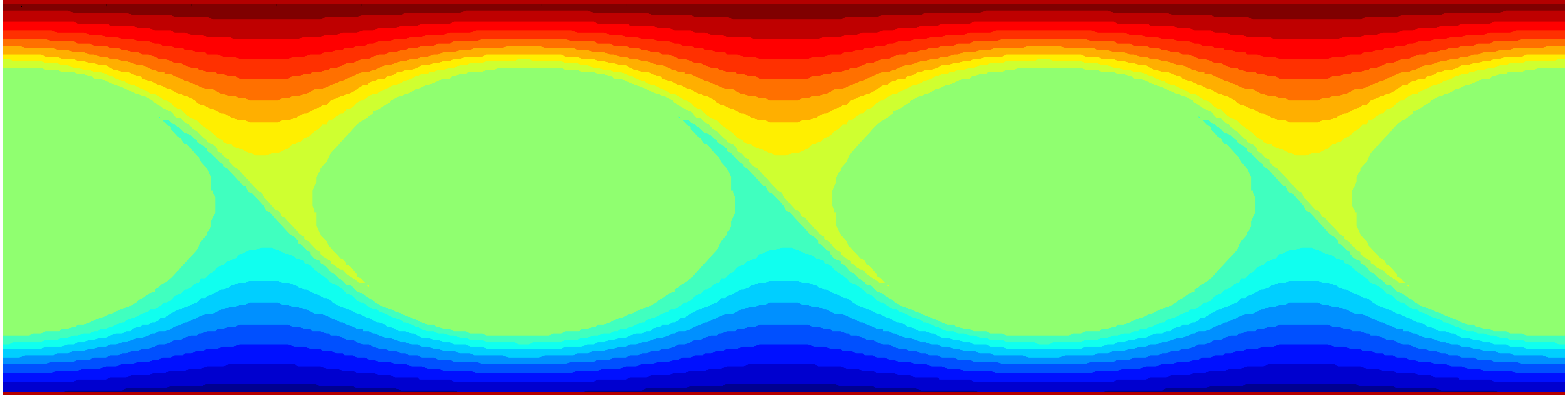
Numerical schemes for a neoclassical pedestal



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Numerical schemes for a neoclassical pedestal

- Local neoclassical calculations with the full linearized Fokker-Planck collision operator
- Nonlocal (pedestal) neoclassical calculations
  - Formulation of the drift-kinetic equation
  - Operator splitting approach
  - Need for sources
- Questions for you

Local neoclassical calculations with the full linearized Fokker-Planck operator

Local drift-kinetic equation

$$\upsilon_{\|}\mathbf{b}\cdot\nabla\theta\left(\frac{\partial f_1}{\partial\theta}\right)_{\mu,\upsilon,\psi}+\mathbf{v}_d\cdot\nabla\psi\frac{\partial f_M}{\partial\psi}=C\{f_1\}$$

$$f=f_M+f_1$$

Equivalent form better suited for numerical work:

$$a_1\frac{\partial f_1}{\partial\theta}+a_2\frac{\partial f_1}{\partial\xi}-\nu_*\left[a_3\frac{\partial}{\partial\xi}a_4\frac{\partial f_1}{\partial\xi}+\frac{1}{\upsilon^2}\frac{\partial}{\partial\upsilon}a_5\frac{\partial}{\partial\upsilon}\frac{f_1}{e^{-\upsilon^2}}+a_6 f_1+a_7 H+a_8\frac{\partial^2 G}{\partial\upsilon^2}\right]=a_9$$

where $f_1=f_1(\theta,\ \upsilon,\ \xi)$, $\qquad \xi=\upsilon_{\|}/\upsilon$

$$\nabla^2 H=-4\pi f_1,$$
$$\nabla^2 G=2H,$$

$$\nabla^2=\frac{1}{\upsilon^2}\frac{\partial}{\partial\upsilon}\upsilon^2\frac{\partial}{\partial\upsilon}+\frac{1}{\upsilon^2}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi},$$

a_1, ..., a_9 are known.

- These are 3 coupled linear 3D partial differential equations.
- Even as $\nu_*\longrightarrow 0$, details of C matter.
- Pitch-angle scattering is expected to be a poor approximation for C in the pedestal.

Discretization scheme

$$f_1(\theta,\ \xi_i,\ \upsilon_j) = \sum_m f_{m,i,j} \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix}$$

$$H(\theta,\ \xi,\ \upsilon_j) = \sum_{\ell,m} H_{\ell,m,j} \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} P_\ell(\xi) \qquad G(\theta,\ \xi,\ \upsilon_j) = \sum_{\ell,m} G_{\ell,m,j} \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} P_\ell(\xi)$$

DKE:

$$\upsilon_\parallel \nabla_\parallel f_1 - C\{f_1\} = -\mathbf{v}_d \cdot \nabla f_0$$

$$\nabla^2 H + 4\pi f_1 = 0$$

$$\nabla^2 G - 2H = 0$$

$$\left(\begin{array}{c|c|c} & & \\ \hline & & 0 \\ \hline 0 & & \end{array}\right) \left(\begin{array}{c} f_1 \\ \hline H \\ \hline G \end{array}\right) = \left(\begin{array}{c} -\mathbf{v}_d \cdot \nabla f_0 \\ \hline 0 \\ \hline 0 \end{array}\right)$$

Also a few rows for boundary conditions:

- Regularity at $\upsilon = 0$
- Derivatives of H & G at $\upsilon_{\max}$

Pitch-angle-scattering approximation is quantitatively poor for realistic ε

Bootstrap current: $$\left\langle j_{\|} B \right\rangle = -\alpha_1 c R B_{tor} \left(\frac{dp_e}{d\psi} + \frac{dp_i}{d\psi} - \alpha_2 n_e \frac{dT_e}{d\psi} - k n_e \frac{dT_i}{d\psi} \right)$$

Poloidal flow: $$V_{\theta i} = k \frac{cI}{Ze} \frac{B_\theta}{\left\langle B^2 \right\rangle} \frac{dT_i}{d\psi}$$

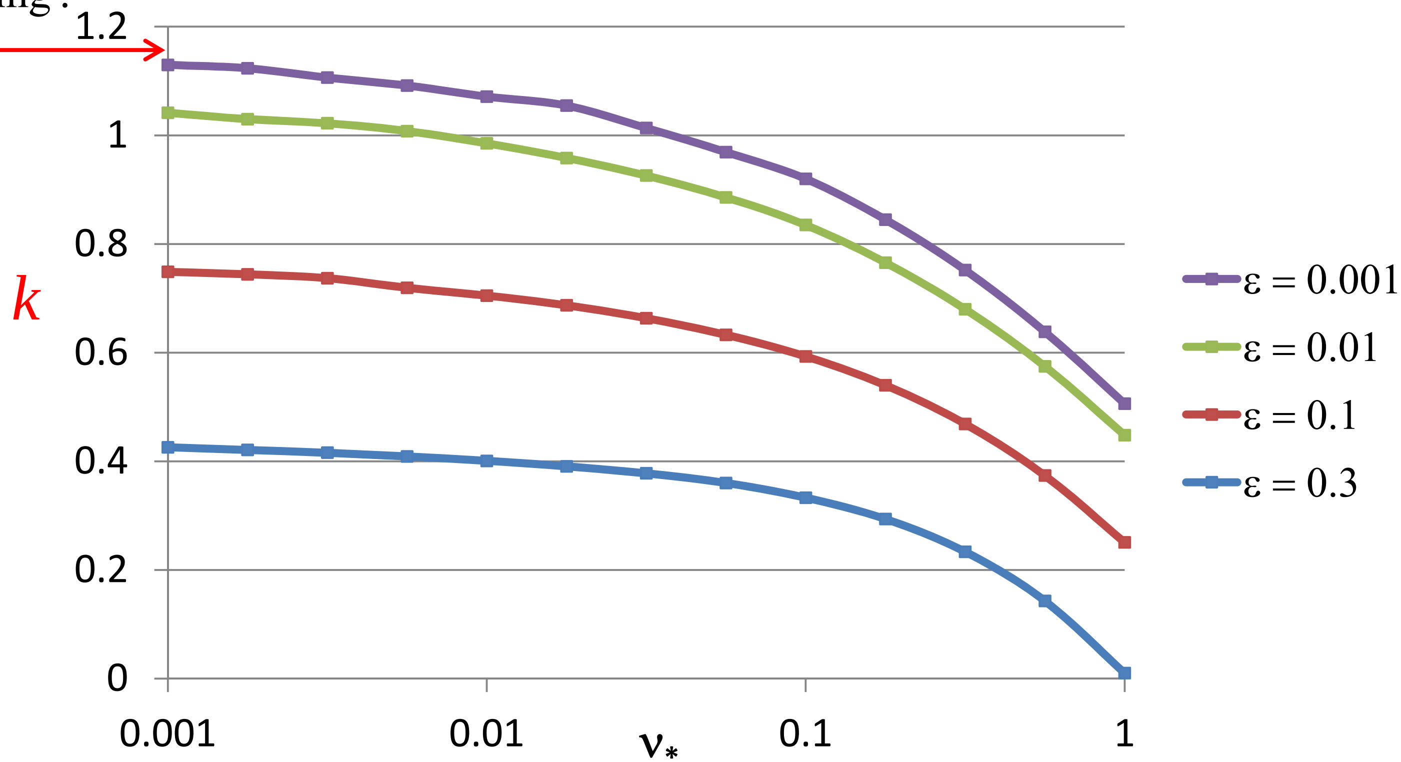
Numerical coefficient

Analytic limit for

$\nu_* \to 0$ &

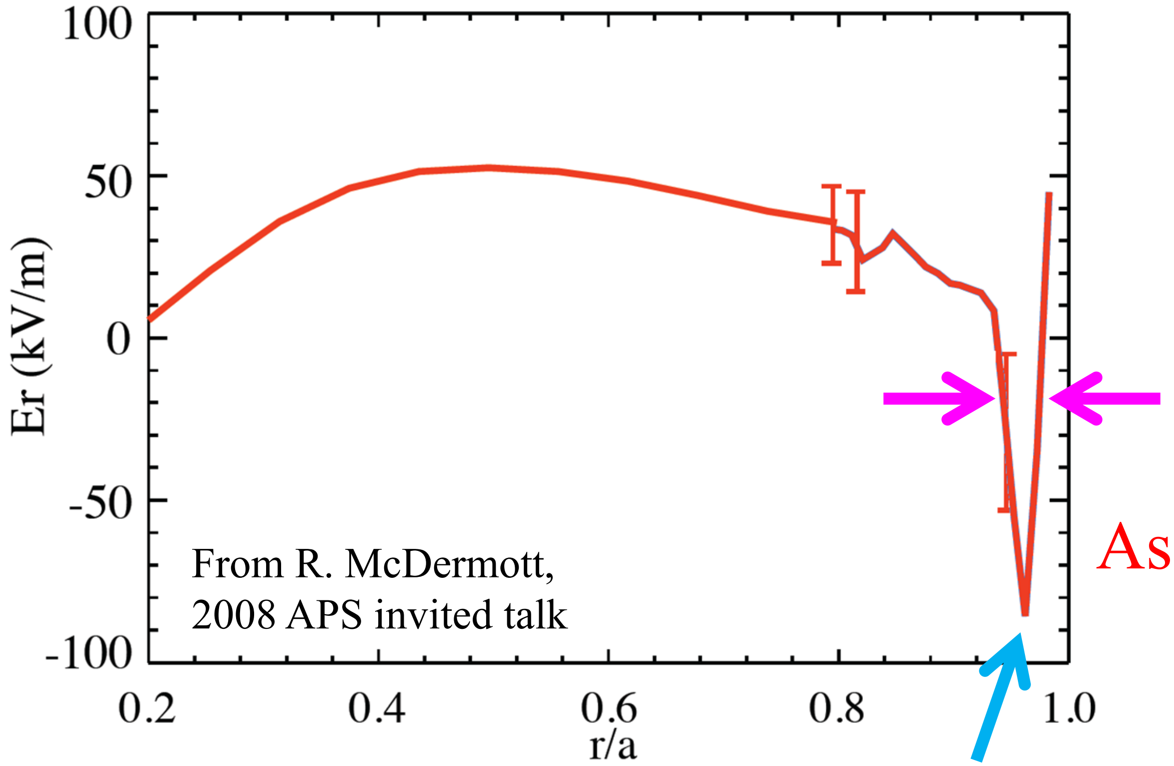
pitch-angle scattering :

1.17



Nonlocal (pedestal) neoclassical calculations

In a pedestal, standard (local) neoclassical theory breaks down



$\sim \rho_{pol}$

Assume $B_p \ll B.$

$\left|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right| \ll \upsilon_{th,i}$

but $\mathbf{v}_{\mathbf{E}\times\mathbf{B}} \cdot \nabla\theta \sim \upsilon_{th,i}\mathbf{b}\cdot\nabla\theta.$

Standard neoclassical: $\dfrac{f_1}{f_0} \sim \dfrac{\rho_{pol}}{r_{n,T}} \sim 1.$

Standard bootstrap current calculations are formally not valid in the pedestal.

Nonlocal drift-kinetic equation

- Suppose $f \approx f_M = n(\psi)\left[\frac{m}{2\pi T(\psi)}\right]\exp\left(-\frac{m\upsilon^2}{2T(\psi)}\right)$. Let $\eta(x) = n(x)\exp\left(\frac{e\Phi(x)}{T(x)}\right)$.
- Introduce $f_* = \eta(\psi_*)\left[\frac{m}{2\pi T(\psi_*)}\right]\exp\left(-\frac{W}{T(\psi_*)}\right)$ where $\psi_* = \psi - \frac{RB_{tor}\upsilon_\parallel}{\Omega}$,

  $$W = \frac{m\upsilon^2}{2} + e\Phi.$$

- $f_* \approx f_M - \frac{RB_{tor}\upsilon_\parallel}{\Omega}\frac{\partial f_M}{\partial \psi}$. Then $f \approx f_*$ if $1 \gg \left|\frac{f_* - f_M}{f_M}\right| = \max\left(\frac{\rho_{pol}}{r_T}, \frac{\rho_{pol}}{r_\eta}\right)$,

  but r_n and r_Φ can still be $\sim \rho_{pol}$ if $n \approx \exp\left(-\frac{e\Phi}{T}\right)$.
- Notice $\left(\upsilon_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla f_* = 0.$
- Kinetic equation: $\left(\upsilon_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla f = C\{f\} + S.$

  $\Rightarrow$ C can be linearized about f_M if r_T and r_η are $\gg \rho_{pol}$.
- Introduce $g = f - f_*$. $\Rightarrow$ $\boxed{\left(\upsilon_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla g - C\{g\} = C\{f_*\} + S.}$

Nonlocal drift-kinetic equation

$$\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right)\cdot\nabla\theta\frac{\partial g}{\partial\theta}+\left[-\upsilon B+\xi cRB_{tor}\frac{d\Phi}{d\psi}\right]\frac{\left(1-\xi^{2}\right)}{2B^{2}}\nabla_{\parallel}B\frac{\partial g}{\partial\xi}$$

$$-\mathbf{v}_{m}\cdot\nabla\psi\frac{e}{m\upsilon}\frac{d\Phi}{d\psi}\frac{\partial g}{\partial\upsilon}+\mathbf{v}_{m}\cdot\nabla\psi\frac{\partial g}{\partial\psi}-C\{g\}=C\{f_{*}\}+S$$

Change variables from (μ, W) to (υ, ξ)

Assume $B=B(\theta)$ and $RB_{tor}=\text{const}$ so $\mathbf{v}_{m}\cdot\nabla\theta=0.$

$$\boxed{\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_{d}\right)\cdot\nabla g-C\{g\}=C\{f_{*}\}+S.}$$

To calculate nonlocal transport, you must solve a 4D integro-differential equation

$$\underbrace{\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right)\cdot\nabla\theta\frac{\partial g}{\partial\theta}+\left[-\upsilon B+\xi cRB_{tor}\frac{d\Phi}{d\psi}\right]\frac{\left(1-\xi^2\right)}{2B^2}\nabla_{\parallel}B\frac{\partial g}{\partial\xi} -\mathbf{v}_m\cdot\nabla\psi\frac{e}{m\upsilon}\frac{d\Phi}{d\psi}\frac{\partial g}{\partial\upsilon}}_{\mathcal{L}_{Loc}}+\underbrace{\mathbf{v}_m\cdot\nabla\psi\frac{\partial g}{\partial\psi}}_{\mathcal{L}_{NLoc}}-C\{g\}=C\{f_*\}+S$$

$$f=f_*+g,\qquad f_*=\eta(\psi_*)\left[\frac{m}{2\pi T(\psi_*)}\right]^{3/2}\exp\left(-\frac{m\upsilon^2}{2T(\psi_*)}-\frac{e\Phi}{T(\psi_*)}\right),$$

$$\psi_*=\psi-\frac{RB_{tor}\upsilon_{\parallel}}{\Omega},\qquad g=g(\psi,\ \theta,\ \upsilon,\ \xi),\qquad \xi=\upsilon_{\parallel}/\upsilon$$

The nonlocal kinetic equation can be solved using operator splitting plus a *local* neoclassical code

$$\frac{\partial g}{\partial t}+\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right)\cdot\nabla\theta\frac{\partial g}{\partial\theta}+\left[-\upsilon B+\xi cRB_{tor}\frac{d\Phi}{d\psi}\right]\frac{\left(1-\xi^{2}\right)}{2B^{2}}\nabla_{\parallel}B\frac{\partial g}{\partial\xi}$$

$$-\mathbf{v}_{m}\cdot\nabla\psi\frac{e}{m\upsilon}\frac{d\Phi}{d\psi}\frac{\partial g}{\partial\upsilon}+\mathbf{v}_{m}\cdot\nabla\psi\frac{\partial g}{\partial\psi}-C\{g\}=C\{f_{*}\}+S$$

$\mathcal{L}_{Loc}$ $\mathcal{L}_{NLoc}$

$$\frac{g^{t+(1/2)}-g^{t}}{\Delta t}+\mathcal{L}_{NLoc}\left\{g^{t+(1/2)}\right\}=0$$

$$+\quad\frac{g^{t+1}-g^{t+(1/2)}}{\Delta t}+\mathcal{L}_{Loc}\left\{g^{t+1}\right\}=C\{f_{*}\}+S$$

$$\frac{g^{t+1}-g^{t}}{\Delta t}+\mathcal{L}_{Loc}\left\{g^{t+1}\right\}+\mathcal{L}_{NLoc}\left\{g^{t+(1/2)}\right\}=C\{f_{*}\}+S$$

The nonlocal kinetic equation can be solved using operator splitting plus a *local* neoclassical code

$$\frac{\partial g}{\partial t} + \left(\upsilon_{\parallel}\mathbf{b} + \mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right)\cdot\nabla\theta\frac{\partial g}{\partial\theta} + \left[-\upsilon B + \xi c R B_{tor}\frac{d\Phi}{d\psi}\right]\frac{\left(1-\xi^2\right)}{2B^2}\nabla_{\parallel}B\frac{\partial g}{\partial\xi}$$

$$\mathcal{L}_{Loc} \qquad -\mathbf{v}_m\cdot\nabla\psi\frac{e}{m\upsilon}\frac{d\Phi}{d\psi}\frac{\partial g}{\partial\upsilon} + \mathbf{v}_m\cdot\nabla\psi\frac{\partial g}{\partial\psi} - C\{g\} = C\{f_*\} + S$$

$$\mathcal{L}_{NLoc}$$

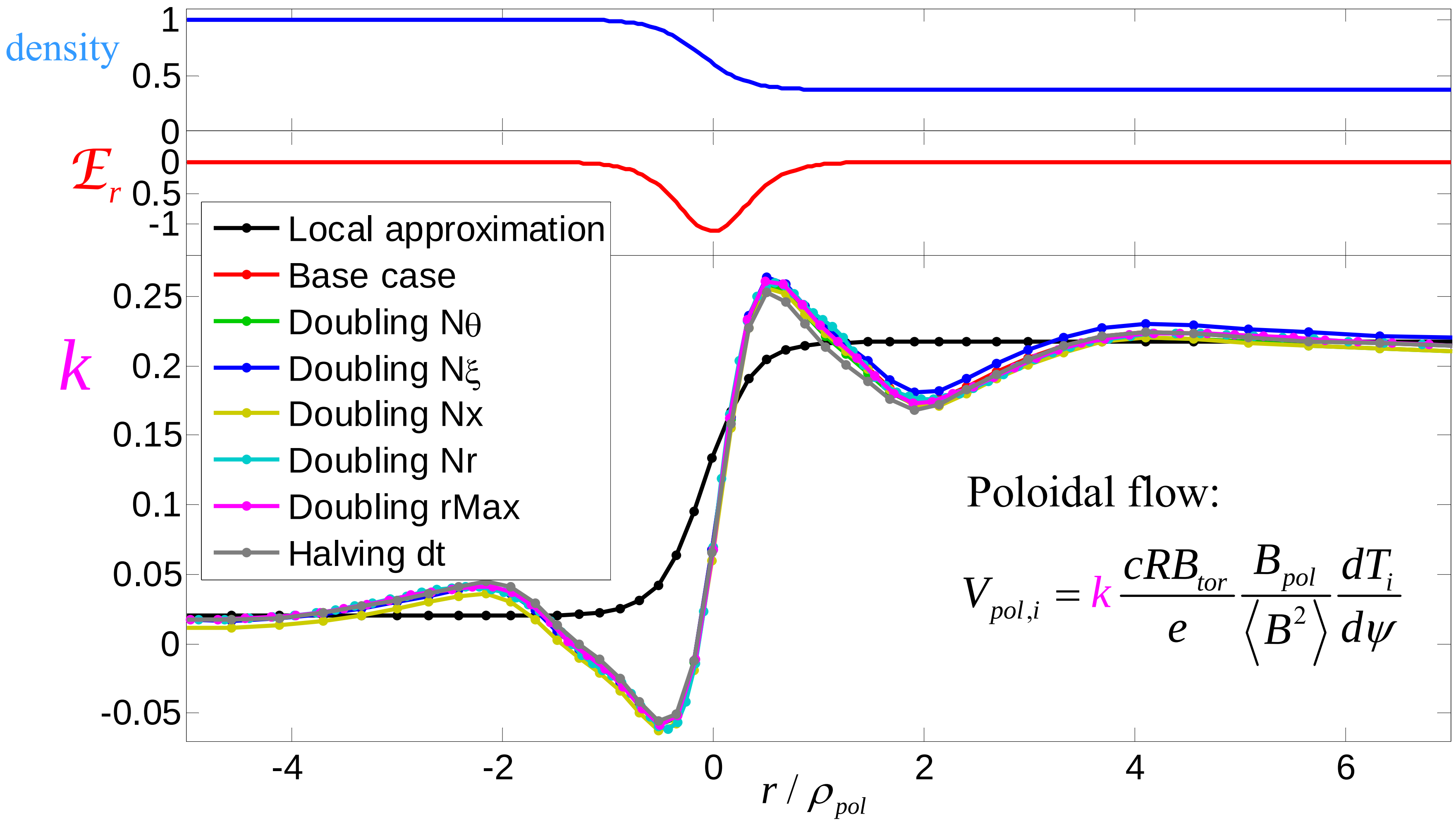
$$\frac{g^{t+(1/2)} - g^{t}}{\Delta t} + \mathcal{L}_{NLoc}\left\{g^{t+(1/2)}\right\} = 0$$

$$+ \qquad \frac{g^{t+1} - g^{t+(1/2)}}{\Delta t} + \mathcal{L}_{Loc}\left\{g^{t+1}\right\} = C\{f_*\} + S$$

$$\frac{g^{t+1} - g^{t}}{\Delta t} + \mathcal{L}_{Loc}\left\{g^{t+1}\right\} + \mathcal{L}_{NLoc}\left\{g^{t+(1/2)}\right\} = C\{f_*\} + S$$

Global code predicts enhanced flow shear & modified j_{BS}

High-order upwinded finite-difference differentiation in r . $\varepsilon = 0.3,\ \nu_* = 1 - 0.3.$



Poloidal flow:

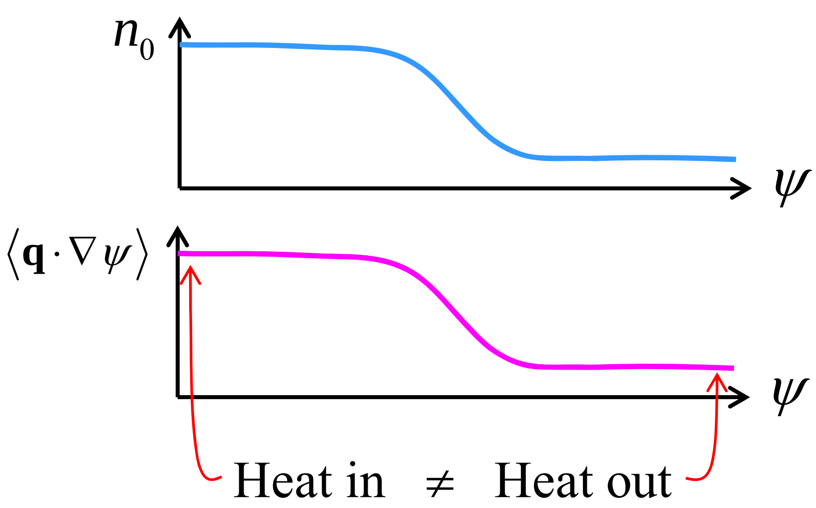
$$V_{pol,i} = k \frac{cRB_{tor}}{e} \frac{B_{pol}}{\langle B^2 \rangle} \frac{dT_i}{d\psi}$$

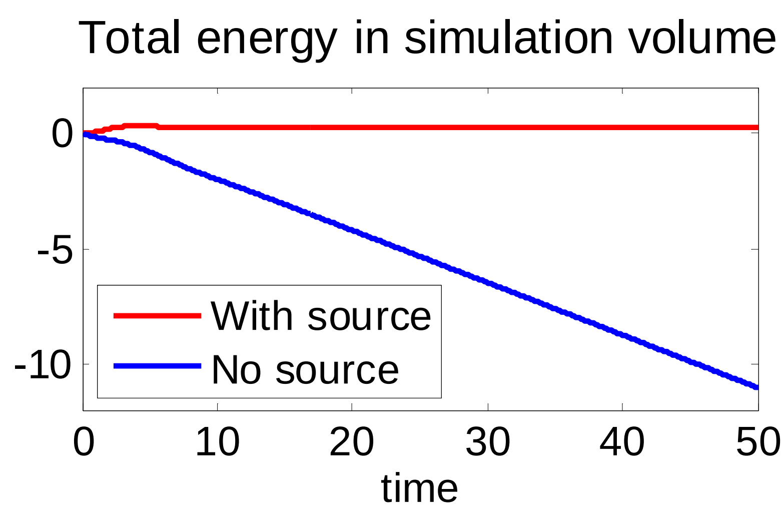
Bootstrap current: $$\langle j_{\|} B \rangle = -\alpha_1 cRB_{tor} \left(\frac{dp_e}{d\psi} + \frac{dp_i}{d\psi} - \alpha_2 n_e \frac{dT_e}{d\psi} - k n_e \frac{dT_i}{d\psi} \right)$$

Problem: without a heat source, no truly time-independent solutions exist.

Apply $\int_{\psi_{\min}}^{\psi_{\max}} d\psi \, V' \left\langle \int d^3 \upsilon \frac{m \upsilon^2}{2} (\quad\quad) \right\rangle$ to kinetic equation with $\partial g / \partial t = 0$:

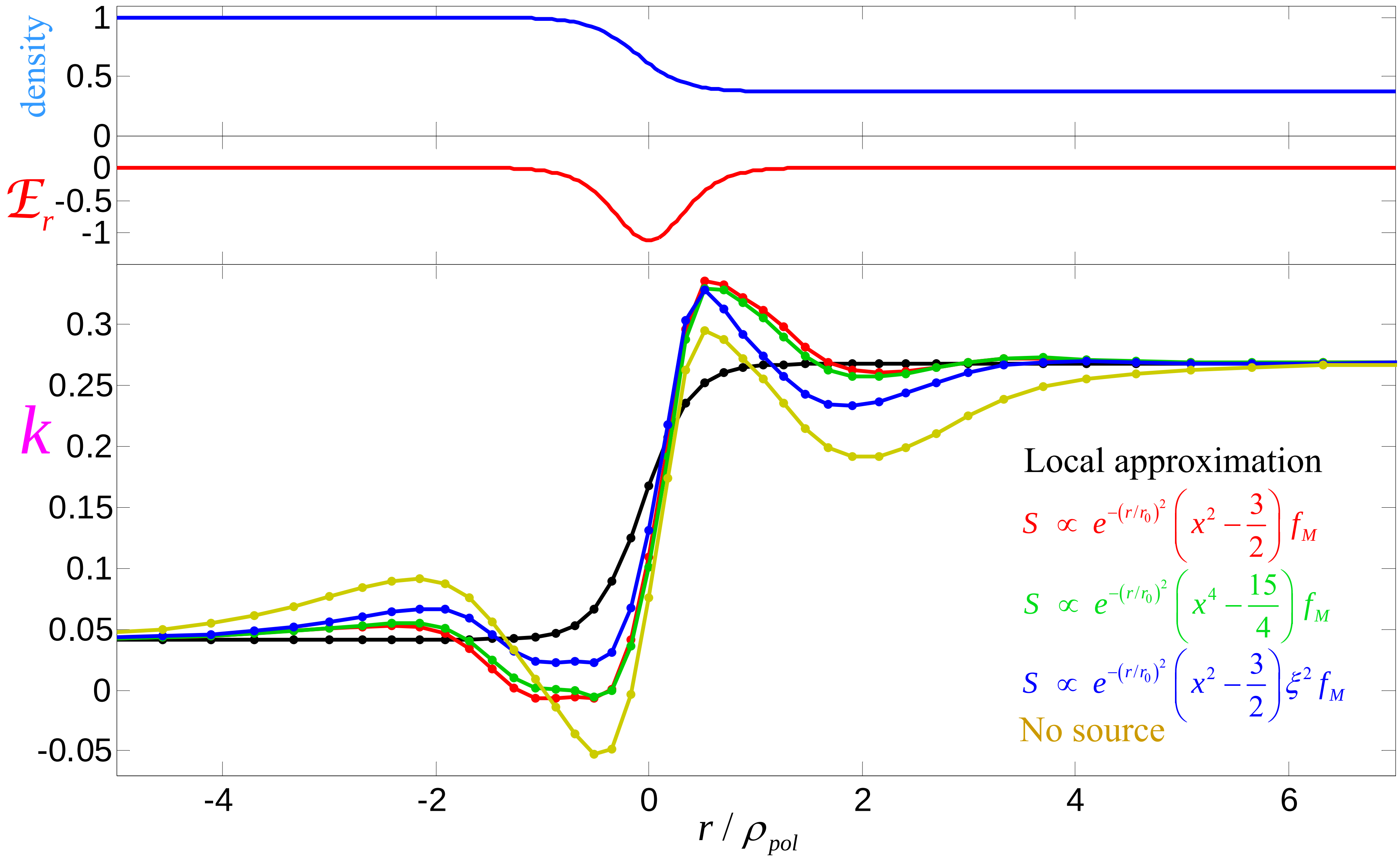
$$\underbrace{\left[V' \left\langle \int d^3 \upsilon \, g \frac{m \upsilon^2}{2} \mathbf{v}_d \cdot \nabla \psi \right\rangle \right]_{\psi=\psi_{\min}}^{\psi_{\max}}}_{\text{Heat out} \; - \; \text{heat in}} = \underbrace{\int_{\psi_{\min}}^{\psi_{\max}} d\psi \, V' \left\langle \int d^3 \upsilon \frac{m \upsilon^2}{2} S \right\rangle}_{\text{Total heat source in the volume}}$$



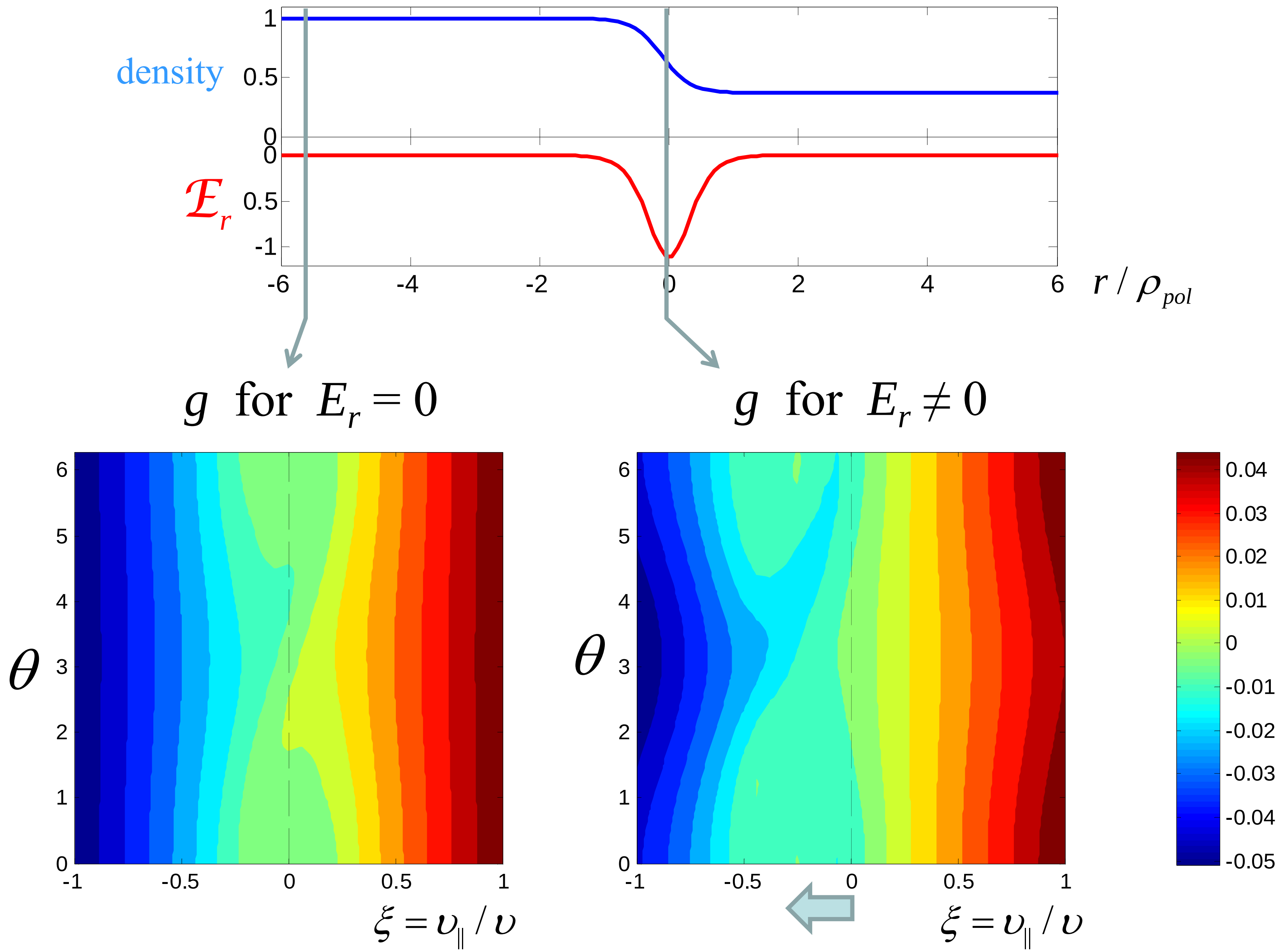


No source needed for mass or momentum because fluxes at ends automatically vanish.

The choice of source has some effect on the results



Shift in trapping region can be seen in the distribution function



Future work

- Iterative schemes with no $\partial f/\partial t$ or $\partial g/\partial t$ term, e.g.

  to solve $\mathcal{L}_{Loc}\{g\}+\mathcal{L}_{NLoc}\{g\}=C\{f_*\}+S$ for g,

  iterate $\mathcal{L}_{Loc}\{g^{i+1}\}=-\mathcal{L}_{NLoc}\{g^{i}\}+C\{f_*\}+S.$

- Numerically solve the *nonlinear* problem:

$$\frac{\partial f}{\partial t}+\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f=C\{f,f\}.$$

  (Allows $r_{Ti}\sim\rho_{pol}$.)

- Rigorous comparisons to finite-$\mathcal{E}_r$ analytic limits.

- Study dependence of ion flow & j_{BS} on ν_*, ε, and depth of Φ well.

- Stellarators.

Summary

- The full linearized Fokker-Planck operator has been included in local neoclassical calculations for finite ε and ν_*.

- A local neoclassical code can be adapted for the pedestal using an operator-splitting time-advance.

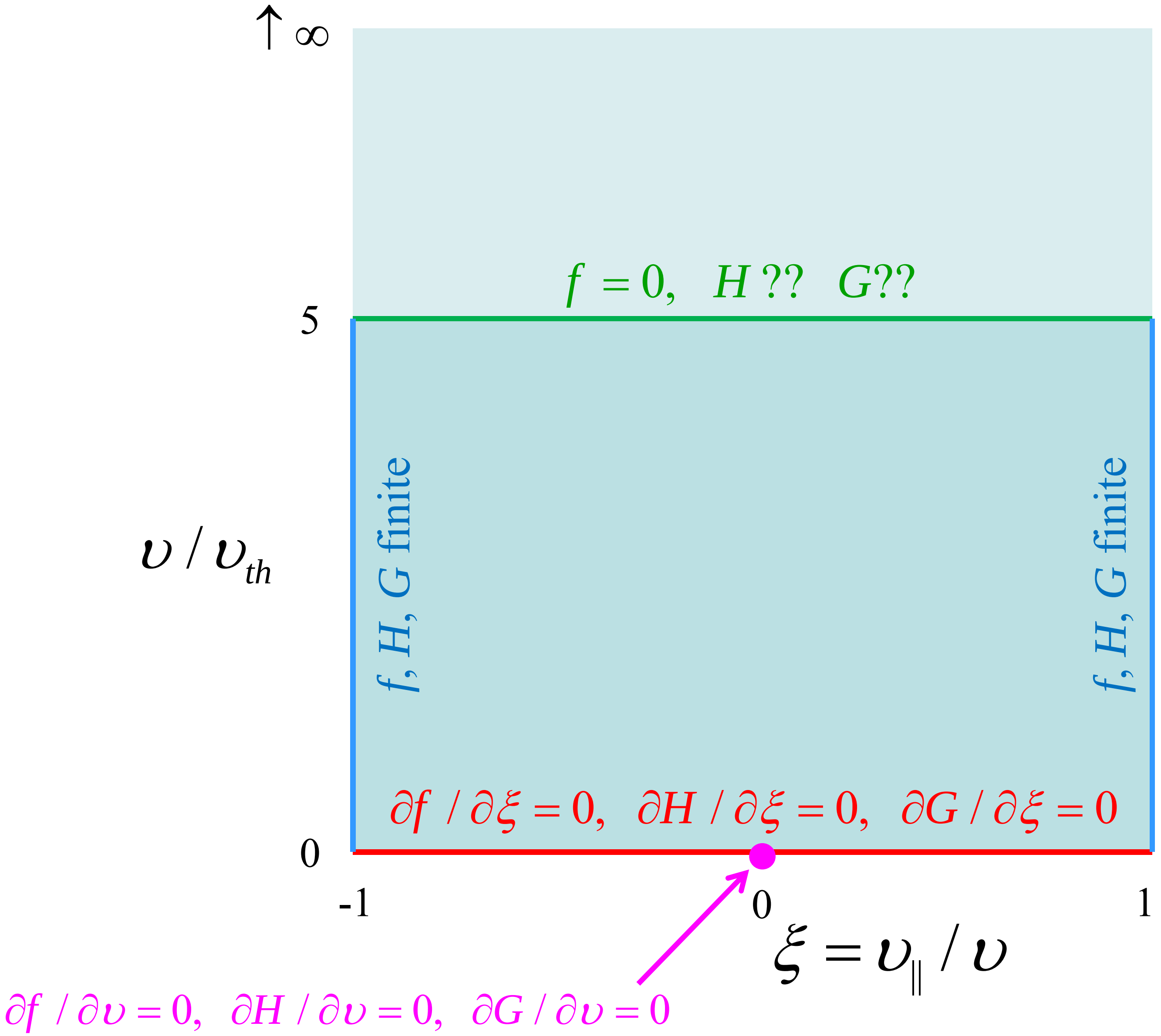
<u>Outstanding questions:</u>

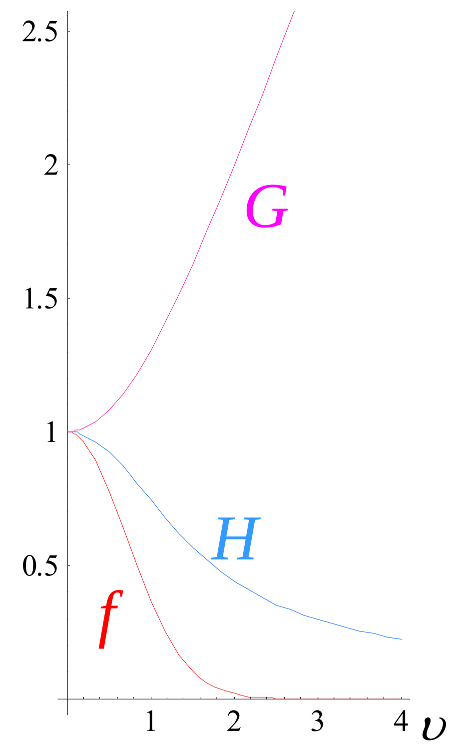
- Is there a better formulation in which a heat sink is not needed?

- Can the iterative scheme $\mathcal{L}_{Loc}\left\{g^{i+1}\right\} = -\mathcal{L}_{NLoc}\left\{g^{i}\right\} + C\left\{f_*\right\} + S$ be made stable?

Extra slides

Geometry is simple, but boundary conditions are tricky



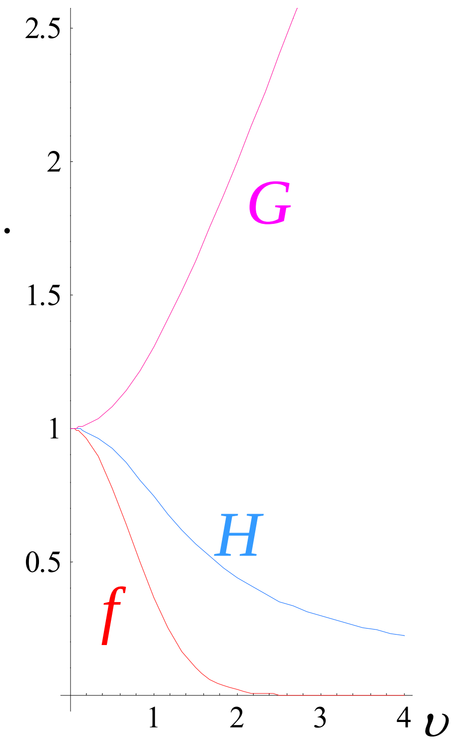


Legendre polynomials are a good basis for the Rosenbluth potentials

$$H(\theta,\xi,\upsilon) = \sum_{\ell=0}^{N} H_\ell(\theta,\upsilon) P_\ell(\xi)$$

$$\nabla^2 H = -4\pi f_1 \qquad \Rightarrow \qquad \frac{1}{\upsilon^2}\frac{d}{d\upsilon}\upsilon^2\frac{dH_\ell}{d\upsilon} - \frac{\ell(\ell+1)}{\upsilon^2}H_\ell = -4\pi f_{1\ell}$$

ODE instead of 2D PDE.



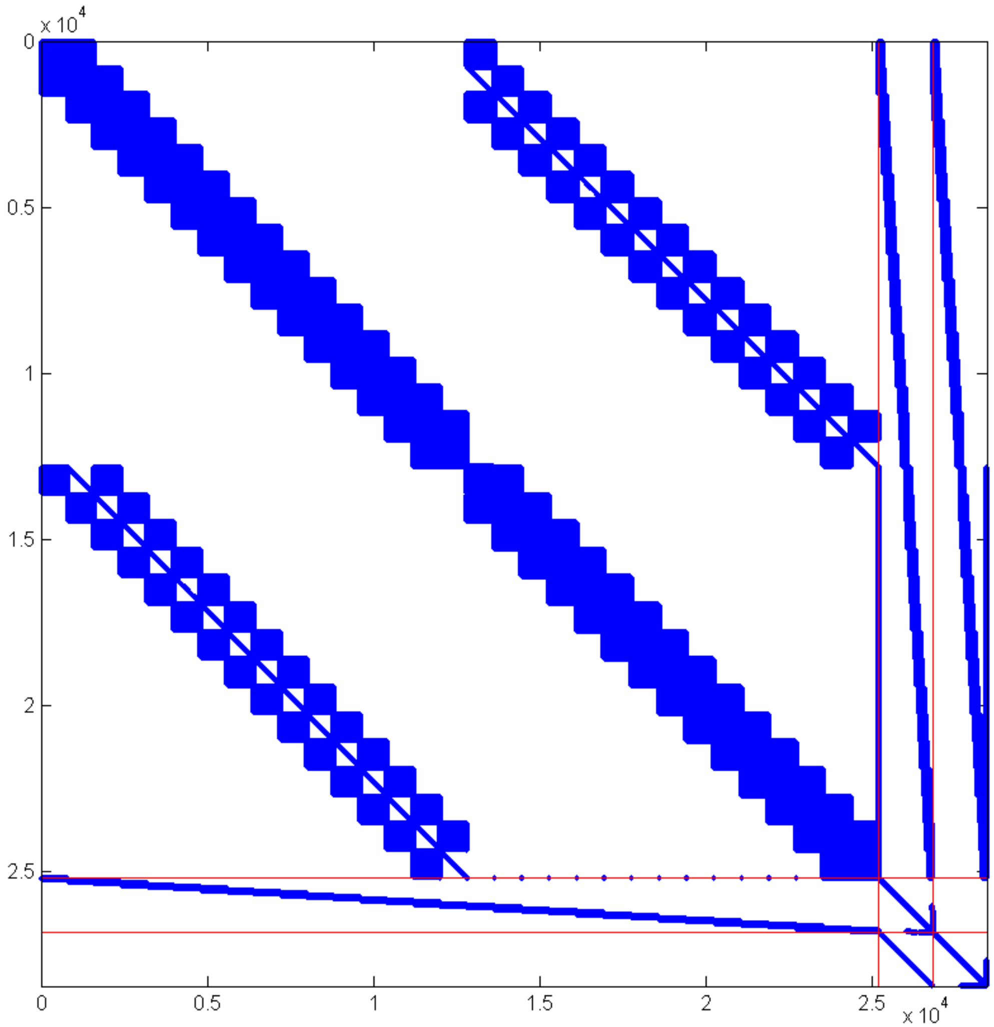
$H_\ell \sim \dfrac{f_{1\ell}}{\ell^2}$, $G_\ell \sim \dfrac{f_{1\ell}}{\ell^4}$, so large ℓ components of H & G are negligible.

$f_1 \approx 0$ for $\upsilon > 5\upsilon_{th}$. $\qquad \Rightarrow \qquad H_\ell \propto 1/\upsilon^{\ell+1}$.

$\Rightarrow$ Gives boundary condition: $\upsilon \dfrac{dH_\ell}{d\upsilon} = -(\ell+1)H_\ell$ at $\upsilon = 5\upsilon_{th}$.

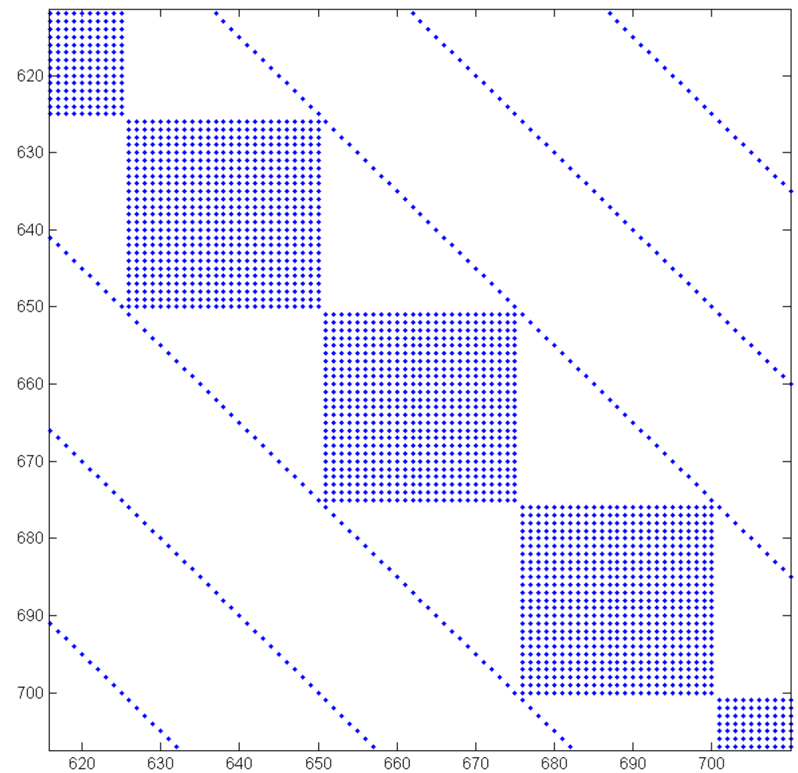
$\Rightarrow$ Don't need to simulate $\upsilon > 5\upsilon_{th}$

Matrix is sparse & asymmetric with complicated structure



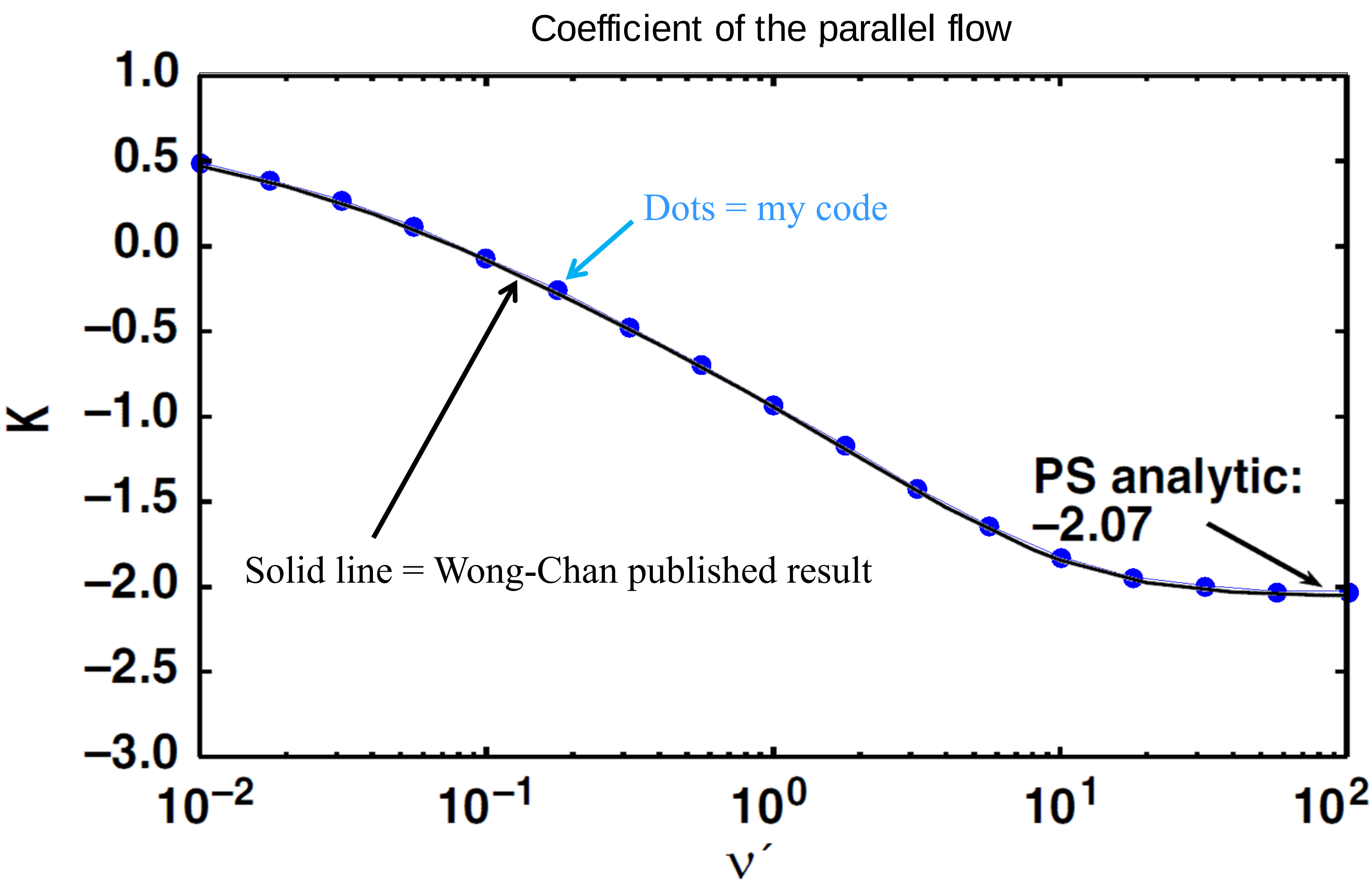
Sparse direct solver.

2-70 seconds to solve on a laptop.



Rate-limiting step is the solver.

My code agrees with code of Wong & Chan



For small ν_*, "nondiamagnetic" distribution function is nearly constant along particle orbits

$$\text{Let } g = f_1 + \frac{I\upsilon_{\parallel}}{\Omega}\frac{\partial f_0}{\partial \psi}.$$

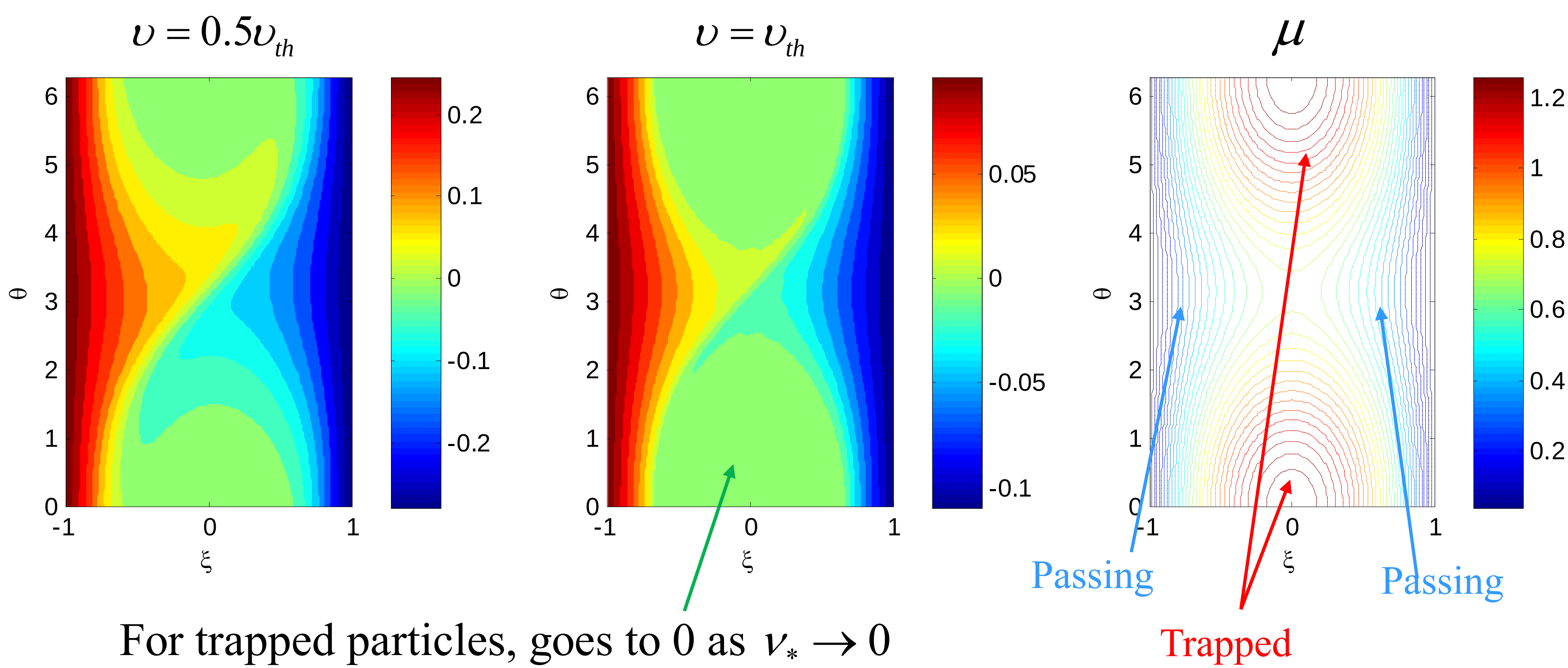
Analytic theory for $\nu_* \to 0$ (banana regime) predicts:

- $g = g(\mu,\ \upsilon)$ (i.e. it is independent of θ.)
- $g = 0$ for trapped particles.

For small ν_*, “nondiamagnetic” distribution function is nearly constant along particle orbits

$$\boxed{\nu_* = \frac{\nu q R}{\upsilon \varepsilon^{3/2}} = 0.01 \text{ at } \upsilon_{th}} \qquad \varepsilon = 0.3$$

Plots of $g = f_1 + \frac{I\upsilon_\parallel}{\Omega}\frac{\partial f_0}{\partial \psi}$

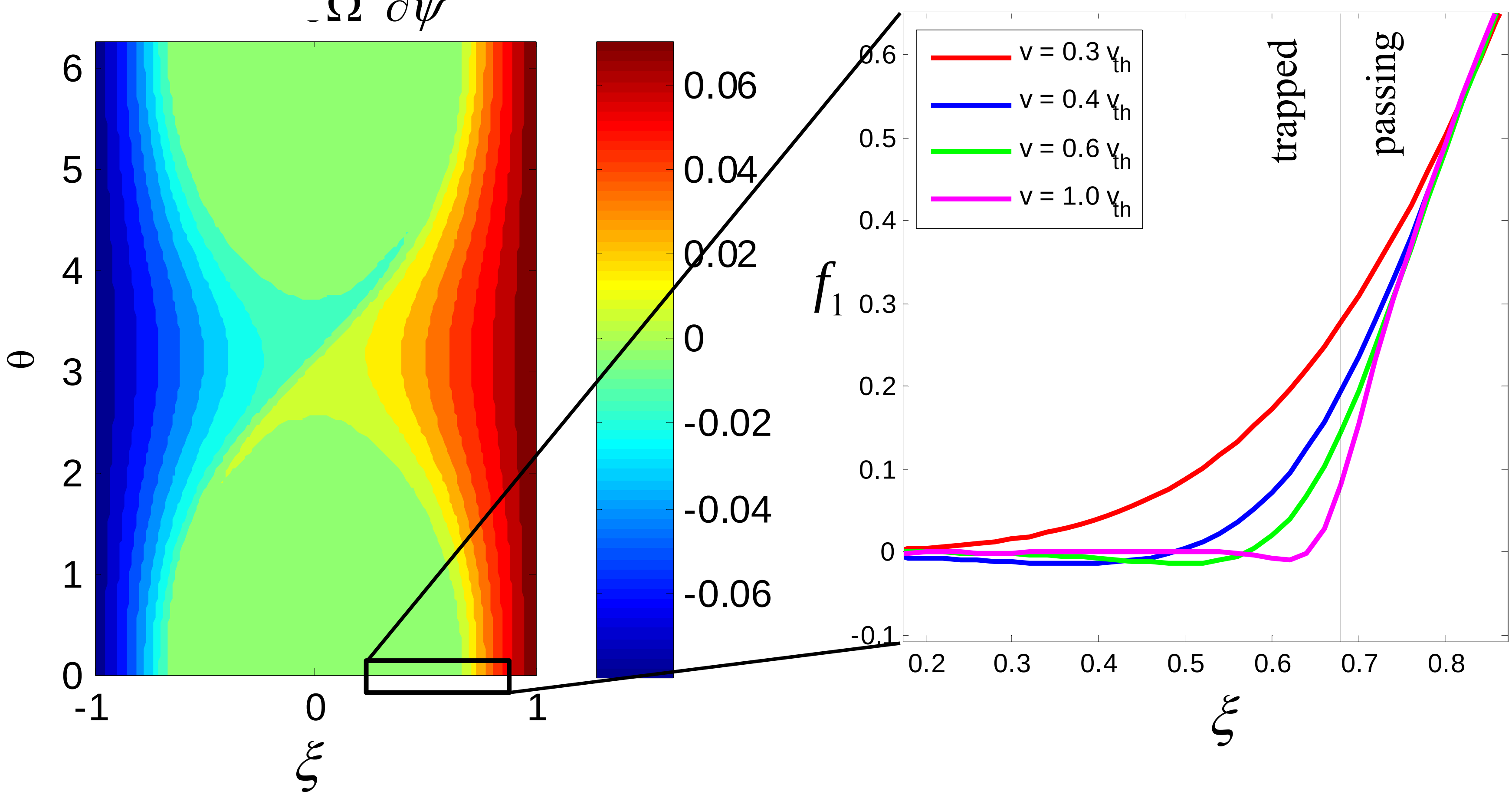


For trapped particles, goes to 0 as $\nu_* \to 0$

Code can resolve the boundary layer between passing and trapped phase-space

$$\nu_* = \frac{\nu q R}{\upsilon \varepsilon^{3/2}} = 0.01 \text{ at } \upsilon_{th} \qquad \varepsilon = 0.3$$

Plot of $g = f_1 + \frac{I\upsilon_\parallel}{\Omega}\frac{\partial f_0}{\partial \psi}$ at $\upsilon = \upsilon_{th}$



Radial ion heat flux

$$\langle \mathbf{q}_i \cdot \nabla \psi \rangle = -q \frac{\sqrt{\varepsilon} n_i I^2 \upsilon_{th}^2}{\tau_i \Omega_0^2} \frac{dT_i}{d\psi}$$