The stability of magnetised vortices in protoplanetary discs

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"*God created woman to tame men*" (Voltaire)

"*God created linear theory to tame numerical simulations*" (anon.)

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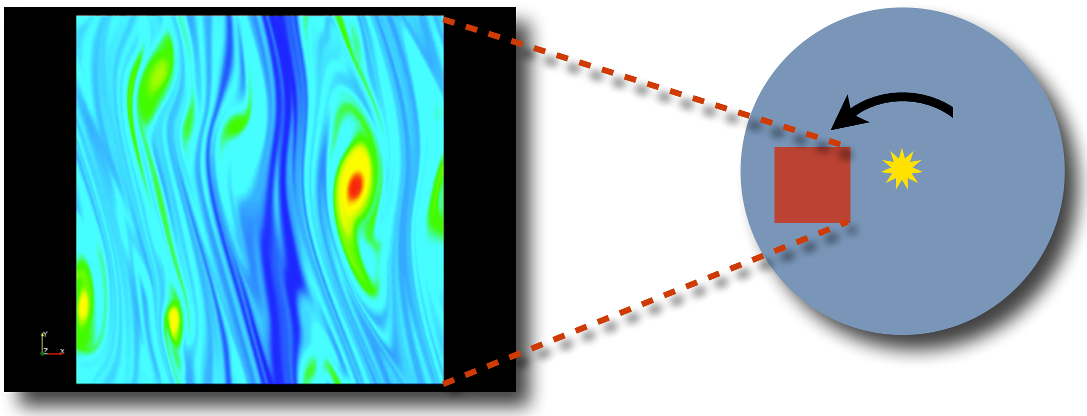
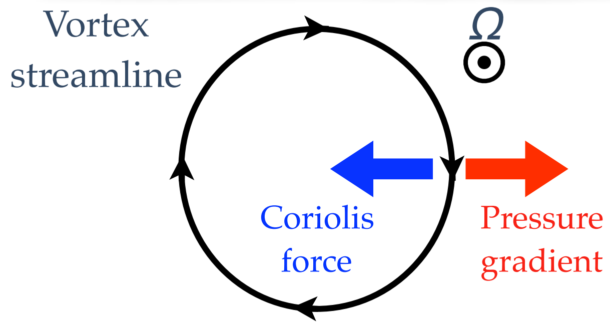
Vortices in accretion discs?

- Initially suggested by von Weizsäcker (1944) to explain planetary formation.
- Reintroduced by Barge & Sommeria (1995) : dust accumulation.
- In discs, only anticyclonic (counter rotating) vortices can survive.

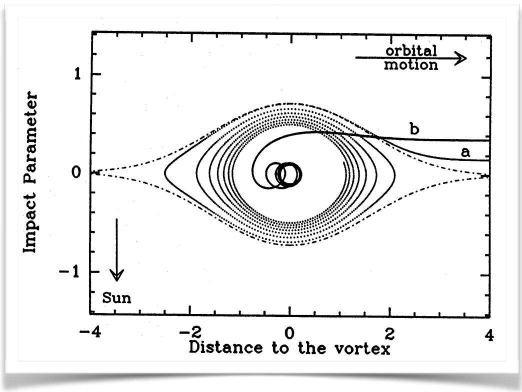


Equilibrium of an anticyclonic vortex



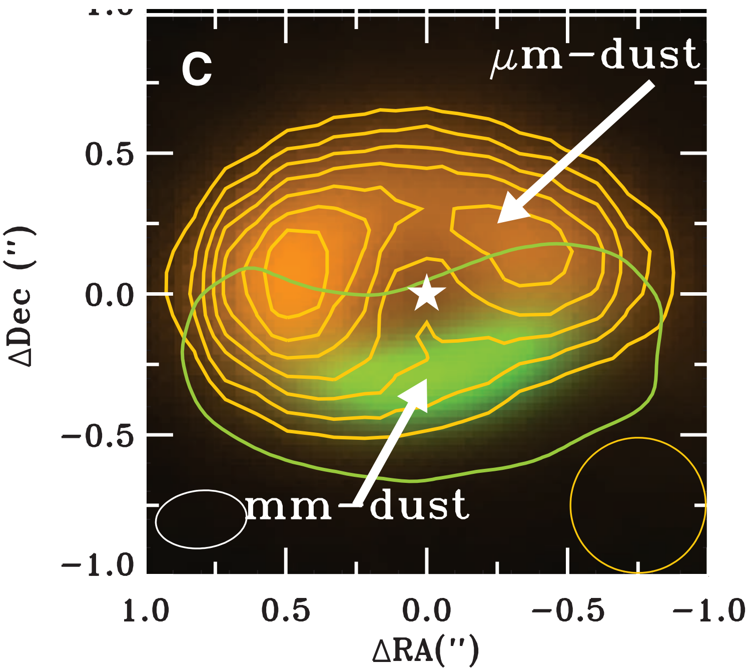
Geostrophic balance: $2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho}$

- Anticyclones are associated with high pressure regions.
- Lagrangian particles tend to accumulate in the centre (they only feel the Coriolis force).

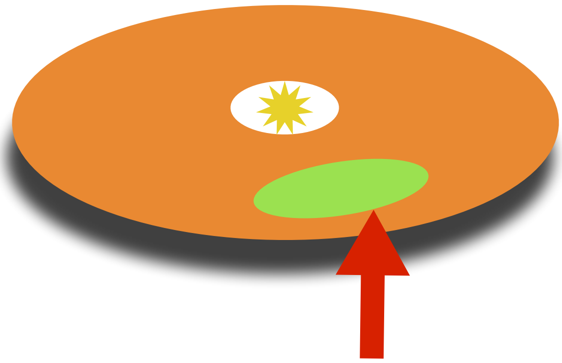


Barge & Sommeria 1995

Evidence of vortices in ppdiscs



[van-der Marel+ (2013)]

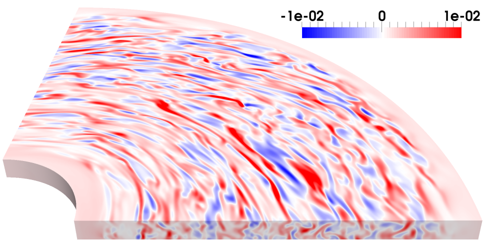


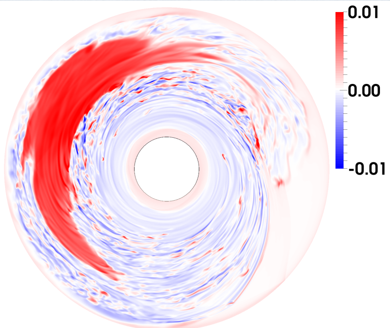
Giant anticyclonic vortex

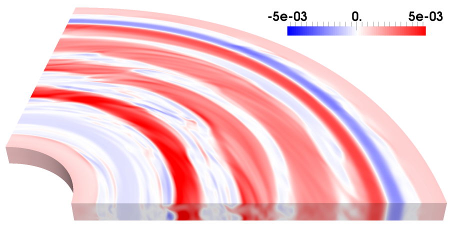
Structures in Hall-dominated discs

Very weak Hall effect

Strong Hall effect







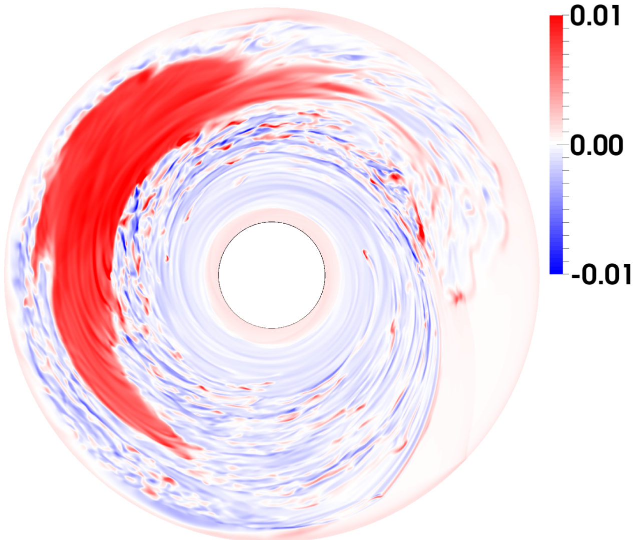
3D turbulence

Large scale vortices

Large scale zonal flows
“rings”

- Hall dominated discs can spontaneously organise [Kunz+ 2013, Bethune+2016]
- Vortices and bands are filled with a strong vertical field

Vortices in Hall-dominated discs



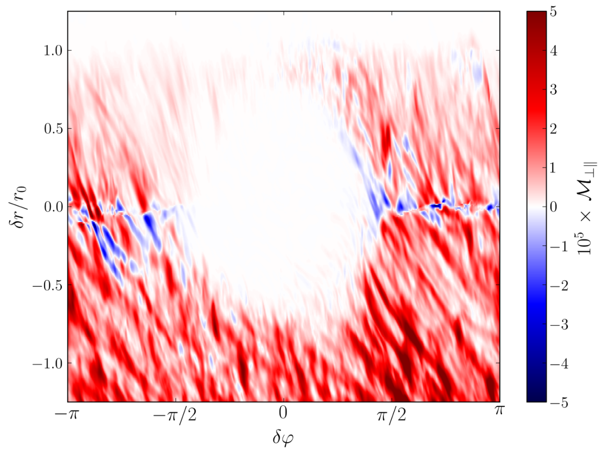


Fig. 14. Projected stress in the (φ, r) plane of run 2πL5, centered on the vortex, averaged in the vertical direction and in time with five snapshots between $250T_0$ and $290T_0$; the vertical axis $\delta r/r_0$ is the radial distance to the measured centre of the vortex.

Vortices look very laminar and stable

Question:
how do vortices survive in a magnetised environment?

Vortex model (Kida 1981)

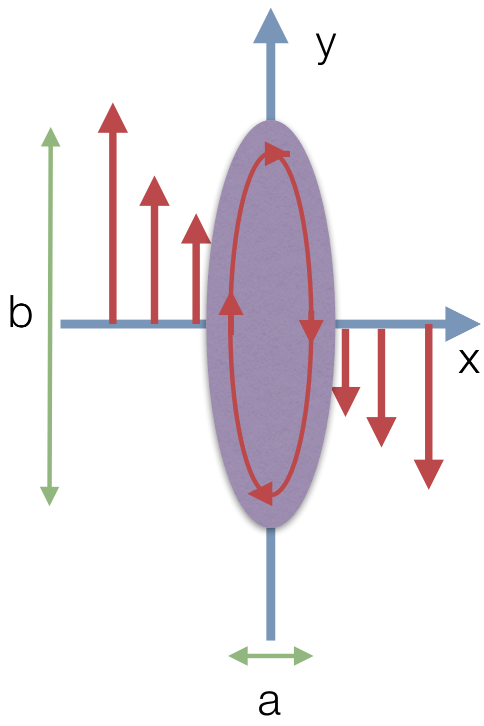
- Consider an elliptical patch of vorticity in a sheared flow
- Pure shear flow $\boldsymbol{U}_0 = -Sx\boldsymbol{e}_y$
- Vortex aspect ratio $X = b/a$
- Total (shear+vortex) velocity field in the vortex

$$\boldsymbol{V}_0 = \frac{S}{X-1}\frac{1}{X}y\boldsymbol{e}_x - \frac{S}{X-1}Xx\boldsymbol{e}_y$$

- Turnover frequency of the vortex

$$\omega_V = \frac{S}{X-1}$$

- Constant vertical field threading the vortex: B_0



Equations

- Full set of equations

$$\partial_t \boldsymbol{u} + \cancel{\boldsymbol{V}_0 \cdot \boldsymbol{\nabla} \boldsymbol{u}} = -\boldsymbol{\nabla} P - 2\boldsymbol{\Omega} \times \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{V}_0 + \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

$$\partial_t \boldsymbol{b} + \cancel{\boldsymbol{V}_0 \cdot \boldsymbol{\nabla} \boldsymbol{b}} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{V}_0 + \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \boldsymbol{u} - \ell_H (\boldsymbol{\nabla} \times \boldsymbol{b}) \times \boldsymbol{B}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

- Horizontal problem $\boldsymbol{u} = \boldsymbol{u}(z)$

$$\boldsymbol{b} = \boldsymbol{b}(z)$$

Horizontal problem

- Normal mode analysis $\propto \exp[i(kz - \omega t)]$

$$
\begin{aligned}
-i\omega u_x &= \left(2\Omega - \frac{\omega_V}{X}\right) u_y + i\omega_A b_x \\
-i\omega u_y &= -(2\Omega - \omega_V X) u_x + i\omega_A b_y \\
-i\omega b_x &= \left(\frac{\omega_V}{X} - \ell_H k\omega_A\right) b_y + i\omega_A v_x \\
-i\omega b_y &= (-\omega_V X + \ell_H k\omega_A) b_x + i\omega_A v_y
\end{aligned}
$$

Modified epicyclic

Modified whistler waves

- Introduce natural frequencies

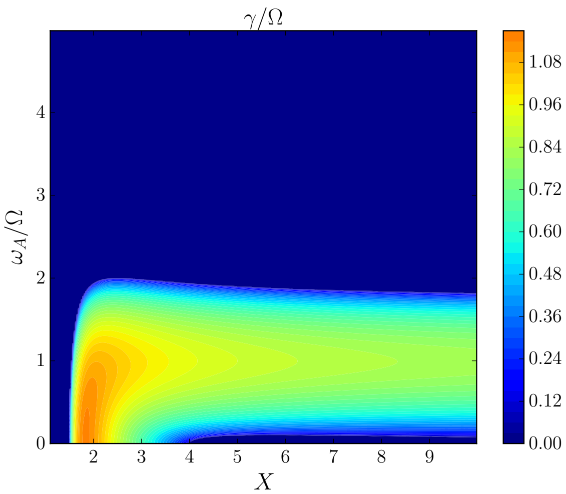
$$
\begin{aligned}
\kappa^2 &= \left(2\Omega - \frac{\omega_V}{X}\right)(2\Omega - \omega_V X) \\
\omega_A &= kB_0 \\
\omega_H^2 &= \left(\frac{\omega_V}{X} - \ell_H k\omega_A\right)(\omega_V X - \ell_H k\omega_A)
\end{aligned}
$$

Dispersion relation: ideal MHD

$$\omega^4 - \omega^2(2\omega_A^2 + \omega_V^2 + \kappa^2) + \omega_A^2\left[\omega_A^2 \boxed{- 2\Omega\omega_V\left(X + \frac{1}{X}\right)} + 2\omega_V^2\right] + \omega_V^2\kappa^2 = 0$$

Instability term

- Recover the MRI when $X \to \infty$
- No instability in the weak field limit $\omega_A \to 0$

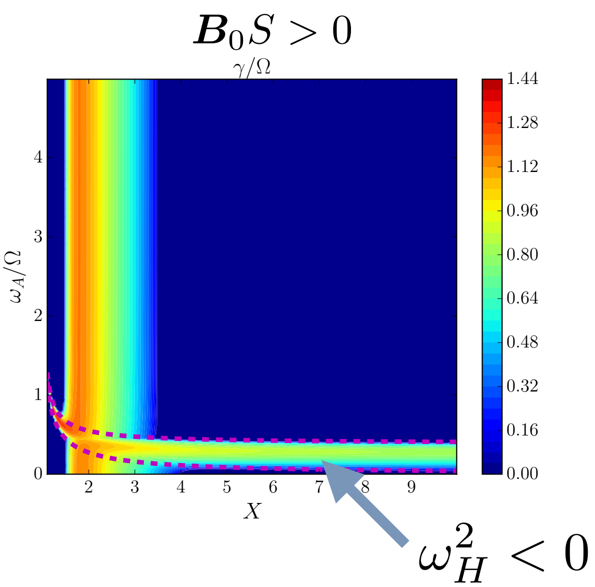


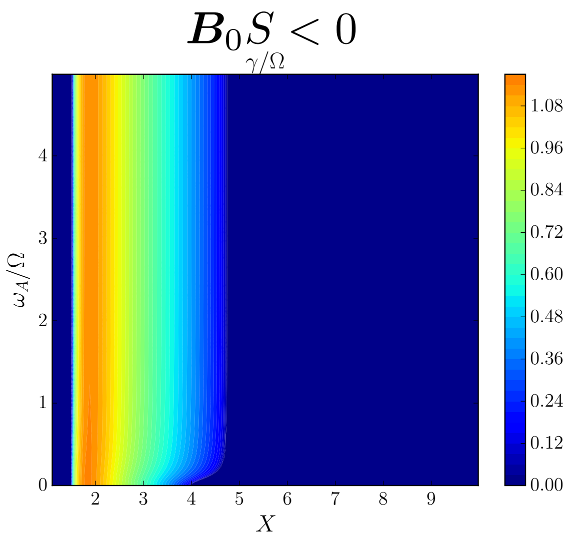
"elliptic MRI"

Hall-MHD dispersion relation

- Hall-shear instability: $\omega_H^2 < 0$
- Full dispersion relation

$$\omega^4 - \omega^2\left[2\omega_A^2 + \omega_H^2 + \kappa^2\right] + \omega_A^2\left[\omega_A^2 - 2\Omega\omega_V\left(X + \frac{1}{X}\right) - \ell_H\omega_A\omega_V\left(X + \frac{1}{X}\right) + 4\Omega\ell_H k\omega_A + 2\omega_V^2\right] + \kappa^2\omega_H^2 = 0$$





The full problem: 3D arbitrary perturbations

$$
\begin{aligned}
\partial_t \boldsymbol{u} + \boldsymbol{V}_0 \cdot \boldsymbol{\nabla} \boldsymbol{u} &= -\boldsymbol{\nabla} P - 2\boldsymbol{\Omega} \times \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{V}_0 + \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \boldsymbol{b} \\
\partial_t \boldsymbol{b} + \boldsymbol{V}_0 \cdot \boldsymbol{\nabla} \boldsymbol{b} &= \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{V}_0 + \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \boldsymbol{u} - \ell_H (\boldsymbol{\nabla} \times \boldsymbol{b}) \times \boldsymbol{B} \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \\
\boldsymbol{\nabla} \cdot \boldsymbol{b} &= 0
\end{aligned}
$$

- Assume a decomposition of the form $\psi(\boldsymbol{x}, t) = \hat{\psi}(t) \exp(i \boldsymbol{k}(t) \cdot \boldsymbol{x})$
- So that $\partial_t \psi + \boldsymbol{V}_0 \cdot \boldsymbol{\nabla} \psi = \partial_t \hat{\psi} + i \hat{\psi} \partial_t k_j x_j + i V_{0_j} k_j \hat{\psi}$

$$
= \partial_t \hat{\psi} + i \hat{\psi} \Big(\partial_t k_j + A_{mj} k_m \Big) x_j
$$

- Choose k(t) so that $\partial_t k_j + A_{mj} k_m = 0$

$$
\Rightarrow \quad \boldsymbol{k}(t) = k_0 \Big[X \tan(\theta) \sin(\omega_V t) \boldsymbol{e}_x + \tan(\theta) \cos(\omega_V t) \boldsymbol{e}_y + \boldsymbol{e}_{\boldsymbol{z}} \Big]
$$

Floquet theory

- The linear system involves time dependent, periodic coefficients

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{b} \end{pmatrix} = M(t) \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{b} \end{pmatrix}$$

- Gaston Floquet says that the solution is written

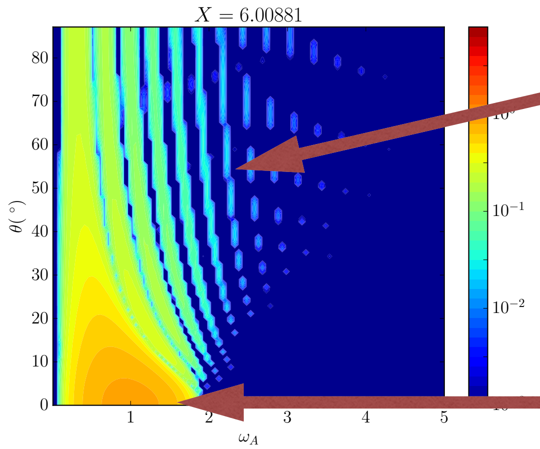
$$\boldsymbol{u} = \boldsymbol{P}(t)\exp(\gamma t)$$

where ***P*** has the same periodicity as *M(t)*

It is sufficient to integrate the linear equations for one period with a set of linearly independent initial conditions to deduce the stability properties of the system

Application: ideal MHD case

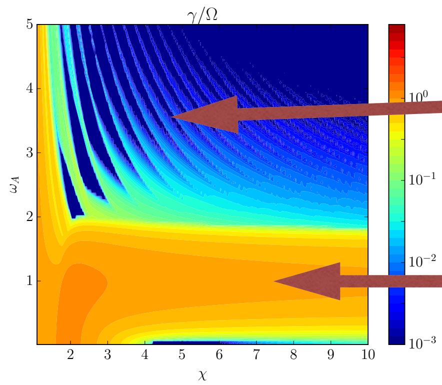
- Growth rate in ideal MHD



Parametric resonances (with Alfvén waves)

elliptic MRI

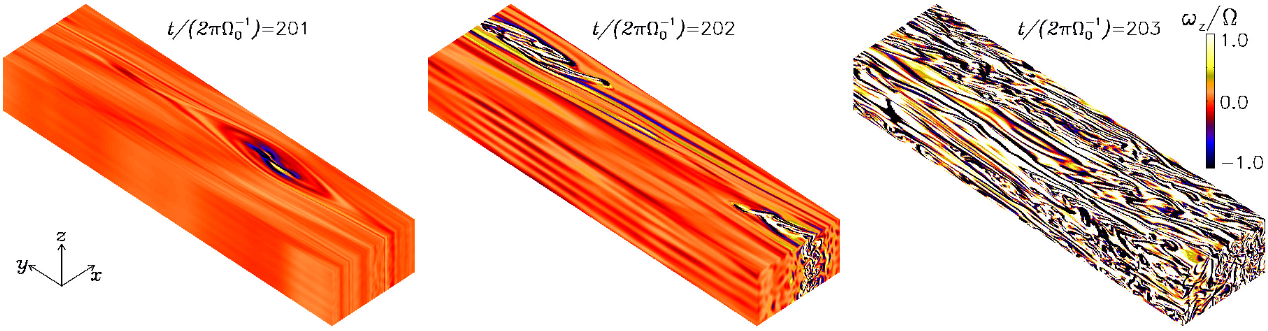
- Maximum growth rate (with respect to θ)



Parametric resonances

elliptic MRI

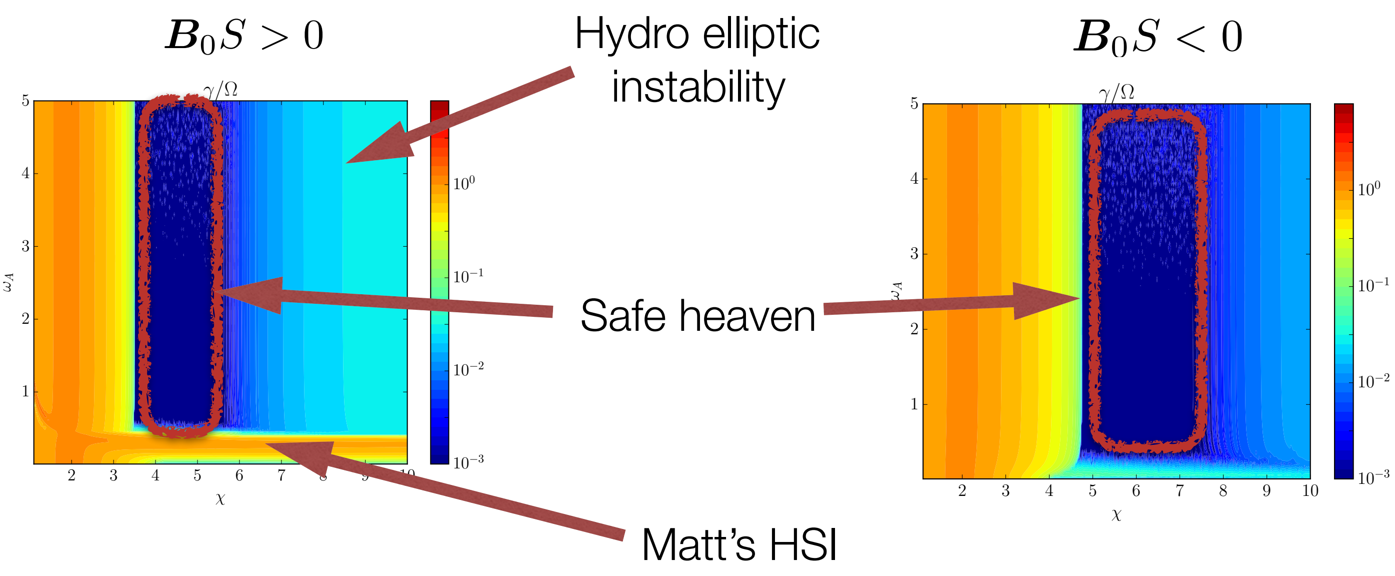
Application: ideal MHD



Lyra & Klahr 2011

Vortex very rapidly destroyed in ideal MHD (3 orbits)

Application: Hall-MHD case



- Adding the Hall effect stabilises the flow for sufficiently large ω_A
- Whistlers can't resonate with the vortex due to frequency mismatch
- The vortex stability becomes very similar to hydro when Hall is included!

Conclusions

- Vortices are strongly unstable in ideal MHD due to elliptic MRI and Alfvén waves in resonance with the vortex turnover frequency

- Adding Hall stabilises the structure for sufficiently strong B0 (in this case, the stability properties are those of hydro vortices!)

- Another way to stabilise these structure is to completely damp the MRI by resistivity/AD