The effect of tangential drifts on neoclassical transport in stellarators close to omnigeneity

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Stellarator drift-kinetic equation at low collisionality: the $1/\nu$ regime.

In a generic stellarator the expansion of the distribution function around a Maxwellian breaks down at lower collisionalities and the drift-kinetic equation becomes radially non-local.

Omnigeneous stellarators and stellarators close to omnigeneity.

In stellarators close to omnigeneity the expansion around a Maxwellian can be carried out for collisionalities below the $1/\nu$ regime and a radially local drift-kinetic equation can be derived.

Neoclassical transport in stellarators close to omnigeneity below the $1/\nu$ regime:

- $\sqrt{\nu}$ regime and superbanana-plateau regime.

The ideas in this work can be used to build fast neoclassical codes. An example: the $\sqrt{\nu}$ regime in LHD.

**Remark:** A mass ratio expansion $\sqrt{m_e/m_i} \ll 1$ is assumed. Hence, ion-electron collisions are dropped. We focus on ion transport.

**Remark:** The effect of large aspect ratio is not studied.
Drift-kinetic equation in stellarators

- Spatial coordinates \( \{\psi, \alpha, l\} \): \( \psi \) is a radial coordinate, \( \alpha \) is a periodic coordinate that labels magnetic field lines and \( l \) is the length along the line.

- Velocity coordinates \( \{v, \lambda, \sigma\} \): \( v \) is the magnitude of the velocity, \( \lambda = v^2_\perp / (v^2B) \) is the pitch-angle coordinate and \( \sigma = v_\parallel / |v_\parallel| \).

- Define the normalized gyroradius as \( \rho_i^* = \rho_i / L_0 \ll 1 \), where \( L_0 \) is the typical variation length of \( B \). In the standard drift-kinetic expansion
  - the distribution function is expanded as \( F_i = F_{Mi} + F_{i1} + \ldots \), where \( F_{i1} \sim \rho_i^* F_{Mi} \) and \( F_{Mi} \) is a Maxwellian distribution with zero flow, and with density \( n_i(\psi) \) and temperature \( T_i(\psi) \) constant on flux surfaces;
  - the electrostatic potential is expanded as \( \varphi(\psi, \alpha, l) = \varphi_0(\psi) + \varphi_1(\psi, \alpha, l) + \ldots \), with \( \varphi_0 \sim T_i / (Z_i e) \) and \( \varphi_1 / \varphi_0 \sim \rho_i^* \).

- The drift-kinetic equation for \( G_{i1} = F_{i1} + (Z_i e \varphi_1 / T_i) F_{Mi} \) is
  \[
  v_\parallel \partial_l G_{i1} + \Upsilon_i v_{M, i} \cdot \nabla \psi F_{Mi} = C_{ii}^\ell [G_{i1}],
  \]
  where \( v_{M, i} \) is the ion magnetic drift, \( \Upsilon_i \) involves the gradients of \( n_i, T_i \) and \( \varphi_0 \), and \( C_{ii}^\ell \) is the linearized ion-ion collision operator.
Drift-kinetic equation at low collisionality: the $\frac{1}{\nu}$ regime

\[ v_|| \partial_t G_{i1} + \Upsilon_i v_{M,i} \cdot \nabla \psi F_{Mi} = C_{ii}^\ell [G_{i1}] \]

- Define the ion collisionality as $\nu_{i*} = \nu_{ii} L_0 / v_{ti} \ll 1$, where $v_{ti} = \sqrt{T_i / m_i}$ is the thermal speed. **If $\nu_{i*} \ll 1$, we can expand in the collisionality.**

- To $O(\nu_i^{-1})$ one finds that $G_{i1}$ is constant on the lowest-order orbits.

- $G_{i1}$ is found by averaging the $O(\nu_i^0)$ equation:
  - For trapped trajectories we take the orbit average
    \[
    \sum_\sigma \int_{l_{b1}}^{l_{b2}} \frac{1}{|v_||} C_{ii}^\ell [G_{i1}] \, dl = \left( 2 \int_{l_{b1}}^{l_{b2}} \frac{1}{|v_||} v_{M,i} \cdot \nabla \psi \, dl \right) \Upsilon_i F_{Mi},
    \]
    where $l_{b1}$ and $l_{b2}$ are the bounce points of the orbit.
  - For passing particles we take the flux surface average
    \[
    \langle B |v_||^{-1} C_{ii}^\ell [G_{i1}] \rangle_\psi = 0.
    \]

- These equations imply $G_{i1} \sim \nu_i^{-1} \rho_i F_{Mi}$, which is fine as long as $\rho_{i*} \ll \nu_{i*} \ll 1$. That is, if the expansion in $\nu_{i*}$ is subsidiary with respect to the expansion in $\rho_{i*}$. 

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Breakdown of the expansion when $\nu_{i*} \lesssim \rho_{i*}$

- $G_{i1} \sim \nu_{i*}^{-1} \rho_{i*} F_{Mi}$ in the $1/\nu$ regime.

- The expansion around the Maxwellian breaks down if $\nu_{i*} \lesssim \rho_{i*}$ because $G_{i1}$ becomes as large as $F_{Mi}$ (and $\varphi_1$ as large as $\varphi_0$).

- In addition, terms like

$$ (v_{M,i} + v_E) \cdot \nabla \psi \partial_\psi G_{i1} \quad \text{and} \quad (v_{M,i} + v_E) \cdot \nabla \alpha \partial_\alpha G_{i1}, $$

where $v_E$ is the $E \times B$ drift, have to be taken into account, and the drift-kinetic equation becomes radially non-local (at least, one cannot guarantee that it remains local).

- Collisionality regimes below the $1/\nu$ regime are relevant in stellarator plasmas.

- Do we have to live with radially non-local neoclassical equations?
In general, the orbit average of the radial magnetic drift, \( v_{M,i} \cdot \nabla \psi \), does not vanish for trapped particles in a stellarator.

Stellarators in which the average of \( v_{M,i} \cdot \nabla \psi \) vanishes for all trajectories are called omnigeneous. They exhibit neoclassical transport levels similar to those of tokamaks.

**The idea:** In the \( 1/\nu \) regime the deviation from the Maxwellian distribution is proportional to the averaged radial magnetic drift. In stellarators close to omnigeneity this average is small, by definition. **This might introduce in the problem a small parameter that restores radial locality.**
Formal definition of omnigeneity

The second adiabatic invariant is defined for each trapped trajectory as [Cary and Shasharina (1997), Parra et al. (2015)]

\[ J = 2 \int_{l_{b_1}}^{l_{b_2}} |v||dl. \]

- A stellarator is omnigeneous if and only if \( \partial_\alpha J = 0 \) for every trapped trajectory.

- Equivalent and useful definition: a stellarator is omnigeneous if and only if

\[ \partial_\alpha \int_{l_{b_1}}^{l_{b_2}} \Lambda(\psi, B(\psi, \alpha, l), v, \lambda)dl = 0, \]

for any function \( \Lambda \) that depends on \( \alpha \) and \( l \) only through \( B \).

In what follows we deal with stellarators whose magnetic field has the form

\[ B = B_0 + \delta B_1, \]

where \( B_0 \) is omnigeneous, \( B_1 \sim B_0 \) and \( 0 \leq \delta \ll 1 \). We also assume that

\[ |\nabla \ln B_0|^{-1} \sim |\nabla \ln B_1|^{-1} \sim L_0. \]
Assuming $\nu_i \sim \rho_i$, the expansion in $\delta \ll 1$ allows to prove that

- $F_i = F_{Mi} + \delta F_{i0}^{(1)} + \ldots$, where $F_{Mi}$ is a Maxwellian with zero flow and constant on flux surfaces and $F_{i0}^{(1)} \sim F_{Mi}$.

- The non-adiabatic component $G_{i0}^{(1)}$ can be written as $G_{i0}^{(1)} = h_i^{(1)}(\psi, v, \lambda, \sigma) + g_i^{(1)}(\psi, \alpha, v, \lambda)$, where $g_i^{(1)}$ vanishes in the passing region and can be chosen so that $\int_0^{2\pi} g_i^{(1)} d\alpha = 0$.

- $h_i^{(1)}$ is Maxwellian and can be absorbed in the definition of $F_{Mi}$.

- $\varphi = \varphi_0 + \delta \varphi_1^{(1)} + \ldots$, where $\varphi_0$ is a flux function and $\varphi_1^{(1)} \sim \varphi_0$.

Hence, we only need to find a drift-kinetic equation for $g_i^{(1)}$.

Remark: From now on, a superindex $(0)$ refers to quantities computed using $B_0$, and a superindex $(1)$ to perturbed quantities.
Drift-kinetic equation for $\nu_i \lesssim \rho_i$ in stellarators close to omnigeneity

Expanding in $\delta$ we get a **radially local equation** (compare to [Sugama PoP 2016] and [Landreman PoP 2014]),

$$- \partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i F_{Mi} = \sum_\sigma \frac{Z_i e \Psi'_t}{m_i c} \int_{l_{b10}}^{l_{b20}} \frac{dl}{|v^{(0)}|} C_{ii}^{\ell(0)} [g_i^{(1)}],$$

where $\Psi_t$ is the toroidal magnetic flux over $2\pi$, the prime stands for differentiation with respect to $\psi$,

$$\partial_\psi J = - \frac{Z_i e \Psi'_t \tau_b}{m_i c} (\mathbf{v}_{d,i} \cdot \nabla \alpha)$$

and

$$\partial_\alpha J = \frac{Z_i e \Psi'_t \tau_b}{m_i c} (\mathbf{v}_{d,i} \cdot \nabla \psi).$$

Here, $\mathbf{v}_{d,i} = \mathbf{v}_{M,i} + \mathbf{v}_E$, the overline denotes orbit average and $\tau_b$ is the corresponding orbit time.
Solution of the drift-kinetic equation when $\nu_{i*} \ll \rho_{i*}$

\[- \partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i F_{Mi} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b10}}^{l_{b20}} \frac{dl}{\|v\|} C_{ii}^{\ell(0)} [g_i^{(1)}] \]

- Expanding in $\nu_{i*}/\rho_{i*} \ll 1$ is the same as expanding in $\nu_{ii}/\omega_\alpha \ll 1$, where $\omega_\alpha = m_i c \partial_\psi J^{(0)}/(Z_i e \Psi_t' \tau_b^{(0)}) \sim \rho_{i*}v_{ti}/L_0$ is the precession frequency due to the tangential drifts.

- To lowest order in the $\nu_{ii}/\omega_\alpha$ expansion one obtains $g_i^{(1)} = g_0 + \ldots$, with

\[g_0 = \frac{1}{\partial_\psi J^{(0)}} \left( J^{(1)} - \frac{1}{2\pi} \int_0^{2\pi} J^{(1)} d\alpha \right) \Upsilon_i F_{Mi}.\]

- It is easy to realize that $g_0$ does not contribute to the energy flux, $Q_i$.

- **Neoclassical transport when $\nu_{i*} \ll \rho_{i*}$ is dominated by two small layers in phase space.**
Discontinuity at the boundary between trapped and passing particles: the $\sqrt{\nu}$ regime

- The distribution function is zero in the passing region, but $g_i^{(1)}$ at the boundary of the trapped region is given by $g_+ := g_0(\lambda_c) \neq 0$, with $\lambda_c = 1/B_{0,\text{max}}$.

- This discontinuity is the consequence of dropping the collision term, and points at the existence of a small boundary layer around $\lambda_c$ where the distribution function develops large variations in $\lambda$.

- Write $g_i^{(1)} = g_0 + g_{b1} + \ldots$, where $g_{b1}$ is the solution in the layer.

- The equation for $g_{b1}$ is

$$\hat{\partial}_\psi \hat{J}^{(0)} \partial_\alpha g_{b1} + \nu \lambda \xi \partial^2_\lambda g_{b1} = -\nu \lambda \xi \partial^2_\lambda \hat{g}_0, \quad g_{b1}(\lambda_c) = -g_+, \quad g_{b1}(\lambda = \infty) = 0.$$ 

where

$$\hat{\partial}_\psi \hat{J}^{(0)} = a_1 \ln(\tilde{a}_2(\lambda - \lambda_c)), \quad \hat{g}_0 = \frac{1}{\hat{\partial}_\psi \hat{J}^{(0)}} \left( \hat{J}^{(1)} - \frac{1}{2\pi} \int_0^{2\pi} \hat{J}^{(1)} d\alpha \right) \Upsilon_i F_{Mi},$$

$$\hat{J}^{(1)} = c_1 \ln(\tilde{c}_2(\lambda - \lambda_c)), \quad \xi := \frac{Z_i e \Psi'_t}{m_i c} \frac{2\lambda_c}{v} \int_{l_{b10}}^{l_{b20}} B_0^{-1} \sqrt{1 - \lambda_c B_0} dl.$$
Discontinuity at the boundary between trapped and passing particles: the $\sqrt{\nu}$ regime

$$\partial_\psi J^{(0)} \partial_\alpha g_{bl} + \nu \xi \partial_\lambda^2 g_{bl} = -\nu \xi \partial_\lambda^2 \hat{g}_0, \quad g_{bl}(\lambda_c) = -g_+, \quad g_{bl}(\lambda = \infty) = 0.$$  

- It is straightforward so see that the typical size of the layer is

$$B_0 \Delta \lambda \sim \left(\frac{\nu_{ii}}{\omega_\alpha}\right)^{1/2}$$

**up to quantitatively important logarithmic corrections!**

- Noting that the coefficients of the homogeneous equation do not depend on $\alpha$, the equation can be easily solved by Fourier transformation.

- The energy flux can be expressed as

$$Q_{i,\sqrt{\nu}} = -\delta^2 \frac{2\pi^2 m_i^2 c}{Ze} \sum_{n=-\infty}^{\infty} in \int_0^{\infty} dv v^3 \left( \frac{v^2}{2} + \frac{Ze \phi_0}{m_i} \right) \int_{\lambda_c}^{\infty} d\lambda \bar{J}^{(1)}_{-n} g_{bl,n},$$

which has a typical size

$$Q_{i,\sqrt{\nu}} \sim \delta^2 \nu_{ii}^{1/2} \frac{1}{\omega_\alpha^{3/2}} \rho_{i*} n_i m_i v_{ti}^4 L^{-1} S_\psi,$$

where $S_\psi$ is the area of the flux surface.
Zeros of $\omega_\alpha$: the superbanana-plateau regime

$$g_0 = \frac{1}{\partial_\psi J(0)} \left( J^{(1)} - \frac{1}{2\pi} \int_0^{2\pi} J^{(1)} d\alpha \right) \Upsilon_i F_{Mi}.$$  

- When the precession frequency $\omega_\alpha$ vanishes, $g_0$ diverges.
- Denote by $\lambda_r(\psi, v)$ the values of $\lambda$ where $\omega_\alpha = 0$.
- Write $g_i^{(1)} = g_0 + g_{r1} + \ldots$, where $g_{r1}$ will be localized in the coordinate $\lambda$ around $\lambda = \lambda_r$.
- The equation for $g_{r1}$ is

$$\omega'_{\alpha,r}(\lambda - \lambda_r) \partial_\alpha g_{r1} + \nu_\lambda \chi_r \partial^2_\lambda g_{r1} = S_r,$$

with

$$\chi_r(\psi, v) := \frac{2\lambda_r}{\tau_{b,r}^{(0)}} \int_{l_{b10}}^{l_{b20}} B_0^{-1}(\psi, \alpha, l) \sqrt{1 - \lambda_r B_0(\psi, \alpha, l)},$$

$$\tau_{b,r}^{(0)}(\psi, v) := \tau_{b}^{(0)}(\psi, v, \lambda_r(\psi, v)), \quad \omega'_{\alpha,r}(\psi, v) := \partial_\lambda \omega_\alpha(\psi, v, \lambda)|_{\lambda = \lambda_r(\psi, v)}$$

and $S_r(\psi, \alpha, v) := m_i c/(Ze^{\Psi_t^{(0)}}) \partial_\alpha J^{(1)}|_{\lambda = \lambda_r(\psi, v)} \Upsilon_i F_{Mi}.$
Zeros of $\omega_\alpha$: the superbanana-plateau regime

$\omega'_{\alpha,r}(\lambda - \lambda_r) \partial_\alpha g_{rl} + \nu \lambda \chi_r \partial^2_\lambda g_{rl} = S_r$

Again, observing that the coefficients of the homogeneous equation do not depend on $\alpha$ we can Fourier transform and solve the equation, obtaining

$$g_{rl,n} = -\frac{S_{r,n}}{\omega'_{\alpha,r}n^{2/3}\lambda_r \beta} \int_0^\infty \exp \left( i \frac{n^{1/3}}{\beta} \frac{\lambda - \lambda_r}{\lambda_r} z - \frac{1}{3} z^3 \right) dz,$$

where

$$\beta := \left( \frac{\nu \lambda \chi_r}{\omega'_{\alpha,r} \lambda_r^3} \right)^{1/3} \ll 1$$

gives the width of the layer.

The energy flux in this case is independent of the collisionality and reads

$$Q_{i,sb-p} = -\delta^2 \frac{4\pi^3 m_i^3 c^2}{Z_i e^2 \Psi_t} \sum_{n=1}^{\infty} \int_{v_{min}}^{v_{max}} \frac{nv^3}{\omega'_{\alpha,r} \tau_{b,r}^{(0)}} \left( \frac{v^2}{2} + \frac{Z_i e \varphi_0}{m_i} \right) |J_{n,r}^{(1)}|^2 \Upsilon_i F_{M_i} dv,$$

with $J_{n,r}^{(1)} := J_{n}^{(1)}(\psi, v, \lambda_r(\psi, v))$. The minimum and maximum values of $v$ for which $\lambda_r$ exists are denoted by $v_{min}, v_{max}$, respectively.
Additive formula for the ion energy flux when $\nu_{i*} \ll \rho_{i*}$

Since both layers are small and are located around different points of phase space, their contributions to transport are additive. Then, for $\nu_{i*} \ll \rho_{i*}$,

$$Q_i = Q_{i,\sqrt{\nu}} + Q_{i,sb-p}.$$

The weight of each term is determined by the value of $v_{\text{min}}$:

- If $v_{\text{min}} \lesssim v_{ti}$, then the superbanana-plateau regime dominates over the $\sqrt{\nu}$ regime.
- If, on the contrary, $v_{\text{min}} \gg v_{ti}$, then the superbanana-plateau regime will be subdominant with respect to the $\sqrt{\nu}$ regime.
A glance to numerical applications based on all the above: $D_{11}$ neoclassical coefficient in LHD at low collisionalities

Discharge number 127689, ECH phase, $R_0 = 3.67\text{m}$.

- One point with DKES (squares) takes about 1 hour of CPU time.
- One point with the code that José Luis Velasco is building (points joined by solid lines) takes about 1 minute of CPU time.
Conclusions and further work

- We have started a line of research that allows to deal in a systematic way with stellarators close to omnigeneity.

- In this work we have focused on neoclassical transport for collisionalities below the $1/\nu$ regime, and we have found expressions for the fluxes in the $\sqrt{\nu}$ and the superbanana-plateau regimes.
  - A linear equation that determines the component of the electrostatic potential that is non-constant on the flux surface can be deduced (not addressed in this talk).

- Concepts and results of this work can be used to build fast neoclassical codes, that might be included in optimization loops.
Thank you for your attention!

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