Kinetic theory of ions in the magnetic presheath

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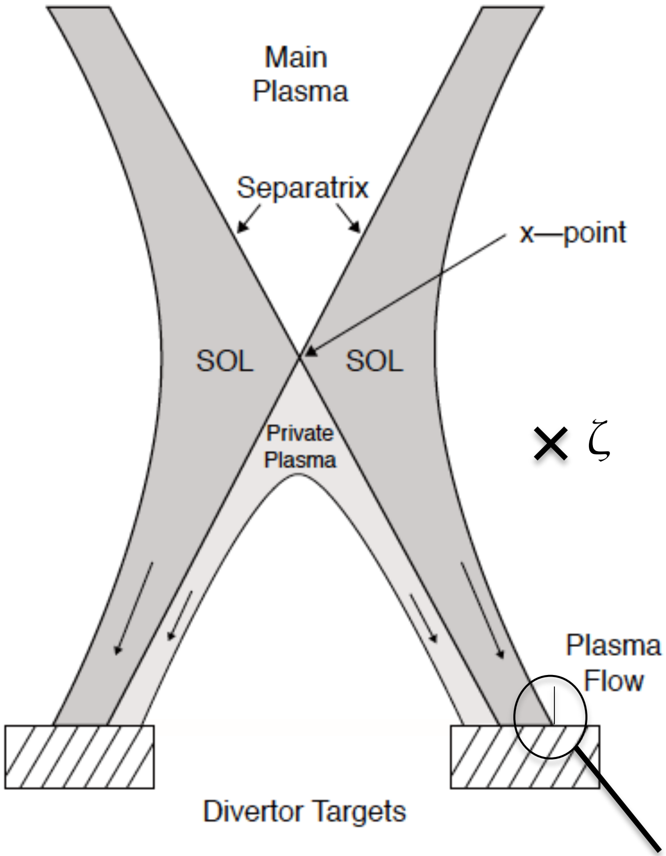
9th Plasma kinetic working group meeting (Vienna, 2016)

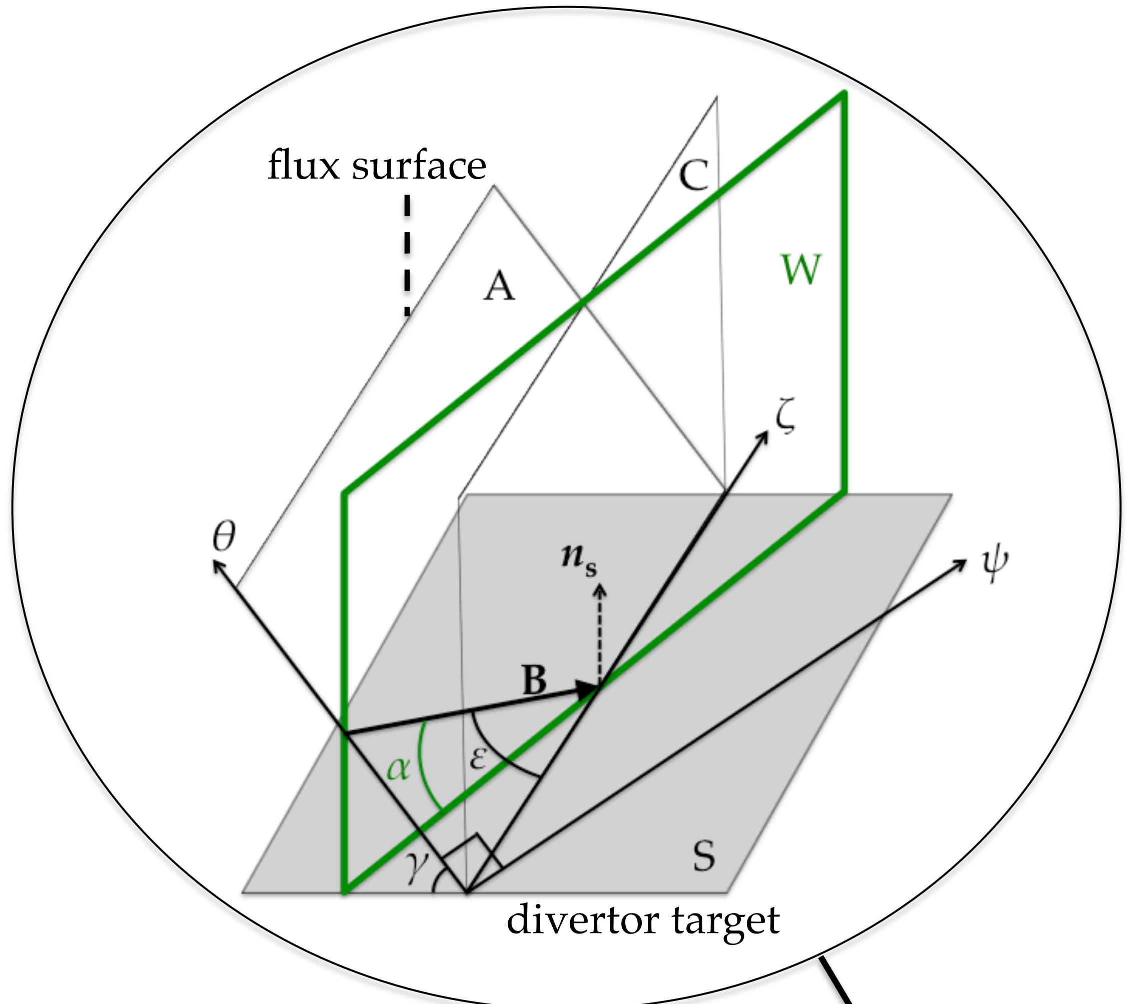
Introduction

- Boundary conditions for fluid codes used to simulate the SOL plasma
- Ones currently used* are obtained using fluid equations: <u>aim is to obtain boundary conditions using a kinetic treatment</u>
- Could be used with future drift kinetic codes of SOL
- Interesting problem from a purely theoretical point of view: <u>generalizing gyrokinetics to strongly distorted orbits in the magnetic presheath geometry</u>

* J. Loizu, P. Ricci, F.D. Halpern and S. Jolliet, *Phys. Plasmas* **19**, 122307 (2012).

Geometry





through the looking glass…

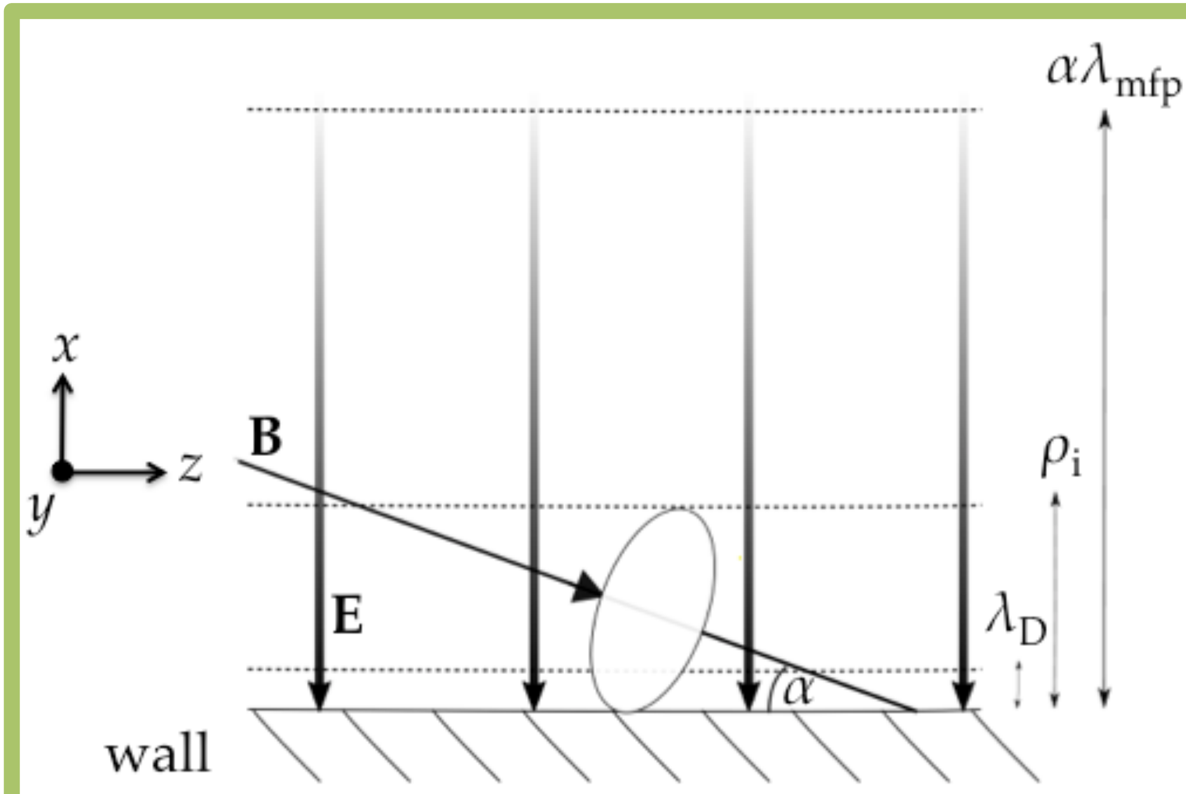
Figure (left) from: P.C. Stangeby, The plasma boundary of magnetic fusion devices (2000)

Boundary layers

	Width	Estimate
Collisional layer	$\alpha\lambda_{\mathrm{mfp}}$	100 mm
Magnetic presheath	ρ_{i}	0.7 mm
Debye sheath	λ_{D}	0.02 mm

$$\Rightarrow \lambda_{\mathrm{D}} << \rho_{\mathrm{i}} << \alpha\lambda_{\mathrm{mfp}}$$

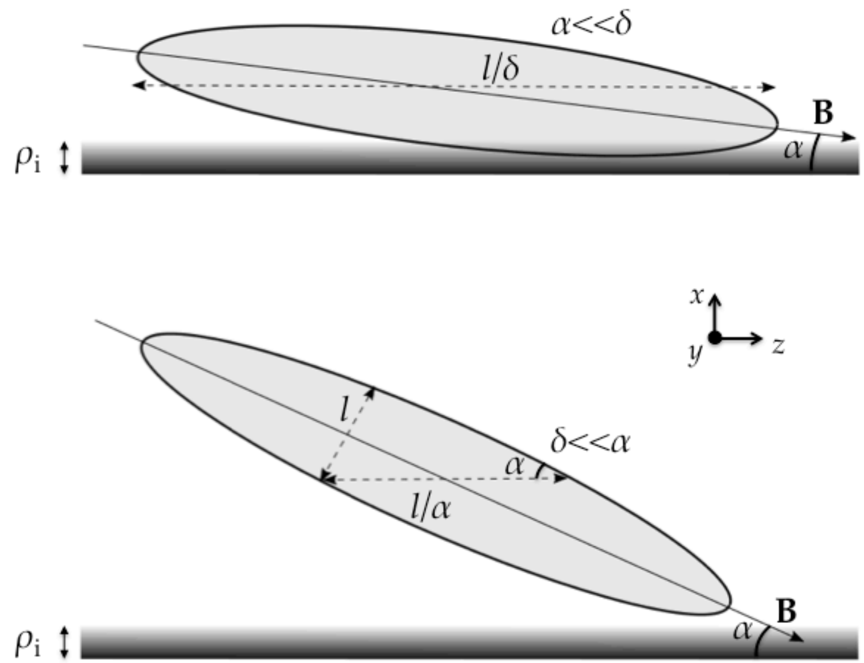
Estimates using data from: F. Militello and W. Fundamenski, Plasma Phys. Control. Fusion, **53**, 095002 (2011)



N.B. wall = divertor target

Assumptions

- Focus on ion gyroradius scale ρ_i => derive **magnetic presheath equations**
- $\lambda_D << \rho_i \sim \alpha\lambda_{mfp}$ => include collisions but keep quasineutral
- Angle α satisfies (1°=) $0.02 \sim (m_e/m_i)^{1/2} << \alpha << 1$
- Magnetic field constant and presheath electrostatic
- Turbulent gradients parallel to the wall inherited from outside magnetic presheath



neglect

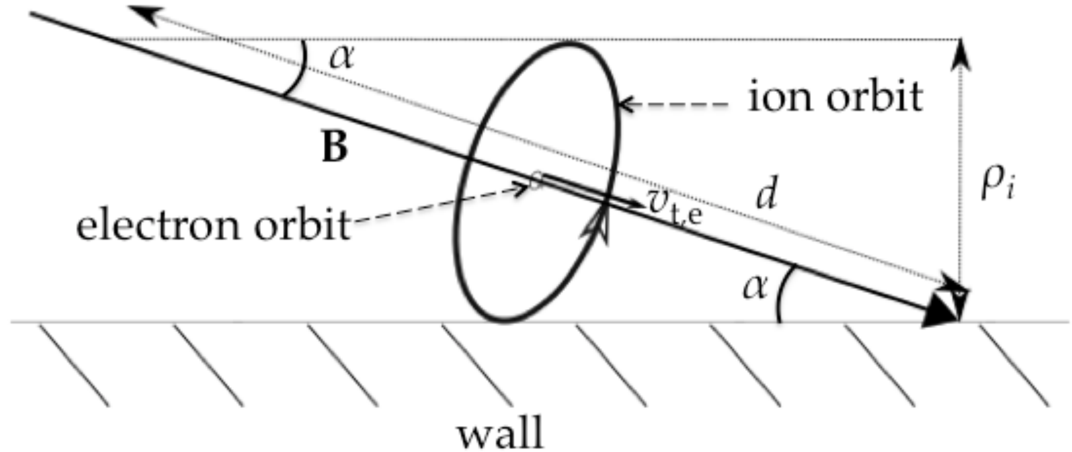
- $\partial/\partial y \sim 1/l \sim \delta/\rho_i$ and $\cancel{\boxed{\partial/\partial z \sim \delta/l \sim \delta^2/\rho_i}}$
- Order $\cancel{\boxed{\partial/\partial t \sim \delta^2\Omega}}$ << presheath timescale $\delta\Omega$
- Maximal ordering $\alpha \sim \delta << 1$

$\alpha << 1$ and $\delta = \rho_i/l << 1$

N.B. $l \sim 1cm$ => $\delta \sim 0.1$

Assumption: (not too) small angle

- Q: When does it take an ion and an electron the same time to reach the wall?



- Ion intersects wall after a single orbit: time~$\rho_i/v_{t,i}$
- Electrons travel along field at $v_{t,e}$ reaching wall after time $d/v_{t,e}$
- Times equal when $\alpha \approx \rho_i/d = \sqrt{(m_e/m_i)}$
- In order to assume **negatively charged wall**, require **electrons to reach wall much faster than ions**
- Corresponds to $\boldsymbol{\alpha >> \sqrt{(m_e/m_i)}}$

The zeroth order problem: α=δ=0

- Equations of motion of single particle are

$$\dot{v}_x = -\frac{Ze}{m_\mathrm{i}}\frac{d\phi(x)}{dx} + \Omega v_y$$

$$\dot{v}_y = -\Omega v_x$$

$$\dot{v}_z = 0$$

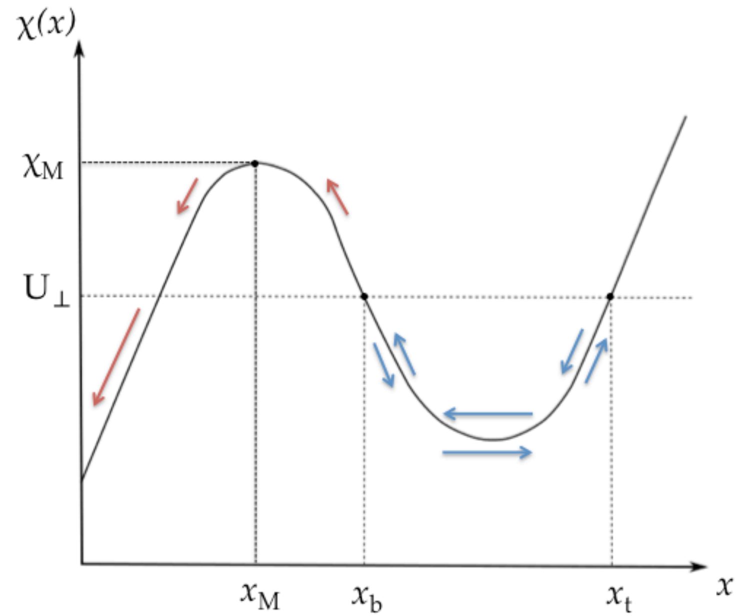
- Constants of motion:

Orbit position	$\bar{x} = x + (1/\Omega)v_y$
Perpendicular energy	$U_\perp = \frac{1}{2}v_x^2 + \frac{1}{2}v_y^2 + Ze\phi/m_\mathrm{i}$
Total energy	$U = U_\perp + \frac{1}{2}v_z^2$

See also: R.H. Cohen and D.D. Ryutov, *Phys. Plasmas* **5**, 808 (1998)

Gyrophase

- The zeroth order motion periodic when magnetic force large enough to make the ion turn
- Can write $v_x = \sigma_x[2(U_\perp - \chi(x))]^{1/2}$ where $\sigma_x = \pm 1$
- Both x and v_x are periodic if particle is trapped around a minimum of the effective potential $\chi(x) = \frac{1}{2}\Omega^2(x-\bar{x})^2 + Ze\phi(x)/m_i$



- Modified ion gyrofrequency Ω_{mod}

$$\frac{2\pi}{\Omega_{\mathrm{mod}}} = 2\int_{x_{\mathrm{b}}}^{x_{\mathrm{t}}} \frac{dx}{|v_x|} = 2\int_{x_{\mathrm{b}}}^{x_{\mathrm{t}}} \frac{dx}{\sqrt{2\left(U_\perp - \chi\left(x\right)\right)}}$$

- Gyrophase φ

$$\varphi = -\pi + \Omega_{\mathrm{mod}} t = \sigma_x \Omega_{\mathrm{mod}} \int_{x_{\mathrm{b}}}^{x} \frac{1}{\sqrt{2\left(U_\perp - \chi\left(x'\right)\right)}} dx'$$

Change of variables

- Can describe particle motion entirely using new set of variables

$$(x, y, z, v_x, v_y, v_z) \rightarrow (\bar{x}, y, z, \mathrm{U}_\perp, \varphi, \mathrm{U}, \sigma_\parallel)$$

- $\bar{x}$, $\mathrm{U}_\perp$ and U are constant and y, z are symmetry directions
- Define gyroaverage of a quantity as an average over φ while holding all other variables fixed

$$\langle \ldots \rangle_\varphi \equiv \frac{1}{2\pi} \oint (\ldots)\, d\varphi = \frac{\Omega_{\mathrm{mod}}}{\pi} \int_{x_{\mathrm{b}}}^{x_{\mathrm{t}}} \frac{(\ldots)}{\sqrt{2\left(U_\perp - \chi(x)\right)}} dx$$

Single particle motion in system with α~δ<<1

- The exact equations of motion are

$$\dot{v}_x = -\frac{Ze}{m_\mathrm{i}}\frac{\partial\phi}{\partial x}(x, y, \cancelto{\text{neglect}}{z, t}) + \Omega v_y \cancelto{\approx 1}{\cos\alpha}$$

$$\dot{v}_y = -\frac{Ze}{m_\mathrm{i}}\frac{\partial\phi}{\partial y}(x, y, \cancelto{\text{neglect}}{z, t}) - \Omega v_x \cancelto{\approx 1}{\cos\alpha} - \Omega v_z \underbrace{\sin\alpha}_{\approx\alpha}$$

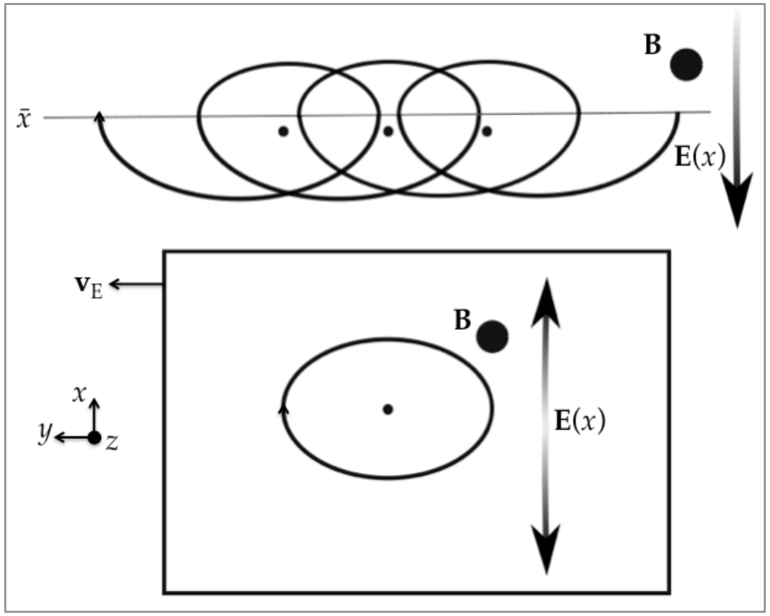
$$\dot{v}_z = \cancelto{\text{small}}{-\frac{Ze}{m_\mathrm{i}}\frac{\partial\phi}{\partial z}(x, y, z, t)} + \Omega v_y \underbrace{\sin\alpha}_{\approx\alpha}$$

- Changes in the orbit parameters:

ExB drift parallel to wall	$\langle \dot{y} \rangle_\varphi = \frac{1}{B}\left\langle \frac{\partial\phi}{\partial x} \right\rangle_\varphi$
Orbit drift normal to wall	$\langle \dot{\bar{x}} \rangle_\varphi = -\alpha v_\parallel - \frac{1}{B}\left\langle \frac{\partial\phi}{\partial y} \right\rangle_\varphi$
Perpendicular energy change	$\left\langle \dot{U}_\perp \right\rangle_\varphi = -\alpha\Omega v_\parallel \frac{1}{B}\left\langle \frac{\partial\phi}{\partial x} \right\rangle_\varphi$
Total energy change	$\left\langle \dot{U} \right\rangle_\varphi = 0$

First order constants: y_* *and* μ

- Frame exists in which zeroth order is periodic →
- => An adiabatic invariant μ exists, with $\langle d\mu/dt \rangle = O(\alpha^2\mu)$ when $\alpha \sim \delta \ll 1$
- Quantity y_* proportional to z-canonical momentum derived by integrating $dv_z/dt = \alpha\Omega v_y$ to obtain $v_z = \alpha\Omega(y - y_*)$, with $dy_*/dt = O(\alpha^2 y_*)$



Adiabatic invariant*	$\mu = \frac{1}{\pi}\int_{x_b}^{x_t}\sqrt{2\left(U_\perp - \chi(x)\right)}dx$	$\langle\dot{\mu}\rangle_\varphi = 0$	replaces $U_\perp$
y-star §	$y_\star = y - \frac{1}{\alpha\Omega}v_\parallel$	$\langle\dot{y}_\star\rangle_\varphi = 0$	replaces y

new change of variables: $(\bar{x}, y, z, U_\perp, \varphi, U, \sigma_\parallel)$ → $(\bar{x}, y_*, z, \mu, \varphi, U, \sigma_\parallel)$

* R.H. Cohen and D.D. Ryutov, *Phys. Plasmas* **5**, 808 (1998)

§ D.L. Holland, B.D. Fried and G.J. Morales, *Phys Fluids B*, **5**, 1723 (1993)

Gyrokinetic equation

$$F \approx <F>_{\varphi}$$

↓ first order $\sim \alpha\Omega F$ ↓

$$\dot{\bar{x}}\frac{\partial F}{\partial \bar{x}} + \dot{y}\frac{\partial F}{\partial y} + \dot{U}_{\perp}\frac{\partial F}{\partial U_{\perp}} + \dot{\varphi}\frac{\partial F}{\partial \varphi} = \mathcal{C}[F]$$

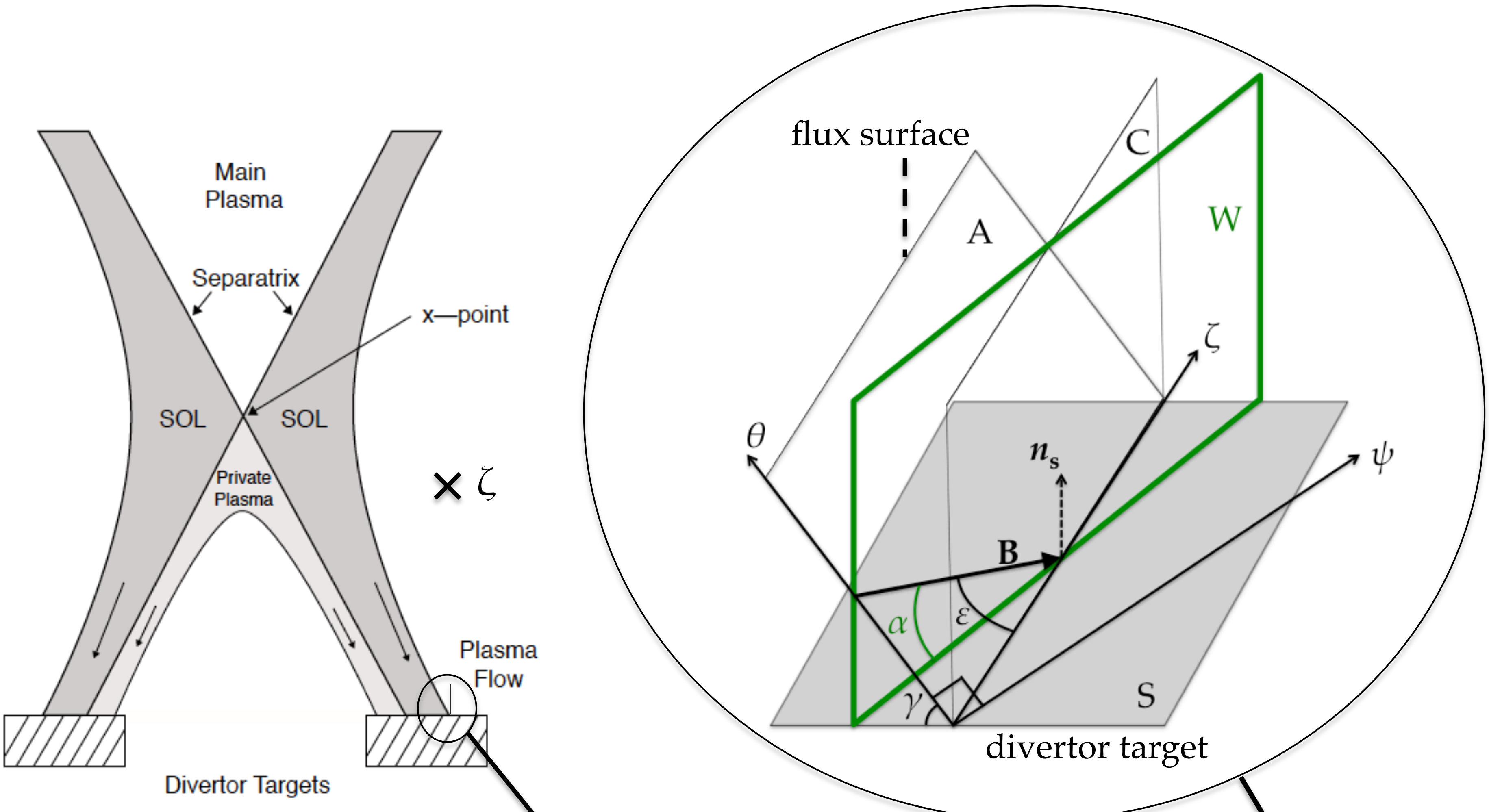
↓ gyroaverage ↓

$$\boxed{\langle\dot{\bar{x}}\rangle_{\varphi}\frac{\partial F}{\partial \bar{x}} + \langle\dot{y}\rangle_{\varphi}\frac{\partial F}{\partial y} + \left\langle\dot{U}_{\perp}\right\rangle_{\varphi}\frac{\partial F}{\partial U_{\perp}} = \langle\mathcal{C}[F]\rangle_{\varphi}}$$

To solve it, require a form for the potential $\phi(x,y)$ (that we will determine) and boundary conditions:

1. $F = 0$ at $y \rightarrow \pm\infty$ (corresponds to outside SOL)
2. $F = 0$ at $U_{\perp} \rightarrow \infty$
3. $F = F^{\infty}(y, U_{\perp}, U)$ for $<d\bar{x}/dt>_{\varphi} < 0$ at $\bar{x} \rightarrow \infty$
4. $F = 0$ for open orbits (ones that intersect wall)

Geometry



through the looking glass…

Figure (left) from: P.C. Stangeby, The plasma boundary of magnetic fusion devices (2000)

Gyrokinetic equation

$$F \approx \langle F \rangle_\varphi$$

first order $\sim \alpha \Omega F$

$$\dot{\bar{x}}\frac{\partial F}{\partial \bar{x}} + \dot{y}_\star \frac{\partial F}{\partial y_\star} + \dot{\mu}\frac{\partial F}{\partial \mu} + \dot{\varphi}\frac{\partial F}{\partial \varphi} = \mathcal{C}[F]$$

gyroaverage

$$\langle \dot{\bar{x}} \rangle_\varphi \frac{\partial F_0}{\partial \bar{x}} + \underbrace{\cancel{\langle \dot{y}_\star \rangle_\varphi \frac{\partial F_0}{\partial y_\star}}}_{\text{small}} + \underbrace{\cancel{\langle \dot{\mu} \rangle_\varphi \frac{\partial F_0}{\partial \mu}}}_{\text{small}} = \langle \mathcal{C}[F_0] \rangle_\varphi$$

$$\boxed{\langle \dot{\bar{x}} \rangle_\varphi \frac{\partial F_0}{\partial \bar{x}} = \langle \mathcal{C}[F_0] \rangle_\varphi}$$

Boundary conditions are:

- $F_0 = F_0^\infty (y_*, \mu, U)$ for $\langle d\bar{x}/dt \rangle_\varphi < 0$ at $\bar{x} \to \infty$
- $F_0 = 0$ for open orbits

Quasineutrality

- The assumption $\alpha >> (m_e/m_i)^{1/2}$ => Boltzmann electrons $n_e = n_{e,\infty}\exp[e(\phi - \phi_\infty)/T_e]$
- $\lambda_D << \rho_i$ => quasineutrality holds throughout magnetic presheath
- To obtain ion density, integrate F_0 in velocity space using ($\bar{x}$, $U_\perp$, U) instead of (v_x, v_y, v_z)*

$$n_{e\infty}(y)\exp\left(\frac{e(\phi(x,y)-\phi_\infty(y))}{T_e}\right) =$$

$$Z\sum_{\sigma_\parallel=\pm 1}\int_{\bar{x}_m(x,y)}^{\infty} d\bar{x}\int_{\chi(x,\bar{x},y)}^{\chi_M(\bar{x},y)}\frac{2\Omega dU_\perp}{\sqrt{2\left(U_\perp-\chi(x,\bar{x})\right)}}\int_{U_\perp}^{\infty}\frac{F(\bar{x},y,U_\perp,U,\sigma_\parallel)}{\sqrt{2\left(U-U_\perp\right)}}dU$$

- Quasineutrality (above) and the gyrokinetic equation allow to solve for F and ϕ self consistently in the magnetic presheath

* M.J. Gerver, S.E. Parker and K. Theilhaber, *Phys. Fluids B*, **2**, 1069 (1990)

Collisionless magnetic presheath

- If $\rho_{\rm i} << \alpha\lambda_{\rm mfp}$, gyrokinetic equation becomes (using green variables)

$$\frac{\partial F}{\partial \bar{x}} = 0$$

- The solution is therefore

$$F = \begin{cases} F^{\infty}\left(y_{\star}, \mu, U\right) & \text{for } \textit{closed} \text{ orbits,} \\ 0 & \text{for } \textit{open} \text{ orbits.} \end{cases}$$

will improve on this!

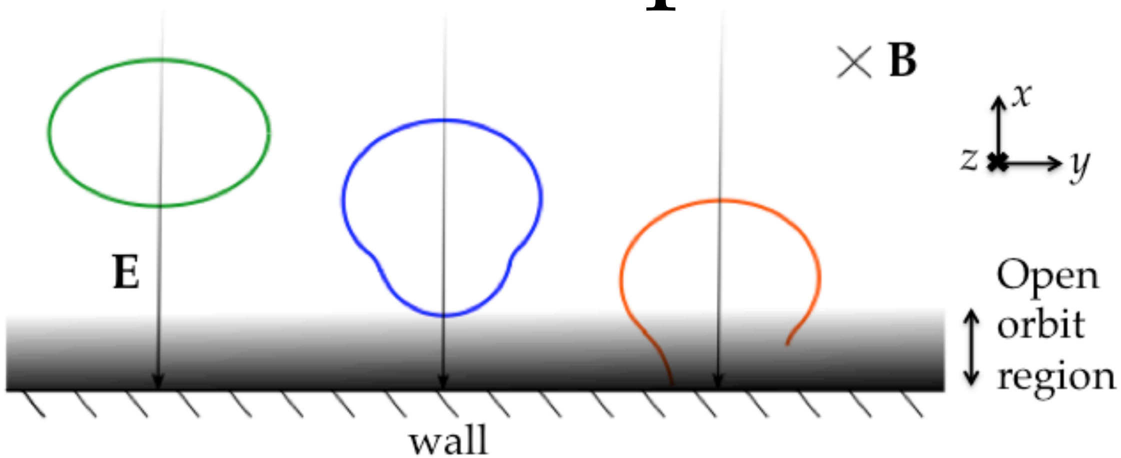
- An iteration scheme with quasineutrality would allow to determine the self-consistent $\phi(x,y)$

$$n_{\rm e\infty}\left(y\right)\exp\left(\frac{e\left(\phi\left(x,y\right)-\phi_{\infty}\left(y\right)\right)}{T_e}\right) =$$

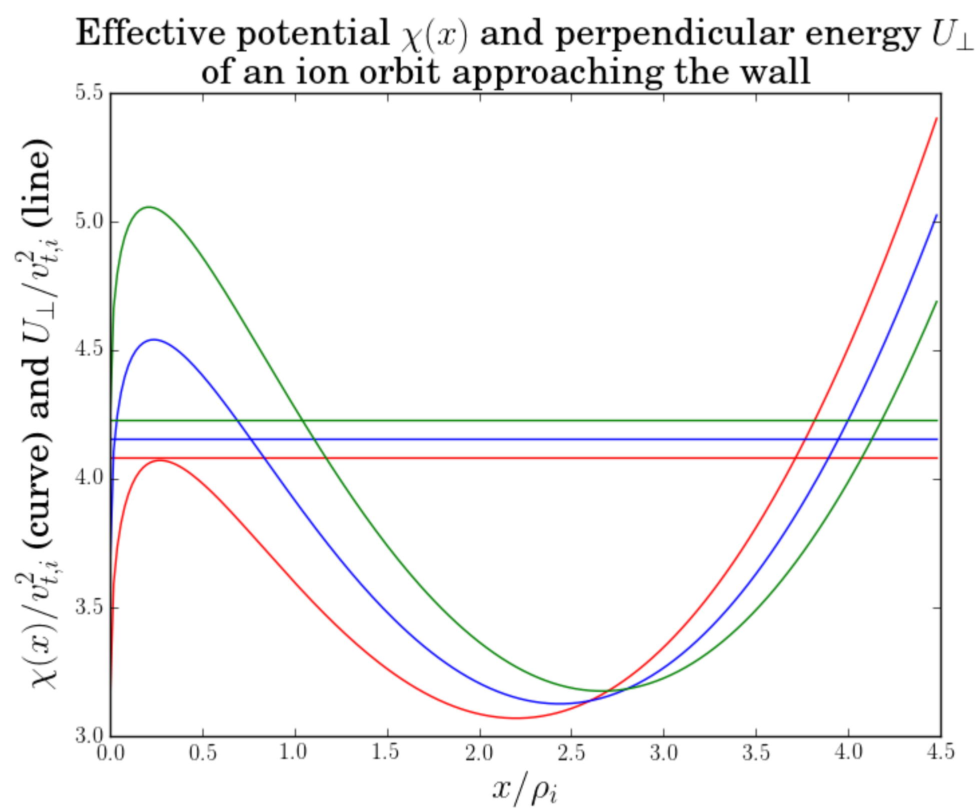
$$Z\sum_{\sigma_{\parallel}=\pm 1}\int_{\bar{x}_m(x,y)}^{\infty} d\bar{x}\int_{\chi(x,\bar{x},y)}^{\chi_M(\bar{x},y)}\frac{2\Omega dU_{\perp}}{\sqrt{2\left(U_{\perp}-\chi\left(x,\bar{x}\right)\right)}}\int_{U_{\perp}}^{\infty}\frac{F^{\infty}\left(y_{\star},\mu,U\right))}{\sqrt{2\left(U-U_{\perp}\right)}}dU$$

See also: R.H. Cohen and D.D. Ryutov, *Phys. Plasmas* **5**, 808 (1998).

Open orbits

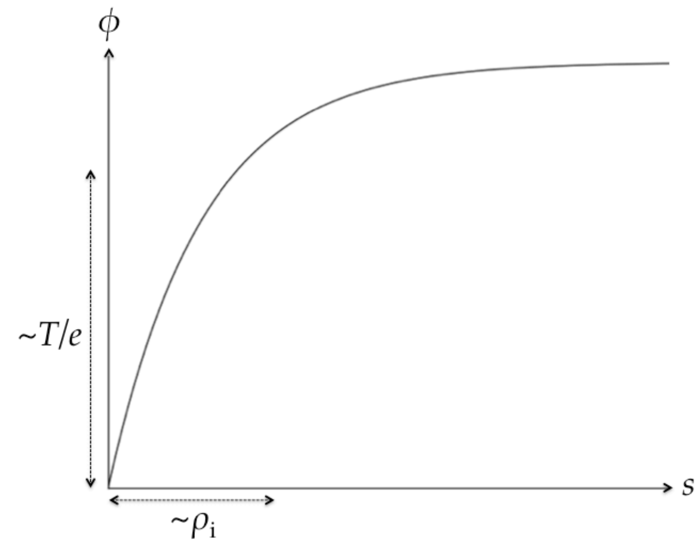


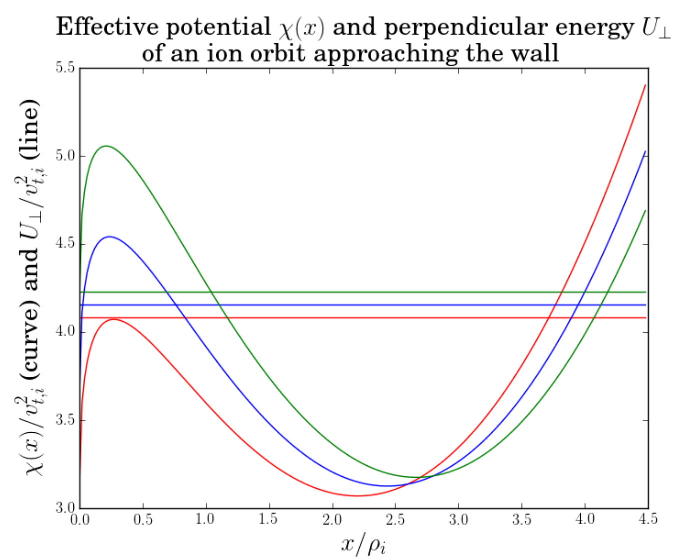
Orbit opening



Why are open orbits important close to the wall?

- At exit of quasineutral presheath, electric field diverges (breakdown of quasineutrality) but potential does not ($\phi \sim \sqrt{x}$)*
- Expect potential of this form →
- Electric force always overcomes magnetic force ($=Zev_yB$) close to x=0
- =>Effective potential always has maximum near wall →
- No closed orbits => need open orbit density

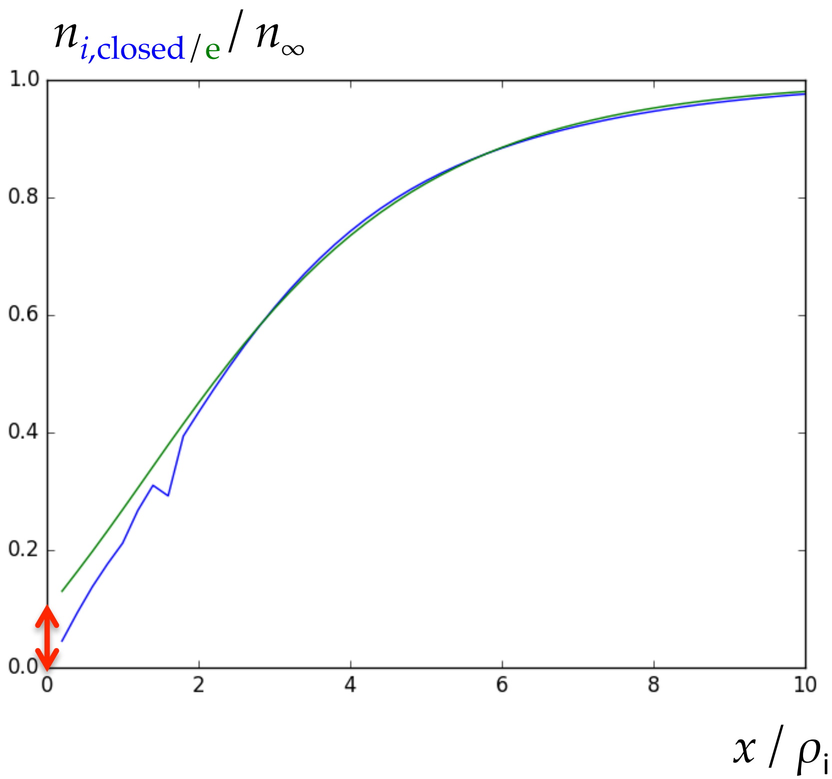




* K.-U. Riemann, *Journal of Physics D: Applied Physics* **24**, 493 (1991)

Quasineutrality without open orbits

- Closed orbit density $n_{\mathrm{i,closed}}$ goes to zero at wall
- Electron density $n_e \sim n_\infty \exp[e(\phi-\phi_\infty)/T_e] \sim n_\infty \exp[e\Delta\phi_{\mathrm{MPS}}/T_e]$
- Gives $\Delta\phi_{\mathrm{MPS}}=-\infty$
- Density of ion open orbits required to obtain finite potential jump
- $Zn_{\mathrm{i,open}} \approx n_e$ near wall

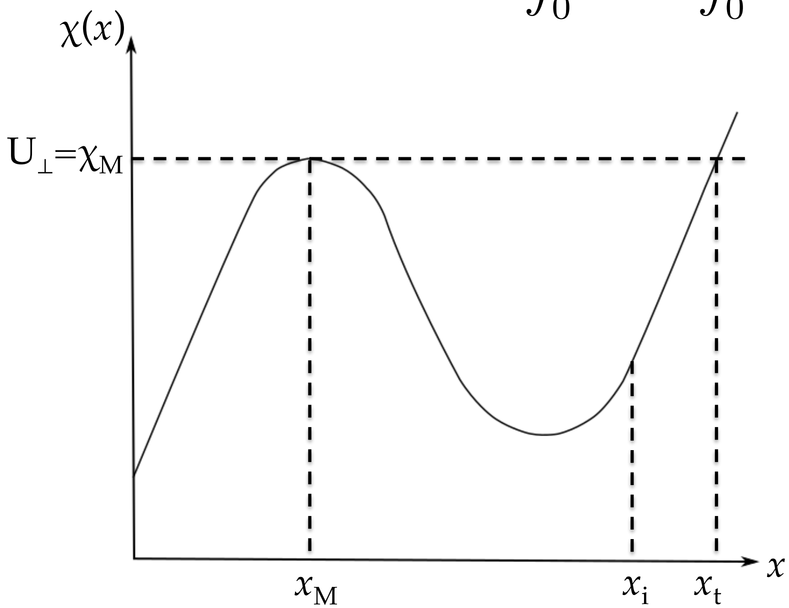


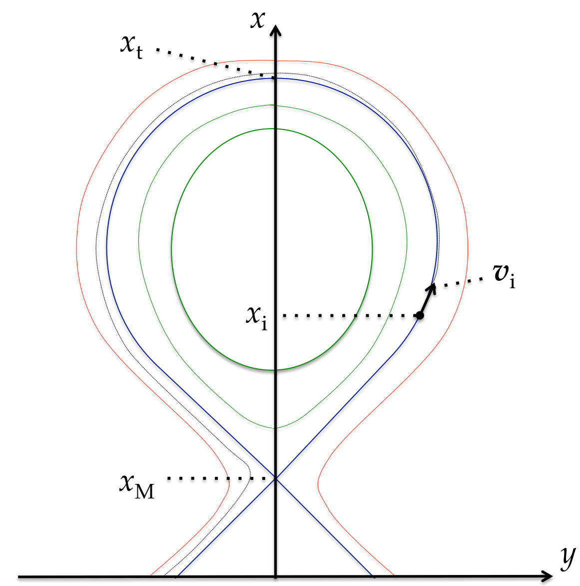
Open orbit density

- Conservation of distribution function F in phase space

$$f_{\mathrm{open}} v_x \Big|_{U_\perp=\chi_{\mathrm{M}}} dv_x dv_y dv_z = \frac{\Omega}{v_{x\mathrm{i}} v_{\|}} F(U, U_\perp = \chi_{\mathrm{M}}, \bar{x}) \left(\dot{\chi}_{\mathrm{M}} - \dot{U}_\perp \right) \Big|_{U_\perp=\chi_{\mathrm{M}}} dx_{\mathrm{i}} d\bar{x} dU$$

- Density is $n_{\mathrm{i,open}} = \alpha \Omega^3 \int_0^\infty dU \int_0^\infty d\bar{x} \int_{x_{\mathrm{M}}}^{x} dx_{\mathrm{i}} \frac{(x_{\mathrm{i}} - x_M)}{v_x v_{x\mathrm{i}}} F^\infty (\bar{x}, U_\perp = \chi_M, U)$

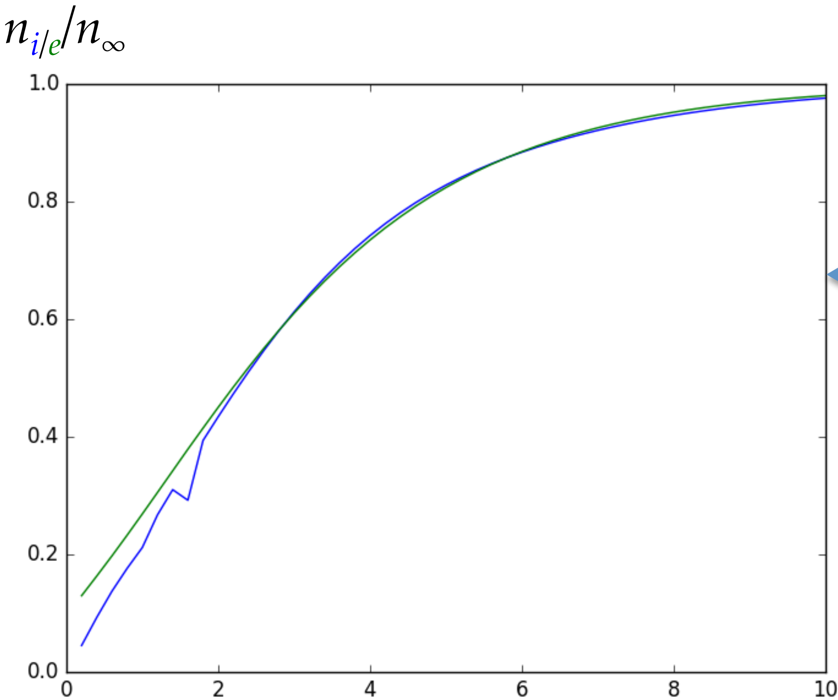




- Confirmed by calculation of velocity corrections near X-point
- Divergence at v_x=0: calculate correction at X point

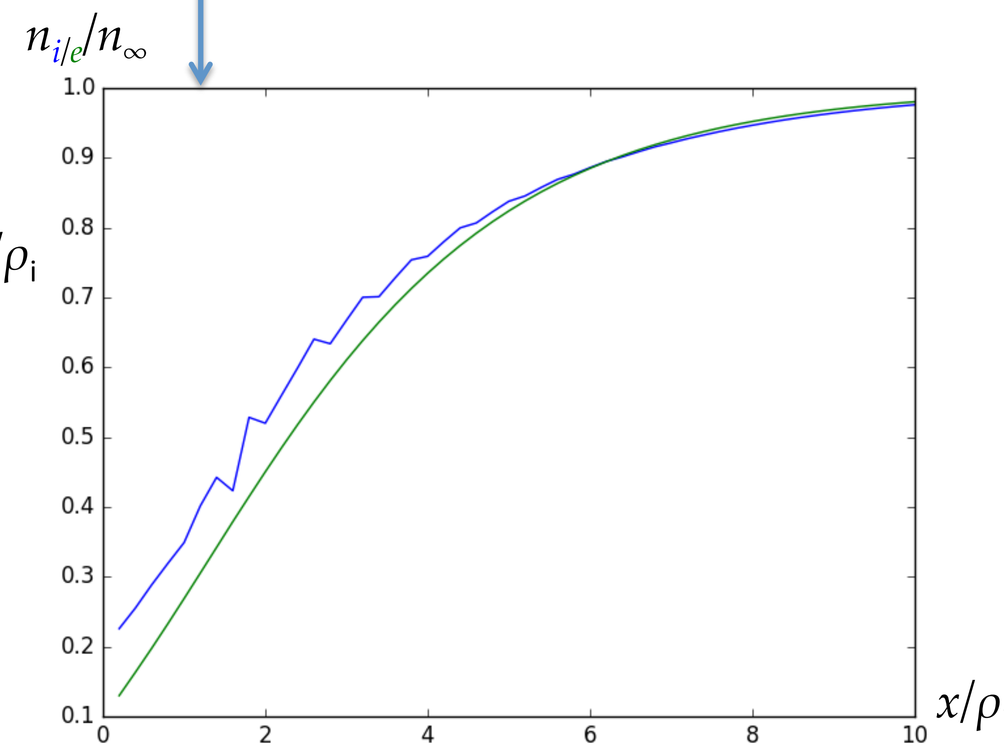
$$v_x = \sqrt{V_{\mathrm{M}}^2 + 2\left(\chi_M - \chi(x)\right)} \quad \text{with} \quad V_{\mathrm{M}}^2 = 2\alpha\Omega^2 v_{\|} \int_{x_{\mathrm{i}}}^{x_{\mathrm{M}}} \frac{s - x_M}{\sqrt{2\left(\chi_{\mathrm{M}} - \chi(s)\right)}} |ds|$$

Preliminary numerical results



Green: electron density
Blue: ion density

Without
With
open orbits



- Analytically, $n_{i,open} \sim \alpha n_\infty$
- **=> $\boldsymbol{\Delta\phi_{MPS} \sim (T/e) \ln \alpha}$**

Numerical results shown in plots obtained using:

- $\alpha=0.1$
- $F^\infty \sim v_\parallel^2 \exp(-m(v_\parallel^2+2\mu B)/2T_i)$
- $\phi=1.05*\ln(1/\alpha)*\exp(-0.19\sqrt{x}-0.49x)$

Conclusions

- Derived gyrokinetic equations of ions in magnetic presheath
- Assumed small magnetic field to wall angle α, small gradients parallel to the wall (δ<<1) and constant **B** field
- Assumed electron repelling wall => Boltzmann electrons
- Derived form of quasineutrality
- Proposed and currently applying iteration scheme that could solve for collisionless magnetic presheath
- Solution valid to lowest order in α and δ
- Open orbits are important close to the wall
- In progress: quantifying effects of open orbits near the wall + including open orbit density in numerical work
- Future work: study the purely collisional layer $\alpha\lambda_{mfp}$ wide

BACKUP SLIDES

Assumption: turbulent gradients

- We include weak gradients parallel to the wall of the electrostatic potential ϕ and the ion and electron densities, due to the width of turbulent structures $l=\rho_i/\delta$ with $\delta\sim\alpha<<1$
- These gradients are in the y direction, across the magnetic field
- The direction (almost) parallel to the magnetic field is associated with even smaller gradients because turbulent structures are elongated along this direction: $v_z\partial/\partial z\sim\alpha^2\Omega\approx 0$
- Time derivatives are ordered very small: $\partial/\partial t\sim\alpha^2\Omega\approx 0$

neglect

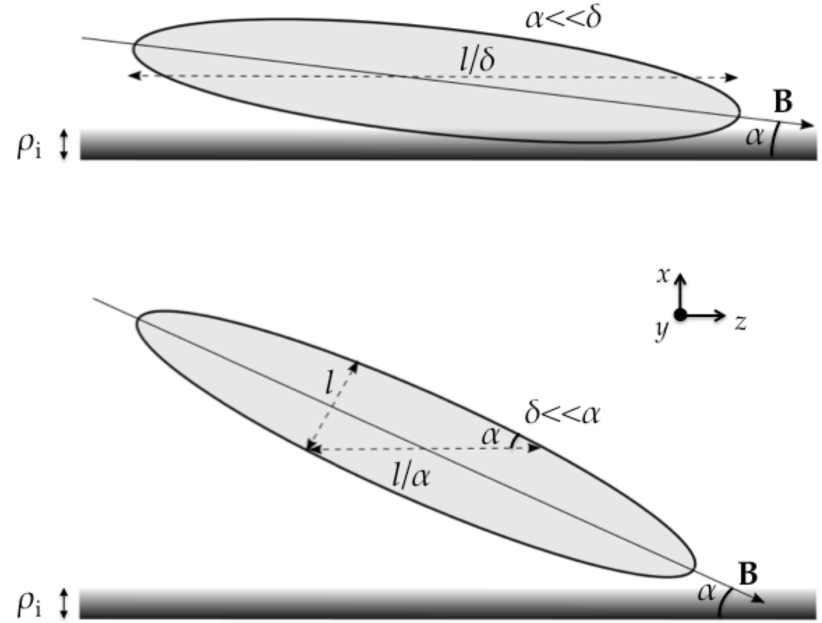


Figure: *The elongation of a turbulent structure along the magnetic field causes the gradients in the z direction to be very small. The characteristic length scale in the y direction (out of the page) is l.*

Assumption: (not too) small angle

- When does it take the same time for an ion and an electron to reach the wall?
- Suppose we are very close to the wall
- It takes an ion a single orbit to intersect the wall $\sim\rho_{\mathrm{i}}/v_{\mathrm{t,i}}$
- Electrons have a much smaller Larmor radius, so they travel parallel to the field at $v_{\mathrm{t,e}}$ and reach the wall after a time $d/v_{\mathrm{t,e}}$
- Times equal when $\alpha\approx\rho_i/d=\sqrt{(m_{\mathrm{e}}/m_{\mathrm{i}})}$

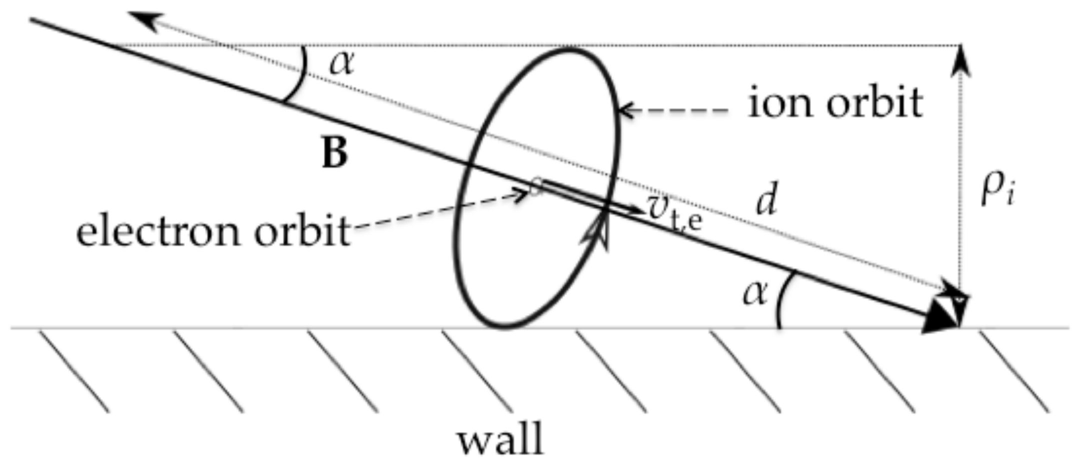


Figure: *A schematic which shows that, when very close to the wall, ions intersect the wall during their gyromotion while electrons are tied much more closely to the field line and have to drift along* **B** *with speed* $\sim v_{t,e}$ *much faster than the characteristic ion speed. At small enough angle α, the ions reach the wall more quickly than thefaster-moving electrons, which travel almost parallel to the wall.*

Assumption: (not too) small angle (2)

- In order to assume a negatively charged wall, we require electrons to reach it much faster than ions
- Corresponds to $\alpha >> \sqrt{(m_e/m_i)}$
- *Provided this is satisfied, almost all of the electrons are repelled by the wall*
- We can therefore assume that electrons are in equilibrium and are **Boltzmann distributed**

$$n_e = n_{e,\infty} \exp[e(\phi - \phi_\infty)/T_e]$$

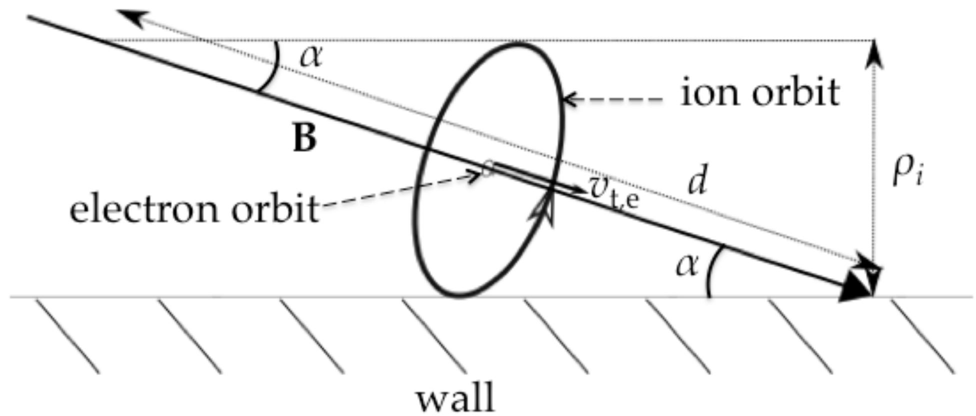


Figure: *A schematic which shows that, when very close to the wall, ions intersect the wall during their gyromotion while electrons are tied much more closely to the field line and have to drift along* **B** *with speed ~$v_{t,e}$ much faster than the characteristic ion speed. At small enough angle α, the ions reach the wall more quickly than thefaster-moving electrons, which travel almost parallel to the wall.*

Orbit parameters

- $dv_y/dt = - \Omega v_x \Rightarrow v_y = \Omega(\bar{x} - x)$ where $\bar{x}$ is the orbit position
- Perpendicular energy $U_\perp$ and total energy U are conserved

Orbit position	$\bar{x} = x + (1/\Omega)v_y$
Perpendicular energy	$U_\perp = \frac{1}{2}v_x^2 + \frac{1}{2}v_y^2 + Ze\phi/m_\mathrm{i}$
Total energy	$U = U_\perp + \frac{1}{2}v_z^2$

Kinetic equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = C\left[f\right]$$

change variables

$$\cancel{\frac{\partial F}{\partial t}} + \dot{\bar{x}}\frac{\partial F}{\partial \bar{x}} + \dot{y}\frac{\partial F}{\partial y} + \cancel{\dot{z}\frac{\partial F}{\partial z}} + \dot{U}_\perp \frac{\partial F}{\partial U_\perp} + \cancel{\dot{U}\frac{\partial F}{\partial U}} + \dot{\varphi}\frac{\partial F}{\partial \varphi} = \mathcal{C}[F]$$

small small small

zeroth order $\sim \Omega F$

expand $F = F_0 + F_1 + \mathrm{O}(\alpha^2 F)$ to get $\bar{\Omega}\dfrac{\partial F_0}{\partial \varphi} = 0$

=> $F_0 = \langle F_0 \rangle_\varphi$

Changes in the orbit parameters

Ex**B** drift parallel to wall	$$\langle \dot{y} \rangle_{\varphi} = \frac{1}{B} \left\langle \frac{\partial \phi}{\partial x} \right\rangle_{\varphi}$$
Orbit drift normal to wall	$$\langle \dot{x} \rangle_{\varphi} = -\alpha v_{\parallel} - \frac{1}{B} \left\langle \frac{\partial \phi}{\partial y} \right\rangle_{\varphi}$$
Perpendicular energy change	$$\left\langle \dot{U}_{\perp} \right\rangle_{\varphi} = -\alpha \Omega v_{\parallel} \frac{1}{B} \left\langle \frac{\partial \phi}{\partial x} \right\rangle_{\varphi}$$
Total energy change	$$\left\langle \dot{U} \right\rangle_{\varphi} = 0$$

Gyrokinetic equation

$$\frac{\partial F}{\partial t} + \dot{\bar{x}}\frac{\partial F}{\partial \bar{x}} + \dot{y}_\star \frac{\partial F}{\partial y_\star} + \dot{z}\frac{\partial F}{\partial z} + \dot{\mu}\frac{\partial F}{\partial \mu} + \dot{U}\frac{\partial F}{\partial U} + \dot{\varphi}\frac{\partial F}{\partial \varphi} = \mathcal{C}[F]$$

expand F and recover $F_0 = \langle F_0 \rangle$

first order $\sim \alpha\Omega F$ and gyroaverage

$$\langle \dot{\bar{x}} \rangle_\varphi \frac{\partial F_0}{\partial \bar{x}} + \cancel{\langle \dot{y}_\star \rangle_\varphi \frac{\partial F_0}{\partial y_\star}} + \cancel{\langle \dot{\mu} \rangle_\varphi \frac{\partial F_0}{\partial \mu}} = \langle \mathcal{C}[F_0] \rangle_\varphi$$

small small

$$\boxed{\langle \dot{\bar{x}} \rangle_\varphi \frac{\partial F_0}{\partial \bar{x}} = \langle \mathcal{C}[F_0] \rangle_\varphi}$$

Boundary conditions are:

- $F_0 = F_0^\infty(y_*, \mu, \mathrm{U})$ for $\langle d\bar{x}/dt \rangle_\varphi < 0$ at $\bar{x} \to \infty$
- $F_0 = 0$ for open orbits