The effect of tangential drifts on neoclassical transport in stellarators close to omnigeneity

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Summary

- Stellarator drift-kinetic equation at low collisionality: the $1/\nu$ regime.
- In a generic stellarator the expansion of the distribution function around a Maxwellian breaks down at lower collisionalities and the drift-kinetic equation becomes radially non-local.
- Omnigeneous stellarators and stellarators close to omnigeneity.
- In stellarators close to omnigeneity the expansion around a Maxwellian can be carried out for collisionalities below the $1/\nu$ regime and a radially local drift-kinetic equation can be derived.
- Neoclassical transport in stellarators close to omnigeneity below the $1/\nu$ regime:
  - $\sqrt{\nu}$ regime and superbanana-plateau regime.
- The ideas in this work can be used to build fast neoclassical codes. An example: the $\sqrt{\nu}$ regime in LHD.

Remark: A mass ratio expansion $\sqrt{m_e/m_i} \ll 1$ is assumed. Hence, ion-electron collisions are dropped. We focus on ion transport.

Remark: The effect of large aspect ratio is not studied.

Drift-kinetic equation in stellarators

- Spatial coordinates $\{\psi, \alpha, l\}$: ψ is a radial coordinate, α is a periodic coordinate that labels magnetic field lines and l is the length along the line.

  Velocity coordinates $\{v, \lambda, \sigma\}$: v is the magnitude of the velocity, $\lambda = v_\perp^2/(v^2 B)$ is the pitch-angle coordinate and $\sigma = v_{||}/|v_{||}|$.
- Define the normalized gyroradius as $\rho_{i*} = \rho_i/L_0 \ll 1$, where L_0 is the typical variation length of $\mathbf{B}$. In the standard drift-kinetic expansion
  - the distribution function is expanded as $F_i = F_{Mi} + F_{i1} + \ldots$, where $F_{i1} \sim \rho_{i*} F_{Mi}$ and F_{Mi} is a Maxwellian distribution with zero flow, and with density $n_i(\psi)$ and temperature $T_i(\psi)$ constant on flux surfaces;
  - the electrostatic potential is expanded as $\varphi(\psi, \alpha, l) = \varphi_0(\psi) + \varphi_1(\psi, \alpha, l) + \ldots$, with $\varphi_0 \sim T_i/(Z_i e)$ and $\varphi_1/\varphi_0 \sim \rho_{i*}$.
- The drift-kinetic equation for $G_{i1} = F_{i1} + (Z_i e \varphi_1/T_i) F_{Mi}$ is

$$v_{||}\partial_l G_{i1} + \Upsilon_i \mathbf{v}_{M,i} \cdot \nabla\psi F_{Mi} = C_{ii}^{\ell}[G_{i1}],$$

  where $\mathbf{v}_{M,i}$ is the ion magnetic drift, Υ_i involves the gradients of n_i, T_i and φ_0, and C_{ii}^{ℓ} is the linearized ion-ion collision operator.

Drift-kinetic equation at low collisionality: the $1/\nu$ regime

$$v_{||}\partial_l G_{i1} + \Upsilon_i \mathbf{v}_{M,i} \cdot \nabla\psi F_{Mi} = C_{ii}^{\ell}[G_{i1}]$$

- Define the ion collisionality as $\nu_{i*} = \nu_{ii} L_0 / v_{ti} \ll 1$, where $v_{ti} = \sqrt{T_i/m_i}$ is the thermal speed. **If $\nu_{i*} \ll 1$, we can expand in the collisionality.**
- To $O(\nu_{i*}^{-1})$ one finds that G_{i1} is constant on the lowest-order orbits.
- G_{i1} is found by averaging the $O(\nu_{i*}^{0})$ equation:
  - For trapped trajectories we take the orbit average

$$\sum_{\sigma} \int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|} C_{ii}^{\ell}[G_{i1}]\,\mathrm{d}l = \left(2\int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|} \mathbf{v}_{M,i} \cdot \nabla\psi\,\mathrm{d}l \right) \Upsilon_i F_{Mi},$$

    where l_{b_1} and l_{b_2} are the bounce points of the orbit.
  - For passing particles we take the flux surface average

$$\left\langle B|v_{||}|^{-1} C_{ii}^{\ell}[G_{i1}] \right\rangle_{\psi} = 0.$$

- **These equations imply $G_{i1} \sim \nu_{i*}^{-1} \rho_{i*} F_{Mi}$, which is fine as long as $\rho_{i*} \ll \nu_{i*} \ll 1$. That is, if the expansion in ν_{i*} is subsidiary with respect to the expansion in ρ_{i*}.**

Breakdown of the expansion when $\nu_{i*} \lesssim \rho_{i*}$

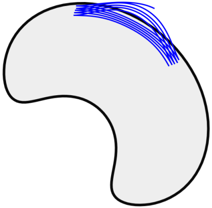
- $G_{i1} \sim \nu_{i*}^{-1} \rho_{i*} F_{Mi}$ in the $1/\nu$ regime.
- The expansion around the Maxwellian breaks down if $\nu_{i*} \lesssim \rho_{i*}$ because G_{i1} **becomes as large as** F_{Mi} (and φ_1 as large as φ_0).
- In addition, terms like

$$(\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\psi \partial_\psi G_{i1} \quad \text{and} \quad (\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\alpha \partial_\alpha G_{i1},$$

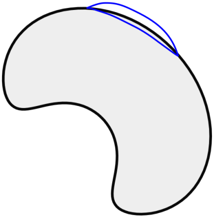
  where $\mathbf{v}_E$ is the $E \times B$ drift, have to be taken into account, and **the drift-kinetic equation becomes radially non-local** (at least, one cannot guarantee that it remains local).
- Collisionality regimes below the $1/\nu$ regime are relevant in stellarator plasmas.
- Do we have to live with radially non-local neoclassical equations?

Orbit-averaged radial magnetic drift in stellarators

- In general, the orbit average of the radial magnetic drift, $\mathbf{v}_{M,i} \cdot \nabla\psi$, does not vanish for trapped particles in a stellarator.



- Stellarators in which the average of $\mathbf{v}_{M,i} \cdot \nabla\psi$ vanishes for all trajectories are called *omnigeneous*. They exhibit neoclassical transport levels similar to those of tokamaks.



The idea: In the $1/\nu$ regime the deviation from the Maxwellian distribution is proportional to the averaged radial magnetic drift. In stellarators close to omnigeneity this average is small, by definition. **This might introduce in the problem a small parameter that restores radial locality.**

Formal definition of omnigeneity

The second adiabatic invariant is defined for each trapped trajectory as [Cary and Shasharina (1997), Parra *et al.* (2015)]

$$J = 2\int_{l_{b1}}^{l_{b2}} |v_{||}|\mathrm{d}l.$$

- A stellarator is omnigeneous if and only if $\partial_\alpha J = 0$ for every trapped trajectory.
- Equivalent and useful definition: a stellarator is omnigeneous if and only if

$$\partial_\alpha \int_{l_{b1}}^{l_{b2}} \Lambda(\psi, B(\psi,\alpha,l), v, \lambda)\mathrm{d}l = 0,$$

  for any function Λ that depends on α and l only through B.

In what follows we deal with stellarators whose magnetic field has the form

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}_1,$$

where $\mathbf{B}_0$ is omnigeneous, $\mathbf{B}_1 \sim \mathbf{B}_0$ and $0 \le \delta \ll 1$. We also assume that $|\nabla \ln B_0|^{-1} \sim |\nabla \ln B_1|^{-1} \sim L_0$.

Drift-kinetic equation for $\nu_{i*} \lesssim \rho_{i*}$ in stellarators close to omnigeneity

Assuming $\nu_{i*} \sim \rho_{i*}$, **the expansion in $\delta \ll 1$ allows to prove that**

- $F_i = F_{Mi} + \delta F_{i0}^{(1)} + \ldots$, where F_{Mi} is a Maxwellian with zero flow and constant on flux surfaces and $F_{i0}^{(1)} \sim F_{Mi}$.
- The non-adiabatic component $G_{i0}^{(1)}$ can be written as $G_{i0}^{(1)} = h_i^{(1)}(\psi, v, \lambda, \sigma) + g_i^{(1)}(\psi, \alpha, v, \lambda)$, where $g_i^{(1)}$ vanishes in the passing region and can be chosen so that $\int_0^{2\pi} g_i^{(1)} \mathrm{d}\alpha = 0$.
- $h_i^{(1)}$ is Maxwellian and can be absorbed in the definition of F_{Mi}.
- $\varphi = \varphi_0 + \delta\varphi_1^{(1)} + \ldots$, where φ_0 is a flux function and $\varphi_1^{(1)} \sim \varphi_0$.

Hence, we only need to find a drift-kinetic equation for $g_i^{(1)}$.

Remark: From now on, a superindex (0) refers to quantities computed using $\mathbf{B}_0$, and a superindex (1) to perturbed quantities.

Drift-kinetic equation for $\nu_{i*} \lesssim \rho_{i*}$ in stellarators close to omnigeneity

Expanding in δ we get a **radially local equation** (compare to [Sugama PoP 2016] and [Landreman PoP 2014]),

$$- \partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i F_{Mi} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\mathrm{d}l}{|v_{||}^{(0)}|} C_{ii}^{\ell(0)}[g_i^{(1)}],$$

where Ψ_t is the toroidal magnetic flux over 2π, the prime stands for differentiation with respect to ψ,

$$\partial_\psi J = - \frac{Z_i e \Psi_t' \tau_b}{m_i c} \overline{(\mathbf{v}_{d,i} \cdot \nabla\alpha)}$$

and

$$\partial_\alpha J = \frac{Z_i e \Psi_t' \tau_b}{m_i c} \overline{(\mathbf{v}_{d,i} \cdot \nabla\psi)}\,.$$

Here, $\mathbf{v}_{d,i} = \mathbf{v}_{M,i} + \mathbf{v}_E$, the overline denotes orbit average and τ_b is the corresponding orbit time.

Solution of the drift-kinetic equation when $\nu_{i*} \ll \rho_{i*}$

$$-\partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i F_{Mi} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\mathrm{d}l}{|v_{||}^{(0)}|} C_{ii}^{\ell(0)}[g_i^{(1)}]$$

- Expanding in $\nu_{i*}/\rho_{i*} \ll 1$ is the same as expanding in $\nu_{ii}/\omega_\alpha \ll 1$, where $\omega_\alpha = m_i c \partial_\psi J^{(0)} / (Z_i e \Psi_t' \tau_b^{(0)}) \sim \rho_{i*} v_{ti}/L_0$ is the precession frequency due to the tangential drifts.
- To lowest order in the ν_{ii}/ω_α expansion one obtains $g_i^{(1)} = g_0 + \ldots$, with

$$g_0 = \frac{1}{\partial_\psi J^{(0)}} \left(J^{(1)} - \frac{1}{2\pi} \int_0^{2\pi} J^{(1)} \mathrm{d}\alpha \right) \Upsilon_i F_{Mi}.$$

- It is easy to realize that g_0 does not contribute to the energy flux, Q_i.
- **Neoclassical transport when $\nu_{i*} \ll \rho_{i*}$ is dominated by two small layers in phase space.**

Discontinuity at the boundary between trapped and passing particles: the $\sqrt{\nu}$ regime

- The distribution function is zero in the passing region, but $g_i^{(1)}$ at the boundary of the trapped region is given by $g_+ := g_0(\lambda_c) \neq 0$, with $\lambda_c = 1/B_{0,\max}$.
- This discontinuity is the consequence of dropping the collision term, and points at the existence of a small boundary layer around λ_c where the distribution function develops large variations in λ.
- Write $g_i^{(1)} = g_0 + g_{\rm bl} + \ldots$, where $g_{\rm bl}$ is the solution in the layer.
- The equation for $g_{\rm bl}$ is

$$\widehat{\partial_\psi J^{(0)}}\partial_\alpha g_{\rm bl} + \nu_\lambda \xi \partial_\lambda^2 g_{\rm bl} = -\nu_\lambda \xi \partial_\lambda^2 \widehat{g_0}, \qquad g_{\rm bl}(\lambda_c) = -g_+,\ g_{\rm bl}(\lambda = \infty) = 0.$$

where

$$\widehat{\partial_\psi J^{(0)}} = a_1 \ln(\tilde{a}_2(\lambda - \lambda_c)), \quad \widehat{g_0} = \frac{1}{\widehat{\partial_\psi J^{(0)}}}\left(\widehat{J^{(1)}} - \frac{1}{2\pi}\int_0^{2\pi} \widehat{J^{(1)}} \mathrm{d}\alpha\right)\Upsilon_i F_{Mi},$$

$$\widehat{J^{(1)}} = c_1 \ln(\tilde{c}_2(\lambda - \lambda_c)), \quad \xi := \frac{Z_i e \Psi_t'}{m_i c}\frac{2\lambda_c}{v}\int_{l_{b_{10}}}^{l_{b_{20}}} B_0^{-1}\sqrt{1 - \lambda_c B_0}\, \mathrm{d}l.$$

Discontinuity at the boundary between trapped and passing particles: the $\sqrt{\nu}$ regime

$$\widehat{\partial_\psi J^{(0)}}\partial_\alpha g_{\rm bl} + \nu_\lambda \xi \partial^2_\lambda g_{\rm bl} = -\nu_\lambda \xi \partial^2_\lambda \widehat{g_0},\ \ g_{\rm bl}(\lambda_c) = -g_+, \qquad g_{\rm bl}(\lambda = \infty) = 0.$$

- It is straightforward so see that the typical size of the layer is

$$B_0 \Delta\lambda \sim (\nu_{ii}/\omega_\alpha)^{1/2}$$

  up to quantitatively important logarithmic corrections!

- Noting that **the coefficients of the homogeneous equation do not depend on α**, the equation can be easily solved by Fourier transformation.

- The energy flux can be expressed as

$$Q_{i,\sqrt{\nu}} = -\delta^2 \frac{2\pi^2 m_i^2 c}{Z_i e} \sum_{n=-\infty}^{\infty} in \int_0^\infty \mathrm{d}v v^3 \left(\frac{v^2}{2} + \frac{Z_i e \varphi_0}{m_i} \right) \int_{\lambda_c}^\infty \mathrm{d}\lambda\, \widehat{J^{(1)}}_{-n}\, g_{\mathrm{bl},n},$$

  which has a typical size

$$Q_{i,\sqrt{\nu}} \sim \delta^2 \frac{\nu_{ii}^{1/2}}{\omega_\alpha^{3/2}} \rho_{i*}^2 n_i m_i v_{ti}^4 L_0^{-1} S_\psi,$$

  where S_ψ is the area of the flux surface.

Zeros of ω_α: the superbanana-plateau regime

$$g_0 = \frac{1}{\partial_\psi J^{(0)}}\left(J^{(1)} - \frac{1}{2\pi}\int_0^{2\pi} J^{(1)}\mathrm{d}\alpha\right)\Upsilon_i F_{Mi}.$$

- When the precession frequency ω_α vanishes, g_0 diverges.
- Denote by $\lambda_r(\psi, v)$ the values of λ where $\omega_\alpha = 0$.
- Write $g_i^{(1)} = g_0 + g_{\mathrm{rl}} + \dots$, where g_{rl} will be localized in the coordinate λ around $\lambda = \lambda_r$.
- The equation for g_{rl} is

$$\omega'_{\alpha,r}(\lambda - \lambda_r)\partial_\alpha g_{\mathrm{rl}} + \nu_\lambda \chi_r \partial^2_\lambda g_{\mathrm{rl}} = S_r,$$

with

$$\chi_r(\psi, v) := \frac{2\lambda_r}{\tau_{b,r}^{(0)}}\int_{l_{b_{10}}}^{l_{b_{20}}} B_0^{-1}(\psi,\alpha,l)\sqrt{1-\lambda_r B_0(\psi,\alpha,l)},$$

$\tau_{b,r}^{(0)}(\psi, v) := \tau_b^{(0)}(\psi, v, \lambda_r(\psi, v))$, $\omega'_{\alpha,r}(\psi, v) := \partial_\lambda \omega_\alpha(\psi, v, \lambda)|_{\lambda=\lambda_r(\psi,v)}$ and $S_r(\psi,\alpha,v) := m_i c/(Z_i e \Psi'_t \tau_b^{(0)})\partial_\alpha J^{(1)}|_{\lambda=\lambda_r(\psi,v)}\Upsilon_i F_{Mi}$.

Zeros of ω_α: the superbanana-plateau regime

$$\omega'_{\alpha,r}(\lambda-\lambda_r)\partial_\alpha g_{\rm rl} + \nu_\lambda \chi_r \partial^2_\lambda g_{\rm rl} = S_r$$

- Again, observing that the coefficients of the homogeneous equation do not depend on α we can Fourier transform and solve the equation, obtaining

$$g_{{\rm rl},n} = -\frac{S_{r,n}}{\omega'_{\alpha,r} n^{2/3}\lambda_r \beta}\int_0^\infty \exp\left(i\frac{n^{1/3}}{\beta}\frac{\lambda-\lambda_r}{\lambda_r}z - \frac{1}{3}z^3\right)\mathrm{d}z,$$

  where

$$\beta := \left(\frac{\nu_\lambda\chi_r}{\omega'_{\alpha,r}\lambda_r^3}\right)^{1/3} \ll 1$$

  gives the width of the layer.

- The energy flux in this case is independent of the collisionality and reads

$$Q_{i,{\rm sb-p}} = -\delta^2 \frac{4\pi^3 m_i^3 c^2}{Z_i^2 e^2 \Psi'_t}\sum_{n=1}^\infty \int_{v_{\rm min}}^{v_{\rm max}} \frac{n v^3}{\omega'_{\alpha,r}\tau^{(0)}_{b,r}}\left(\frac{v^2}{2}+\frac{Z_i e\varphi_0}{m_i}\right)|J^{(1)}_{n,r}|^2 \Upsilon_i F_{Mi}\mathrm{d}v,$$

  with $J^{(1)}_{n,r} := J^{(1)}_n(\psi, v, \lambda_r(\psi,v))$. The minimum and maximum values of v for which λ_r exists are denoted by $v_{\rm min}$, $v_{\rm max}$, respectively.

Additive formula for the ion energy flux when $\nu_{i*} \ll \rho_{i*}$

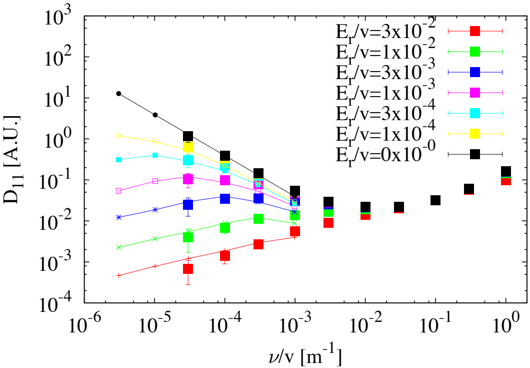
- Since both layers are small and are located around different points of phase space, their contributions to transport are additive. Then, for $\nu_{i*} \ll \rho_{i*}$,

$$Q_i = Q_{i,\sqrt{\nu}} + Q_{i,\mathrm{sb-p}}.$$

- The weight of each term is determined by the value of $v_{\min}$:
  - If $v_{\min} \lesssim v_{ti}$, then the superbanana-plateau regime dominates over the $\sqrt{\nu}$ regime.
  - If, on the contrary, $v_{\min} \gg v_{ti}$, then the superbanana-plateau regime will be subdominant with respect to the $\sqrt{\nu}$ regime.

A glance to numerical applications based on all the above: D_{11} neoclassical coefficient in LHD at low collisionalities

Discharge number 127689, ECH phase, $R_0 = 3.67\text{m}$.



- One point with DKES (squares) takes about 1 hour of CPU time.
- One point with the code that José Luis Velasco is building (points joined by solid lines) takes about 1 minute of CPU time.

Conclusions and further work

- We have started a line of research that allows to deal in a systematic way with stellarators close to omnigeneity.
- In this work we have focused on neoclassical transport for collisionalities below the $1/\nu$ regime, and we have found expressions for the fluxes in the $\sqrt{\nu}$ and the superbanana-plateau regimes.
  - A linear equation that determines the component of the electrostatic potential that is non-constant on the flux surface can be deduced (not addressed in this talk).
- Concepts and results of this work can be used to build fast neoclassical codes, that might be included in optimization loops.

Thank you for your attention!

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