Gyroviscous effects in Braginskii magnetohydrodynamic flow between parallel planes

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Overview

Braginskii magnetohydrodynamics (MHD)

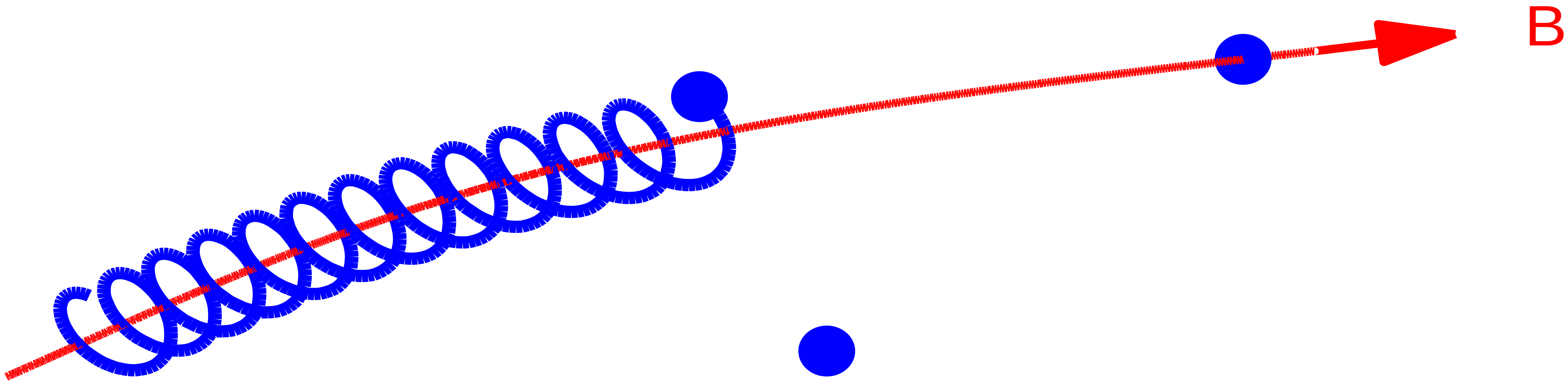
- Fluid description for “strongly magnetised” plasmas
- Valid on lengthscales $\gg$ mean free path $\gg$ gyroradius
- Covers many astrophysical plasmas *e.g.* solar corona, tokamak edges
- Anisotropic viscous stress directed along magnetic field lines
- Gyroviscous stress perpendicular to both strain rate and magnetic field

Flow between two parallel planes

- Problem geometry and formulation
- Hartmann layers
- Current singularities at the walls
- Regularisation by perpendicular stress [see JFM **667** 520]
- Regularisation by gyroviscous stress

Braginskii's magnetohydrodynamics (part 1)

Strongly magnetised plasmas: particles are tied to magnetic field lines.



Effective mean free path perpendicular to field lines is the gyroradius.

Braginskii's (1965) theory: viscous stress aligned with the magnetic field,

$$\mathbf{\Pi}_{\text{visc}} \approx -2\mu_{||}\, \hat{\mathbf{b}}\hat{\mathbf{b}}\, \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} \qquad \text{(where } \hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|\text{)}$$

"Strongly magnetised" when $\omega_{\mathrm{i}}\tau_{\mathrm{i}} \gg 1$.

Braginskii's magnetohydrodynamics (part 2)

Single-fluid description using Braginskii's viscous stress
[see Lifshitz & Pitaevskii *"Physical Kinetics"* or Schekochihin *et al.* 2005]

Momentum equation is

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla\cdot\left(p\mathbf{I} + \tfrac{1}{2}|\mathbf{B}|^2\,\mathbf{I} - \mathbf{BB} + \mathbf{\Pi}_{\text{visc}} \right) = 0.$$

$$\text{total stress} = \text{pressure} + \text{Maxwell stress} + \text{viscous stress}.$$

Take as incompressible, $\rho =$ cst and $\nabla\cdot\mathbf{u} = 0$. (Small Mach number)

Magnetic field $\mathbf{B}$ evolves through Faraday's and Ohm's laws

$$\partial_t \mathbf{B} = \nabla\times\left(\mathbf{u}\times\mathbf{B} - \eta_{\|}\,\mathbf{j}_{\|} - \eta_{\perp}\,\mathbf{j}_{\perp}\right),$$

where $\eta_{\perp} = 1.96\,\eta_{\|}$.

The full Braginskii (1965) viscous stress ...

$$\mathbf{\Pi}_{\text{visc}} = \eta_0 \mathsf{W}^{(0)} + \eta_1 \mathsf{W}^{(1)} + \eta_2 \mathsf{W}^{(2)} + \eta_3 \mathsf{W}^{(3)} + \eta_4 \mathsf{W}^{(4)}.$$

Five separate contributions:

$$\begin{aligned}
\mathsf{W}^{(0)} &= \tfrac{3}{2}(\hat{\mathbf{b}}\hat{\mathbf{b}} - \tfrac{1}{3}\mathbf{I})\hat{\mathbf{b}} \cdot \mathsf{W} \cdot \hat{\mathbf{b}}, \\
\mathsf{W}^{(1)} &= (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \mathsf{W} \cdot (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \tfrac{1}{2}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}})\hat{\mathbf{b}} \cdot \mathsf{W} \cdot \hat{\mathbf{b}}, \\
\mathsf{W}^{(2)} &= (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \mathsf{W} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \mathsf{W} \cdot (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}), \\
\mathsf{W}^{(3)} &= \tfrac{1}{2}\hat{\mathbf{b}}\times\mathsf{W} \cdot (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - \tfrac{1}{2}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \mathsf{W}\times\hat{\mathbf{b}}, \\
\mathsf{W}^{(4)} &= (\hat{\mathbf{b}}\times\mathsf{W} \cdot \hat{\mathbf{b}})\,\hat{\mathbf{b}} - \hat{\mathbf{b}}\,(\hat{\mathbf{b}}\times\mathsf{W} \cdot \hat{\mathbf{b}}),
\end{aligned}$$

where the strain rate is

$$\mathsf{W} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}} - \tfrac{2}{3}\,\mathbf{I}\,\nabla\cdot\mathbf{u}\,.$$

The classical values for the five viscosities:

$$\eta_0 = 0.96\, n_{\text{i}} T_{\text{i}} \tau_{\text{i}}, \;\; \eta_1 = \frac{3}{10}\frac{n_{\text{i}} T_{\text{i}} \tau_{\text{i}}}{(\omega_{\text{i}}\tau_{\text{i}})^2}, \;\; \eta_2 = 4\eta_1, \;\; \eta_3 = \frac{1}{2}\frac{n_{\text{i}} T_{\text{i}} \tau_{\text{i}}}{\omega_{\text{i}}\tau_{\text{i}}}, \;\; \eta_4 = 2\eta_3.$$

A simpler (astrophysical) regularisation

Regularise with a perpendicular viscosity $\mu_\perp \ll \mu_\|$. Write the stress as

$$\mathbf{\Pi}_{\mathrm{visc}} = -\mu_\perp \mathbf{W} - (\mu_\| - \mu_\perp)\hat{\mathbf{b}}\hat{\mathbf{b}}\ \hat{\mathbf{b}}\hat{\mathbf{b}} : \mathbf{W},$$

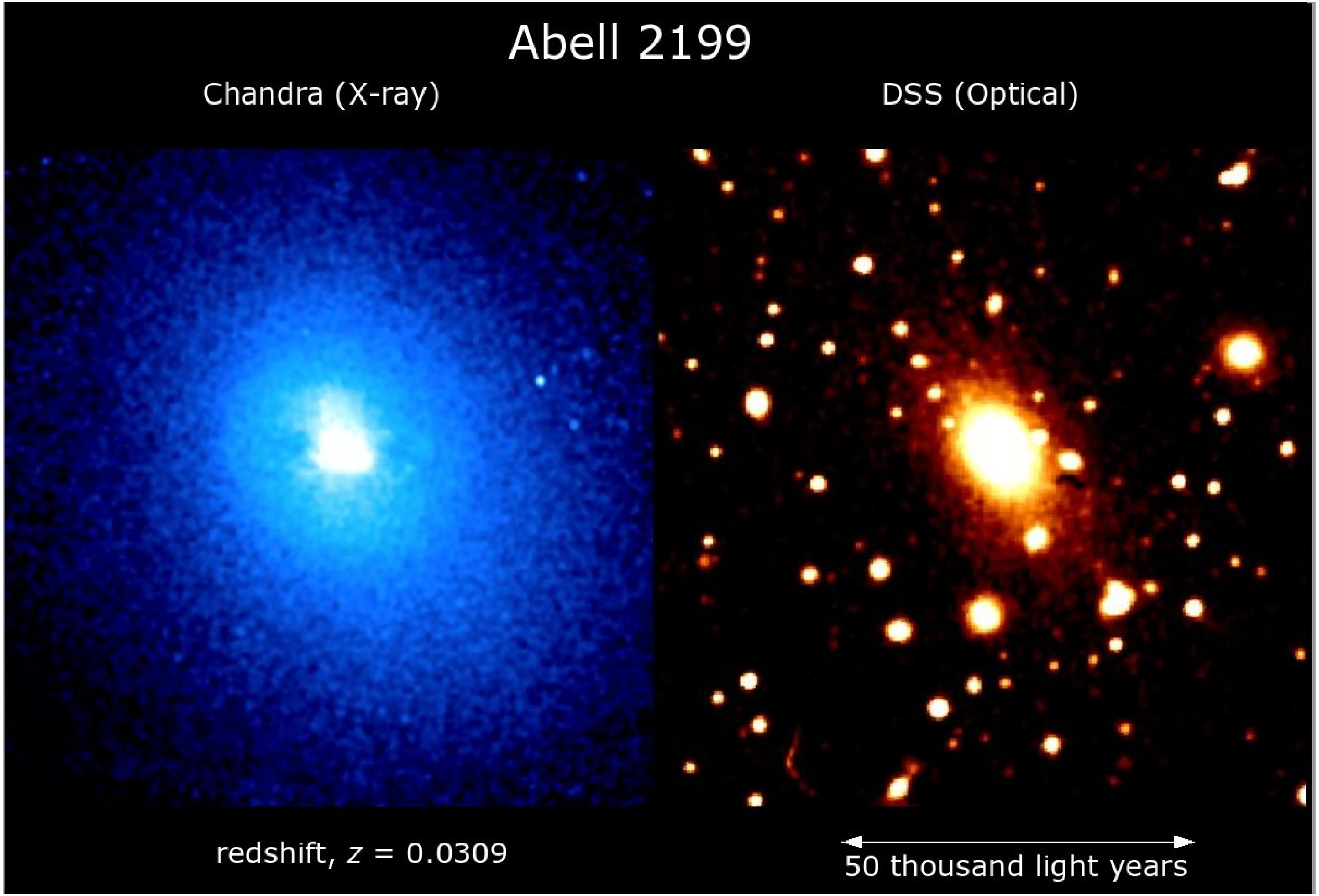
where $\mathbf{W} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}$ for an incompressible fluid.

In axes with the first axis aligned with the direction $\hat{\mathbf{b}}$,

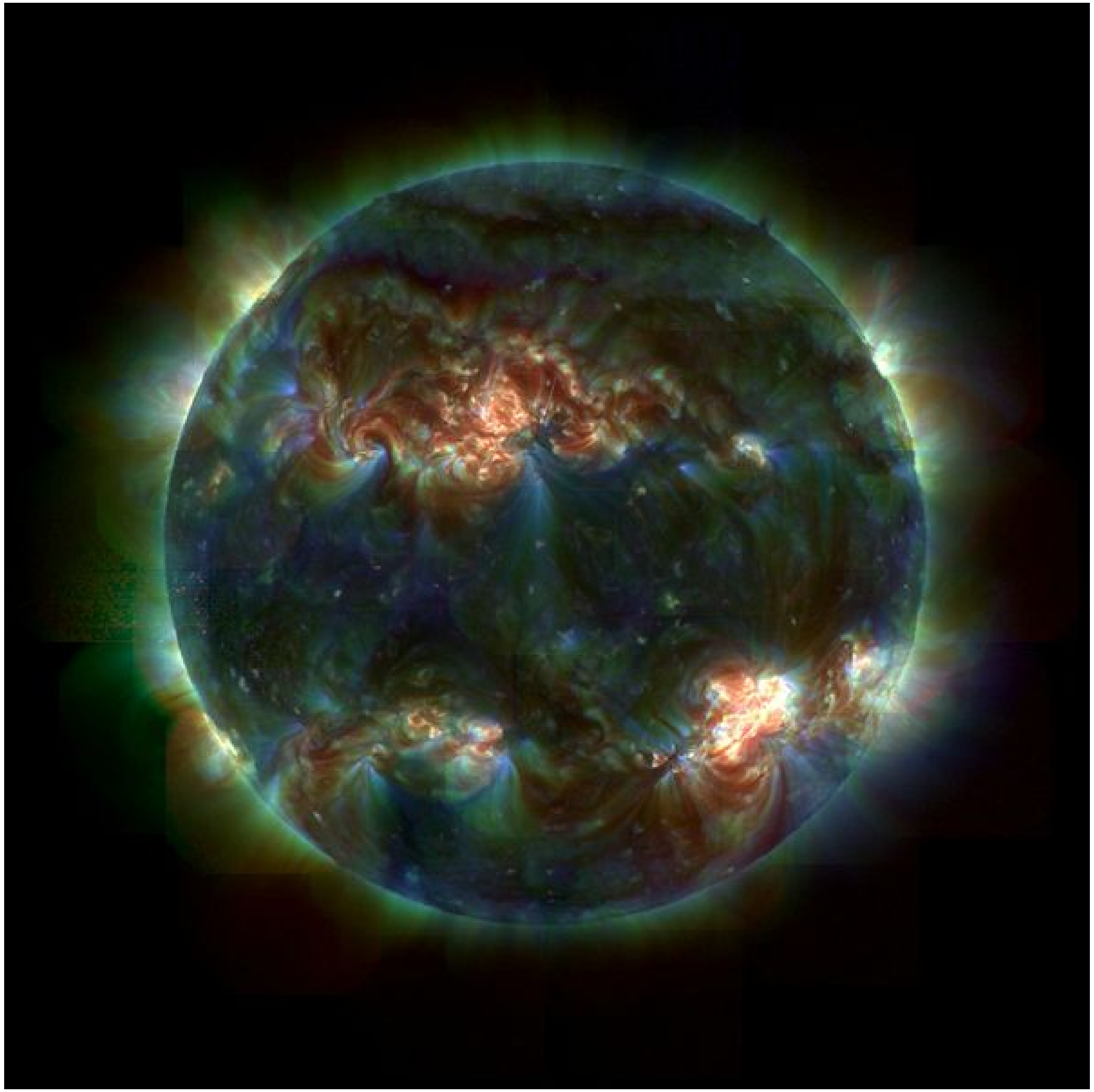
$$\mathbf{\Pi}_{\mathrm{visc}} = -\begin{pmatrix} \mu_\| & & & \\ & \mu_\perp & & \\ & & \ddots & \\ & & & \mu_\perp \end{pmatrix} \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}\right).$$

Similar to liquid crystals, except $\mu_\perp \ll \mu_\|$ instead of $\mu_\perp \sim \mu_\|$.

Astrophysical applications: galaxy clusters

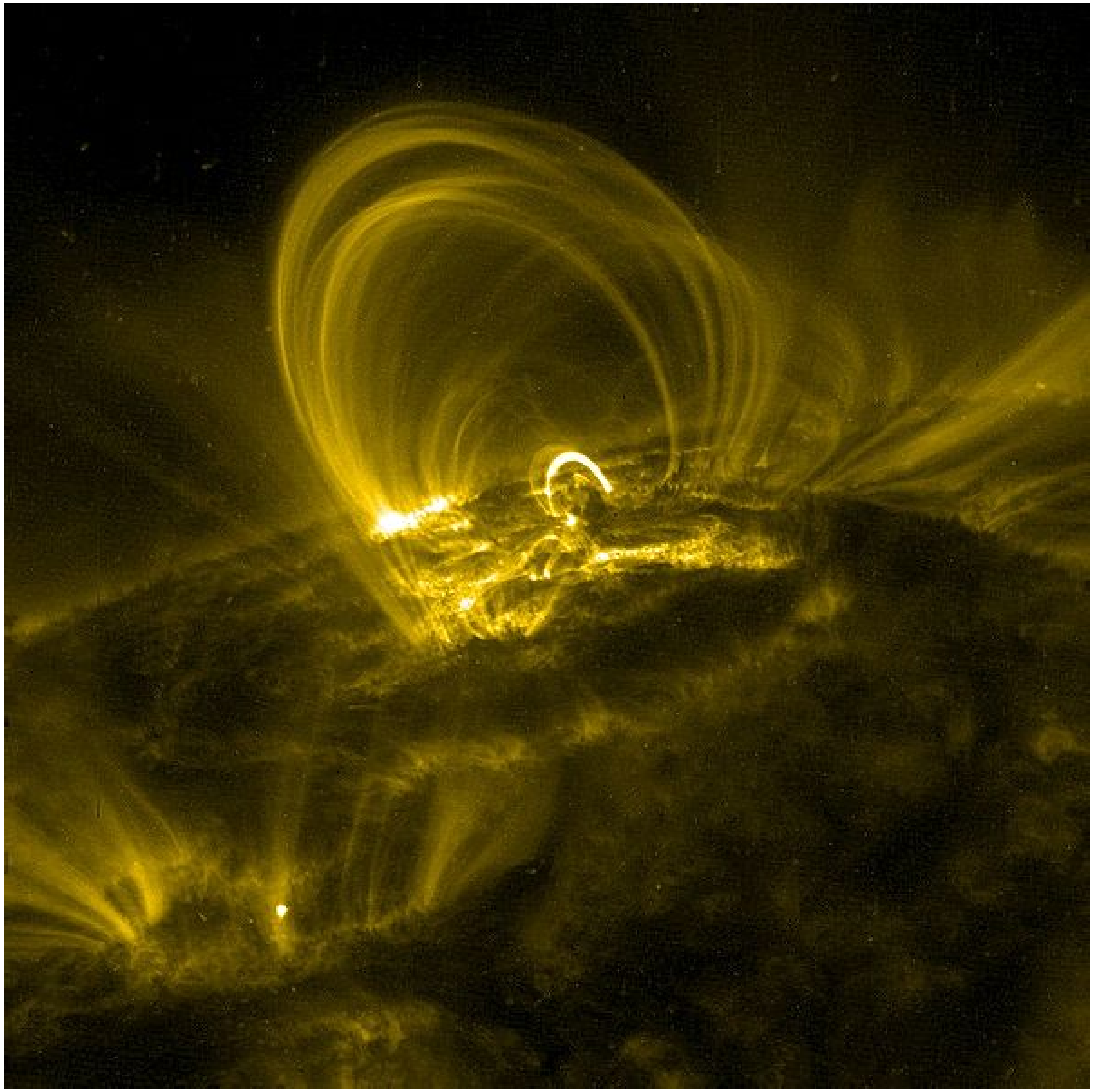


Astrophysical applications: solar corona



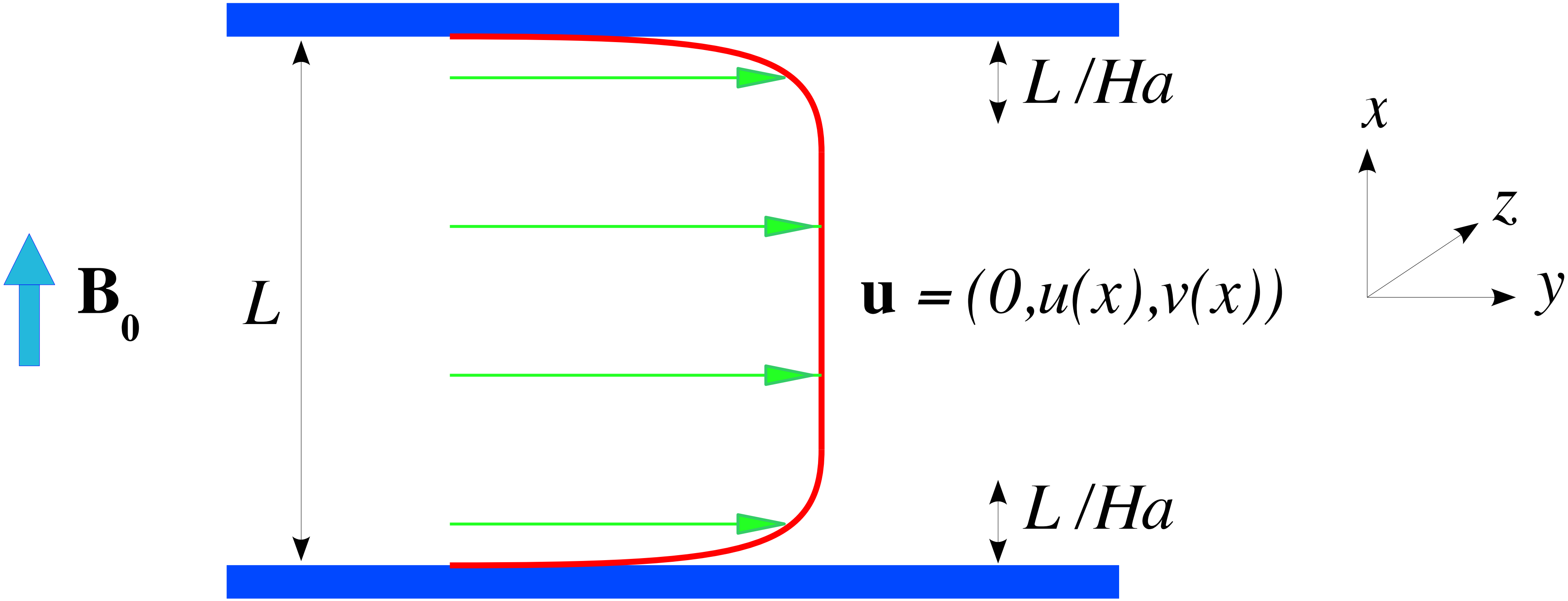
Transition Region and Coronal Explorer (TRACE) composite from 171Å, 195Å, 284Å lines

Astrophysical applications: solar corona in close-up



Transition Region and Coronal Explorer (TRACE) image at 171 Å, roughly 10^6 Kelvin.

Hartmann / channel flow [c.f. Lyutikov 2008 ApJ]



Fields $\mathbf{u} = U_0(0, u(x), v(x))$ and $\mathbf{B} = B_0(1, b(x), c(x))$.

Magnetic field direction is $\hat{\mathbf{b}} = \dfrac{(1, b, c)}{\sqrt{1 + b^2 + c^2}}$.

Strain components $W_{xy} = W_{yx} = U_0 \dfrac{\mathrm{d}u}{\mathrm{d}x}$ and $W_{xz} = W_{zx} = U_0 \dfrac{\mathrm{d}v}{\mathrm{d}x}$.

Governing equations

Induction equation using $\eta_\perp$ in Ohm's law

$$B_0 U_0 \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\eta_\perp B_0}{L}\frac{\mathrm{d}^2 b}{\mathrm{d}x^2} = 0, \quad B_0 U_0 \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\eta_\perp B_0}{L}\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} = 0,$$

Neglect distinction between $\eta_\perp$ and $\eta_\perp = 1.96\,\eta_\parallel$.

Scale x with L, and choose velocity scale $U_0 = \eta_\perp / L$.

Dimensionless induction equations are

$$0 = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}^2 b}{\mathrm{d}x^2}, \quad 0 = \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}^2 c}{\mathrm{d}x^2}.$$

Boundary conditions: $u = v = b = c = 0$ on walls $x = \pm 1/2$.

Momentum equations

Momentum equations with streamwise forcing $F = \mathsf{d}p/\mathsf{d}x$

$$0 = F + \frac{B_0^2}{L}\frac{\mathsf{d}b}{\mathsf{d}x} + \frac{1}{L}\frac{\mathsf{d}}{\mathsf{d}x}\Pi_{xy}, \quad 0 = \frac{B_0^2}{L}\frac{\mathsf{d}c}{\mathsf{d}x} + \frac{1}{L}\frac{\mathsf{d}}{\mathsf{d}x}\Pi_{xz}.$$

Integrate once in x,

$$0 = FLx + B_0^2 b + \Pi_{xy}, \quad 0 = B_0^2 c + \Pi_{xz}.$$

Sum of forcing plus Maxwell stress plus viscous stress is constant.

Now eliminate

$$\frac{\mathsf{d}u}{\mathsf{d}x} = -\frac{\mathsf{d}^2 b}{\mathsf{d}x^2}, \quad \frac{\mathsf{d}v}{\mathsf{d}x} = -\frac{\mathsf{d}^2 c}{\mathsf{d}x^2}.$$

Reconstruct velocities later from the magnetic field,

$$u(x) = \left.\frac{\mathsf{d}b}{\mathsf{d}x}\right|_{\mathrm{wall}} - \frac{\mathsf{d}b}{\mathsf{d}x}, \quad v(x) = \left.\frac{\mathsf{d}c}{\mathsf{d}x}\right|_{\mathrm{wall}} - \frac{\mathsf{d}c}{\mathsf{d}x}.$$

Dimensionless momentum equations

Eliminating u and v and rescaling gives

$$\mathrm{Ha}^2(fx+b) - 2b\,\frac{b\dfrac{\mathrm{d}^2 b}{\mathrm{d}x^2} + c\dfrac{\mathrm{d}^2 c}{\mathrm{d}x^2}}{(1+b^2+c^2)^2} + \epsilon\,\frac{2-b^2-c^2}{(1+b^2+c^2)^{3/2}}\,\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} = O(\epsilon^2),$$

$$\mathrm{Ha}^2 c - 2c\,\frac{b\dfrac{\mathrm{d}^2 b}{\mathrm{d}x^2} + c\dfrac{\mathrm{d}^2 c}{\mathrm{d}x^2}}{(1+b^2+c^2)^2} - \epsilon\,\frac{2-b^2-c^2}{(1+b^2+c^2)^{3/2}}\,\frac{\mathrm{d}^2 b}{\mathrm{d}x^2} = O(\epsilon^2),$$

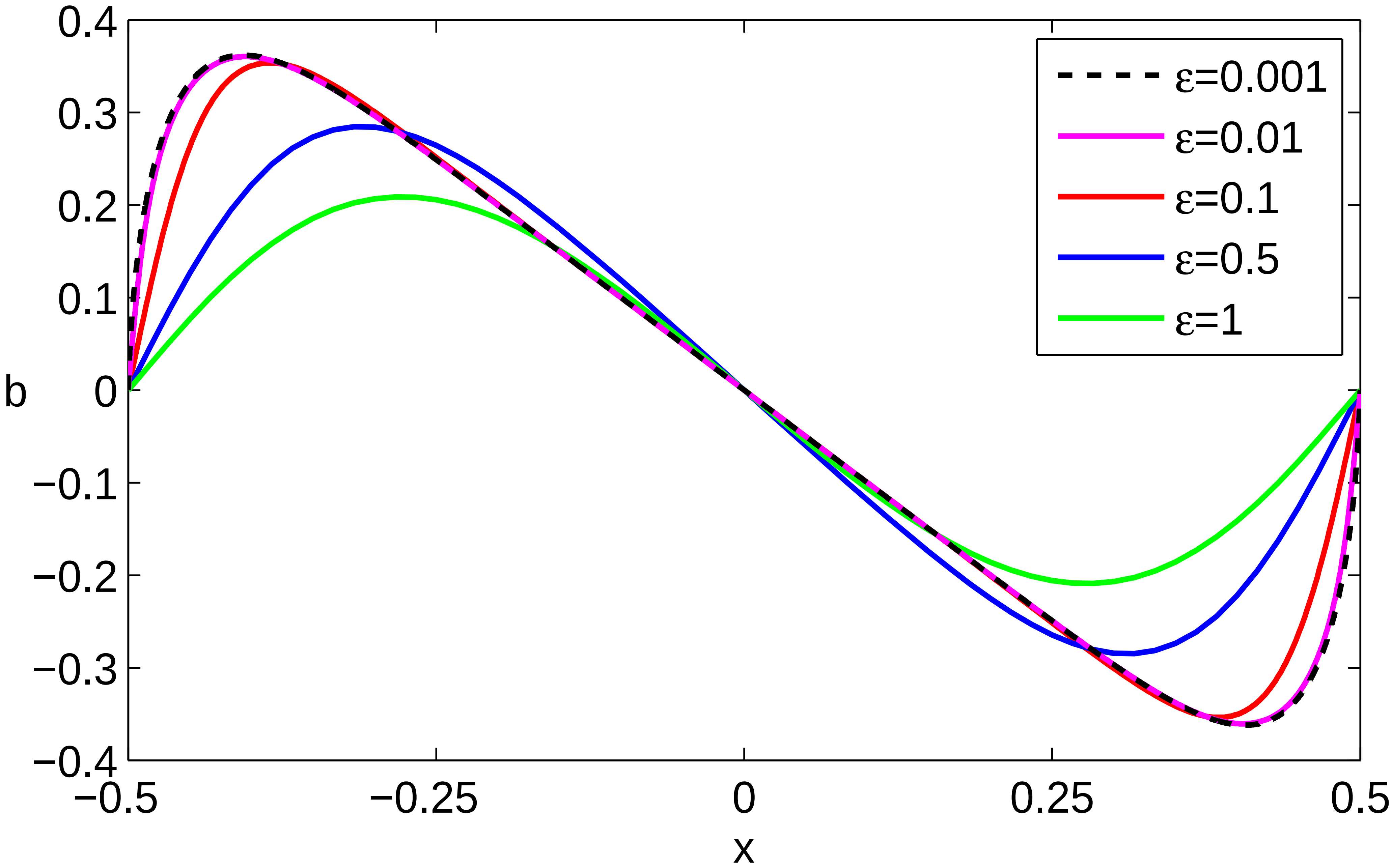
with parameters

$$\mathrm{Ha} = \frac{B_0 L}{(4\pi\eta_\perp\mu_\parallel)^{1/2}}, \quad f = \frac{4\pi\, FL}{B_0^2}, \qquad \epsilon = \frac{\mu_\times}{\mu_\parallel} = \frac{25}{72}(\omega_\mathrm{i}\tau_\mathrm{i})^{-1} \ll 1.$$

Lorentz / viscous   forcing / Lorentz   gyro / parallel

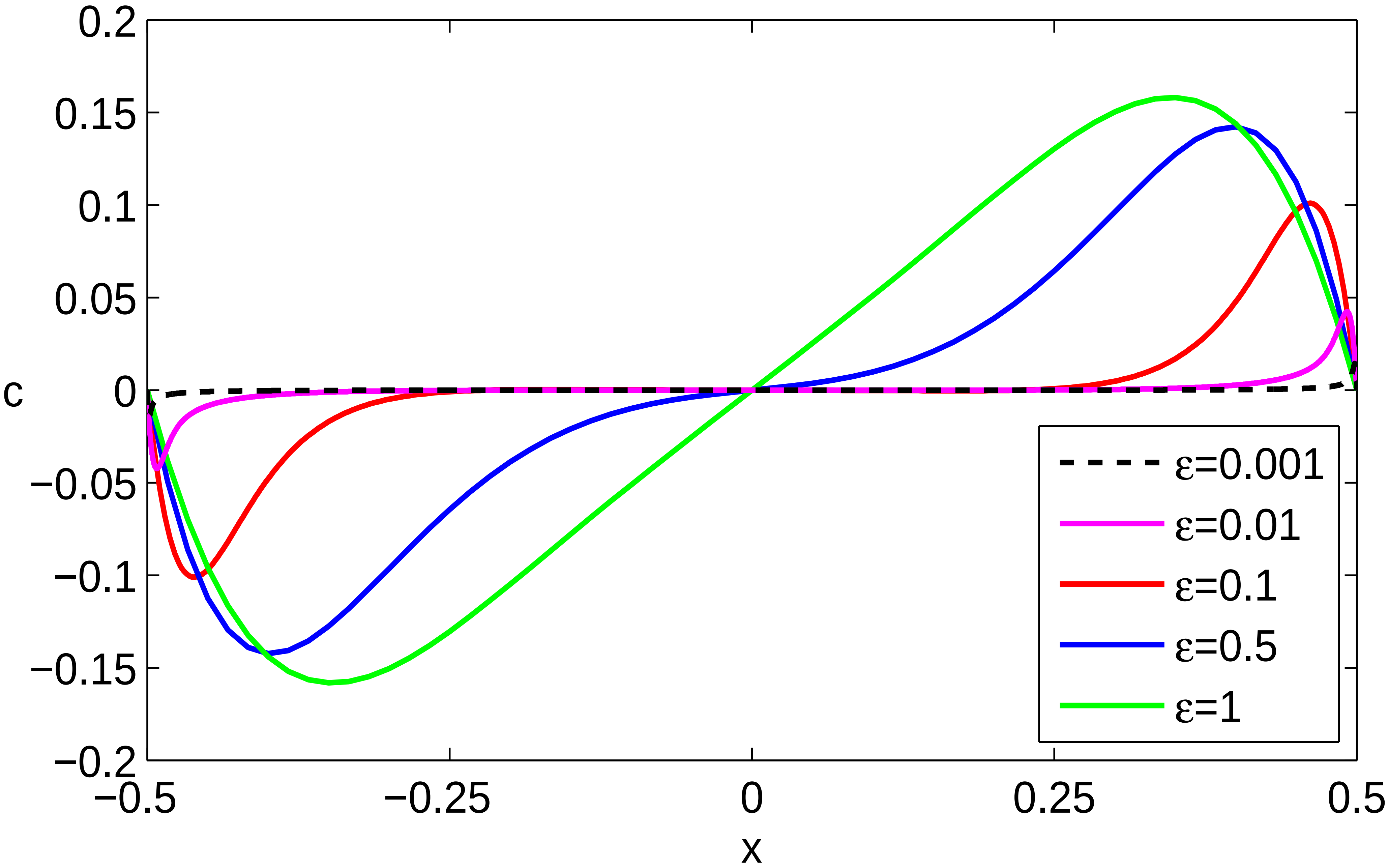
Parallel viscosity $\mu_\parallel = (2/3)\eta_0$ and gyroviscosity $\mu_\times = \eta_3$.

Numerical solution – streamwise field (b)



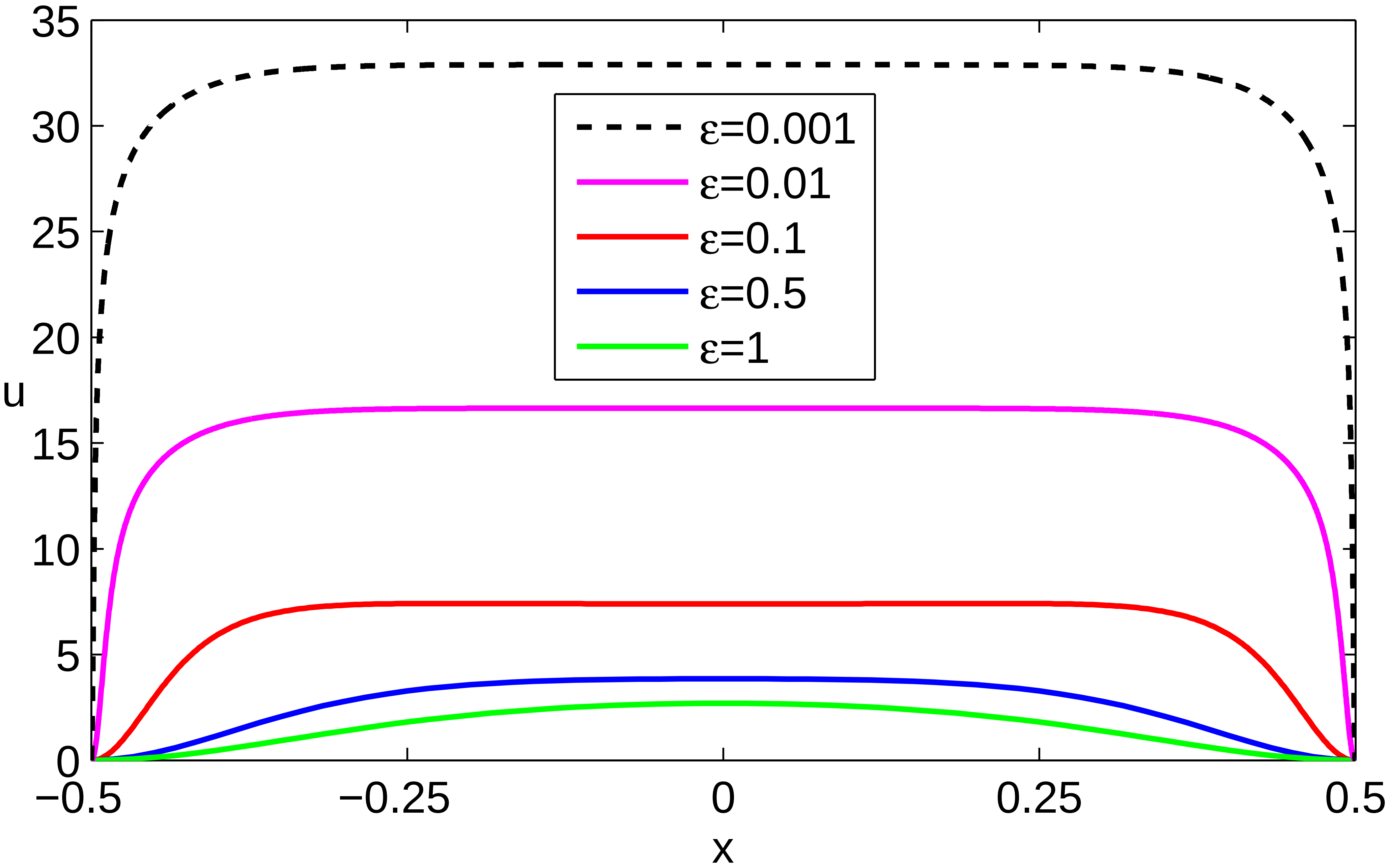
$\mathsf{Ha} = 10, f = 1$ two-point boundary value solver (Cash & Mazzia 2005)

Numerical solution – out of plane field (c)



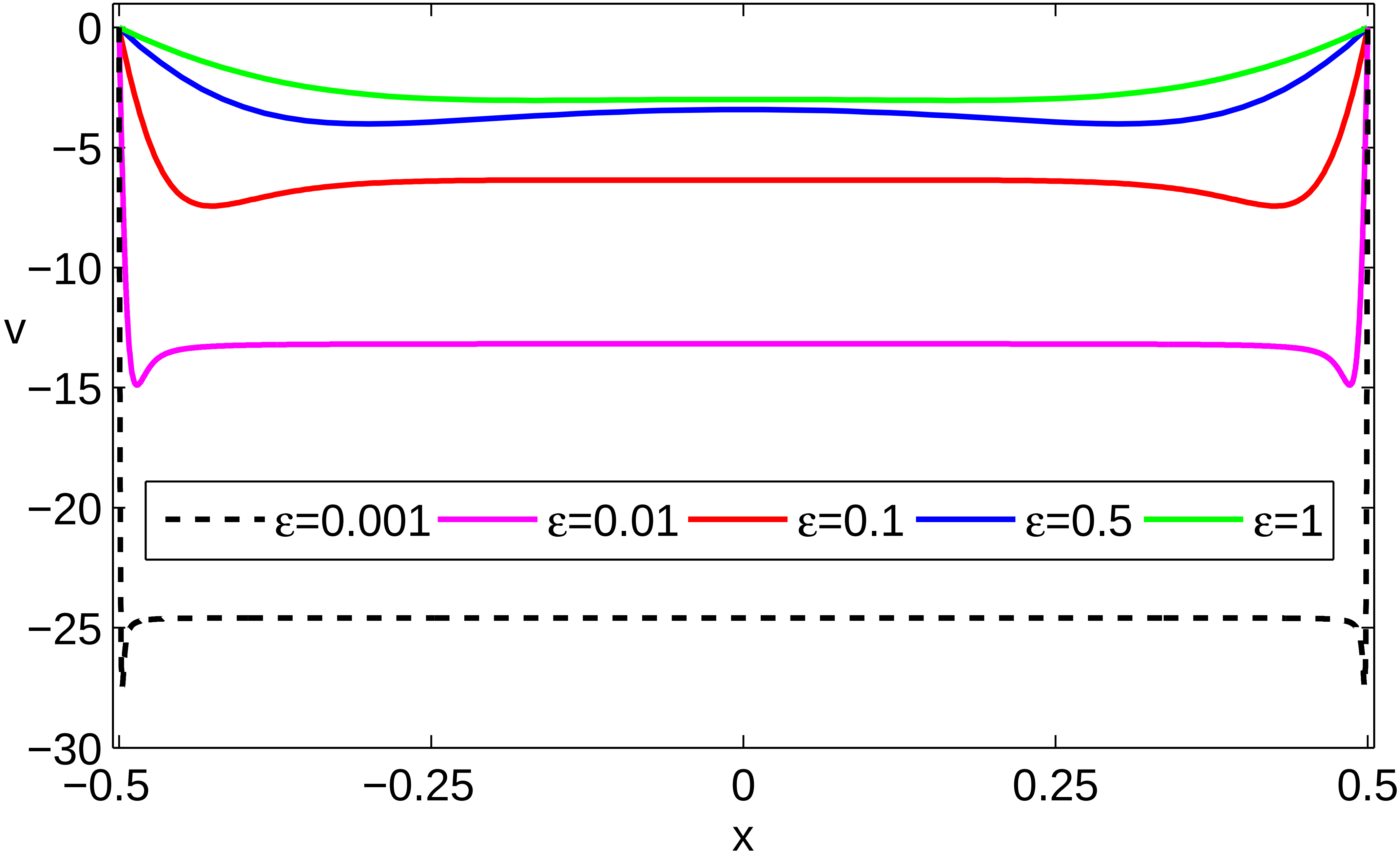
$\mathsf{Ha} = 10, f = 1$ two-point boundary value solver (Cash & Mazzia 2005)

Numerical solution – streamwise velocity (u)



$\mathrm{Ha} = 10, f = 1$ two-point boundary value solver (Cash & Mazzia 2005)

Numerical solution – out of plane velocity (v)



$\mathsf{Ha} = 10, f = 1$ two-point boundary value solver (Cash & Mazzia 2005)

Magnetic field in the core

In most of the domain $b = O(1)$ and $c = O(\epsilon)$.

Leading order equation for b is

$$fx + b = \frac{1}{\mathsf{Ha}^2}\frac{2b^2}{(1+b^2)^2}\frac{\mathsf{d}^2 b}{\mathsf{d}x^2}.$$

When $\mathsf{Ha} \gg 1$ the solution is $b = -fx$ outside $O(1/\mathsf{Ha})$-wide Hartmann boundary layers at the walls.

Having solved for b, the out-of-plane field c is given by the algebraic relation

$$c = \epsilon\,\frac{fx + b}{2bfx}\,(b^2 - 2)\,(1 + b^2)^{1/2},$$

which becomes $c \sim -\epsilon/b$ for $|b| \ll 1$.

The assumption $c \ll b$ breaks down as $b \to 0$ at the walls.

Hartmann layers (near wall, $\epsilon = 0$, $\mathsf{Ha} \gg 1$)

The linear profile $b = -fx$ cannot satisfy $b = 0$ at the walls $x = \pm 1/2$.

The viscous stress reappears in $O(1/\mathsf{Ha})$-wide boundary layers at the walls, as in standard MHD.

Rescale to bring in the viscous stress near the walls:

$$0 = \left(-\frac{1}{2} + \frac{Y}{\mathsf{Ha}}\right) f + b - \frac{2b^2}{(1+b^2)^2} \frac{\mathrm{d}^2 b}{\mathrm{d}Y^2}, \quad \text{where} \quad x = -\frac{1}{2} + \frac{Y}{\mathsf{Ha}}.$$

After dropping the small Y/Ha term, we get an x-independent ODE.

$$\frac{\mathrm{d}^2 b}{\mathrm{d}Y^2} + V'(b) = 0$$

for a particle in a potential.

The Maxwell stress plus the viscous stress is spatially uniform in these scalings.

Hartmann layer solutions

Multiply by db/dY and integrate once

$$\left(\frac{db}{dY}\right)^2 = \mathcal{E} - V(b), \text{ for } V(b) = \left[b + \frac{b^3}{6} - \frac{1}{2b}\right] f - b^2 - \log b - \frac{b^4}{4}.$$

The potential V has a maximum at $b = \frac{1}{2} f$.

The outer linear solution $b = -fx$ tends to $\frac{1}{2} f$ at the left wall.

Matching the Hartmann layer to the outer solution implies $b \to \frac{1}{2} f$ as $Y \to \infty$. We also impose the wall boundary condition $b = 0$ at $Y = 0$.

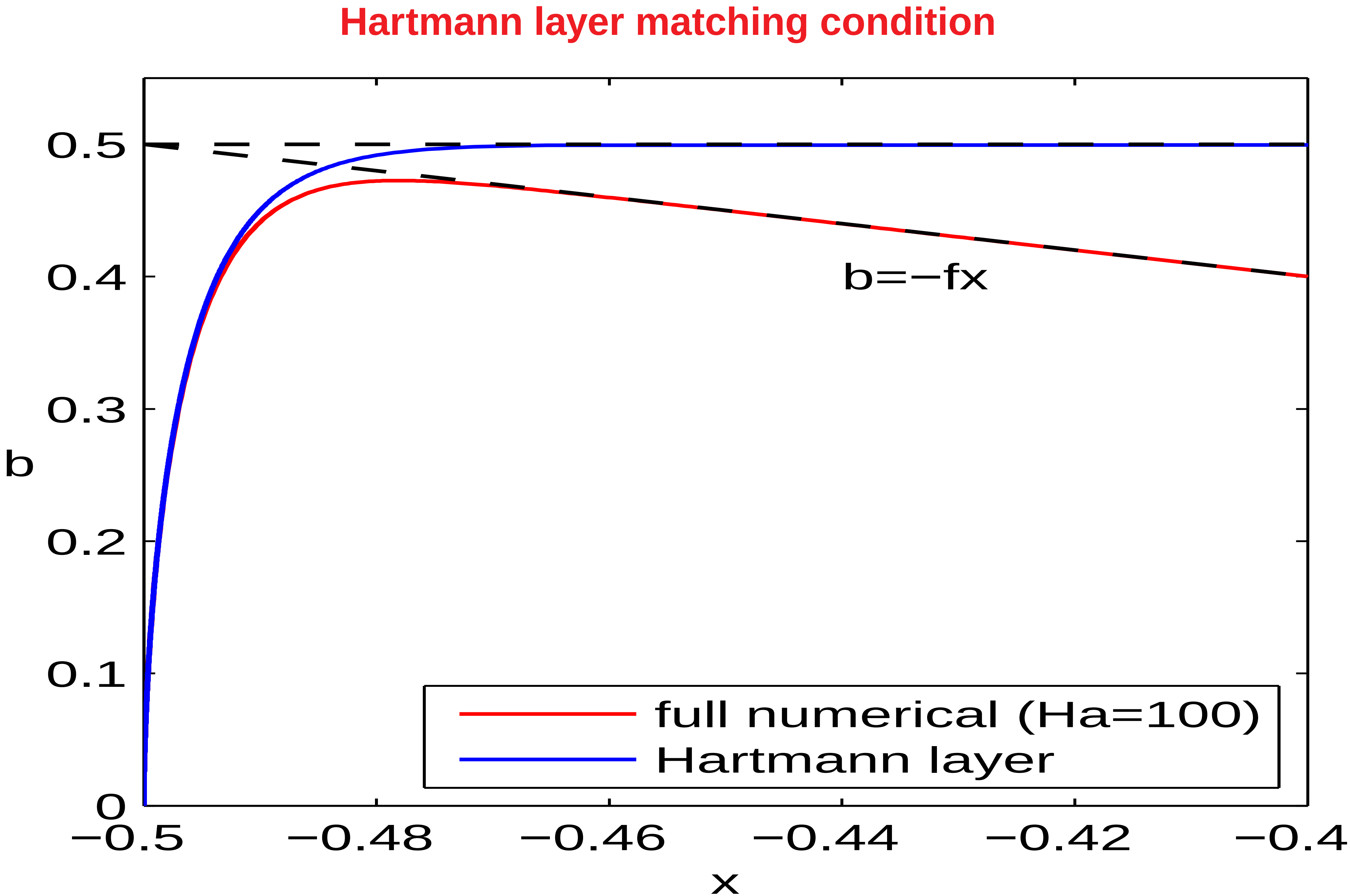
The implicit solution is

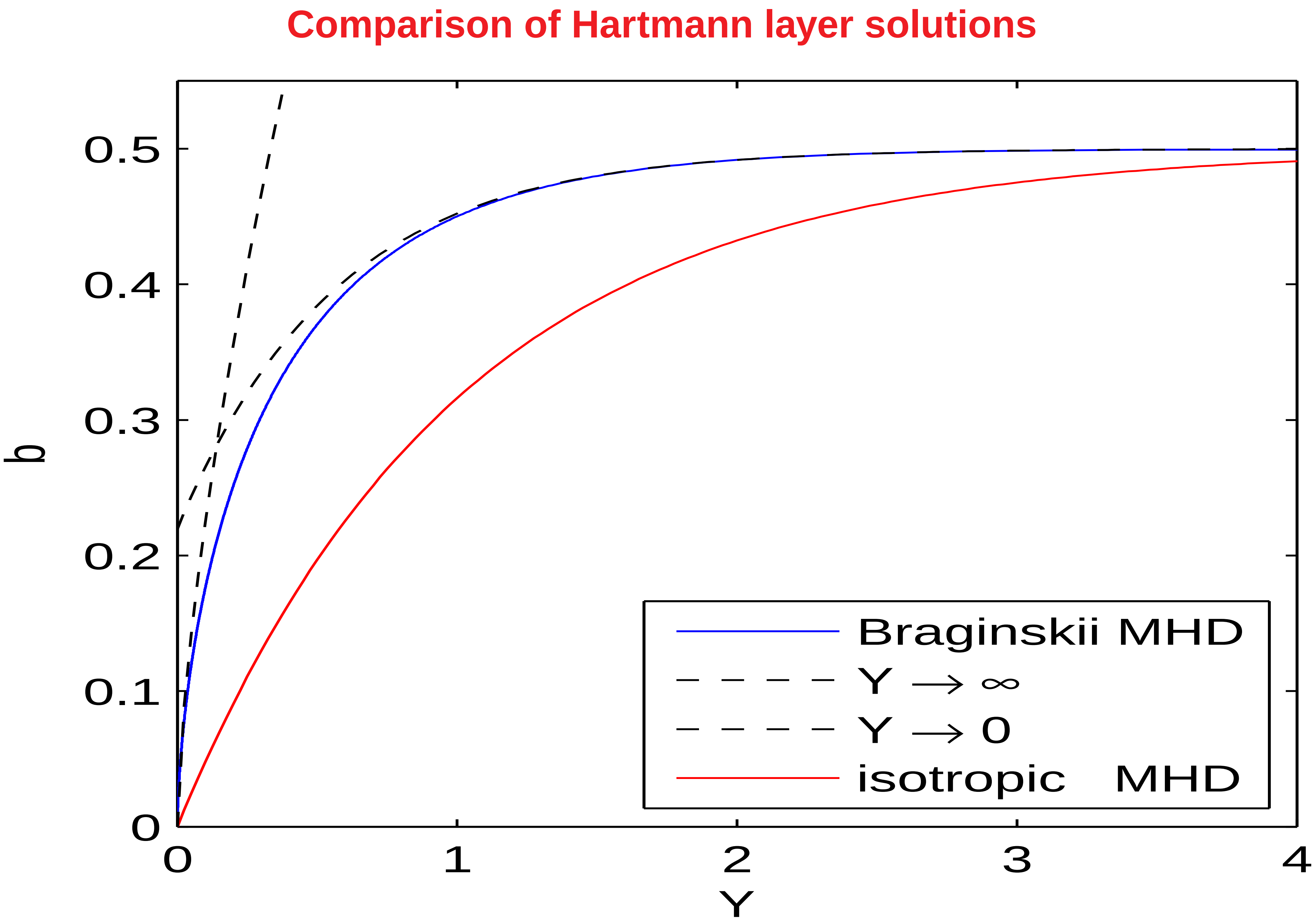
$$Y = \int_0^b \left(V(\tfrac{1}{2} f) - V(s)\right)^{-1/2} ds.$$

Exploiting $s \ll 1$,

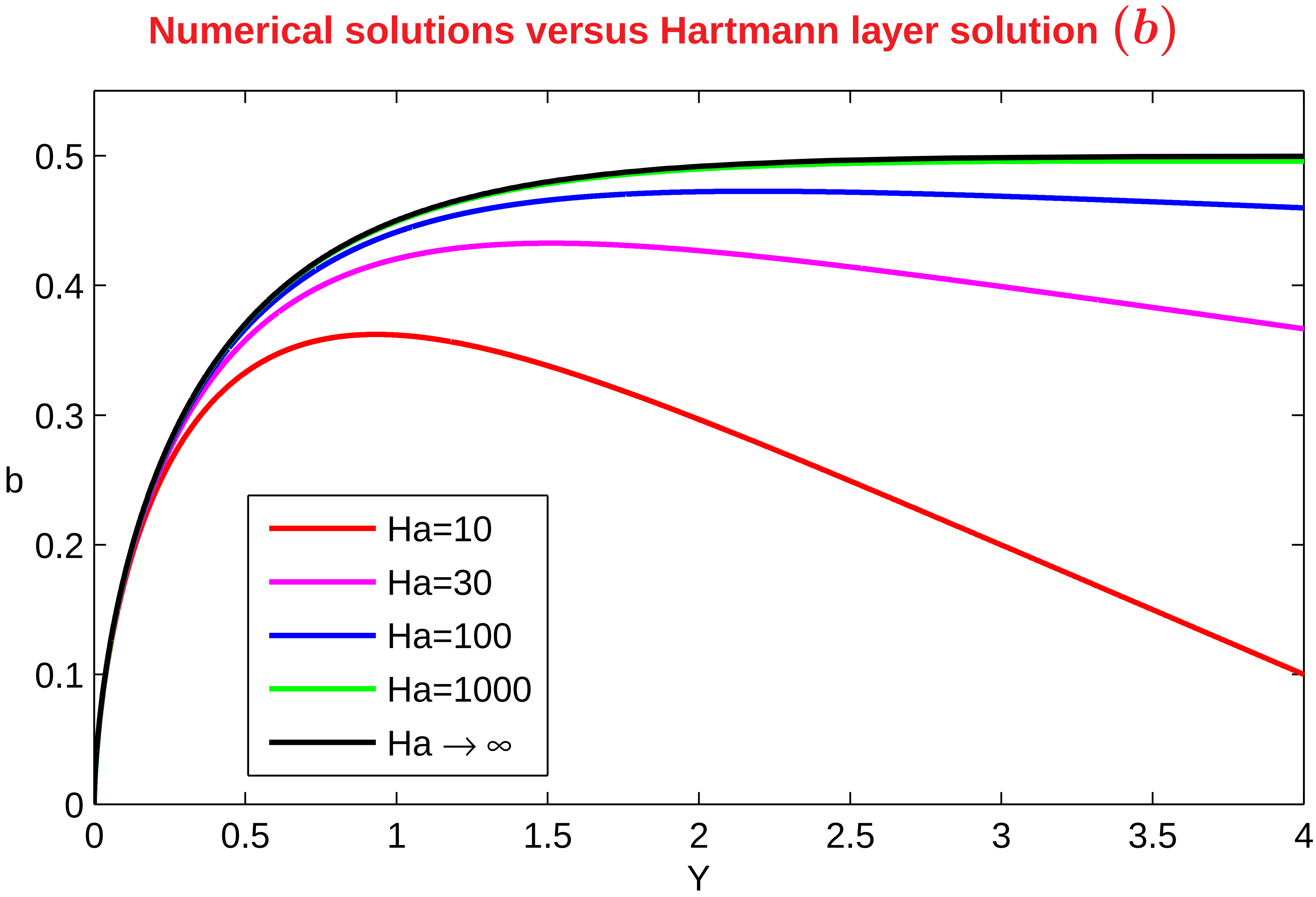
$$b \sim \tfrac{1}{2}\, 3^{2/3} f^{1/3} Y^{2/3} \text{ as } Y \to 0.$$

We satisfy $b = 0$ at the wall, but the current db/dx becomes infinite.


Hartmann layer matching condition
0.5
0.4
0.3
0.2
0.1
0
b
b=−fx
full numerical (Ha=100)
Hartmann layer
−0.5
−0.48
−0.46
−0.44
−0.42
−0.4
x

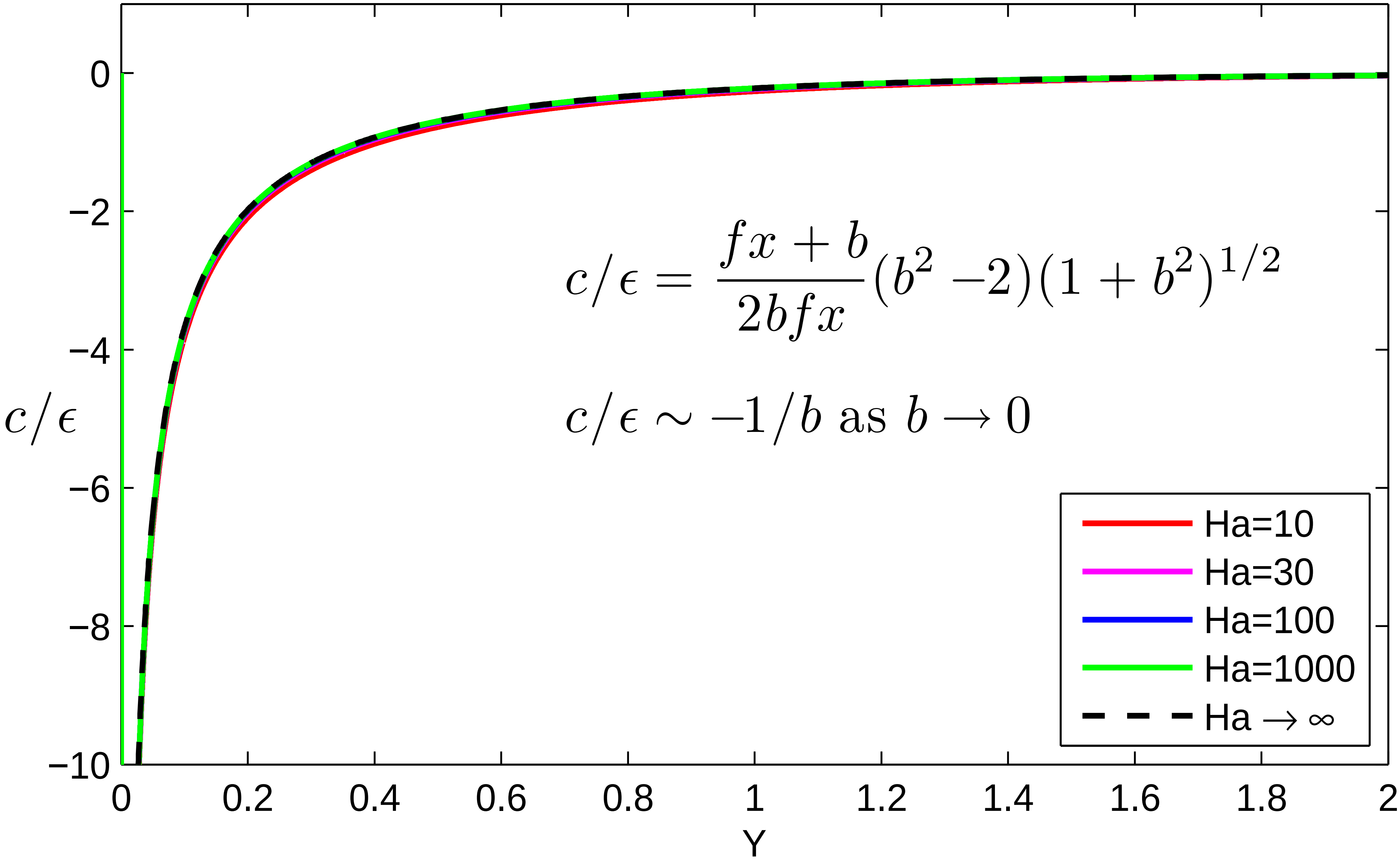


Comparison of Hartmann layer solutions
b
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
Y
Braginskii MHD
Y ⟶ ∞
Y ⟶ 0
isotropic MHD



Numerical solutions versus Hartmann layer solution (b)
0.5
0.4
0.3
0.2
0.1
0
b
0
0.5
1
1.5
2
2.5
3
3.5
4
Y
Ha=10
Ha=30
Ha=100
Ha=1000
Ha $\to \infty$

Numerical solutions versus Hartmann layer solution (c)



Inner wall layers

Outer scaling $b = O(1)$ and $c \sim -\epsilon/b \ll b$ breaks down for $b \sim \epsilon^{1/2}$.

Rescale for $b \sim \epsilon^{1/2}$ and $c \sim \epsilon^{1/2}$ using

$$b = \epsilon^{1/2} B, \quad c = \epsilon^{1/2} C, \quad x = -\frac{1}{2} + \frac{2}{\sqrt{f}\,\mathsf{Ha}} \epsilon^{3/4} X.$$

The $\epsilon^{3/4}$ scaling of x balances the gyroviscous terms with the forcing.

Universal ODE system $\quad \dfrac{\mathrm{d}^2 C}{\mathrm{d}X^2} = 1 + BC, \quad \dfrac{\mathrm{d}^2 B}{\mathrm{d}X^2} = -C^2.$

Boundary conditions:

- $B = C = 0$ on $X = 0$,
- Matching to the Hartmann layer as $X \to \infty$.

Matching conditions

Inside the Hartmann layer

$$c = \epsilon \frac{fx + b}{2bfx}(b^2 - 2)(1 + b^2)^{1/2} \sim -\frac{\epsilon}{b} \text{ as } b \to 0.$$

In wall layer variables (van Dyke's matching rule)

$$C \sim -1/B \text{ as } B \to \infty.$$

Putting $C = -1/B$ into the universal ODEs gives

$$\frac{\mathrm{d}^2 B}{\mathrm{d}X^2} = -\frac{1}{B^2}.$$

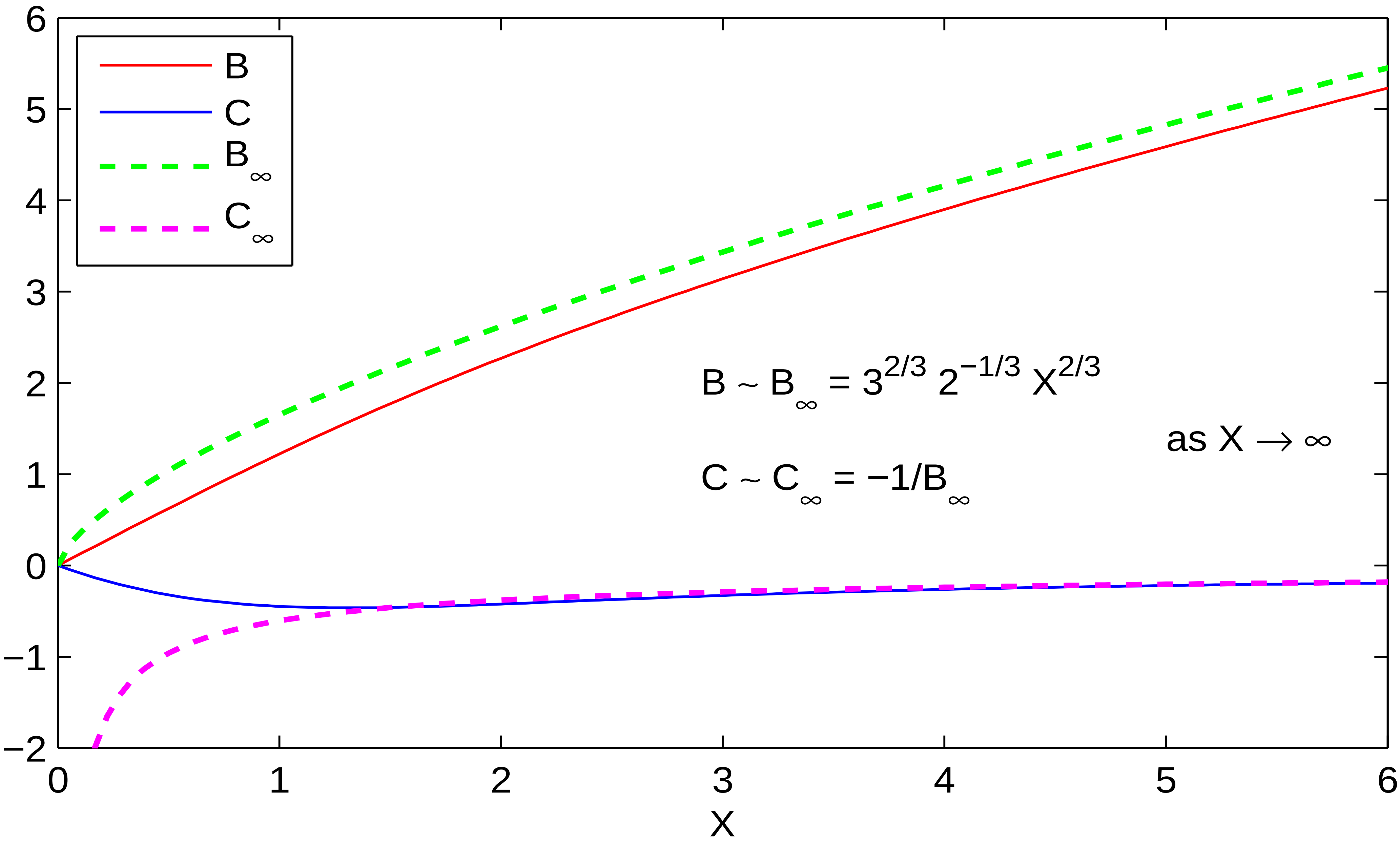
The solution

$$B(X) = 3^{2/3} 2^{-1/3} X^{2/3}$$

coincides with the Hartmann layer solution in the original variables,

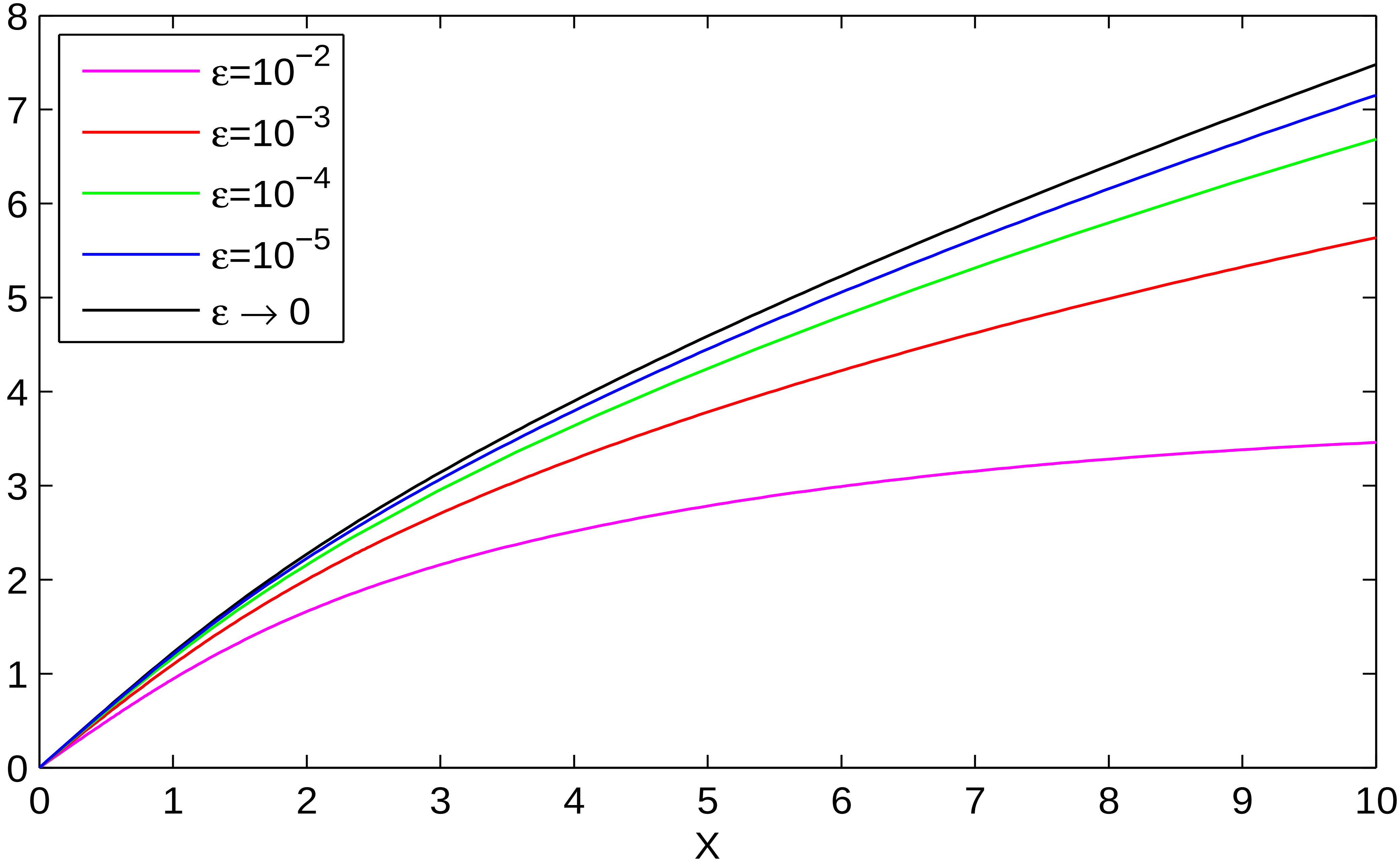
$$b \sim \tfrac{1}{2}\, 3^{2/3} f^{1/3}\, \mathrm{Ha}^{2/3} \left(x + \tfrac{1}{2}\right)^{2/3} \text{ as } x \to 0.$$

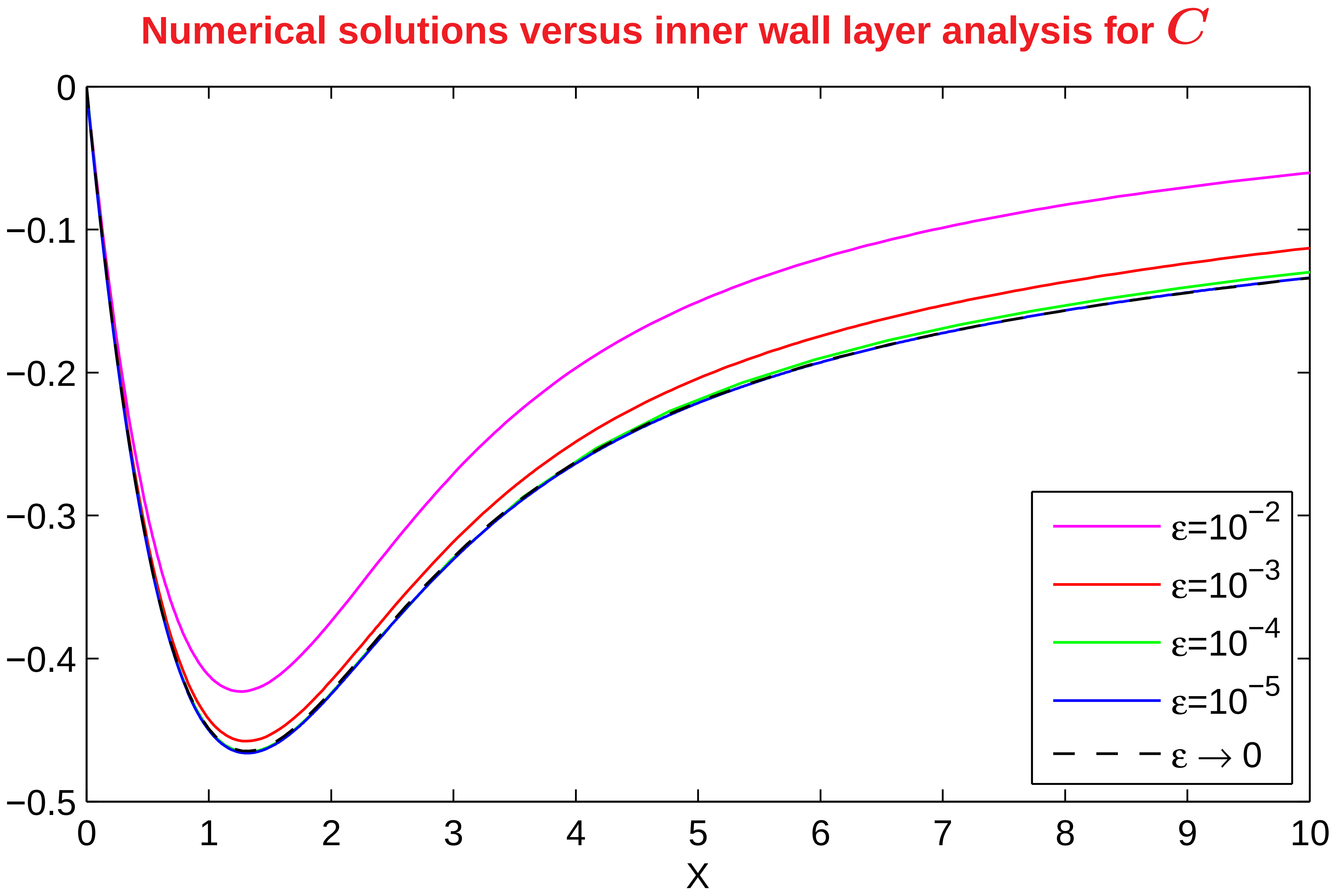
Universal inner wall layer solution (numerical)



$$B_X(0) = 1.253\ldots \quad C_X(0) = -0.886\ldots \quad C_{\text{min}} = -0.465\ldots$$

Numerical solutions versus inner wall layer analysis for B





Numerical solutions versus inner wall layer analysis for C
0
−0.1
−0.2
−0.3
−0.4
−0.5
0
1
2
3
4
5
6
7
8
9
10
X
ε=10^{−2}
ε=10^{−3}
ε=10^{−4}
ε=10^{−5}
ε → 0

Peak velocity scaling

The maximum streamwise velocity is

$$u_{\mathsf{max}} = 1.253\ldots\tfrac{1}{2} f^{1/2}\,\mathsf{Ha}\,\epsilon^{-1/4}.$$

from integrating the dimensionless induction equation

$$\frac{\mathsf{d}u}{\mathsf{d}x} + \frac{\mathsf{d}^2 b}{\mathsf{d}x^2} = 0,$$

to obtain

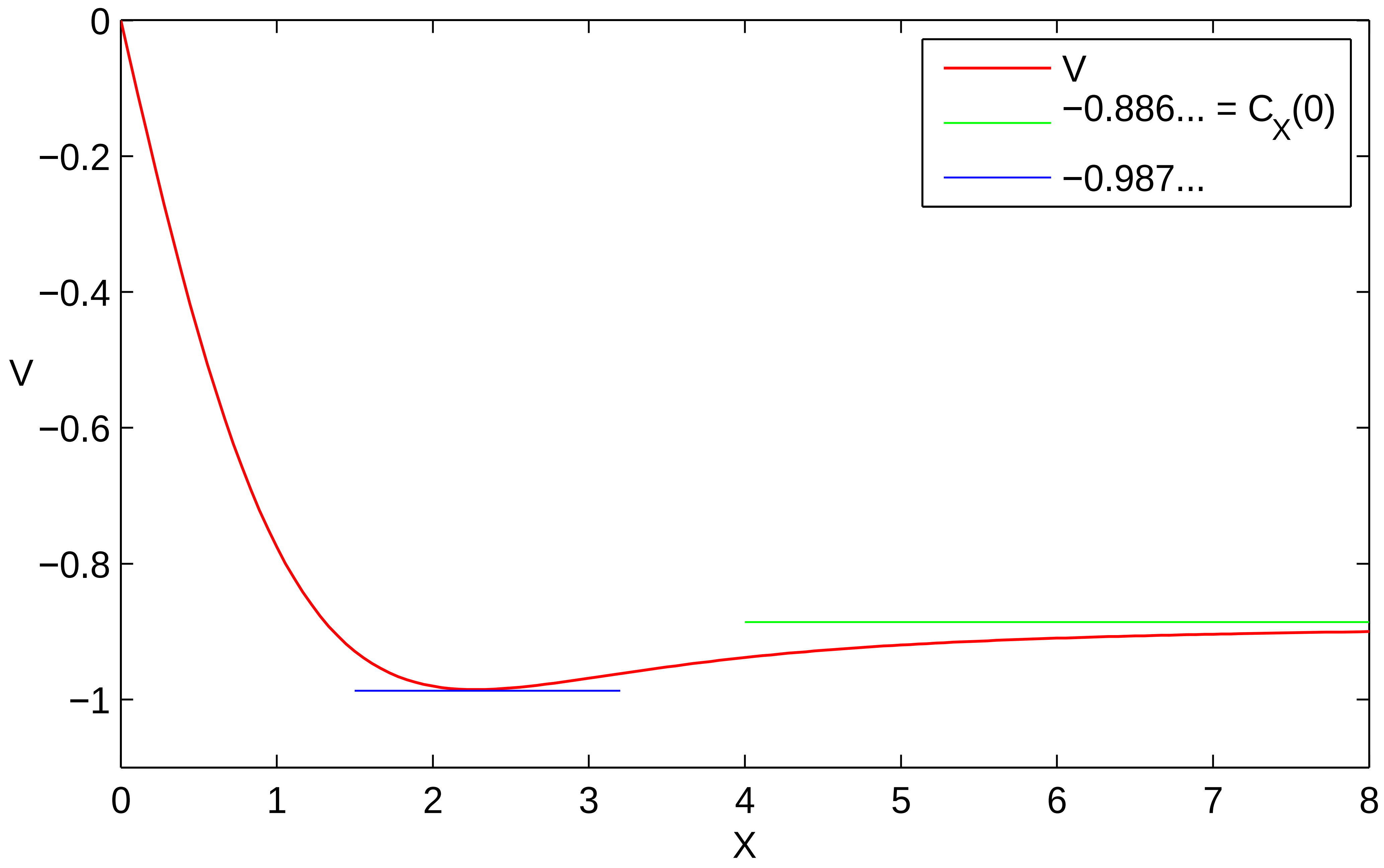
$$u(x) = \left.\frac{\mathsf{d}b}{\mathsf{d}x}\right|_{\text{wall}} - \frac{\mathsf{d}b}{\mathsf{d}x}.$$

Similarly, the maximum out-of-plane velocity is

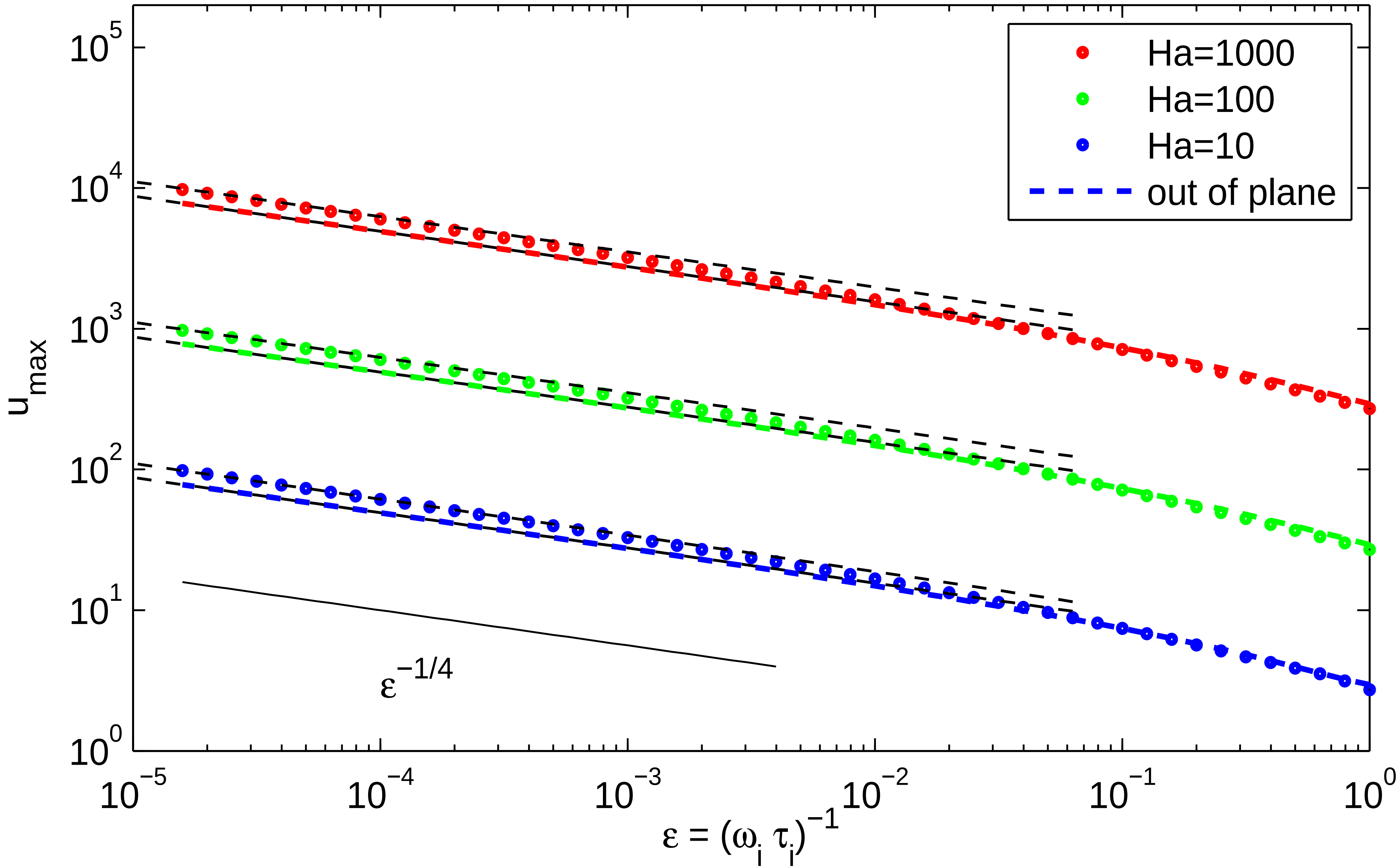
$$|v_{\mathsf{max}}| = 0.987\ldots\tfrac{1}{2} f^{1/2}\,\mathsf{Ha}\,\epsilon^{-1/4}.$$

Neither result depends upon knowing the detailed solution in the core.

Universal inner solution for the out-of-plane velocity (V)



Maximum velocities, theory versus numerical solutions



Conclusions

Braginskii magnetohydrodynamics describes strongly magnetised plasmas on lengthscales $\gg$ mean free path $\gg$ gyroradius.

Three different viscosities: parallel $\mu_{\|} \gg$ gyro $\mu_{\times} \gg$ perpendicular $\mu_{\perp}$.

In Hartmann flow the parallel viscous stress vanishes on the walls, so Braginskii MHD needs regularising, *e.g.* by gyroviscous stresses

Asymptotic solution contains

- Hartmann layers, thickness $\sim \mathsf{Ha}^{-1}$.
- Inner wall layers, thickness $\sim (\mu_{\times}/\mu_{\|})^{3/4}\,\mathsf{Ha}^{-1} \sim (\Omega_{\mathrm{i}}\tau_{\mathrm{i}})^{-3/4}\,\mathsf{Ha}^{-1}$.

Regular solution with gyroviscosity alone. No perpendicular viscosity needed.

Peak velocities and peak currents scale as $(\mu_{\times}/\mu_{\|})^{-1/4} \sim (\Omega_{\mathrm{i}}\tau_{\mathrm{i}})^{1/4}$.

No well-defined limit as $\mu_{\times}/\mu_{\|} \to 0$.

“Planar channel flow in Braginskii magnetohydrodynamics” JFM **667** 520

“Lattice Boltzmann formulation for Braginskii MHD” Comput. & Fluids **46** 201