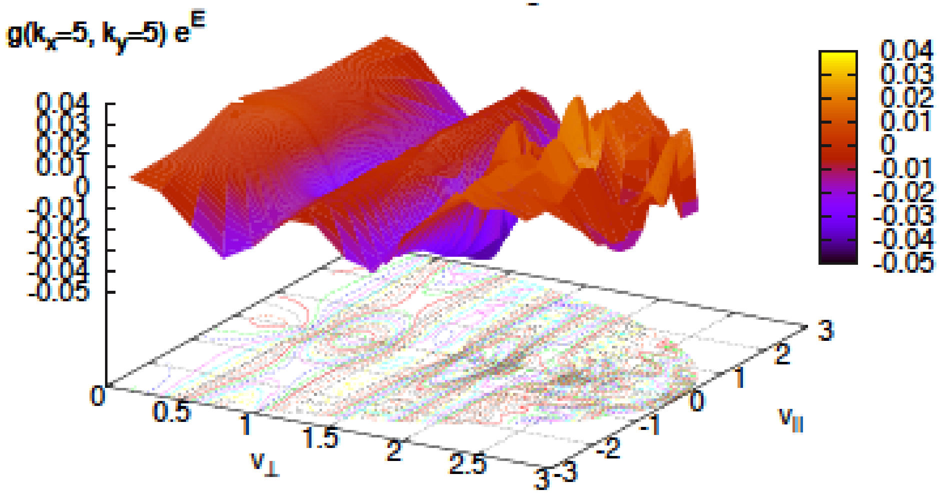
$$\tau_{\rho_e}^{-1} \sim \Omega_e\varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}.$$

"Dorland number"

[AAS et al. 2008, PPCF 50, 24024]

Linear (∥) vs. Nonlinear (⊥) Phase Mixing

Quick treatment:



NONLINEAR (perpendicular):

$$\boxed{\frac{\delta v_{\perp c}}{v_{\mathrm{the}}} \sim (\nu_e\tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5} \ll 1}$$

Since cascade is nonlinear,
mixing occurs in one
turnover time (fast)

Collisional cutoff:

$$\frac{1}{k_{\perp c}\rho_e} \underset{\uparrow}{\sim} \frac{\delta v_{\perp c}}{v_{\mathrm{the}}} \sim \underset{\uparrow}{(\nu_e\tau_{\rho_e})^{3/5}} \equiv \underset{\uparrow}{\mathrm{Do}^{-3/5}}$$

NB: spatial and velocity
resolution are linked!

nonlinear time
at Larmor scale

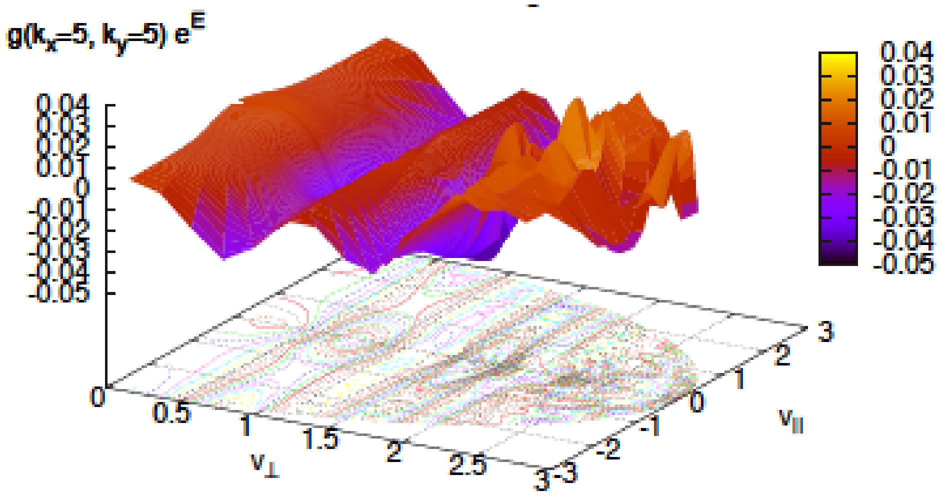
"Dorland number"

$$\tau_{\rho_e}^{-1} \sim \Omega_e\varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}$$

[AAS et al. 2008, PPCF 50, 24024]

Linear ($\|$) vs. Nonlinear ($\perp$) Phase Mixing

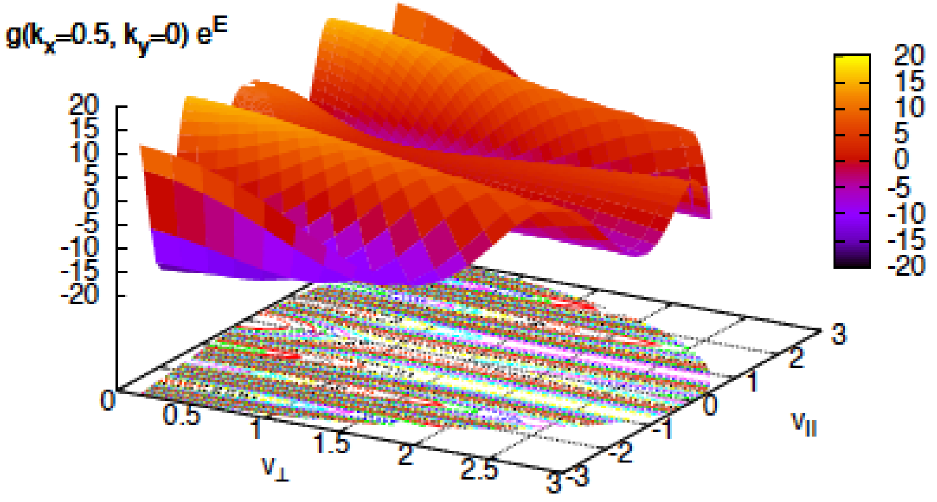
Quick treatment:



NONLINEAR (perpendicular):

$$\boxed{\frac{\delta v_{\perp c}}{v_{\rm the}} \sim (\nu_e \tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5} \ll 1}$$

Since cascade is nonlinear,
mixing occurs in one
turnover time (fast)



LINEAR (parallel):

"ballistic response"

$$\frac{\partial h}{\partial t} + v_\| \nabla_\| h = \dots \quad \Rightarrow \quad h \propto e^{-ik_\| v_\| t}$$

$$\boxed{\frac{\delta v_\|}{v_{\rm the}} \sim \frac{1}{k_\| v_{\rm the}\, t} \sim 1}$$

after one turnover time

if *"critical balance"* holds,
so linear phase mixing is slow

[AAS et al. 2008, PPCF 50, 24024]

Linear Phase Mixing and Critical Balance

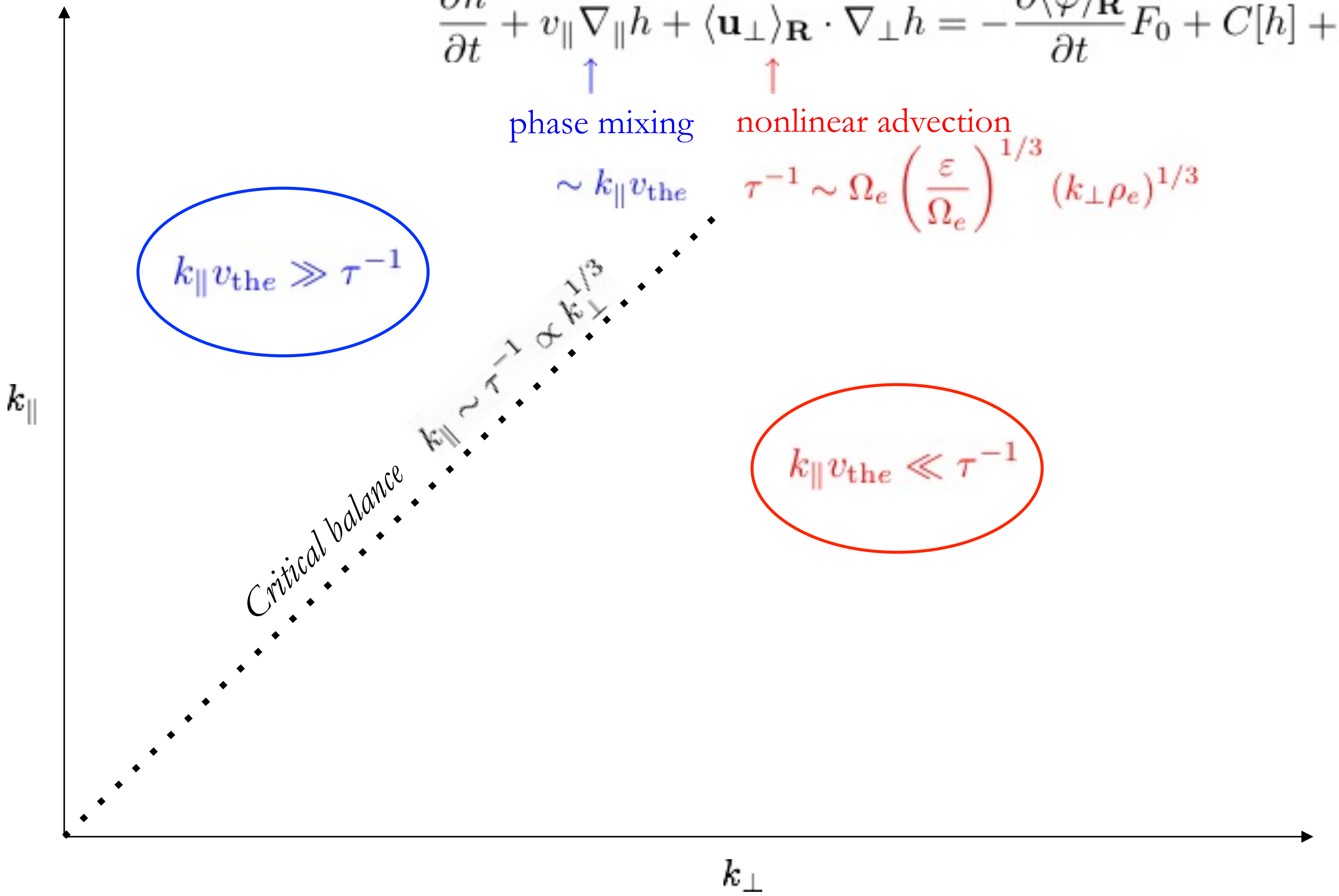
$$\frac{\partial h}{\partial t} + \underset{\uparrow}{v_\parallel \nabla_\parallel h} + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h} = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

phase mixing

$\sim k_\parallel v_{\rm the}$

nonlinear advection

$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$



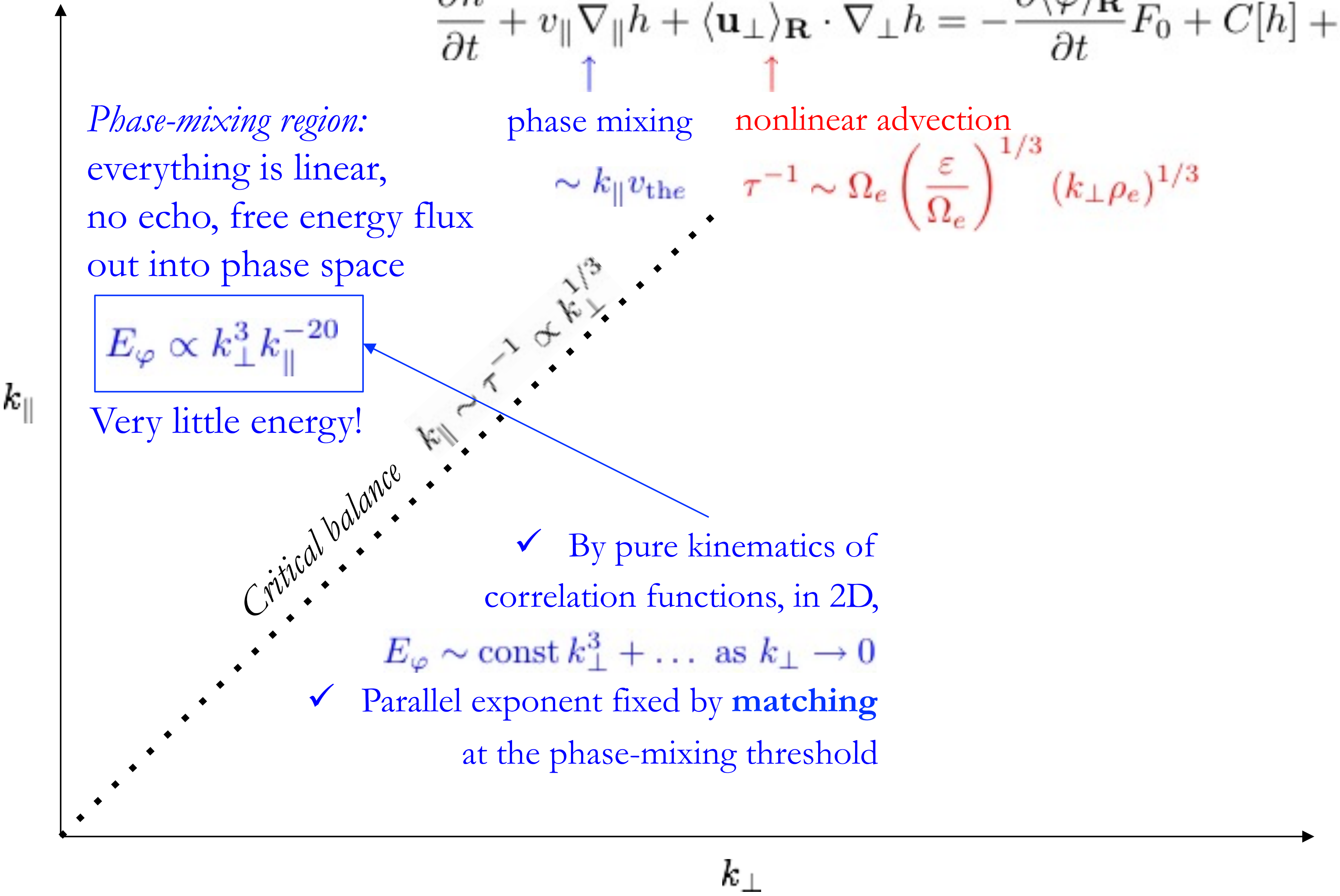
[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

Linear Phase Mixing and Critical Balance

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$\uparrow$ phase mixing $\sim k_\parallel v_{\mathrm{th}e}$

$\uparrow$ nonlinear advection $\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$



Phase-mixing region:
everything is linear,
no echo, free energy flux
out into phase space

$$\boxed{E_\varphi \propto k_\perp^3 k_\parallel^{-20}}$$

Very little energy!

✓ By pure kinematics of correlation functions, in 2D,

$$E_\varphi \sim \text{const}\, k_\perp^3 + \dots \text{ as } k_\perp \to 0$$

✓ Parallel exponent fixed by **matching** at the phase-mixing threshold

[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

Linear Phase Mixing and Critical Balance

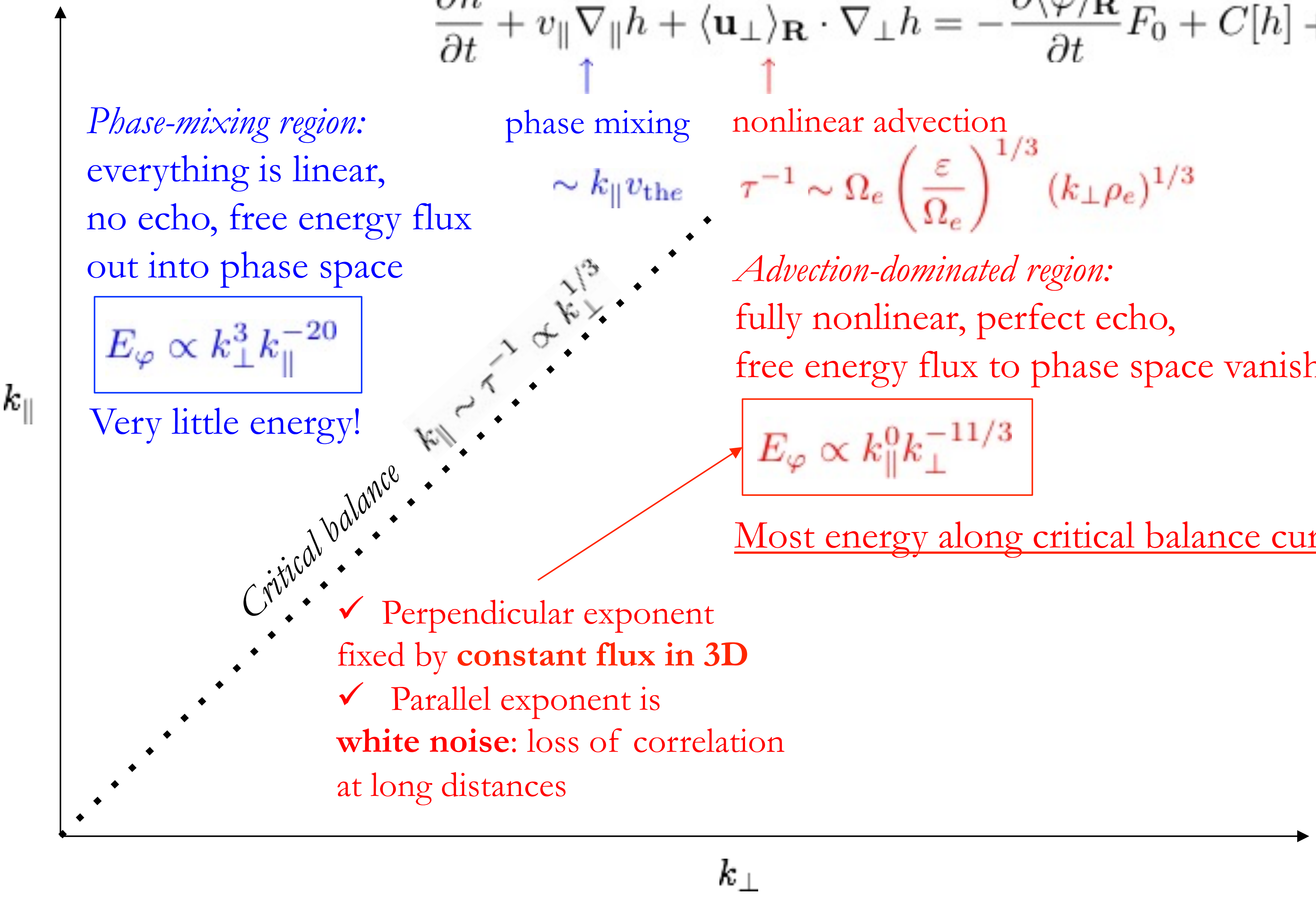
$$\frac{\partial h}{\partial t} + \underset{\uparrow}{v_\parallel \nabla_\parallel h} + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h} = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

phase mixing

$\sim k_\parallel v_{\mathrm{the}}$

nonlinear advection

$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e} \right)^{1/3} (k_\perp \rho_e)^{1/3}$



Phase-mixing region:
everything is linear,
no echo, free energy flux
out into phase space

$$\boxed{E_\varphi \propto k_\perp^3 k_\parallel^{-20}}$$

Very little energy!

Advection-dominated region:
fully nonlinear, perfect echo,
free energy flux to phase space vanishes

$$\boxed{E_\varphi \propto k_\parallel^0 k_\perp^{-11/3}}$$

Most energy along critical balance curve

✓ Perpendicular exponent fixed by **constant flux in 3D**

✓ Parallel exponent is **white noise**: loss of correlation at long distances

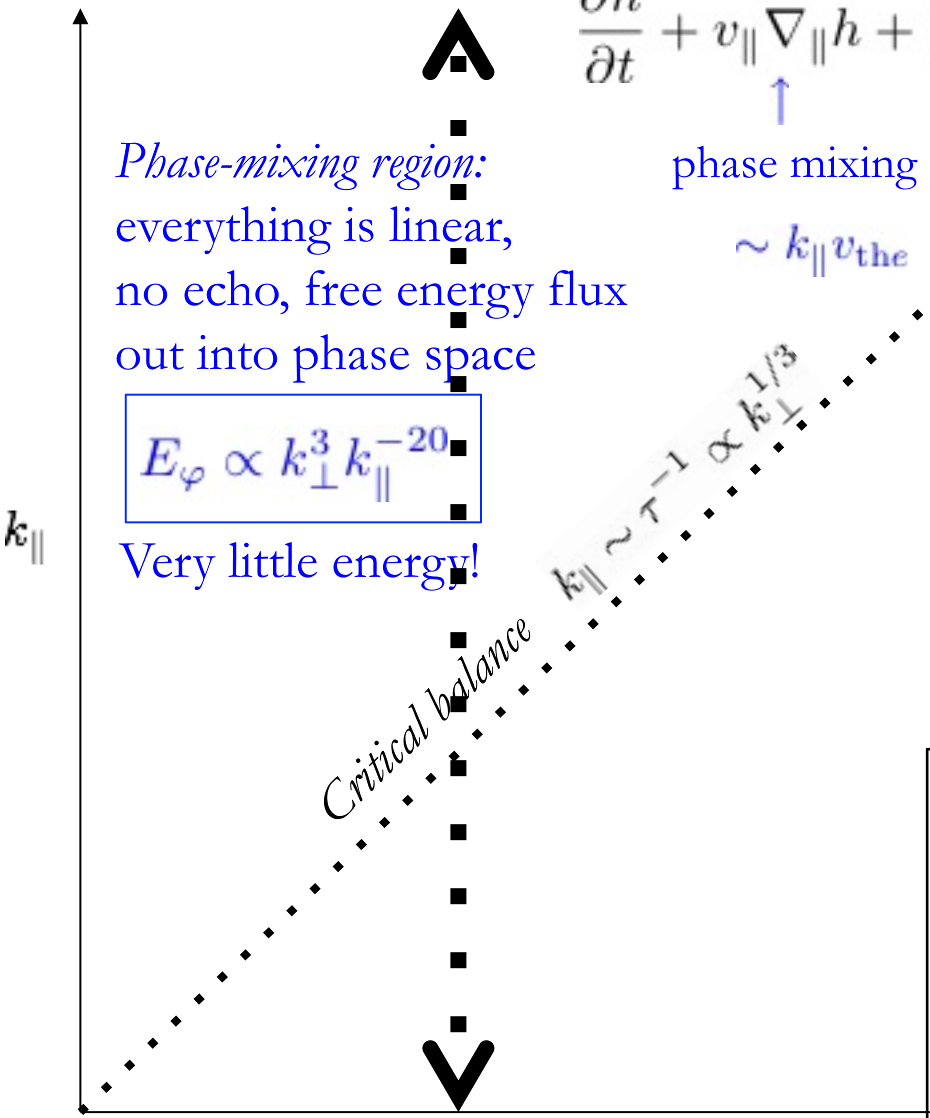
[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

2D Spectra

$$\frac{\partial h}{\partial t} + \underset{\substack{\uparrow \\ \text{phase mixing}}}{v_\parallel \nabla_\parallel h} + \underset{\substack{\uparrow \\ \text{nonlinear advection}}}{\langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h} = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

$\sim k_\parallel v_{\mathrm{the}}$

$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e} \right)^{1/3} (k_\perp \rho_e)^{1/3}$



Phase-mixing region:
everything is linear,
no echo, free energy flux
out into phase space

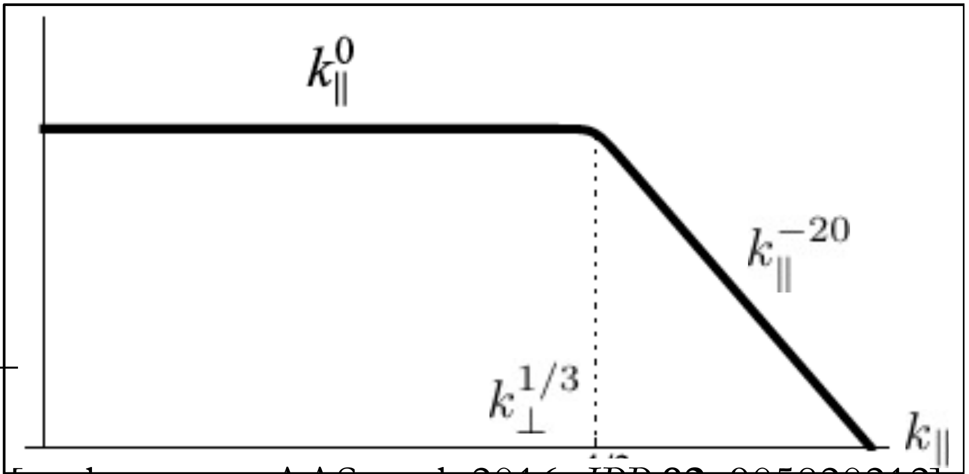
$$\boxed{E_\varphi \propto k_\perp^3 k_\parallel^{-20}}$$

Very little energy!

Advection-dominated region:
fully nonlinear, perfect echo,
free energy flux to phase space vanishes

$$\boxed{E_\varphi \propto k_\parallel^0 k_\perp^{-11/3}}$$

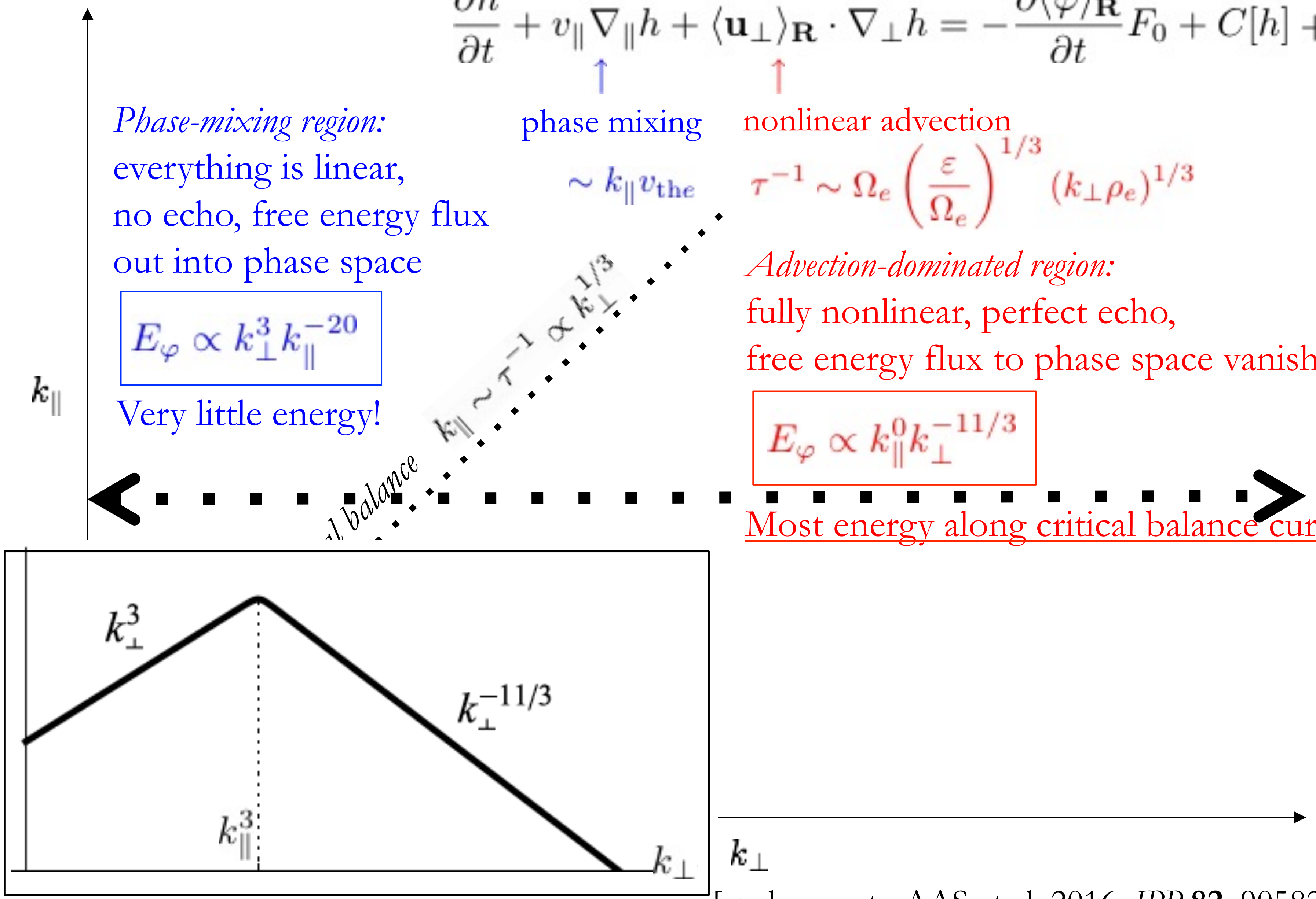
Most energy along critical balance curve



[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

2D Spectra

$$\frac{\partial h}{\partial t} + \underset{\uparrow}{v_\parallel \nabla_\parallel h} + \underset{\uparrow}{\langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h} = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi$$



phase mixing

$\sim k_\parallel v_\mathrm{the}$

nonlinear advection

$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$

Phase-mixing region:
everything is linear,
no echo, free energy flux
out into phase space

$\boxed{E_\varphi \propto k_\perp^3 k_\parallel^{-20}}$

Very little energy!

Advection-dominated region:
fully nonlinear, perfect echo,
free energy flux to phase space vanishes

$\boxed{E_\varphi \propto k_\parallel^0 k_\perp^{-11/3}}$

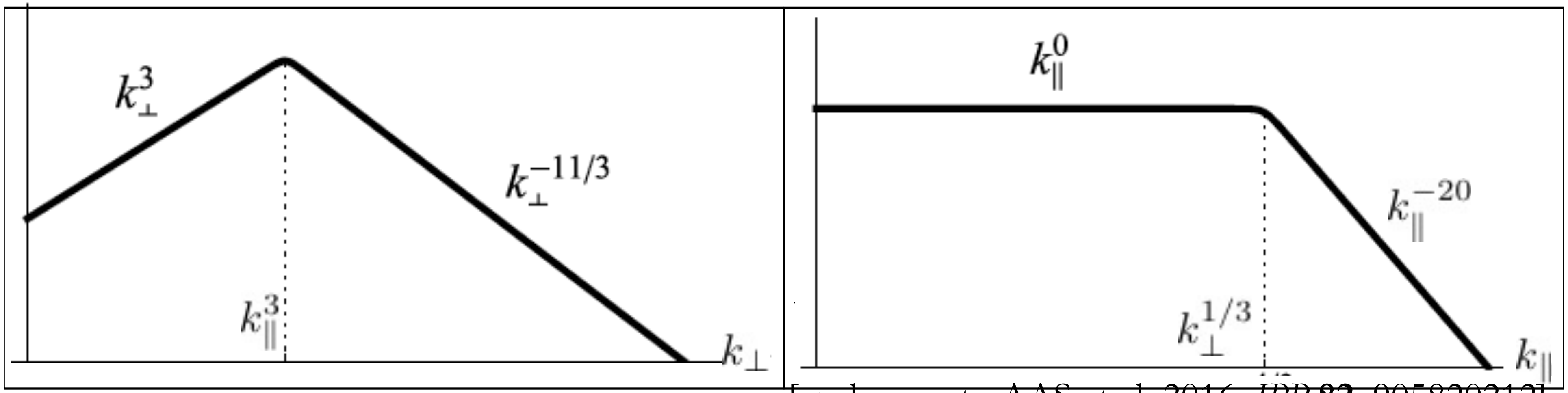
Most energy along critical balance curve

[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

2D Spectra

These are “2D spectra” of φ.

- ➢ Magnetic-field spectra are $E_B(k_\parallel, k_\perp) \propto \dfrac{E_\varphi(k_\parallel, k_\perp)}{k_\perp^2}$



[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

2D Spectra

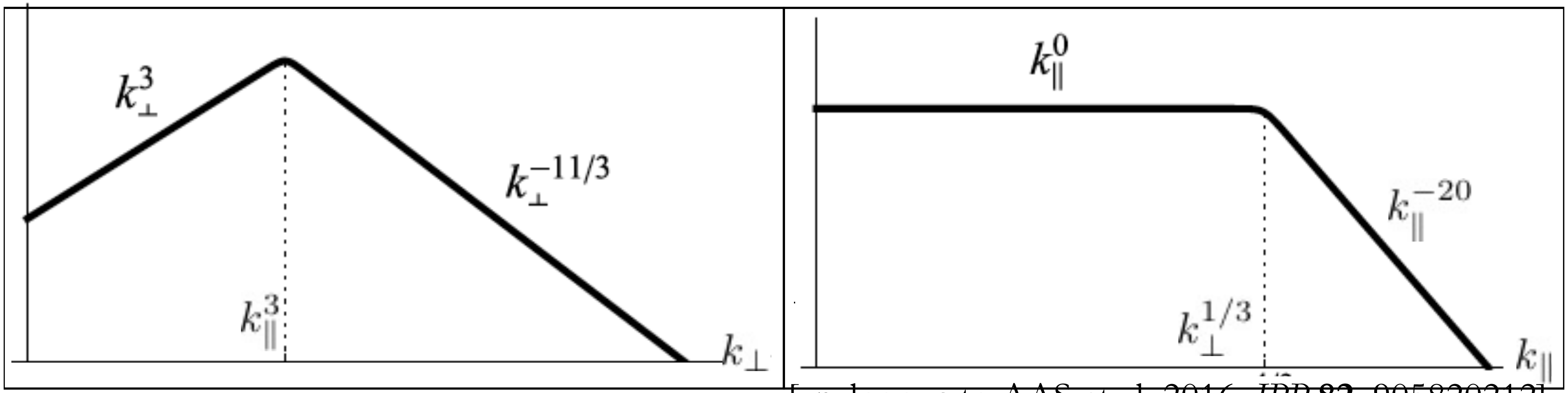
These are “2D spectra” of φ.

- Magnetic-field spectra are $E_B(k_\|, k_\perp) \propto \dfrac{E_\varphi(k_\|, k_\perp)}{k_\perp^2}$
- To get “1D spectra,” integrate over wavenumber ranges bounded by critical balance:

$$E_\varphi^{(1\mathrm{D})}(k_\perp) \propto \int_0^{k_\perp^{1/3}} dk_\| k_\|^0 k_\perp^{-11/3} \sim k_\perp^{-10/3}, \; E_\varphi^{(1\mathrm{D})}(k_\|) \propto \int_{k_\|^3}^{\infty} dk_\perp k_\|^0 k_\perp^{-11/3} \sim k_\|^{-8}$$

(same as derived above)

very steep!
NB: this is also the
frequency spectrum



[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

Phase-Space Spectra

These are "2D spectra" of φ.

- Magnetic-field spectra are $E_B(k_\parallel, k_\perp) \propto \dfrac{E_\varphi(k_\parallel, k_\perp)}{k_\perp^2}$
- To get "1D spectra," integrate over wavenumber ranges bounded by critical balance:

$$E_\varphi^{(1\mathrm{D})}(k_\perp) \propto \int_0^{k_\perp^{1/3}} dk_\parallel k_\parallel^0 k_\perp^{-11/3} \sim k_\perp^{-10/3}, \quad E_\varphi^{(1\mathrm{D})}(k_\parallel) \propto \int_{k_\parallel^3}^{\infty} dk_\perp k_\parallel^0 k_\perp^{-11/3} \sim k_\parallel^{-8}$$

- This all the tip of a larger iceberg – **PHASE-SPACE TURBULENCE:**

Hermite spectrum: $\boxed{E_h(m, k_\parallel) \propto m^{-19/2}}$

$m \sim (\delta v_\parallel / v_{\mathrm{the}})^{-2}$

Spectrum of parallel
phase-mixing:
super-steep, so
Landau damping
is heavily reduced!

Hankel spectrum: $\boxed{E_h(p) \propto p^{-4/3}}$

$p \sim (\delta v_\perp / v_{\mathrm{the}})^{-1}$

Spectrum of perpendicular
phase-mixing (entropy cascade)
[Plunk et al. 2010, JFM, 664, 407]

← Cf. linear case: $E_h \propto m^{-1/2}$
[Kanekar et al. 2014, JPP 81, 305810104]

Details: another talk… or (<u>exercise</u>) derive this yourself by analogy with this paper

↓

[analogous to AAS et al. 2016, *JPP* **82**, 905820212]

Conclusions

- Turbulence associated with the kinetic species at sub-Larmor scales can be understood in terms of entropy cascade, intimately associated with nonlinear perpendicular phase mixing (small-scale spatial structure imprints itself on the velocity space due to Larmor gyration of particles).

- Spectra at electron sub-Larmor scales:

  density $E_n \propto k_\perp^{-10/3}$, electric field $E_E \propto k_\perp^{-4/3}$, magnetic field $E_B \propto k_\perp^{-16/3}$

  These appear to have numerical, experimental and perhaps observational support.

- Parallel phase-mixing is a subdominant effect (but this has **not** been checked!)

- Phase-space dynamics, statistics, scalings, etc. remain largely unexplored.
  THIS IS THE NEW FRONTIER (imho): both for theoreticians & for observers.

"Turbulent Dissipation Challenge"
what it <u>should</u> be about:
cascade via phase space or position space?

THOR?
velocity-space
structure!

PPCF 50, 24024 (2008)
ApJS 182, 310 (2009), sec. 7.12
PRL 103, 015003 (2009)
JPP 82, 905820212 (2016)