Accurate representation of velocity space using truncated Hermite expansions.

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Test problem for GENE, simplified gyrokinetic equations:

- 1D slab, \((z, v_\parallel)\)
- \(B = B_z\) homogeneous and static
- Electrostatic: \(E = -\nabla \Phi\)
-Uniform density and temperature gradients in background
- Perturbation about Maxwellian \(F_0(v_\parallel) = \pi^{-1/2} \exp(-v_\parallel^2)\)
- Nonlinearity dropped

May also derive directly from Vlasov equation and quasineutrality.

Electrostatic limit of Belli & Hammett (2005) model
Model for ITG instabilities (from Pueschel et al., 2010)

Linearized kinetic equation

$$\frac{\partial F_1}{\partial t} + \alpha_i v_\parallel \frac{\partial F_1}{\partial z} + \left[ \omega_n + \omega_{Ti} \left( v_\parallel^2 - \frac{1}{2} \right) \right] i k_y \Phi F_0 + \tau_e \alpha_i v_\parallel \frac{\partial \Phi}{\partial z} F_0 = 0$$

Quasineutrality

$$\Phi = \int_{-\infty}^{\infty} F_1 \, dv_\parallel.$$ 

Parameters

- $\omega_n = 1$, normalized density gradient
- $\omega_{Ti} = 10$, normalized ion temperature gradient
- $k_y = 0.3$, perpendicular wavenumber
- $\tau_e = 1$, species temperature ratio ($T_i = T_e$)
- $\alpha_i = 0.34$, nondimensional constant: velocity and length scales
Wave-like solutions

\[ F_1 = f(v_{\parallel}) \exp(i (k_{\parallel} z - \omega t)) \]

Leads to eigenvalue problem,

\[ L_0 f = \omega f \]

where

\[ L_0 f = \alpha_i k v_{\parallel} f + \left[ \omega_n + \omega_{T_i} \left( v_{\parallel}^2 - \frac{1}{2} \right) \right] k_y \Phi F_0 + \tau_e \alpha_i k v_{\parallel} \Phi F_0 \]

\( L \) real \( \implies \) eigenvalues in complex conjugate pairs \( \implies \) no damping

So actually want to solve,

\[
\lim_{\epsilon \to 0} L_{\epsilon} f = \omega f, \quad \text{for} \quad L_{\epsilon} = L_0 + i \epsilon C
\]

where \( C \) is collisions, e.g. Lénard–Bernstein or BGK.

Breaks symmetry, so decay rate may tend to nonzero limit as \( \epsilon \to 0 \).
Dispersion relation

parallel wavenumber, $k_{\parallel}$

growth rate, $\gamma$

Dispersion relation

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Velocity space using Hermite expansions

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Discretization

- Keep \( \exp(ik_z) \) behaviour, represent \( v_{||} \) on a uniform grid,

\[
\Phi = \sum_{i=1}^{N} \Delta v f(v_i)
\]

- Obtain linear ODE system

\[
\frac{\partial f}{\partial t} = iMf, \quad M \text{ real } N \times N \text{ matrix}
\]

- **Matrix eigenvalue problem** (quicker than running IVPs):

\[
-i\omega f = \frac{\partial f}{\partial t} = iMf
\]

Growth rate is \( \gamma = \max \mathfrak{R}(\omega) \), for \( \omega \in \text{spectrum}(M) \).

Matrix \( M \) depends on:

- All parameters, particularly **parallel wavenumber**.
- **Discretization** and **resolution** of grid in \( v_{||} \).
• Capture growth, but convergence slow with $N$.
• No decay.
  ▶ $\epsilon \to 0$ at fixed $N$ resolution.
  ▶ Need to restore collisions.
Distribution function in velocity space

Fine scales develop as $\mathcal{F}(\omega) \to 0$. 
Hermite expansion in velocity space

Use expansion in asymmetric Hermite functions,

\[ f(v) = \sum_{m=0}^{\infty} a_m \phi_m(v), \quad \phi^m(v) = \frac{H_m(v)}{\sqrt{2^m m!}}, \quad \phi_m(v) = F_0(v)\phi^m(v) \]

- Bi-orthogonal polynomials with Maxwellian as weight function

\[ \int_{-\infty}^{\infty} \phi_m \phi^n \, dv = \delta_{mn} \]

- Recurrence relation

\[ v\phi_m = \sqrt{\frac{m+1}{2}} \phi_{m+1} + \sqrt{\frac{m}{2}} \phi_{m-1} \]

\[ \implies \text{particle streaming becomes mode coupling} \]

- Represent velocity space scales,

\[ H_m(v) \propto \cos \left( v\sqrt{2m} - \frac{m\pi}{2} \right), \quad \text{as} \quad m \to \infty \]

large \( m \) \implies \text{fine scales}
Relative Entropy and Free Energy

Relative entropy is

\[ R[F|F_0] \equiv \int_{-\infty}^{\infty} F \log \left( \frac{F}{F_0} \right) - F + F_0 \, dv \]

\[ = \int_{-\infty}^{\infty} F \log F \, dv + 2U + \text{function}(\text{density}) \]

Expansion \( F = F_0 + \epsilon F_1 \) gives (at leading order)

\[ R[F|F_0] = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \frac{F_1^2}{F_0} \, dv = \frac{\epsilon^2}{2} \sum_{m=0}^{\infty} a_m^2 + O(\epsilon^3) \]

Define free energy of each Hermite mode, \( E_m \),

\[ E = \sum_{m=0}^{\infty} E_m, \quad \text{with} \quad E_m = \frac{1}{2} |a_m|^2 \]

cf. energy spectra in Fourier space for Navier–Stokes turbulence.
Moment system

Replace $F_0 \equiv \phi_0$ and $\nu$ with Hermite functions

$$\frac{\partial F_1}{\partial t} + i\alpha_k \nu F_1 + ik_y \Phi \left[ \omega_n \phi_0 + \frac{\omega_T}{\sqrt{2}} \phi_2 \right] + i \frac{k\tau_e \alpha_i \Phi}{\sqrt{2}} \phi_1 = 0$$

Put $F_1 = a_m \phi_m$ (implicit sum over repeated $m$)

$$\frac{\partial a_m}{\partial t} \phi_m + i\alpha_k \nu a_m \phi_m + ik_y \Phi \left[ \omega_n \phi_0 + \frac{\omega_T}{\sqrt{2}} \phi_2 \right] + i \frac{k\tau_e \alpha_i \Phi}{\sqrt{2}} \phi_1 = 0$$

Use recurrence relation on particle streaming

$$\frac{\partial a_m}{\partial t} \phi_m + i\alpha_k \nu a_m \left( \sqrt{\frac{m+1}{2}} \phi_{m+1} + \sqrt{\frac{m}{2}} \phi_{m-1} \right) + \ldots = 0$$

Gives infinite set of coupled algebraic equations for $\{a_n\}$. 
System of equations for expansion coefficients

\[ \omega a_m = \left( \sqrt{m + 1} a_{m+1} + \sqrt{m} a_{m-1} \right) \]

\[ + \frac{k_y}{\alpha_i k_{||}} \left[ \omega n a_0 \delta_{m0} + \frac{\omega T_i a_0}{\sqrt{2}} \delta_{m2} \right] \]

\[ + \frac{\tau_e a_0}{\sqrt{2}} \delta_{m1} \]

We have also used \( \phi^0 \equiv 1 \), so that,

\[ \Phi = \int_{-\infty}^{\infty} F_1 \phi^0 \, dv = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} a_m \phi_m \phi^0 \, dv = a_0 \]
Matrix equation

\[
\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k_{\|}} & \frac{1}{\sqrt{2}} & 1 \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & 1 & \sqrt{\frac{3}{2}} \\
\frac{k_y \omega T_i}{\sqrt{2} \alpha_i k_{\|}} & \sqrt{\frac{3}{2}} & \sqrt{2}
\end{pmatrix}
\]

\[a = \omega a\]
Simple Truncation

\[
\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k_{||}} & \frac{1}{\sqrt{2}} & 1 \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & 1 & \sqrt{\frac{3}{2}} \\
\frac{k_y \omega_{T_i}}{\sqrt{2} \alpha_i k_{||}} & \sqrt{\frac{3}{2}} & \sqrt{2}
\end{pmatrix}
\]

\[a = \omega a\]

- Assume \(a_{N+1} = 0\) at point of truncation. Last row of matrix:

\[
\sqrt{\frac{N+1}{2}} a_{N+1} + \sqrt{\frac{N}{2}} a_{N-1} = \omega a_N
\]

- Equivalent to discretization on Gauss–Hermite points \(\{v_j\}\)
Growing mode: $a_m$ decay as $m$ increases $\Rightarrow a_{N+1} \approx 0$ is okay

(Not) decaying mode not resolved
Collisions

- Free energy should cascade to large $m$ (fine scales in $v$).
- Free energy has nowhere to go when $a_{N+1} = 0$.
- Restore collisions:

$$\frac{\partial F_1}{\partial t} + \left[ \omega_n + \omega_{Ti} \left(v_2^2 - \frac{1}{2}\right) \right] ik_y \Phi F_0 + i\alpha_i k v \parallel F_1 + i\tau e \alpha_i k v \parallel \Phi F_0 = C[F_1]$$

Desirable properties for $C[F_1]$

- conserves mass, momentum and energy
  $$\int C[F_1] \, dv = \int vC[F_1] \, dv = \int v^2 C[F_1] \, dv = 0$$
- satisfies a linearized $H$ theorem:
  $$\frac{dR}{dt} = \epsilon^2 \int_{-\infty}^{\infty} \frac{F_1 C[F_1]}{F_0} \, dv \leq 0$$
- represents small-angle collisions (contains $v$-derivatives)
Lénard–Bernstein (1958) collisions

Simple member of linearized Landau/Fokker–Planck class:

\[ C[F_1] = \nu \frac{\partial}{\partial v} \left[ vF_1 + \frac{1}{2} \frac{\partial F_1}{\partial v} \right] \]

- collision frequency \( \nu \)
- Hermite functions are eigenfunctions:

\[ C[a_m \phi_m] = -\nu m a_m \phi_m \]

⇒ easy to implement in Hermite space

- Conserves mass, but not momentum or energy
- Satisfies a linearized \( R \) theorem:

\[ \frac{dR}{dt} = -\epsilon^2 \nu \sum_m m|a_m|^2 \leq 0 \]
Lénard–Bernstein in Hermite space

\[
\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k} & \frac{1}{\sqrt{2}} & 1 \\
\frac{T_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & -i\nu & 1 \\
\frac{k_y \omega_{T_i}}{\sqrt{2} \alpha_i k} & 1 & -2i\nu & \sqrt{3} \\
& \sqrt{3} & -3i\nu & \cdots
\end{pmatrix}
\]

\[a = \omega a\]

- matrix now complex
- roots not in complex-conjugate pairs \(\Rightarrow\) can find negative growth rates
- (we can manually conserve momentum and energy)
Lénard–Bernstein in Hermite space

\[
\begin{pmatrix}
\frac{\omega_n k_y}{\alpha_i k_i} & \frac{1}{\sqrt{2}} & 1 \\
\frac{\tau_e}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \sqrt{2} & \sqrt{3} \\
\frac{k_y \omega T_i}{\sqrt{2} \alpha_i k_i} & -2i\nu & \sqrt{3} \\
\sqrt{3} & -3i\nu & \ddots \\
\sqrt{3} & \ddots & \ddots \\
1 & \sqrt{N} & -iN\nu \\
0 & \sqrt{N/2} & -iN\nu
\end{pmatrix}
\]

\[\mathbf{a} = \omega \mathbf{a}\]

- matrix now complex
- roots *not* in complex-conjugate pairs \(\Rightarrow\) can find negative growth rates
- (we can manually conserve momentum and energy)
If collision frequency $\nu$ large enough to get $k_{\parallel} > 4$ correct, then shape for $k_{\parallel} < 4$ distorted.
• resolves dissipative scales
• expensive: 300 modes
Appreciable damping along whole spectrum.
Hypercollisions

Would prefer:
- low modes undamped
- high modes strongly damped

... whatever the truncation point $N$

Iterate Lénard–Bernstein collisions:

$$\mathcal{L}[F_1] = \frac{\partial}{\partial v} \left( v F_1 + \frac{1}{2} \frac{\partial F_1}{\partial v} \right), \quad C[F_1] = -\nu (-N)^{-n} \mathcal{L}^n[F_1]$$

(like hyperdiffusion: $-(-\nabla^2)^n$ in physical space)

Hermite functions are still eigenfunctions:

$$C [a_m \phi_m] = -\nu \left( \frac{m}{N} \right)^n a_m \phi_m$$

- $\nu$ sets the decay rate of the highest mode
- linearized $R$-theorem: $\frac{dR}{dt} = -\epsilon^2 \nu \sum_{m=0}^{\infty} \left( \frac{m}{N} \right)^n |a_m|^2 \leq 0$
- $n = 1$ corresponds to Lénard–Bernstein collisions.
- two parameters: $n \approx 6$, $\nu \approx 10$ (robust to variation)
Captures decaying parts of spectrum.
Excellent fit, fast convergence.
Appropriate for nonlinear problems.
Two parameters, $n$, $\nu$, robust to variation.
Hermite mode, $m+1$

- Low moments largely undamped
- Damping at high $m$, for any $N$. 
Low moments largely undamped

Damping at high $m$, for any $N$. 
Energy equations and theoretical spectra

Equation for coefficients, valid for \( m \geq 3 \),

\[
\omega a_m = \left( \sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) + \text{driving + Boltzmann response}
\]

Treat as a finite difference approximation in continuous \( m \).

Energy equation for \( E_m = |a_m|^2/2 \) \( \quad \text{(Zocco & Schekochihin, 2011)} \)

\[
\frac{\partial E_m}{\partial t} + \frac{\partial}{\partial m} \left( \sqrt{2mE_m} \right) = -2\nu \left( \frac{m}{N} \right)^n E_m.
\]

For a mode with growth rate \( \gamma \),

\[
E_m = \frac{C}{\sqrt{2m}} \exp \left( -\frac{\gamma}{|\gamma|} \left( \frac{m}{m_\gamma} \right)^{1/2} - \left( \frac{m}{m_c} \right)^{n+1/2} \right),
\]

with the cutoffs,

\[
m_\gamma = \frac{1}{8\gamma^2}, \quad m_c^{(n+1/2)} = \left[ \frac{N^n (n + 1/2)}{\nu \sqrt{2}} \right]
\]
Energy equations and theoretical spectra

Equation for coefficients, valid for $m \geq 3$,

$$\omega a_m = \left( \sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) + \text{hypercollisions}$$

Treat as a finite difference approximation in continuous $m$.

Energy equation for $E_m = |a_m|^2 / 2$ (Zocco & Schekochihin, 2011)

$$\frac{\partial E_m}{\partial t} + \frac{\partial}{\partial m} \left( \sqrt{2mE_m} \right) = -2\nu \left( \frac{m}{N} \right)^n E_m.$$

For a mode with growth rate $\gamma$,

$$E_m = \frac{C}{\sqrt{2m}} \exp \left( -\frac{\gamma}{|\gamma|} \left( \frac{m}{m_\gamma} \right)^{1/2} - \left( \frac{m}{m_c} \right)^{n+1/2} \right),$$

with the cutoffs,

$$m_\gamma = \frac{1}{8\gamma^2}, \quad m_c^{(n+1/2)} = \left[ \frac{N^n (n + 1/2)}{\nu \sqrt{2}} \right].$$
Growing mode, $k = 2$

\[
E_m = \frac{C}{\sqrt{2m}} \exp \left( -\frac{\gamma}{|\gamma|} \left( \frac{m}{m_\gamma} \right)^{1/2} - \left( \frac{m}{m_c} \right)^{n+1/2} \right)
\]
Decaying mode, $k = 6$

\[
E_m = \frac{C}{\sqrt{2m}} \exp \left( -\frac{\gamma}{|\gamma|} \left( \frac{m}{m_\gamma} \right)^{1/2} - \left( \frac{m}{m_c} \right)^{n+1/2} \right)
\]
How strong should collisions be?

- We can find the collision strength required for a given resolution.
- Write hypercollisions as,

\[ C[F_1] = -\nu(-\mathcal{L})^n[F_1] \]

(i.e. remove \( N^{-n} \), damping strength expressed just by \( \nu \).)

- Need to resolve collisional cutoff,

\[ N > m_c = \left( \frac{n + 1/2}{\nu \sqrt{2}} \right)^{1/(n+1/2)} \rightarrow \infty \quad \text{as} \quad \nu \rightarrow 0 \]

- Need infinite resolution to resolve collisionless case.
Weaker collisions $\implies$ finer scales in spectra
a range of $\nu$ give the correct growth rate
range extends to smaller $\nu$ as $N$ increases
Hypercollisions on other grids

- Easiest to implement in Hermite space:
  \[
  C[a_m] = -\nu (m/N)^n a_m \\
  C[a] = Da
  \]

- But may be used on any grid.

- Map function values \( f \) to Hermite space by \( a = Mf \), collide and map back,
  \[
  C[f] = M^{-1}DMf
  \]

- Only need to calculate \( M^{-1}DM \) once.

Example: hypercollisions implemented on a uniform grid.
Summary

- **1D model for ITG instability**
  - velocity space discretization $\Rightarrow$ eigenvalue problem
  - finite composition of normal modes $\Rightarrow$ no Landau damping
- **Hermite representation**
  - in decaying modes, have energy pile-up at small scales
- **Damping with collisions**
  - Lénard–Bernstein collisions
    - finds damping, but requires $O(100)$ terms
  - **hypercollisions**
    - excellent agreement with dispersion relation
    - theoretical expression for eigenfunctions
    - only $\sim 10$ terms
    - robust parameters
    - easy to implement in Hermite space
    - can use on any grid

- “Vanishing collisions” is different from “collisionless”.

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References


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Drift approximation in the Vlasov equation

\[ \mathbf{E}_\perp + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0, \quad \mathbf{v} = v_\parallel \hat{\mathbf{z}} - \frac{c}{B^2} \nabla \Phi \times \mathbf{B}, \quad \mathbf{B} = B\hat{\mathbf{z}}, \quad \mathbf{E} = -\nabla \Phi. \]

Vlasov equation becomes

\[ \frac{\partial f}{\partial t} + \left( v_\parallel \hat{\mathbf{z}} - \frac{c}{B^2} \nabla \Phi \times \mathbf{B} \right) \cdot \nabla f - \frac{q}{m} \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_\parallel} = 0. \]

Perturb about stationary equilibrium, \( f(x, v, t) = F_0(x, v) + \epsilon F_1(x, v, t) \)

Impose density and temperature gradients in \( x \),

\[ \frac{\partial F_0}{\partial x} = -\frac{1}{L} \left[ \omega_n + \omega_{Ti} \left( \frac{v_\parallel^2 + v_\perp^2}{v_{th}^2} - \frac{3}{2} \right) \right] F_0 \]

Assume a Fourier mode in \( y \): \( \partial_y \mapsto ik_y \).

Integrate out perpendicular directions in \( v \).

\[ \frac{\partial F_1}{\partial t} + \left[ \omega_n + \omega_{Ti} \left( v_\parallel^2 - \frac{1}{2} \right) \right] ik_y \Phi F_0 + \alpha_i v_\parallel \frac{\partial F_1}{\partial z} + \tau_e \alpha_i v_\parallel \frac{\partial \Phi}{\partial z} F_0 = 0 \]
Quasineutrality

Poisson's equation

\[ \nabla^2 \Phi = 4\pi q (n_i - n_e), \]

Boltzmann electrons

\[ n_e = \bar{n}_e \exp \left( \frac{q\Phi}{T_e} \right) \]

Nondimensionalize (with \( \epsilon = \rho_s / L \))

\[ \frac{\epsilon T_e}{4\pi \bar{n}_i q^2 \rho_s L^2} \nabla^2 \Phi = \left( 1 + \epsilon \int_{-\infty}^{\infty} F_1 \, dv \right) - \frac{\bar{n}_e}{\bar{n}_i} \exp (\epsilon \Phi). \]

\[ \epsilon^2 \nabla^2_\perp \Phi = \left( 1 - \frac{\bar{n}_e}{\bar{n}_i} \right) + \epsilon \left( \int_{-\infty}^{\infty} F_1 \, dv - \frac{\bar{n}_e}{\bar{n}_i} \Phi \right) + \mathcal{O}(\epsilon^2). \]

\( \Phi \), potential for electrostatic perturbation

\[ \Phi = \int_{-\infty}^{\infty} F_1 \, dv_\parallel. \]