Gyro-kinetic Theory of the Tearing Mode and Internal Kink Mode

J W Connor, R J Hastie

& A Zocco

CCFE, Abingdon, Oxon, OXI4 3DB, UK

4th Fusion Theory Working Group Meeting,
WPI Vienna, 19-30 March 2012

Background

- **In hot tokamaks the reconnecting layer is narrower than the ion Larmor radius**

  $\Rightarrow$ gyro-kinetic model for ions,

  i.e. fluid model inappropriate

- **Use semi-collisional model for electrons**

- **Earlier work:**

  Cowley et al – gyro-kinetic ions, both collisionless and semi-collisional electrons

  Pegoraro, Porcelli & Schep – gyro-kinetic ions, no temperature effects on electrons

  Drake et al – cold ions, semi-collisional electrons

- **Apply to tearing and internal kink mode stability**

Equations

Quasi-neutrality – from electron continuity eqn, Ohm's Law and ion FLR density response

$$-\frac{x}{\delta}\left(A-\frac{x}{\delta}\varphi\right)\frac{\left(\sigma_0+\sigma_1(x/\delta)^2\right)}{\left(1+d_0(x/\delta)^2+d_1(x/\delta)^4\right)}=\int_{-\infty}^{\infty}dp\exp(ipx)\hat{\varphi}(p)F(p\rho_i)$$

Ampere's Law – current from electron continuity and quasi-neutrality

$$\frac{1}{\hat{\omega}^2\hat{\beta}}\frac{d^2A}{dx^2}=\frac{1}{\delta^2}\frac{\delta}{x}\int_{-\infty}^{\infty}dp\exp(ipx)\hat{\varphi}(p)F(p\rho_i),$$

$$F(p\rho_i)=\left(\frac{1}{\hat{\omega}}+\tau\right)(\Gamma_0-1)-\frac{1}{\hat{\omega}}\frac{\eta_i}{2}(p\rho_i)^2\left(\Gamma_0-\Gamma_1\right);\ \Gamma_n=\exp\left(-p^2\rho_i^2/2\right)I_n\left(p^2\rho_i^2/2\right),\quad \hat{\omega}=\frac{\omega}{\omega_{*e}}$$

$$F(k)\to F_\infty+\frac{f_1}{k};\ F_\infty=-\left(\tau+\frac{1}{\hat{\omega}}\right),\quad f_1=\frac{1}{\sqrt{\pi}}\left(\tau+\frac{(1-\eta_i/2)}{\hat{\omega}}\right)\ \text{for}\ k\to\infty \qquad F(k)\to-\left(\tau+\left(1+\eta_i\right)/\hat{\omega}\right)k^2/2\ \text{for}\ k\to0$$

Parameters

$$\frac{\delta_0}{\rho_i},\ \left(\delta^2=e^{-i\pi/2}\frac{\omega\nu_{ei}L_s^2}{k_y^2v_{the}^2}\equiv e^{-i\pi/2}\hat{\omega}\delta_0^2\right),\ \hat{\beta}=\frac{\beta_e}{2}\frac{L_s^2}{L_n^2}\ \left(\beta_e=\frac{2\mu_0nT_e}{B^2}\right),\ \eta_{e,i},\ \Delta'\rho_i,\text{(large x boundary condition)}$$

Semi-collisional conductivity

$$\sigma_0=\left(1-(1+1.71\eta_e)/\hat{\omega}\right),\ \sigma_1=d_1(1-1/\hat{\omega}),\ d_0=5.08\ \ d_1=2.13$$

Regions

- Simplify equations by considering two regions
- Region 1 - ‘Ion region’: $x \sim \rho_i \gg \delta$
- Region 2 - ’Electron region’ $x \sim \delta \ll \rho_i$
- Asymptotic matching of solutions in the two regions: involves interplay between k-space and x-space
- At large x match to MHD boundary condition involving tearing mode stability parameter: Δ'

Region 1

- Convenient to Fourier Transform and introduce current

$$J = -\frac{d^2A}{dX^2}, \quad \text{or} \quad \hat{J} = k^2\hat{A} \qquad\qquad X = x/\rho_i$$

In Region 1:

$$\frac{d}{dk}\left(\frac{G(k)}{F(k)}\frac{d\hat{J}}{dk}\right) + \hat{\omega}^2\hat{\beta}\frac{\sigma_1}{d_1}\frac{\hat{J}}{k^2} = 0; \quad G(k) = -\frac{\sigma_1}{d_1} + F(k)$$

$$\text{Since } F \to F_\infty = -(1/\hat{\omega} + \tau) \quad \text{as} \quad k \to \infty,$$

$$\hat{J}(k) \sim \hat{a}_+ k^{1/2+\mu} + \hat{a}_- k^{1/2-\mu}$$

$$1/4 - \mu^2 = \hat{\omega}^2\hat{\beta}(1/\hat{\omega} + \tau)(1 - 1/\hat{\omega})/(1+\tau)$$

Low k boundary condition $\hat{J}(k) \sim 1 + \frac{\pi\hat{\beta}}{3\Delta'\rho_i}\hat{\omega}^2\left(\tau + \frac{1+\eta_i}{\hat{\omega}}\right)k^3, \quad k \to 0$

Region 2

- Fourier transform produces 4th order equation: best to back-transform

$$\frac{d^2}{ds^2}\left(\frac{\left(1+\bar{d}_0 s^2+\bar{d}_1 s^4\right)}{\left(\sigma_0+\sigma_1 s^2\right)}\right)J+\hat{\omega}^2\hat{\beta}J=0 \qquad s=e^{i\pi/4}(\rho_i/\hat{\omega}^{1/2}\delta_0)X$$

$$\bar{d}_0(\hat{\omega})=d_0+\frac{\sigma_0}{(\tau+1/\hat{\omega})}, \quad \bar{d}_1(\hat{\omega})=d_1+\frac{\sigma_1}{(\tau+1/\hat{\omega})}=d_1\frac{(1+\tau)}{(\tau+1/\hat{\omega})}$$

- Thus

$$J(s)\sim b_+ s^{-3/2+\mu}+b_- s^{-3/2-\mu} \quad \text{as} \quad s\to\infty$$

$$\text{so that} \quad \hat{J}\sim\hat{c}_+ t^{1/2+\mu}+\hat{c}_- t^{1/2-\mu} \qquad k=e^{i\pi/4}(\rho_i/\hat{\omega}^{1/2}\delta_0)t$$

$$\text{where} \quad \frac{\hat{c}_-}{\hat{c}_+}=\frac{\Gamma(-1/2+\mu)}{\Gamma(-1/2-\mu)}\tan\left(\pi\left(1/4+\mu/2\right)\right)\frac{b_+}{b_-}$$

- Matching condition

$$\frac{\hat{c}_-}{\hat{c}_+}=\frac{\hat{a}_-}{\hat{a}_+}\left(\frac{e^{-i\pi/4}\hat{\omega}^{1/2}\delta_0}{\rho_i}\right)^{2\mu}$$

Low $\hat{\beta} = (\beta_e/2)L_s^2/L_n^2$ Solution (1)

Region 1: Expand in $\hat{\beta}$, matching to MHD at low k

$$\hat{J}(k) = \exp\left(\frac{\hat{\omega}^2\hat{\beta}\sigma_1}{d_1}\int_0^k \frac{du}{u}\frac{F(u)}{G(u)}\right) + \left(\frac{\hat{\omega}^2\hat{\beta}\sigma_1}{d_1}\right)\left(\frac{\pi}{\Delta'\rho_i}\right)\int_0^k du\frac{F(u)}{G(u)}$$

$$-\left(\hat{\omega}^2\frac{\sigma_1}{d_1}\hat{\beta}\right)^2\int_0^k du\frac{F(u)}{G(u)}\int_0^u dv\frac{F(v)}{v^2G(v)} \quad .$$

Thus for $k \to \infty$

$$\hat{J}(k) \sim k^{1/2-\mu} + \frac{\pi}{\Delta'\rho_i}\left(\frac{1}{4}-\mu^2\right)k - \frac{\pi}{\Delta'\rho_i}\left(\frac{1}{4}-\mu^2\right)\frac{(\hat{\omega}-1)}{(1+\tau)}\frac{f_1}{(\hat{\omega}\tau+1)}\ell n\, k$$

$$+\frac{\pi}{\Delta'\rho_i}\left(\frac{1}{4}-\mu^2\right)\frac{\hat{\omega}(1+\tau)}{(1+\tau\hat{\omega})}\bar{I}\,k^0 - \left(\frac{1}{4}-\mu^2\right)^2\frac{\hat{\omega}(1+\tau)}{(1+\tau\hat{\omega})}I(\hat{\omega},\tau,\eta_i)\,k$$

where

$$I(\hat{\omega},\tau,\eta_i) = \int_0^\infty dk\frac{F(k)}{k^2G(k)}$$

$$\bar{I}(\hat{\omega},\tau,\eta_i) = \int_0^\infty dk\left(\frac{F(k)}{G(k)} - \frac{(1+\hat{\omega}\tau)}{\hat{\omega}(1+\tau)} + \frac{(\hat{\omega}-1)(\hat{\omega}\tau+1-\eta_i/2)}{\sqrt{\pi}(1+k)\hat{\omega}^2(1+\tau)^2}\right) \quad .$$

Low $\hat{\beta}$ Solution (2)

Region 2

$$J(s) = J_0\left(\bar{\sigma}(s^2) - \hat{\omega}^2\hat{\beta}\int_0^s ds'(s-s')\bar{\sigma}(s'^2)\right)$$

$$\bar{\sigma}(s^2) = \frac{\sigma_0 + \sigma_1 s^2}{1+\bar{d}_0 s^2 + \bar{d}_1 s^4}$$

At large s

$$J(s) \sim J_0 \frac{\sigma_1}{\bar{d}_1 s^2}\left[1 - \frac{\pi}{2\bar{d}_1}\frac{\hat{\omega}^2\hat{\beta}}{(s_+ + s_-)}\left(\frac{\sigma_0}{s_+ s_-} - \sigma_1\right)s + 2\hat{\omega}^2\hat{\beta}\frac{\sigma_1}{\bar{d}_1}\ell n\, s\right]$$

$$s_\pm^2 = \frac{\bar{d}_0}{2\bar{d}_1} \pm \left(\frac{\bar{d}_0^2}{4\bar{d}_1^2} - \frac{1}{\bar{d}_1}\right)^{1/2}$$

Note at low $\hat{\beta}$, $J(s) \sim s^{-2}\left[b_-\left(1 + 2\hat{\omega}^2\hat{\beta}\frac{\sigma_1}{\bar{d}_1}\ell n\, s^2 + \ldots\right) + b_+(s + \ldots)\right]$

So we can identify $b_\pm$

Unified Low $\hat{\beta}$ Dispersion Relation

$$e^{-i\pi/4}\frac{\delta_0}{\rho_i}A(\hat{\omega})B(\hat{\omega})+\hat{\omega}\sqrt{1+\hat{\omega}\tau}\,C(\hat{\omega})D(\hat{\omega})=0$$

where

$$A(\hat{\omega})=\sqrt{1+\tau}\sqrt{d_0(1+\hat{\omega}\tau)+\hat{\omega}-1-1.71\eta_e+2\sqrt{d_1(1+\tau)\hat{\omega}(1+\hat{\omega}\tau)}}$$

$$B(\hat{\omega})=\frac{\Delta'\rho_i}{\pi\hat{\beta}}\hat{\omega}-\left(\hat{\omega}-1\right)^2\frac{\left(\hat{\omega}\tau+1-\eta_i/2\right)}{\sqrt{\pi}(1+\tau)^2}\left(\frac{i\pi}{4}+\ell n\left(\frac{\rho_i}{\delta_0\hat{\omega}^{1/2}}\right)\right)+\hat{\omega}^2\left(\hat{\omega}-1\right)\overline{I}(\hat{\omega})$$

$$C(\hat{\omega})=1-\frac{\hat{\omega}\left(\hat{\omega}-1\right)\Delta'\rho_i\hat{\beta}}{\pi}I(\hat{\omega},\eta_i,\tau)$$

$$D(\hat{\omega})=\left(\hat{\omega}-1-1.71\eta_e\right)\sqrt{\hat{\omega}(1+\tau)}+\left(\hat{\omega}-1\right)\sqrt{d_1(1+\hat{\omega}\tau)}\quad .$$

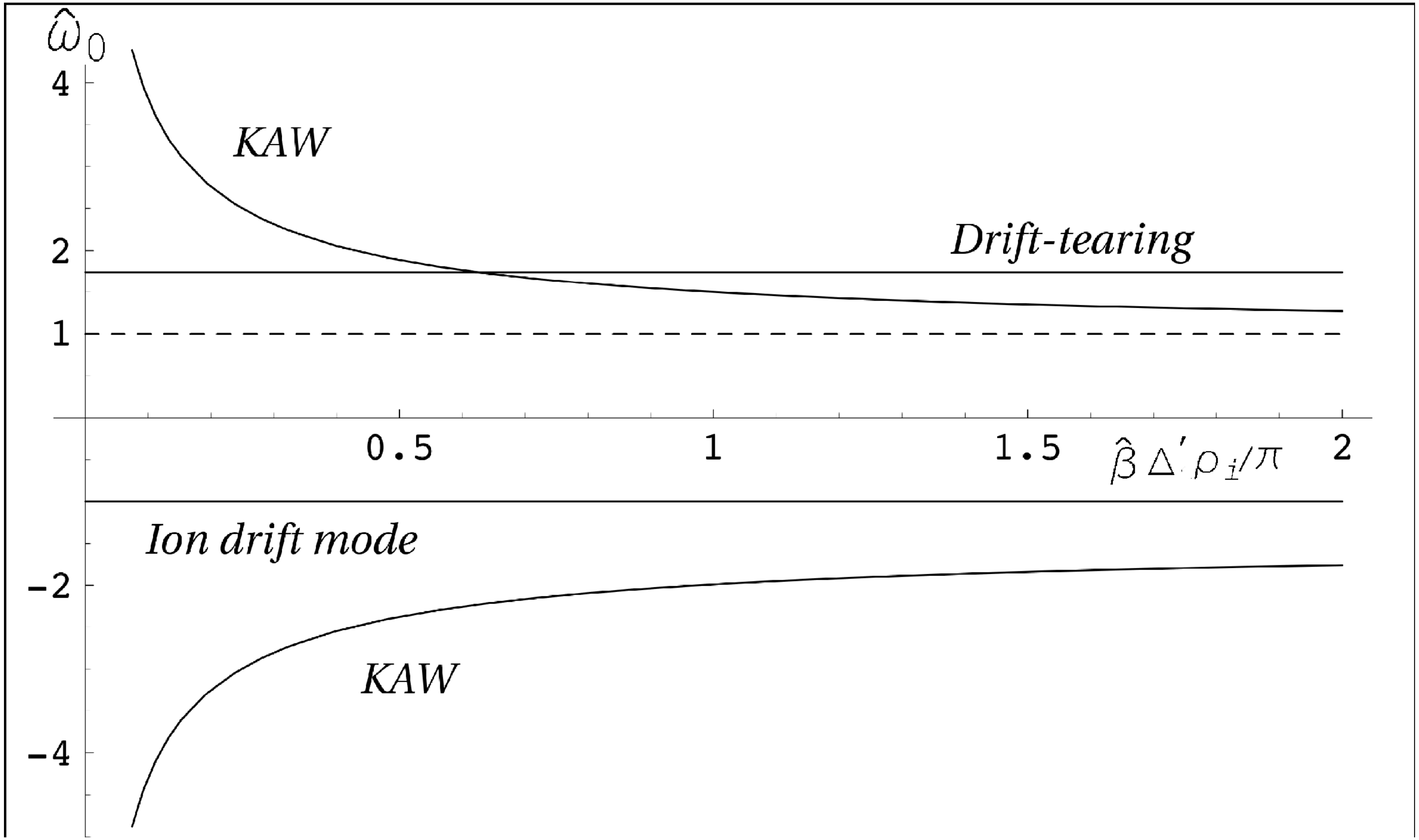
Modes

$D(\hat{\omega})=0,\ \ C\simeq 1$: Drift – tearing mode (DT)

$C(\hat{\omega})=0$: 'Kinetic Alfven waves' (KAW)

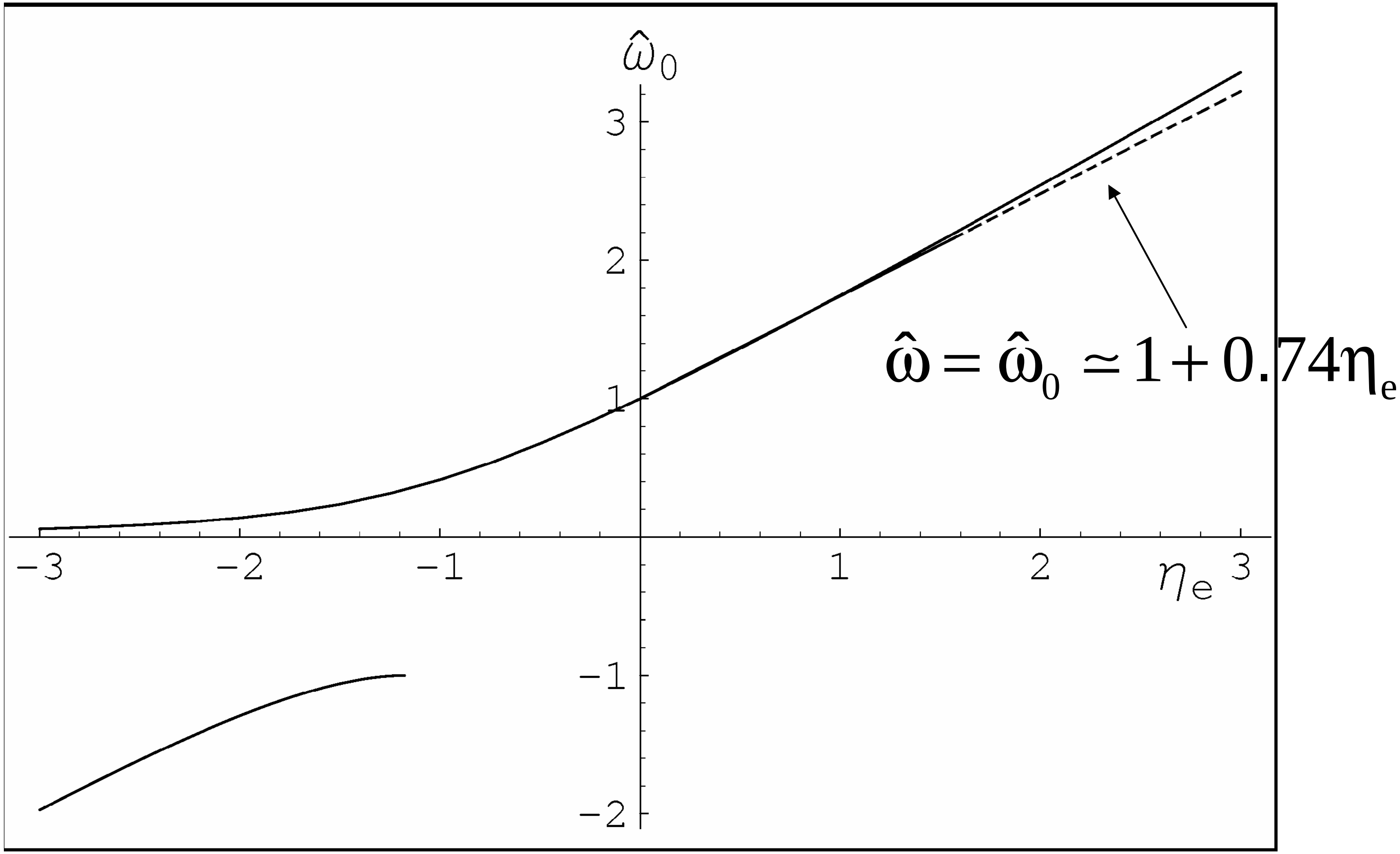
Stability controlled by small δ_0/ρ_i term

Modes



$\hat{\omega}_0$
4
2
1
-2
-4
0.5
1
1.5
2
$\hat{\beta}\,\Delta'\rho_i/\pi$
KAW
Drift-tearing
Ion drift mode
KAW

Drift-tearing Mode (1)



Drift-tearing Mode Stability (1)

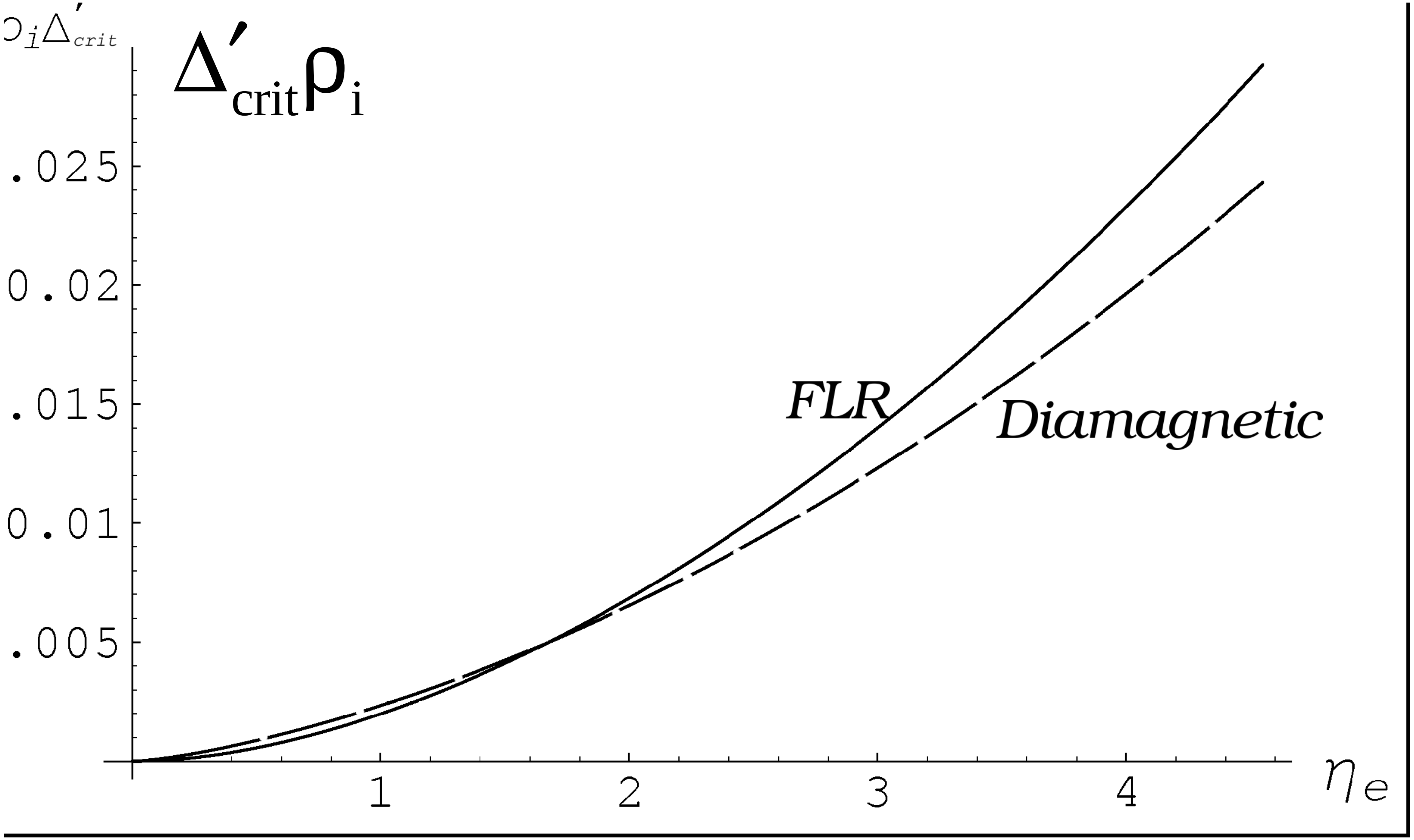
Solving for $\delta_0/\rho_i << 1$ $\qquad \hat{\gamma} \sim \frac{\delta_0 \hat{\omega}_0^{1/2}}{\pi\hat{\beta}}\left(\Delta' - \Delta'_{crit}\right)$

$$\Delta'_{crit} = \frac{\sqrt{\pi}\hat{\beta}}{\rho_i} \frac{1+\hat{\omega}_0\tau}{\hat{\omega}_0^2(1+\tau)^2}\left(1+\hat{\omega}_0\tau - \frac{\eta_i}{2}\right)(\hat{\omega}_0 - 1)^2 \,\ell n\left(\frac{\rho_i e^{-\pi/4}}{\delta_0 \hat{\omega}_0^{1/2}}\right) - \frac{\pi\hat{\beta}}{\rho_i}(\hat{\omega}_0 - 1)\,\bar{I}$$

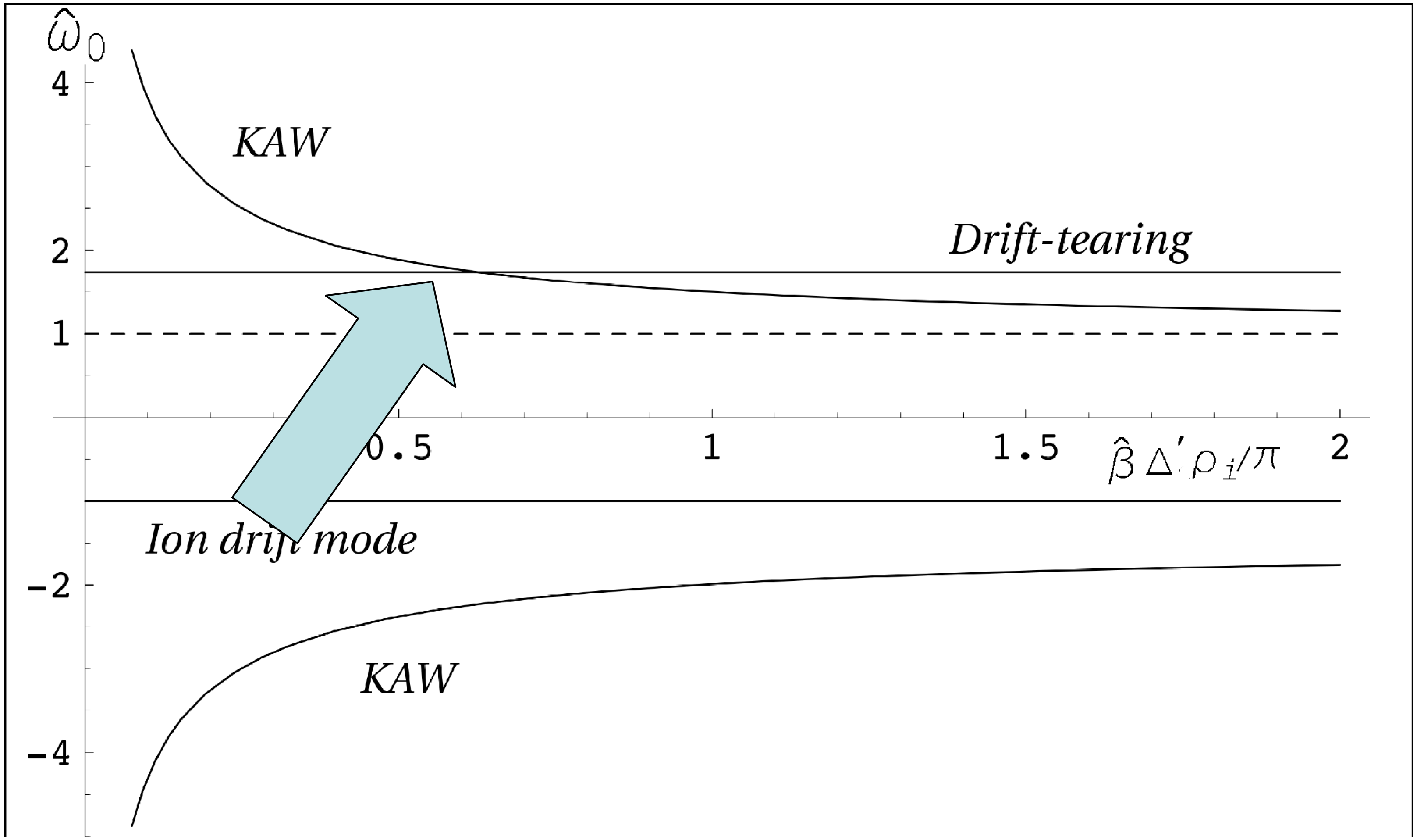
Δ'^{FLR}_{crit} $\qquad$ Δ'^{Dia}_{crit}

Cowley et al $\qquad$ Similar to Drake et al

Drift-tearing Mode Stability (2)



Coupling of DT and KAW Modes



Kinetic Alfven Wave

- When $\Delta'\rho_i\hat{\beta} \sim O(1)$ the drift-tearing mode couples strongly to the KAW and there is an interchange of stability; for higher values of $\Delta'\rho_i\beta$ the DT is stable and the KAW unstable

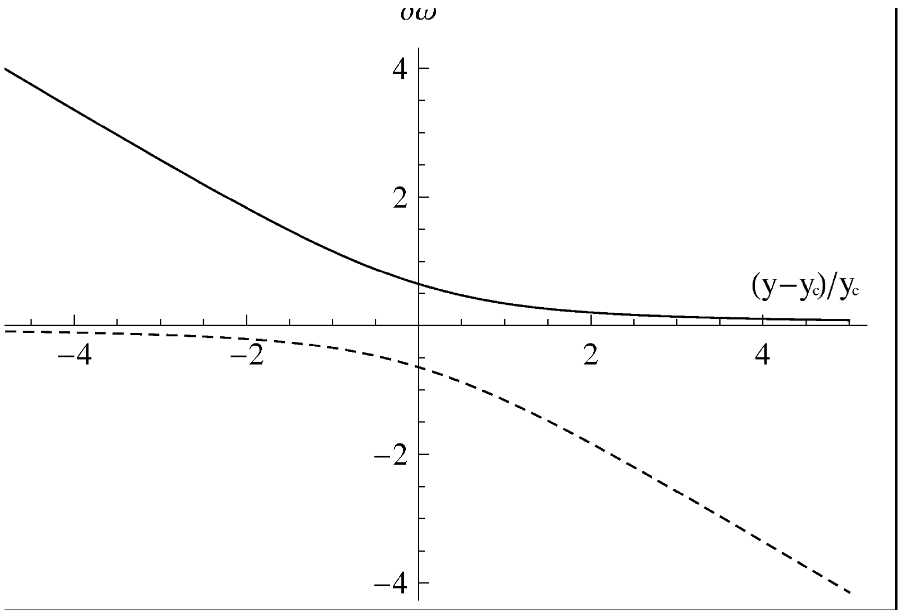
$$\hat{\omega} = 1 + \frac{2}{(1+\tau+\eta_i)}\left(\frac{2}{\Delta'\rho_i\hat{\beta}}\right)^2 - \frac{8\sqrt{2}\left(d_0 + 2\sqrt{d_1} - 1.71/(1+\tau)\right)^{1/2}}{1.71\eta_e(1+\tau+\eta_i)\hat{\beta}^3\Delta'\rho_i}\frac{\delta_0}{\rho_i}$$

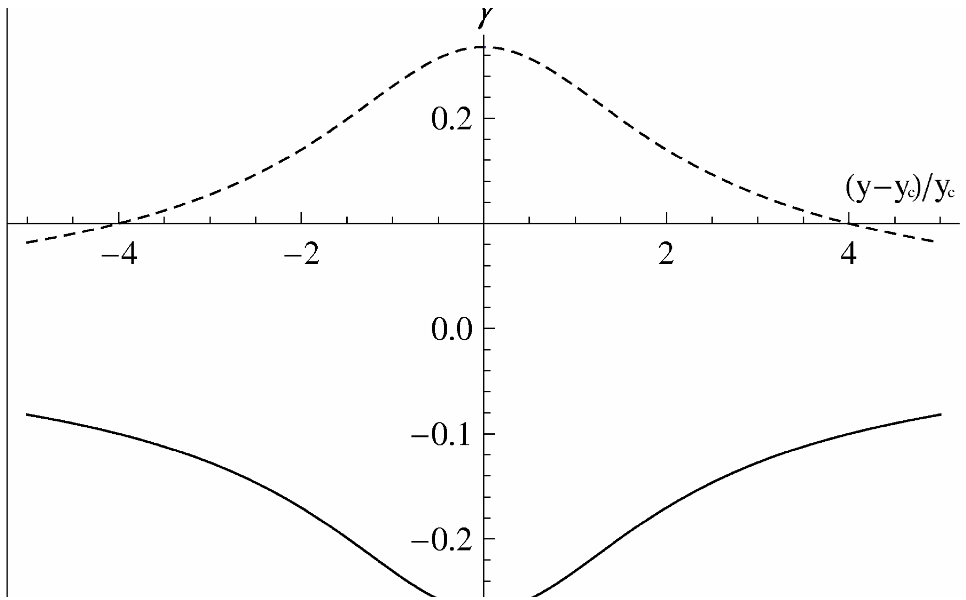
$$\frac{\gamma}{\omega_{*e}} = \frac{16}{\sqrt{2\pi}}\frac{\delta_0}{\rho_i}\frac{\left(\left(d_0 + 2\sqrt{d_1}\right) - 1.71\eta_e/(1+\tau)\right)^{1/2}}{\Delta'\rho_i\hat{\beta}^3(1.71\eta_e)(1+\tau+\eta_i)}\left(1 - \frac{1}{\sqrt{2\pi}}\frac{\delta_0}{\rho_i}\frac{\Delta'\rho_i}{\hat{\beta}}\frac{\left(\left(d_0 + 2\sqrt{d_1}\right) - 1.71\eta_e/(1+\tau)\right)^{1/2}}{(1.71\eta_e)}\right).$$

- At high $\Delta'\rho_i\hat{\beta}$, $\hat{\omega} \to 1$ and there is stability for

$$\Delta'\rho_i > 2.42\pi\frac{\rho_i}{\delta_0}\frac{\eta_e\hat{\beta}}{\left(\left(d_0 + 2\sqrt{d_1}\right) - 1.71\eta_e/(1+\tau)\right)^{1/2}}$$

DT and KAW Coupling





$$y = \Delta' \rho_i \hat{\beta}$$

Larger growth rate: $\hat{\gamma} \sim \left(\delta_0 / \rho_i\right)^{1/2}$ at cross – over

High $\hat{\beta} = (\beta_e / 2) L_s^2 / L_n^2$

At high $\hat{\beta}$, $\hat{\omega} = 1 - 0(1/\hat{\beta})$: simplifies equations

Ion Region: Asymptotic power law solution holds down to low k; Hypergeometric function solution provides transition for matching to MHD boundary condition. Since only small k involved, fluid-like solution

Electron region: Modified Bessel function solution in s shields resonant layer in presence of η_e

Matching solutions provides dispersion relation

High $\hat{\beta}$ Dispersion Relation

$$e^{i\pi\mu/2}R^{\mu} = \frac{(\mu+1/2)\Gamma^2(-\mu)}{(1/2-\mu)\Gamma^2(\mu)}\left[\frac{D-\cot\left(\pi\left(1/4+\mu/2\right)\right)}{D-\cot\left(\pi\left(1/4-\mu/2\right)\right)}\right]$$

$$D = \frac{\sqrt{2}}{\pi}\Delta'\rho_i \frac{\left(1+\tau+\eta_i\right)^{1/2}}{\left(\hat{\omega}-1\right)^{1/2}} \frac{\Gamma\left(5/4-\mu/2\right)\Gamma\left(5/4+\mu/2\right)}{\Gamma\left(3/4-\mu/2\right)\Gamma\left(3/4+\mu/2\right)}$$

$$R = \left(\frac{8d_1}{1.71\eta_e}\right)\frac{\rho_i^2}{\delta_0^2}\frac{\left(1+\tau+\eta_i\right)}{\left(1/4-\mu^2\right)} >> 1$$

Generalises result of Drake et al to finite η_i and τ

Always stable!

At large Δ', $\hat{\gamma} \sim -1/\left(R^{1/2}\Delta'\rho_i\right)$

Intermediate $\hat{\beta}$

KAW branch has $\hat{\omega} \simeq 1$ as $\hat{\beta}$ increases

Can solve electron region for special η_e

Simplify $\sigma(s)$ by considering regions in s:

For s~1 $$\frac{d^2}{ds^2}\left(1+\left(d_0-1.71\eta_e/(1+\tau)\right)s^2+d_1 s^4\right)J-1.71\eta_e\hat{\beta}J=0$$

If $$d_0-1.71\eta_e/(1+\tau)=2\sqrt{d_1} \Rightarrow \eta_e=2.53 \text{ for } \tau=1$$

can solve exactly: $$J=\left(1+u^2\right)^{-3/2}\cos\left(\sqrt{1-\lambda^2}\arctan u\right),$$

$$u=d_1^{1/4}s$$

For s>>1 solution still in terms of Bessel function solutions, but now both I and K needed for matching – reduces stabilising 'shielding'!

Critical $\hat{\beta}$ for Stability (1)

Dispersion Relation $$e^{i\pi\mu/2}R^{\mu} = \frac{(\mu+1/2)\Gamma^2(-\mu)}{(1/2-\mu)\Gamma^2(\mu)}\left[\frac{D-\cot\left(\pi\left(1/4+\mu/2\right)\right)}{D-\cot\left(\pi\left(1/4-\mu/2\right)\right)}\right]\Lambda(\mu,\lambda)$$

Shielding factor $$\Lambda(\mu,\lambda)=\left\{1-\sin\pi\mu\left[1-\frac{\sqrt{\lambda^2-1}}{\lambda}\tanh\left(\frac{\pi\sqrt{\lambda^2-1}}{2}\right)\right]\right\} \qquad \lambda^2=\frac{1.71\eta_e\hat{\beta}}{\sqrt{d_1}}$$

Eigenvalue $$\hat{\omega}\simeq 1+\frac{2}{(1+\tau+\eta_i)}\left(\frac{2}{\hat{\beta}\Delta'\rho_i}\right)^2+\frac{4\sqrt{2}\Lambda}{\hat{\beta}^{3/2}\Delta'\rho_i(1+\tau+\eta_i)}\left(\frac{1.71\eta_e}{d_1}\right)^{1/2}\frac{\delta_0}{\rho_i},$$

$$\hat{\gamma}=-4\sqrt{2}\Lambda\frac{\delta_0}{\rho_i}\left(\frac{1.71\eta_e}{d_1\hat{\beta}}\right)^{1/2}\frac{1}{\Delta'\rho_i\hat{\beta}(1+\tau+\eta_i)}\left(1+\frac{\Lambda\Delta'\rho_i\hat{\beta}^{1/2}}{2\sqrt{2}}\frac{\delta_0}{\rho_i}\left(\frac{1.71\eta_e}{d_1}\right)^{1/2}\right),$$

Implies $\Delta'\rho_i\hat{\beta} >> 1$ for consistency

Agrees with low $\hat{\beta}$ limit:

$$\hat{\gamma}=\frac{32}{\sqrt{2\pi}}\frac{\delta_0}{\rho_i}\frac{d_1^{1/4}}{(1.71\eta_e)}\frac{1}{\Delta'\rho_i\hat{\beta}^3(1+\tau+\eta_i)}\left(1-\frac{\sqrt{2}d_1^{1/4}\Delta'\rho_i}{1.71\eta_e\hat{\beta}}\frac{\delta_0}{\rho_i}\right)$$

Critical $\hat{\beta}$ for Stability (2)

$$\text{Critical } \hat{\beta}: \text{ stability if } \Lambda > 0, \text{ i.e. } \lambda = 1$$

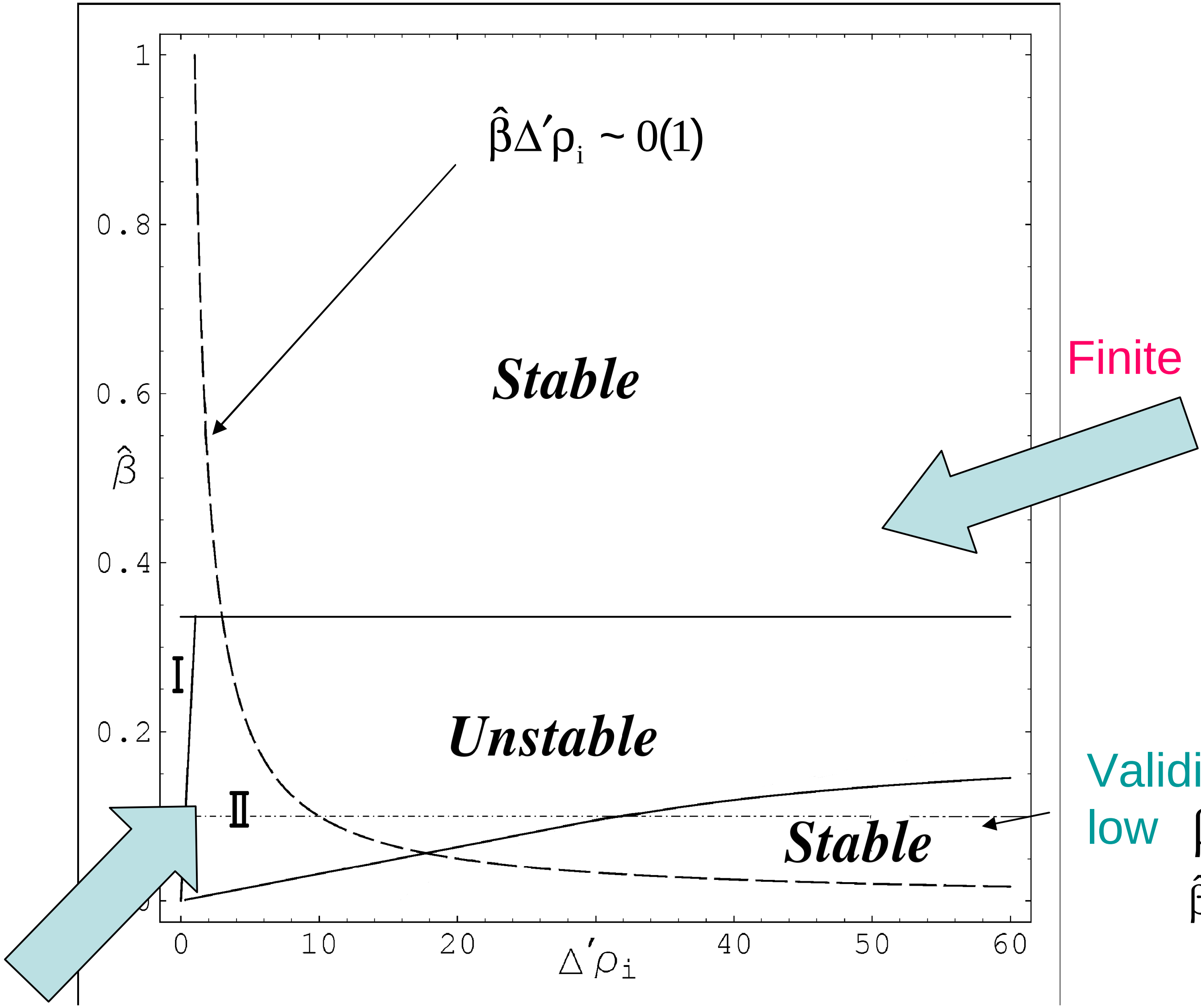
$$\Rightarrow \hat{\beta} > \hat{\beta}_{crit} = \frac{\sqrt{d_1}}{1.71\eta_e} = 0.34 \quad \text{for} \quad \tau = 1$$

$$\text{If } \Lambda < 0 \text{ still stability if}$$

$$\Delta'\delta_0 \hat{\beta}^{1/2} \sqrt{\frac{1.71\eta_e}{d_1}} > \frac{1}{(-\Lambda)}$$

$$\Rightarrow \text{for } \hat{\beta} \to \hat{\beta}_{crit}, \quad \text{stability if}: \quad \Delta'\delta_0 > \frac{\pi}{4\sqrt{2}} \sqrt{\frac{1.71\eta_e}{d_1}} \frac{1}{\hat{\beta}_c - \hat{\beta}}$$

Tearing Mode Stability Diagram



Finite $\hat{\beta}$ theory

$\hat{\beta} \sim (\delta_0 / \rho_i)^{1/2}$

Validity condition for low $\hat{\beta}$ theory when

$\hat{\beta}\Delta'\rho_i \sim 0(1)$

Cowley et al low $\hat{\beta}$ theory

Internal Kink Mode

- Internal kink mode corresponds to $C(\hat{\omega}) = 0$

- Usually introduce $\lambda_H = -\pi / \Delta' r_s,$

$\lambda_H > 0 \Rightarrow$ ideal MHD instability

Low $\hat{\beta}$ theory reproduces Pegoraro et al results

- $\lambda_H < 0 \Rightarrow$ dissipative MHD instability

Our low $\hat{\beta}$ theory in the $|\hat{\omega}| << 1$ limit reproduces Pegoraro et al results, but includes electron thermal effects

Dissipative Internal Kink Stability (1)

- Dispersion relation

$$\hat{\omega}\left\{\ell n\left(\frac{\sqrt{\pi}\hat{\omega}(1+\tau)}{(1-\eta_i/2)}\right)-i\pi-\left(1-\frac{\eta_i}{2}\right)I_2-\frac{\lambda_H}{\sqrt{\pi}\rho_i\hat{\beta}\hat{\omega}}\left(1-\frac{\eta_i}{2}\right)\right\}=\sqrt{\frac{(1+\tau)d(\eta_e)}{\pi d_1}}\left(1-\frac{\eta_i}{2}\right)\frac{\delta_0}{\rho_i\hat{\beta}^2}e^{-i\pi/4}$$

- Unstable mode in ion direction for $\lambda_H = 0$
- Stable mode in electron direction for

$$\delta_0/\rho_i \ll \hat{\beta}\left|\lambda_H\right|/\rho_i \ll \hat{\beta}^2$$

- Marginally stable criterion

$$\lambda_H^{crit}=-\frac{\pi}{\Delta'_{crit}r_s}=-\sqrt{\frac{(1+\tau)d(\eta_e)}{2d_1}}\frac{\delta_0}{\pi\hat{\beta}}\left[\ell n\left(\frac{\rho_i\hat{\beta}^2}{\delta_0}\frac{\pi}{(1+\tau)^{3/2}}\sqrt{\frac{2d_1}{d(\eta_e)}}\right)+\pi-1+\left(1-\frac{\eta_i}{2}\right)I_2(\eta_i)\right]$$

- Validity condition $\left|\hat{\omega}\right| \ll 1 \Rightarrow \hat{\beta} > (\delta_0/\rho_i)^{1/2}$

Dissipative Internal Kink Stability (2)

Can exploit $|\hat{\omega}| << 1$ to develop a finite $\hat{\beta}$ treatment

In electron region

$$\frac{d^2}{ds^2}A - \hat{\omega}\hat{\beta}\sigma(s)A = 0 \ ,$$

$$\sigma(s) = \frac{1 + 1.71\eta_e + d_1 s^2}{1 + \left(d_0 - 1 - 1.71\eta_e\right)s^2 + \left(1+\tau\right)\hat{\omega}s^4}$$

$\Rightarrow$ match simpler subregions : $s \sim 1$ and $s \sim |\hat{\omega}|^{-1/2}$

$s \sim 1$ involves elementary expansion;

$s \sim |\hat{\omega}|^{-1/2}$ involves hypergeometric solutions

Dissipative Internal Kink Stability (3)

In ion region

For $k \sim 1$ use similar expansion as before, based on $|\hat{\omega}| << 1$ rather than $\hat{\beta} << 1$, matched to λ_H

But intermediate region $k \sim |\hat{\omega}|^{-1}$ where k^{-1} tail of $G(k)/F(k) = -\hat{\omega}(1+\tau) + (1-\eta_i/2)/\sqrt{\pi}\,k$ competes before reaching large k electron region

– solution involves hypergeometric solutions

Dissipative Internal Kink Stability

- Dispersion relation follows from matching
- Critical negative λ_H for marginal stability

$$\lambda_H = -\frac{H(\mu_1)}{8}\sqrt{\frac{(1+\tau)\,d(\eta_e)}{d_1}}\,\delta_0$$

$$\times\left\{\frac{\hat{\beta}}{(1+\tau)}\frac{\pi}{\cos(\pi\mu_1)}\left(\frac{\sin(\pi\mu_1/2)}{\sin(3\pi\mu_1/2)}\right)^{1/2} - 1 - \frac{\hat{\beta}}{(1+\tau)}\left[\frac{3}{2} - k_0 - \left(1-\frac{\eta_i}{2}\right)I_2 + \ell n\left(\frac{\delta_0}{\rho_i}\frac{H(\mu_1)}{8}\sqrt{\frac{(1+\tau)d(\eta_e)}{d_1}}\right)\right]\right\}$$

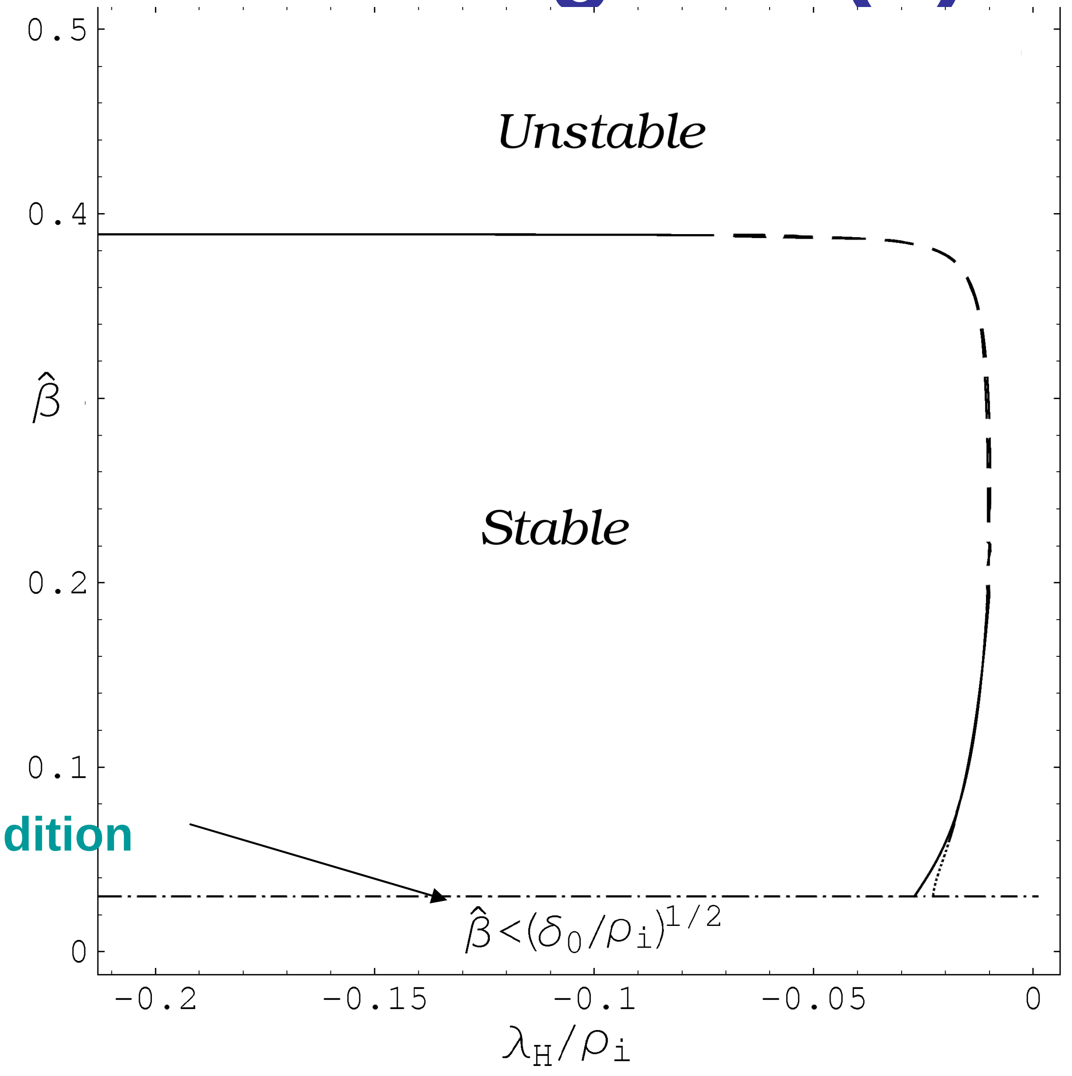
- where

$$H(\mu_1) = \left\{\frac{\Gamma^2(-\mu_1)}{\Gamma^2(\mu_1)}\frac{(\mu_1+1/2)}{(\mu_1-1/2)}\frac{-\cos(\pi\mu_1)}{\left[\cos(\pi\mu_1/2) \pm \left(\sin(\pi\mu_1/2)\sin(3\pi\mu_1/2)\right)^{1/2}\right]}\right\}^{1/2\mu_1} ;\ \mu_1 = \sqrt{1/4 + \hat{\beta}/(1+\tau)},\ d(\eta_e) = d_0 - 1 - 1.71\eta_e$$

- With negative frequency

$$\hat{\omega} = \frac{\delta_0}{\rho_i}\frac{H(\mu_1)}{8}\sqrt{\frac{d_1}{\pi d(\eta_e)}}\frac{(1-\eta_i/2)}{(1+\tau)^{3/2}}$$

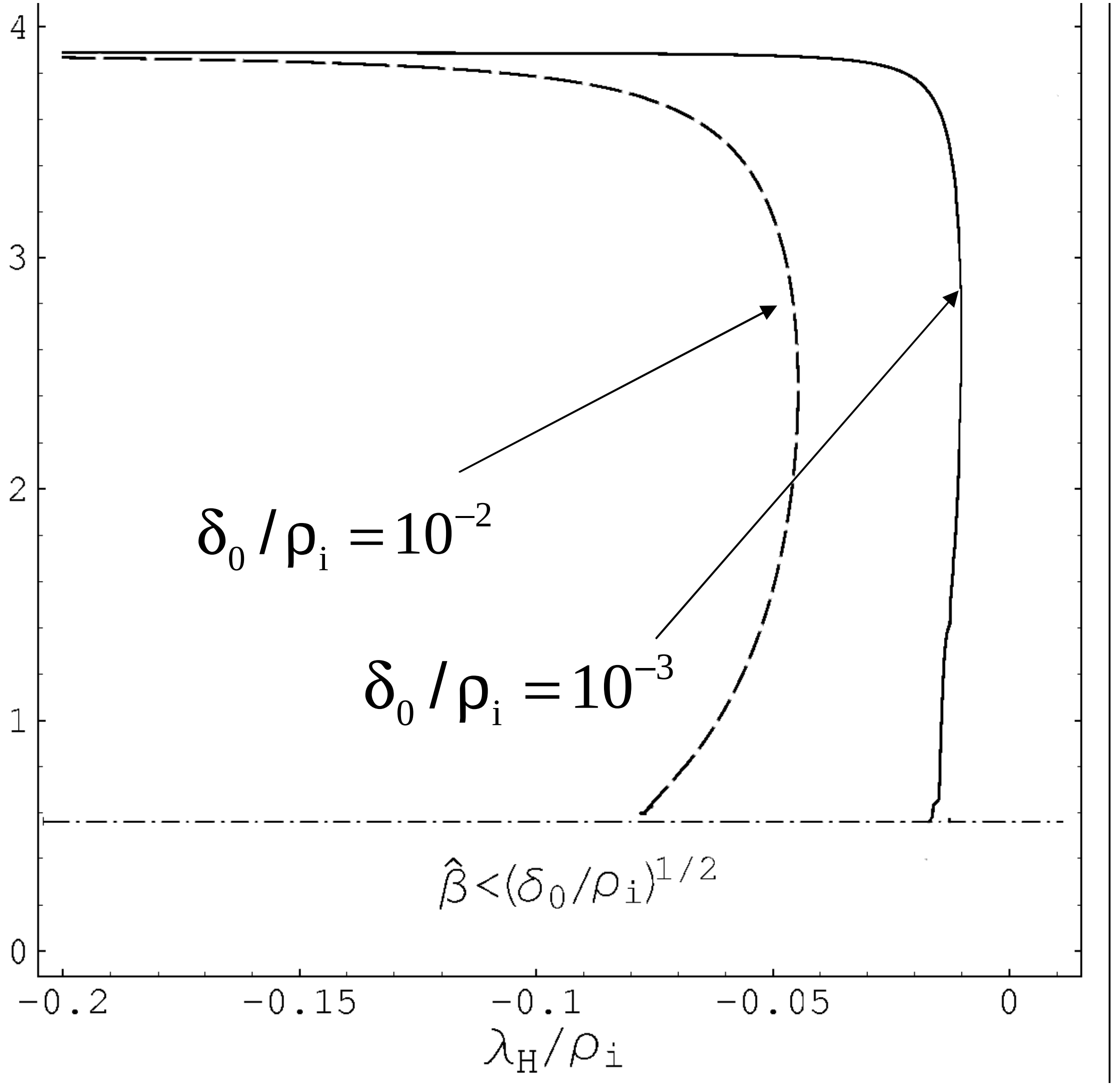
Dissipative Internal Kink Stability Diagram (1)



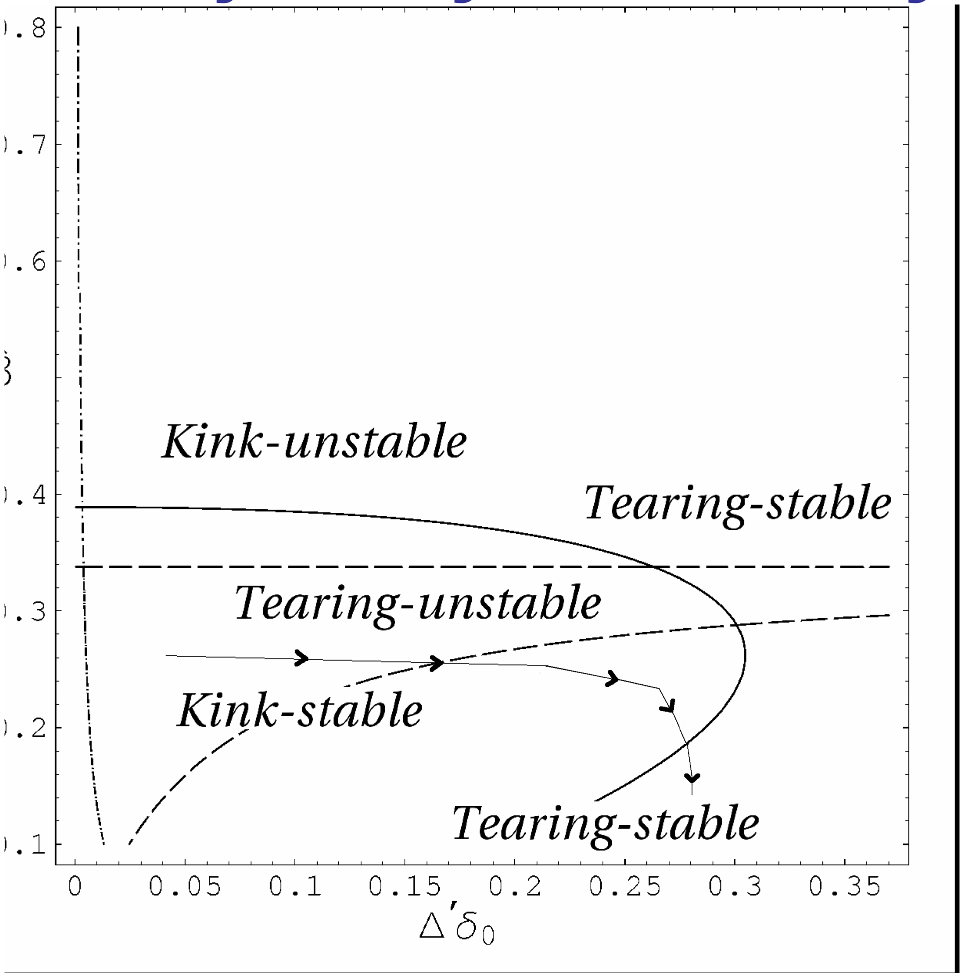
Validity condition

$$\delta_0/\rho_i = 10^{-3}$$

Dissipative Internal Kink Stability Diagram (2)



Sawtooth Trajectory in Stability Diagram



Precise expressions for trigger criteria as functions of $\beta_e, \eta_e, \eta_i, \tau, L_n / L_s \ldots$ for use in sawtooth models such as that of Boucher, Porcelli and Rosenbluth

Conclusions (1)

- Unified treatment of tearing mode and internal kink mode for large ion orbits and semi-collisional electrons using Fourier transform formulation

- For tearing mode, recovered orbit stabilisation of Cowley et al at low $\hat{\beta}$ and stabilisation due to plasma gradients (η_e) of Drake et al at high $\hat{\beta}$
  - determined critical beta for this stabilisation
  - anyway, stable for $\Delta' \geq \hat{\beta} / \delta_0$

Conclusions (2)

- Generalises theory of internal kink mode of Porcelli et al to include electron thermal effects: η_e
- Critical, large $\Delta' \; (= -\pi / \lambda_H)$ for dissipative internal kink instability at low $\hat{\beta}$; always unstable above critical $\hat{\beta}$
- Joint tearing/internal kink stability diagram of relevance to sawtooth modelling; provides precise, quantitative trigger criteria (Jim Hastie)
- Techniques can be applied to issue of reconnection driven by ITG modes (S Cowley, A Zocco)