Accurate representation of velocity space using truncated Hermite expansions.

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Model for ITG instabilities (from Pueschel *et al.*, 2010)

Test problem for GENE, simplified gyrokinetic equations:

- 1D slab, $(z, v_{\|})$
- $\mathbf{B} = B\mathbf{z}$ homogeneous and static
- electrostatic: $\mathbf{E} = -\nabla\Phi$
- adiabatic electrons
- drift-kinetic ions
- quasineutral
- uniform density and temperature gradients in background
- perturbation about Maxwellian $F_0(v_{\|}) = \pi^{-1/2}\exp(-v_{\|}^2)$
- nonlinearity dropped

May also derive directly from Vlasov equation and quasineutrality.

Electrostatic limit of Belli & Hammett (2005) model

Model for ITG instabilities (from Pueschel *et al.*, 2010)

Linearized kinetic equation

$$\frac{\partial F_1}{\partial t} + \alpha_i v_\parallel \frac{\partial F_1}{\partial z} + \left[\omega_n + \omega_{T_i}\left(v_\parallel^2 - \frac{1}{2}\right)\right] ik_y \Phi F_0 + \tau_e \alpha_i v_\parallel \frac{\partial \Phi}{\partial z} F_0 = 0$$

Quasineutrality

$$\Phi = \int_{-\infty}^{\infty} F_1 \, \mathrm{d}v_\parallel .$$

Parameters

$\omega_n = 1,$	normalized density gradient
$\omega_{T_i} = 10,$	normalized ion temperature gradient
$k_y = 0.3,$	perpendicular wavenumber
$\tau_e = 1,$	species temperature ratio $(T_i = T_e)$
$\alpha_i = 0.34$	nondimensional constant: velocity and length scales

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Wave-like solutions

$$F_1 = f(v_\parallel) \exp\left(i\left(k_\parallel z - \omega t\right)\right)$$

Leads to eigenvalue problem,

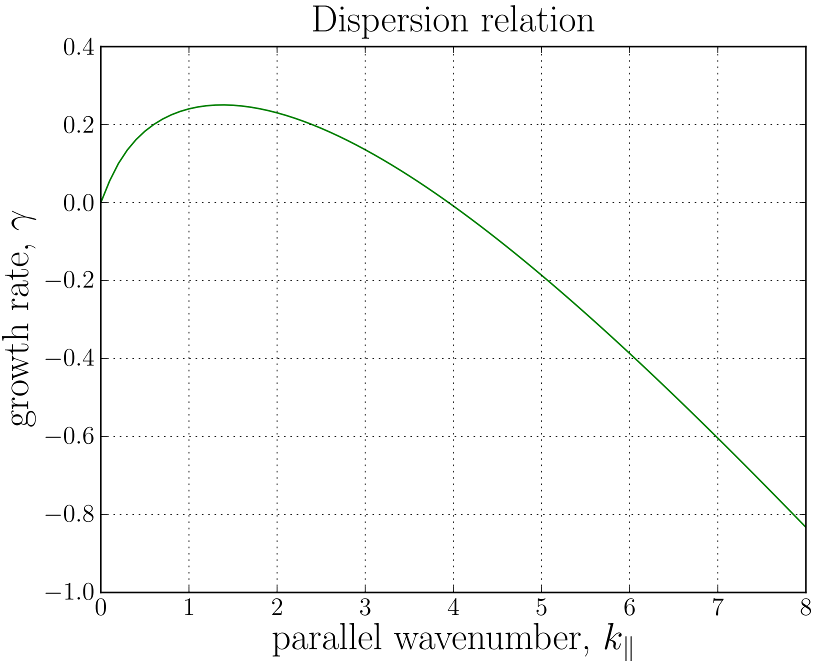
$$L_0 f = \omega f$$

where $$L_0 f = \alpha_i k v_\parallel f + \left[\omega_n + \omega_{T_i}\left(v_\parallel^2 - \frac{1}{2}\right)\right] k_y \Phi F_0 + \tau_e \alpha_i k v_\parallel \Phi F_0$$

L real $\implies$ eigenvalues in complex conjugate pairs $\implies$ **no damping**
So actually want to solve,

$$\lim_{\epsilon \to 0} L_\epsilon f = \omega f, \quad \text{for} \quad L_\epsilon = L_0 + i\epsilon C$$

where C is collisions, *e.g.* Lénard–Bernstein or BGK.
Breaks symmetry, so decay rate may tend to nonzero limit as $\epsilon \to 0$.



Dispersion relation
growth rate, γ
parallel wavenumber, $k_{\|}$
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0
1
2
3
4
5
6
7
8

Discretization

- Keep $\exp(ik_{\|}z)$ behaviour, represent $v_{\|}$ on a uniform grid,

$$\Phi = \sum_{i=1}^{N} \Delta v f(v_i)$$

- Obtain linear ODE system

$$\frac{\partial f}{\partial t} = iMf, \qquad M \text{ real } N \times N \text{ matrix}$$

- **Matrix eigenvalue problem** (quicker than running IVPs):

$$-i\omega f = \frac{\partial f}{\partial t} = iMf$$

  Growth rate is $\gamma = \max \Im(\omega)$, for $\omega \in \text{spectrum}(M)$.

Matrix M depends on:

- All parameters, particularly **parallel wavenumber**.
- **Discretization** and **resolution** of grid in $v_{\|}$.

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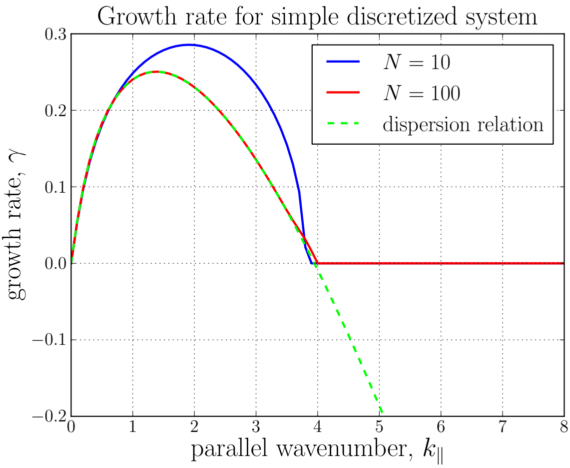
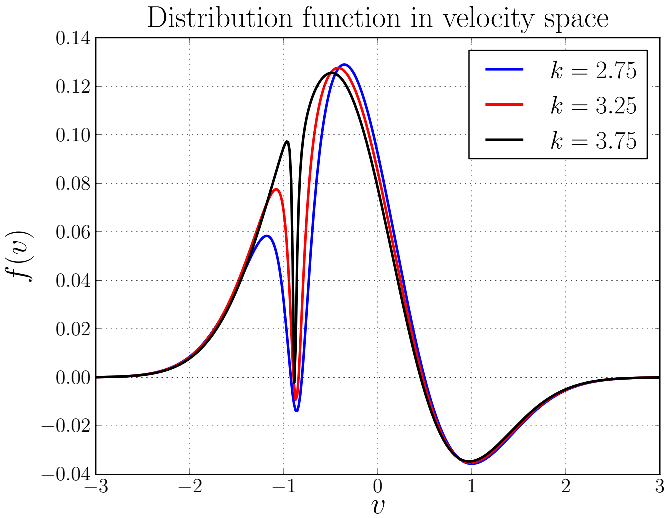


- Capture growth, but convergence slow with N.
- No decay.
  - $\epsilon \to 0$ at fixed N resolution.
  - Need to restore collisions.



Fine scales develop as $\Im(\omega) \to 0$.

Hermite expansion in velocity space

Use expansion in asymmetric Hermite functions,

$$f(v) = \sum_{m=0}^{\infty} a_m \phi_m(v), \qquad \phi^m(v) = \frac{H_m(v)}{\sqrt{2^m m!}}, \qquad \phi_m(v) = F_0(v)\phi^m(v)$$

- Bi-orthogonal polynomials with Maxwellian as weight function

$$\int_{-\infty}^{\infty} \phi_m \phi^n \, \mathrm{d}v = \delta_{mn}$$

- Recurrence relation

$$v\phi_m = \sqrt{\frac{m+1}{2}}\phi_{m+1} + \sqrt{\frac{m}{2}}\phi_{m-1}$$

$\Longrightarrow$ particle streaming becomes mode coupling

- Represent velocity space scales,

$$H_m(v) \propto \cos\left(v\sqrt{2m} - \frac{m\pi}{2}\right), \quad \text{as} \quad m \to \infty$$

large $m \Longrightarrow$ fine scales

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Relative Entropy and Free Energy

Relative entropy is

$$R[F|F_0] \equiv \int_{-\infty}^{\infty} F \log\left(\frac{F}{F_0}\right) - F + F_0 \, \mathrm{d}v$$
$$= \int_{-\infty}^{\infty} F \log F \, \mathrm{d}v + 2U + \text{function(density)}$$

Expansion $F = F_0 + \epsilon F_1$ gives (at leading order)

$$R[F|F_0] = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \frac{F_1^2}{F_0} \, \mathrm{d}v = \frac{\epsilon^2}{2} \sum_{m=0}^{\infty} a_m^2 + \mathcal{O}(\epsilon^3)$$

Define free energy of each Hermite mode, E_m,

$$E = \sum_{m=0}^{\infty} E_m, \qquad \text{with } \; E_m = \frac{1}{2}|a_m|^2$$

cf. energy spectra in Fourier space for Navier–Stokes turbulence.

Moment system

Replace $F_0 \equiv \phi_0$ and v with Hermite functions

$$\frac{\partial F_1}{\partial t} + i\alpha_i k v F_1 + i k_y \Phi \left[\omega_n \phi_0 + \frac{\omega_{T_i}}{\sqrt{2}} \phi_2 \right] + i \frac{k \tau_e \alpha_i \Phi}{\sqrt{2}} \phi_1 = 0$$

Put $F_1 = a_m \phi_m$ (implicit sum over repeated m)

$$\frac{\partial a_m}{\partial t} \phi_m + i\alpha_i k v a_m \phi_m + i k_y \Phi \left[\omega_n \phi_0 + \frac{\omega_{T_i}}{\sqrt{2}} \phi_2 \right] + i \frac{k \tau_e \alpha_i \Phi}{\sqrt{2}} \phi_1 = 0$$

Use recurrence relation on particle streaming

$$\frac{\partial a_m}{\partial t} \phi_m + i\alpha_i k a_m \left(\sqrt{\frac{m+1}{2}} \phi_{m+1} + \sqrt{\frac{m}{2}} \phi_{m-1} \right) + \ldots = 0$$

Gives infinite set of coupled algebraic equations for $\{a_n\}$.

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System of equations for expansion coefficients

$$\omega a_m = \left(\sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) + \underbrace{\frac{k_y}{\alpha_i k_\parallel} \left[\omega_n a_0 \delta_{m0} + \frac{\omega_{T_i} a_0}{\sqrt{2}} \delta_{m2} \right]}_{\text{driving}} + \underbrace{\frac{\tau_e a_0}{\sqrt{2}}}_{\text{Boltzmann response}} \delta_{m1}$$

We have also used $\phi^0 \equiv 1$, so that,

$$\Phi = \int_{-\infty}^{\infty} F_1 \phi^0 \, \mathrm{d}v = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} a_m \phi_m \phi^0 \, \mathrm{d}v = a_0$$

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Matrix equation

$$
\left(\begin{array}{ccc|ccccc}
\frac{\omega_n k_y}{\alpha_i k_\parallel} & \frac{1}{\sqrt{2}} & & & & & & \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & & 1 & & & & & \\
\frac{k_y \omega_{T_i}}{\sqrt{2}\alpha_i k_\parallel} & 1 & & \sqrt{\frac{3}{2}} & & & & \\
\hline
 & & \sqrt{\frac{3}{2}} & & \sqrt{2} & & & \\
 & & & \sqrt{2} & & \ddots & & \\
 & & & & \ddots & & \sqrt{\frac{N}{2}} & \\
 & & & & & \sqrt{\frac{N}{2}} & & \ddots \\
 & & & & & & \ddots &
\end{array}\right) \mathbf{a} = \omega \mathbf{a}
$$

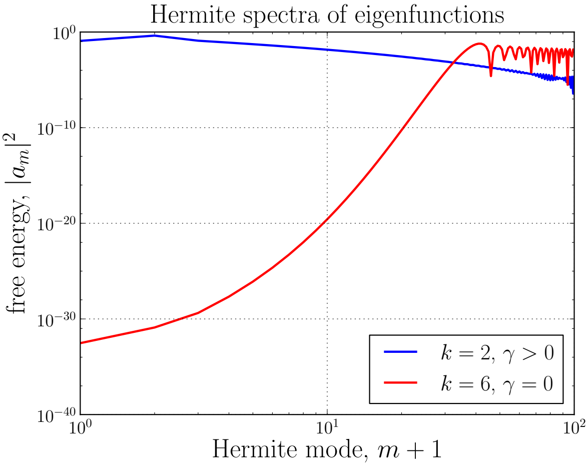
Simple Truncation

$$
\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k_\parallel} & \frac{1}{\sqrt{2}} & & & \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & & 1 & & \\
\frac{k_y \omega_{T_i}}{\sqrt{2}\alpha_i k_\parallel} & 1 & & \sqrt{\frac{3}{2}} & \\
 & & \sqrt{\frac{3}{2}} & & \sqrt{2} \\
 & & & \sqrt{2} &
\end{pmatrix} \mathbf{a} = \omega \mathbf{a}
$$

- Assume $a_{N+1} = 0$ at point of truncation. Last row of matrix:

$$
\cancel{\sqrt{\frac{N+1}{2}} a_{N+1}} + \sqrt{\frac{N}{2}} a_{N-1} = \omega a_N
$$

- Equivalent to discretization on Gauss–Hermite points $\{v_j\}$



- Growing mode: a_m decay as m increases $\implies$ $a_{N+1} \approx 0$ is okay
- (Not) decaying mode **not resolved**

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Collisions

- Free energy should cascade to large m (fine scales in v).
- Free energy has nowhere to go when $a_{N+1} = 0$.
- Restore collisions:

$$\frac{\partial F_1}{\partial t} + \left[\omega_n + \omega_{T_i}\left(v_\parallel^2 - \frac{1}{2}\right)\right] ik_y\Phi F_0 + i\alpha_i k v_\parallel F_1 + i\tau_e\alpha_i k v_\parallel \Phi F_0 = C[F_1]$$

Desirable properties for $C[F_1]$

- conserves mass, momentum and energy
  - $\int C[F_1]\,\mathrm{d}v = \int vC[F_1]\,\mathrm{d}v = \int v^2 C[F_1]\,\mathrm{d}v = 0$
- satisfies a linearized H theorem:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \epsilon^2 \int_{-\infty}^{\infty} \frac{F_1 C[F_1]}{F_0}\,\mathrm{d}v \leqslant 0$$

- represents small-angle collisions (contains v-derivatives)

Lénard–Bernstein (1958) collisions

Simple member of linearized Landau/Fokker–Planck class:

$$C\left[F_1\right] = \nu \frac{\partial}{\partial v}\left[v F_1 + \frac{1}{2}\frac{\partial F_1}{\partial v}\right]$$

- collision frequency ν
- Hermite functions are eigenfunctions:

$$C\left[a_m \phi_m\right] = -\nu m a_m \phi_m$$

$\Longrightarrow$ easy to implement in Hermite space

- Conserves mass, but **not** momentum or energy
- Satisfies a linearized R theorem:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\epsilon^2 \nu \sum_m m |a_m|^2 \leqslant 0$$

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Lénard–Bernstein in Hermite space

$$\begin{pmatrix} \frac{\omega_n k_y}{\alpha_i k_\parallel} & \frac{1}{\sqrt{2}} & & & & \\ \frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & -i\nu & 1 & & & \\ \frac{k_y \omega_{T_i}}{\sqrt{2}\alpha_i k_\parallel} & 1 & -2i\nu & \sqrt{\frac{3}{2}} & & \\ & & \sqrt{\frac{3}{2}} & -3i\nu & \ddots & \\ & & & \ddots & & \sqrt{\frac{N}{2}} \\ & & & & \sqrt{\frac{N}{2}} & -iN\nu \end{pmatrix} \mathbf{a} = \omega \mathbf{a}$$

- matrix now complex
- roots *not* in complex-conjugate pairs $\implies$ can find negative growth rates
- (we can manually conserve momentum and energy)

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Lénard–Bernstein in Hermite space

$$\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k_\parallel} & \frac{1}{\sqrt{2}} & & & & & \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \xcancel{-i\nu} & 1 & & & & \\
\frac{k_y \omega_{T_i}}{\sqrt{2}\alpha_i k_\parallel} & 1 & \xcancel{-2i\nu} & \sqrt{\frac{3}{2}} & & & \\
 & & \sqrt{\frac{3}{2}} & -3i\nu & \ddots & & \\
 & & & \ddots & & \sqrt{\frac{N}{2}} \\
 & & & & \sqrt{\frac{N}{2}} & -iN\nu
\end{pmatrix} \mathbf{a} = \omega \mathbf{a}$$

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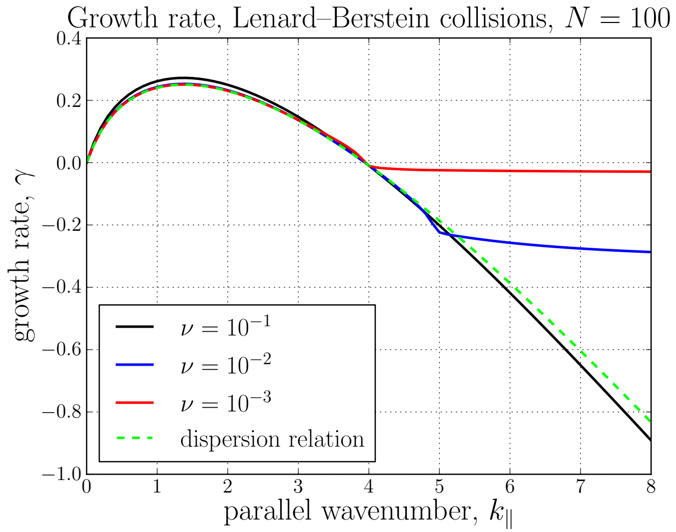
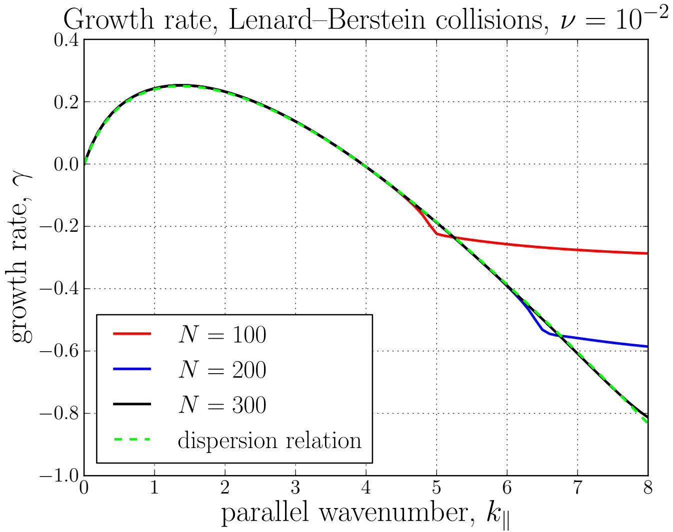
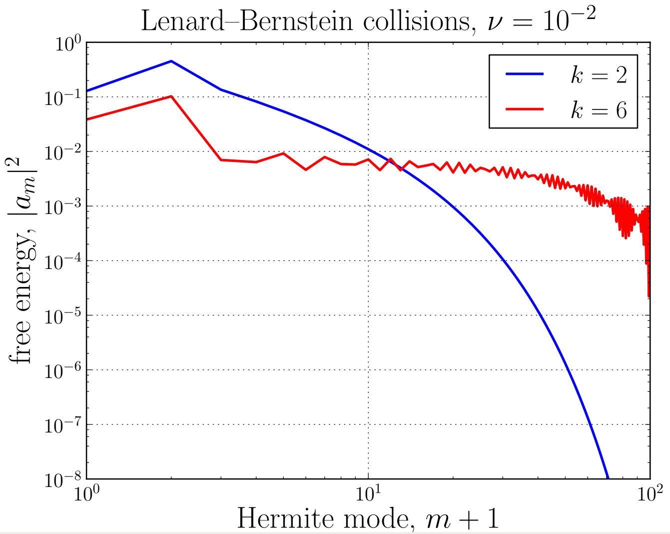
- (we can manually conserve momentum and energy)



If collision frequency ν large enough to get $k_{\parallel} > 4$ correct,
then shape for $k_{\parallel} < 4$ distorted.



- resolves dissipative scales
- expensive: 300 modes



- Appreciable damping along whole spectrum.

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Hypercollisions

Would prefer:

- low modes undamped
- high modes strongly damped

…whatever the truncation point N

Iterate Lénard–Bernstein collisions:

$$\mathcal{L}[F_1] = \frac{\partial}{\partial v}\left(vF_1 + \frac{1}{2}\frac{\partial F_1}{\partial v}\right), \qquad C[F_1] = -\nu(-N)^{-n}\mathcal{L}^n[F_1]$$

(like *hyperdiffusion*: $-(-\nabla^2)^n$ in physical space)

Hermite functions are still eigenfunctions:

$$C\left[a_m\phi_m\right] = -\nu\left(\frac{m}{N}\right)^n a_m\phi_m$$

- ν sets the decay rate of the highest mode
- linearized R-theorem: $\frac{\mathrm{d}R}{\mathrm{d}t} = -\epsilon^2\nu\sum_{m=0}^{\infty}\left(\frac{m}{N}\right)^n |a_m|^2 \leqslant 0$
- $n = 1$ corresponds to Lénard–Bernstein collisions.
- two parameters: $n \approx 6$, $\nu \approx 10$ (robust to variation)

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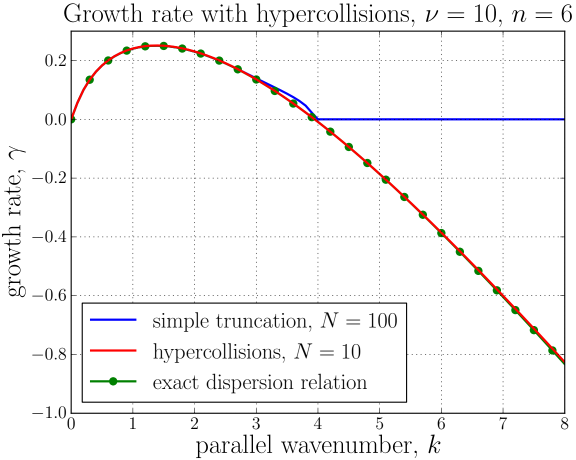
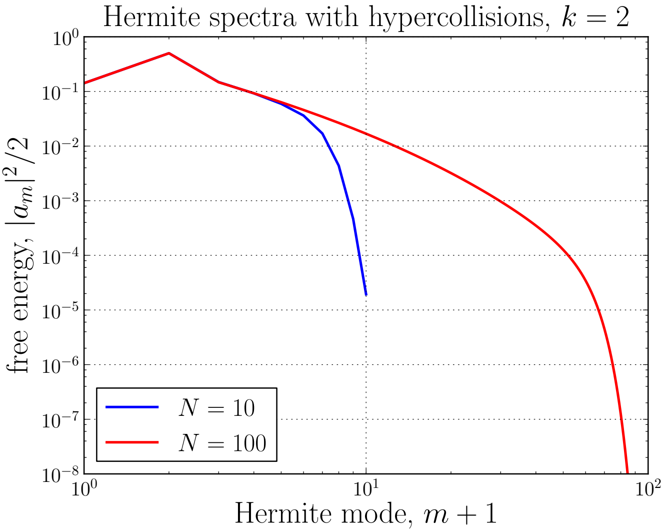
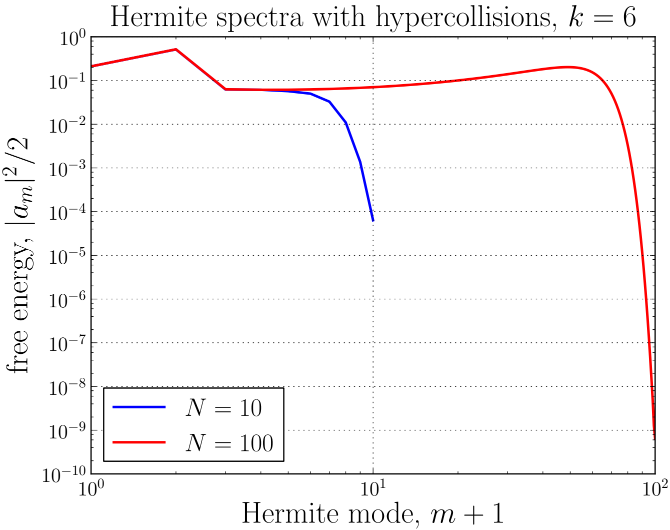


- Captures decaying parts of spectrum.
- Excellent fit, fast convergence.
- Appropriate for nonlinear problems.
- Two parameters, n, ν, robust to variation.



- Low moments largely undamped
- Damping at high m, for any N.



- Low moments largely undamped
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Energy equations and theoretical spectra

Equation for coefficients, valid for $m \geqslant 3$,

$$\omega a_m = \left(\sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) + \text{driving} + \text{Boltzmann response}$$

Treat as a finite difference approximation in continuous m.

Energy equation for $E_m = |a_m|^2/2$ (Zocco & Schekochihin, 2011)

$$\frac{\partial E_m}{\partial t} + \frac{\partial}{\partial m} \left(\sqrt{2m} E_m \right) = -2\nu \left(\frac{m}{N} \right)^n E_m.$$

For a mode with growth rate γ,

$$E_m = \frac{C}{\sqrt{2m}} \exp \left(-\frac{\gamma}{|\gamma|} \left(\frac{m}{m_\gamma} \right)^{1/2} - \left(\frac{m}{m_c} \right)^{n+1/2} \right),$$

with the cutoffs,

$$m_\gamma = \frac{1}{8\gamma^2}, \qquad m_c^{(n+1/2)} = \left[\frac{N^n (n+1/2)}{\nu \sqrt{2}} \right]$$

Energy equations and theoretical spectra

Equation for coefficients, valid for $m \geqslant 3$,

$$\omega a_m = \left(\sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) + \text{hypercollisions}$$

Treat as a finite difference approximation in continuous m.

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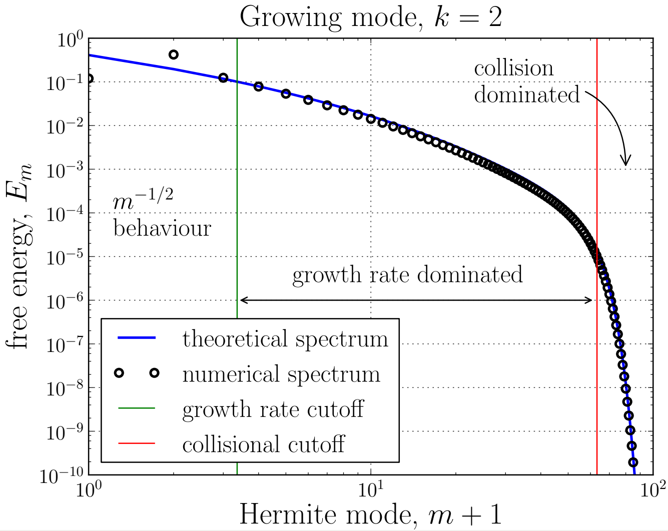

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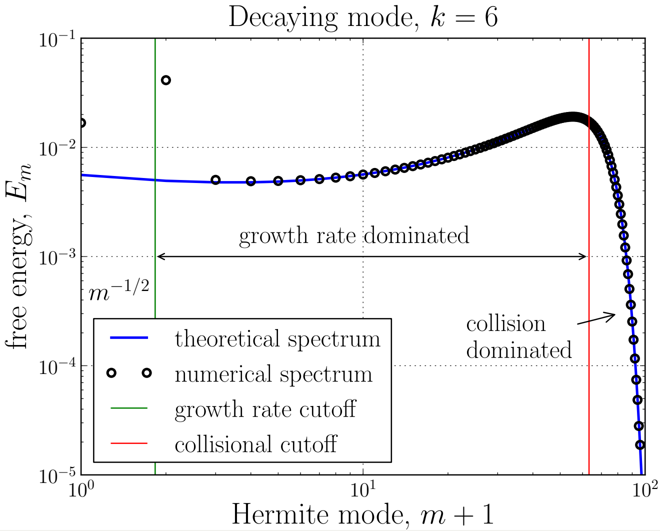

$$E_m = \frac{C}{\sqrt{2m}} \exp\left(-\frac{\gamma}{|\gamma|} \left(\frac{m}{m_\gamma} \right)^{1/2} - \left(\frac{m}{m_c} \right)^{n+1/2} \right)$$

How strong should collisions be?

- We can find the collision strength required for a given resolution.
- Write hypercollisions as,

$$C[F_1] = -\nu(-\mathcal{L})^n[F_1]$$

(*i.e.* remove N^{-n}, damping strength expressed just by ν.)

- Need to resolve collisional cutoff,

$$N > m_c = \left(\frac{(n+1/2)}{\nu\sqrt{2}}\right)^{1/(n+1/2)} \to \infty \quad \text{as} \quad \nu \to 0$$

- Need infinite resolution to resolve collisionless case.

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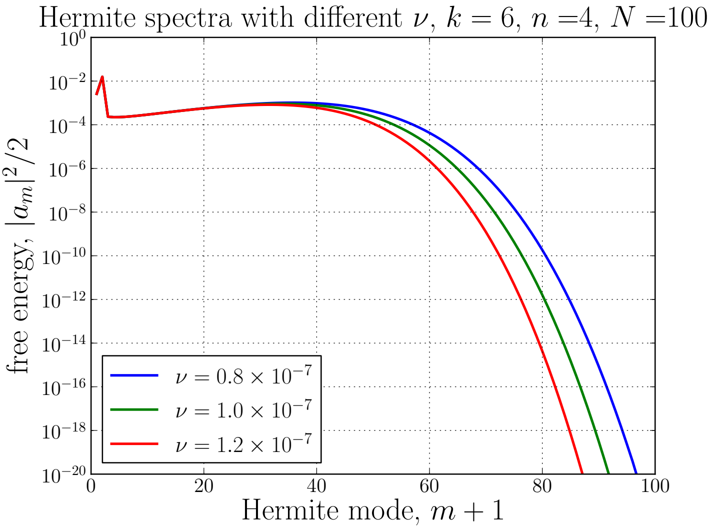
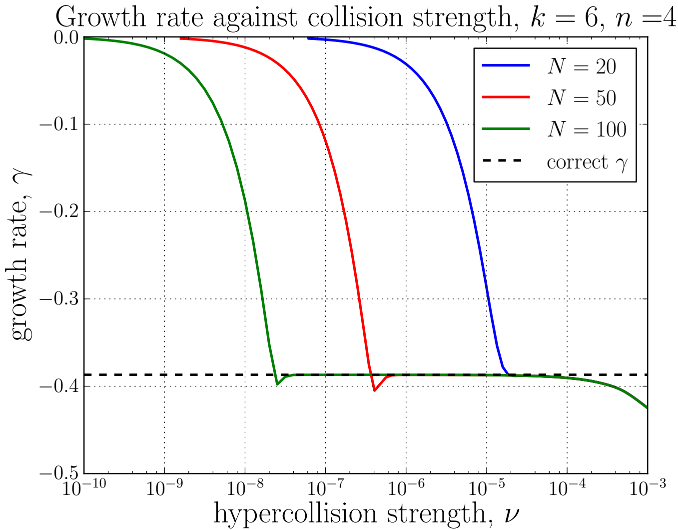


Weaker collisions $\implies$ finer scales in spectra



- a range of ν give the correct growth rate
- range extends to smaller ν as N increases

Hypercollisions on other grids

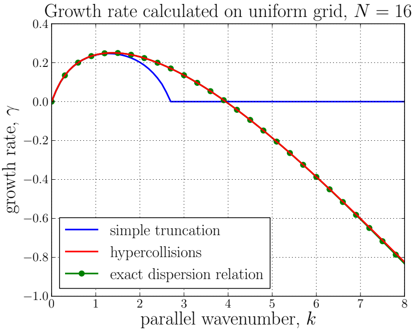
- Easiest to implement in Hermite space:

$$C[a_m] = -\nu(m/N)^n a_m$$
$$C[\mathbf{a}] = D\mathbf{a}$$

- But may be used on any grid.
- Map function values $\mathbf{f}$ to Hermite space by $\mathbf{a} = M\mathbf{f}$, collide and map back,

$$C[\mathbf{f}] = M^{-1}DM\mathbf{f}$$

- Only need to calculate $M^{-1}DM$ *once*.



Example: hypercollisions implemented on a uniform grid.

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Summary

- 1D model for ITG instability
  - velocity space discretization $\Longrightarrow$ eigenvalue problem
  - finite composition of normal modes $\Longrightarrow$ no Landau damping
- Hermite representation
  - in decaying modes, have energy pile-up at small scales
- Damping with collisions
  - Lénard–Bernstein collisions
    - finds damping, but requires $\mathcal{O}(100)$ terms
  - **hypercollisions**
    - excellent agreement with dispersion relation
    - theoretical expression for eigenfunctions
    - only ~ 10 terms
    - robust parameters
    - easy to implement in Hermite space
    - can use on any grid
- "Vanishing collisions" is different from "collisionless".

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Drift approximation in the Vlasov equation

$$\mathbf{E}_\perp + \frac{\mathbf{v}\times\mathbf{B}}{c} = 0, \qquad \mathbf{v} = v_\parallel\hat{\mathbf{z}} - \frac{c}{B^2}\nabla\Phi\times\mathbf{B}, \qquad \mathbf{B} = B\hat{\mathbf{z}}, \qquad \mathbf{E} = -\nabla\Phi.$$

Vlasov equation becomes

$$\frac{\partial f}{\partial t} + \left(v_\parallel\hat{\mathbf{z}} - \frac{c}{B^2}\nabla\Phi\times\mathbf{B}\right)\cdot\nabla f - \frac{q}{m}\frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_\parallel} = 0.$$

Perturb about stationary equilibrium, $f(\mathbf{x},\mathbf{v},t) = F_0(\mathbf{x},\mathbf{v}) + \epsilon F_1(\mathbf{x},\mathbf{v},t)$
Impose density and temperature gradients in x,

$$\frac{\partial F_0}{\partial x} = -\frac{1}{L}\left[\omega_n + \omega_{T_i}\left(\frac{v_\parallel^2 + v_\perp^2}{v_{\mathrm{th}}^2} - \frac{3}{2}\right)\right]F_0$$

Assume a Fourier mode in y: $\partial_y \mapsto ik_y$.
Integrate out perpendicular directions in $\mathbf{v}$.

$$\frac{\partial F_1}{\partial t} + \left[\omega_n + \omega_{T_i}\left(v_\parallel^2 - \frac{1}{2}\right)\right]ik_y\Phi F_0 + \alpha_i v_\parallel\frac{\partial F_1}{\partial z} + \tau_e\alpha_i v_\parallel\frac{\partial\Phi}{\partial z}F_0 = 0$$

Quasineutrality

Poisson's equation

$$\nabla^2\Phi = 4\pi q(n_i - n_e),$$

Boltzmann electrons

$$n_e = \bar{n}_e \exp\left(\frac{q\Phi}{T_e}\right)$$

Nondimensionalize (with $\epsilon = \rho_s/L$)

$$\frac{\epsilon T_e}{4\pi\bar{n}_i q^2 \rho_s L^2}\nabla^2\Phi = \left(1 + \epsilon\int_{-\infty}^{\infty} F_1\,\mathrm{d}v\right) - \frac{\bar{n}_e}{\bar{n}_i}\exp\left(\epsilon\Phi\right).$$

$$\epsilon^2\nabla_\perp^2\Phi = \left(1 - \frac{\bar{n}_e}{\bar{n}_i}\right) + \epsilon\left(\int_{-\infty}^{\infty} F_1\,\mathrm{d}v - \frac{\bar{n}_e}{\bar{n}_i}\Phi\right) + \mathcal{O}(\epsilon^2).$$

Φ, potential for electrostatic perturbation

$$\Phi = \int_{-\infty}^{\infty} F_1\,\mathrm{d}v_\parallel.$$