Stellarator equilibria with non-planar magnetic axis:
The high-beta stellarator expansion

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1 Formulation

The derivation starts with the form of the MHD equilibrium equations which is more convenient for stellarator expansions:

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{B}=0\\
&\vec{\nabla}\times\vec{B}=\mu_0\vec{J}\\
&\vec{J}_\perp=\left(\vec{B}\times\vec{\nabla}p\right)/B^2\\
&\vec{B}\cdot\vec{\nabla}p=0\\
&\vec{\nabla}\cdot\vec{J}=0
\end{aligned}\tag{1}$$

In our description, a stellarator equilibrium consists of a large toroidal magnetic field, small helical and axisymmetric poloidal magnetic fields, and a small pressure. There are five independent dimensionless parameters which appear in our analysis:

$$\begin{array}{ll}
\text{Inverse aspect ratio} & \epsilon=a/R_0\\
\text{Normalized helical field amplitude} & \delta=\left|\vec{B_p}\right|/B_\phi\\
\text{Normalized plasma pressure} & \beta=2\mu_0 p/B_\phi^2\\
\text{Poloidal periodicity mode number} & l\\
\text{Toroidal periodicity mode number} & N
\end{array}\tag{2}$$

Here, B_ϕ is the toroidal field. $\vec{B_p}$ is the poloidal magnetic field, which consists of a sum of helical and axisymmetric harmonics and can be written as follows

$$\vec{B_p}\left(\vec{r}\right)=\sum_{N,l}\vec{B_p}^{(N,l)}(r,l\theta+N\phi)\tag{3}$$

With $\vec{B_p}$ written in this form, this axisymmetric component of the poloidal field corresponds $N=0$.

For our expansion, we assume that the normalized helical field amplitude is very small: $\delta\ll 1$. All the other parameters are ordered with respect to δ, as shown in Table 1 below. For comparison, the well-known Greene-Johnson ordering is also given in this table.

Note that one of the motivations for the new ordering is that it allows N to be of order unity, which is necessary to have a non-planar magnetic axis. Indeed, when N is large, the term $B_\phi\vec{e_\phi}\cdot\vec{\nabla}p$ is larger than any other term in the equation $\vec{B}\cdot\vec{\nabla}p=0$ and the pressure thus is independent of ϕ to lowest order.

Quantity	Non-planar Expansion	Greene-Johnson Expansion
ϵ	δ	δ^2
β	δ	δ^2
N	1	$1/\delta^2$
l	1	1

Table 1: Stellarator expansions

To start the derivation of the new expansion, we introduce the normalized coordinate system (x, y, ϕ), defined using the traditional $(R^{'}, \phi^{'}, Z^{'})$ cylindrical coordinate system:

$$\begin{aligned} R^{'} &= R_0 + ax \\ Z^{'} &= ay \\ \phi^{'} &= -\phi/N_0 \end{aligned} \tag{4}$$

R_0 and a are the average major and minor radii of the plasma. N_0 is the number of helical periods in the stellarator and is related to the toroidal periodicity number N by the relation $N = nN_0$, with $n \sim 1$, n integer. Additionally, by normalizing ϕ by N_0, one ensures that one helical period corresponds to $0 \le \phi \le 2\pi$. Finally, in the new variables, the gradient operator can be written

$$a\overrightarrow{\nabla}' = \overrightarrow{e_x}\frac{\partial}{\partial x} + \overrightarrow{e_y}\frac{\partial}{\partial y} + \overrightarrow{e_\phi}\frac{\epsilon N_0}{1+\epsilon x}\frac{\partial}{\partial \phi} \tag{5}$$

$$= \overrightarrow{\nabla}_\perp + \overrightarrow{\nabla}_\parallel \qquad , \qquad \overrightarrow{\nabla}_\perp \sim 1 \qquad \overrightarrow{\nabla}_\parallel \sim \epsilon \tag{6}$$

In eq.(5), we have defined $\overrightarrow{e_x} = \overrightarrow{e_{R'}}$, $\overrightarrow{e_y} = \overrightarrow{e_{Z'}}$, and $\overrightarrow{e_\phi} = -\overrightarrow{e_{\phi'}}$. Both $(R^{'}, \phi^{'}, Z^{'})$, and (x, y, ϕ) are right-handed coordinate systems.

We now introduce the basic expansion of the magnetic field, current density and pressure consistent with our new ordering.

$$\begin{array}{rll} & O(1) & O(\delta) \\ \overrightarrow{B} = & B_0\overrightarrow{e_\phi} & + (B_{\phi 1} - \epsilon x B_0)\,\overrightarrow{e_\phi} + \overrightarrow{B_{p1}} \\ \overrightarrow{J} = & & J_{\phi 1}\overrightarrow{e_\phi} + \overrightarrow{J_{p1}} \\ p = & & p_1 \end{array}$$

$B_{\phi 1}$ represents the diamagnetic correction to the toroidal field. The correction $-\epsilon x B_0$ to the vacuum toroidal field is found by writing $B_\phi = B_0 R_0/R$.

With all the quantities of interest introduced, we can plug in our expansion in the MHD equations given in eq.(1), one after the other.

2 Asymptotic expansion

1. The $\overrightarrow{\nabla}' \cdot \overrightarrow{B} = 0$ equation

Only $\overrightarrow{B_{p1}}$ comes in to lowest order, since B_0 is a constant:

$$\overrightarrow{\nabla}' \cdot \overrightarrow{B} = 0 \Rightarrow \overrightarrow{\nabla}_\perp \cdot \overrightarrow{B_{p1}} = 0 \tag{7}$$

$\overrightarrow{\nabla}_\perp$ is a two-dimensional operator, so we can introduce a stream function $A_1(x, y, \phi)$ to express $\overrightarrow{B_{p1}}$:

$$\frac{\overrightarrow{B_{p1}}}{B_0} = \overrightarrow{\nabla}_\perp A_1 \times \overrightarrow{e_\phi} \tag{8}$$

For the $\vec{\nabla}' \cdot \vec{B} = 0$ equation to give information about B_ϕ, we would need to go up to second order. This is not necessary in our analysis, as we will later see. Thus, for the moment, we keep the following form for the total field:

$$\frac{\vec{B}}{B_0} = \left(1 - \epsilon x + \frac{B_{\phi 1}}{B_0}\right)\vec{e_\phi} + \vec{\nabla}_\perp A_1 \times \vec{e_\phi} \tag{9}$$

2. The $\mu_0 \vec{J} = \vec{\nabla}' \times \vec{B}$ equation

We start by rewriting this equation as

$$\frac{\mu_0 a \vec{J}}{B_0} = \frac{1}{B_0}\left(\vec{\nabla}_\perp \times \vec{B} + \vec{\nabla}_\parallel \times \vec{B}\right) \tag{10}$$

Since the parallel gradient is down by $\epsilon \sim \delta$, only the part due to $B_0\vec{e_\phi}$ will contribute to lowest order. Using the fact that

$$\frac{\partial \vec{e_\phi}}{\partial \phi} = -\frac{\vec{e_x}}{N_0}$$

we immediately obtain

$$\vec{\nabla}_\parallel \times \frac{\vec{B}}{B_0} \approx -\epsilon \vec{e_y} \tag{11}$$

to lowest order.

Furthermore, using eq.(9), one finds that to lowest order

$$\vec{\nabla}_\perp \times \frac{\vec{B}}{B_0} = \epsilon \vec{e_y} - \vec{e_\phi} \times \vec{\nabla}_\perp \frac{B_{\phi 1}}{B_0} - \vec{e_\phi} \vec{\nabla}^2_\perp A_1 \tag{12}$$

Combining eqs.(11) and (12), one finally obtains the expression for the current density to lowest order:

$$\frac{\mu_0 a \vec{J_1}}{B_0} = -\vec{e_\phi} \times \vec{\nabla}_\perp \frac{B_{\phi 1}}{B_0} - \vec{e_\phi} \vec{\nabla}^2_\perp A_1 \tag{13}$$

3. The $\vec{J_\perp} = (\vec{B} \times \vec{\nabla}' p)/B^2$ equation

To lowest order, only the vacuum magnetic field contributes to this equation, since p is a first-order quantity. Thus we have

$$\frac{\mu_0 a \overrightarrow{J_{p1}}}{B_0} = \frac{1}{2}\vec{e_\phi} \times \vec{\nabla}_\perp \beta_1 \tag{14}$$

where $\beta_1 = 2\mu_0 p_1 / B_0^2$.

After identifying the poloidal current density in eq.(13), eq.(14) leads to the equality

$$\vec{\nabla}_\perp \left(\frac{\beta_1}{2} + \frac{B_{\phi 1}}{B_0}\right) = 0 \tag{15}$$

The solution to this equation consistent with the boundary condition that the pressure vanishes at the plasma edge is:

$$\frac{B_{\phi 1}}{B_0} = -\frac{\beta_1}{2} \tag{16}$$

Eq.(16) is the usual θ-pinch pressure balance relation.

We have now expressed the fields in terms of the two scalar quantities A_1 and β_1:

$$\frac{\vec{B}}{B_0} = \left(1 - \epsilon x - \frac{\beta_1}{2}\right)\vec{e_\phi} + \vec{\nabla}_\perp A_1 \times \vec{e_\phi}$$

$$\frac{\mu_0 a \vec{J_1}}{B_0} = \frac{1}{2}\vec{e_\phi} \times \vec{\nabla}_\perp \beta_1 - \vec{e_\phi}\vec{\nabla}^2_\perp A_1 \tag{17}$$

In the last steps of the calculations, we derive the equations which determine A_1 and β_1.

4. The $\vec{B} \cdot \vec{\nabla}' p = 0$ equation

To lowest order, this means that $\vec{B_0} \cdot \vec{\nabla}_\parallel p_1 + \overrightarrow{B_{p1}} \cdot \vec{\nabla}_\perp p_1 = 0$. Substituting the expression derived previously for $\overrightarrow{B_{p1}}$ and dividing through by $B_0^3/2\mu_0$, this becomes

$$\left(\epsilon N_0 \frac{\partial}{\partial \phi} - \vec{e_\phi} \times \vec{\nabla}_\perp A_1 \cdot \vec{\nabla}_\perp\right)\beta_1 = 0 \tag{18}$$

This is a first relation between A_1 and β_1. We need a second one to close the system, which is obtained with the $\vec{\nabla}' \cdot \vec{J} = 0$ equation.

5. The $\vec{\nabla}' \cdot \vec{J} = 0$ equation

We separate this equation in two parts, by writing

$$\vec{B} \cdot \vec{\nabla}\left(\frac{J_\parallel}{B}\right) + \vec{\nabla} \cdot \vec{J}_\perp = 0 \tag{19}$$

To lowest order, $J_\parallel/B = J_{\phi 1}/B_0$. Thus, as with the pressure equation in Section 4, the first term on the left-hand side of eq. (19) is, to lowest order,

$$\vec{B} \cdot \vec{\nabla}\left(\frac{J_\parallel}{B}\right) \approx \left(\epsilon N_0 \frac{\partial}{\partial \phi} - \vec{e_\phi} \times \vec{\nabla}_\perp A_1 \cdot \vec{\nabla}_\perp\right) J_{\phi 1} \tag{20}$$

Using the expression for $J_{\phi 1}$ in eq.(17), this can be rewritten as:

$$\vec{B} \cdot \vec{\nabla}\left(\frac{J_\parallel}{B}\right) \approx -\frac{B_0}{\mu_0 a}\left(\epsilon N_0 \frac{\partial}{\partial \phi} - \vec{e_\phi} \times \vec{\nabla}_\perp A_1 \cdot \vec{\nabla}_\perp\right)\vec{\nabla}^2_\perp A_1 \tag{21}$$

For the second term on the left-hand side of eq.(19), we use the pressure balance equation:

$$\vec{\nabla} \cdot \vec{J}_\perp = \frac{1}{a}\vec{\nabla} \cdot \left(\frac{\vec{B} \times \vec{\nabla} p}{B^2}\right) = \frac{1}{aB^2}\left[\vec{\nabla} \cdot \left(\vec{B} \times \vec{\nabla} p\right) + \frac{\vec{\nabla} p \times \vec{B} \cdot \vec{\nabla} B^2}{B^2}\right] \tag{22}$$

The first term in the square brackets vanishes because $\vec{J} \cdot \vec{\nabla} p = 0$ and $\vec{\nabla} \times \vec{\nabla} p = \vec{0}$. Thus,

$$\vec{\nabla} \cdot \vec{J}_\perp = \frac{\vec{\nabla} p \times \vec{B} \cdot \vec{\nabla} B^2}{aB^4} \tag{23}$$

From the MHD equilibrium momentum equation $(\vec{\nabla} \times \vec{B}) \times \vec{B} = \mu_0 \vec{\nabla} p$, we know that we have

$$\vec{B} \cdot \vec{\nabla} B \vec{b} + B^2 \vec{\kappa} - \mu_0 \vec{\nabla} p = \frac{\vec{\nabla} B^2}{2} \tag{24}$$

with $\vec{b} = \vec{B}/B$, and $\vec{\kappa} = \vec{b} \cdot \vec{\nabla}\vec{b}$. Taking the dot product of this equation with $\vec{\nabla} p \times \vec{B}$, we find

$$2\vec{\nabla} p \times \vec{B} \cdot \vec{\kappa} = \frac{\vec{\nabla} p \times \vec{B} \cdot \vec{\nabla} B^2}{B^2} \tag{25}$$

so that eq. (23) becomes

$$\overrightarrow{\nabla}\cdot\overrightarrow{J}_{\perp} = 2\frac{\overrightarrow{\nabla}p\times\overrightarrow{B}\cdot\overrightarrow{\kappa}}{aB^2} \tag{26}$$

To lowest order, the curvature is that of the vacuum toroidal field:

$$\overrightarrow{\kappa}' \approx -\frac{1}{R_0}\overrightarrow{e_{R'}} \Rightarrow \overrightarrow{\kappa} \approx -\epsilon\overrightarrow{e_x} \tag{27}$$

Thus, to lowest order, we have

$$\overrightarrow{\nabla}\cdot\overrightarrow{J}_{\perp 1} = -2\epsilon\frac{\overrightarrow{\nabla}p_1\cdot\overrightarrow{e_y}}{aB_0} \tag{28}$$

Collecting all the pieces, the $\overrightarrow{\nabla}'\cdot\overrightarrow{J}=0$ equation can thus be rewritten as:

$$\left(\epsilon N_0\frac{\partial}{\partial\phi} - \overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp A_1\cdot\overrightarrow{\nabla}_\perp\right)\overrightarrow{\nabla}^2_\perp A_1 = -\epsilon\overrightarrow{\nabla}\beta_1\cdot\overrightarrow{e_y} \tag{29}$$

6. Summary

The basic model describing the new stellarator expansion is given by two partial differential equations for the two scalar quantities A_1 and β_1, eq. (18) and eq.(29), repeated here for convenience

$$\left(\epsilon N_0\frac{\partial}{\partial\phi} - \overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp A_1\cdot\overrightarrow{\nabla}_\perp\right)\beta_1 = 0 \tag{30}$$

$$\left(\epsilon N_0\frac{\partial}{\partial\phi} - \overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp A_1\cdot\overrightarrow{\nabla}_\perp\right)\overrightarrow{\nabla}^2_\perp A_1 = -\epsilon\overrightarrow{\nabla}\beta_1\cdot\overrightarrow{e_y} \tag{31}$$

We renormalize the two unknowns A_1 and β_1 as

$$A = -\frac{A_1}{\epsilon N_0} \sim 1 \qquad\qquad \beta = \frac{\beta_1}{\epsilon N_0^2} \tag{32}$$

so that eqs.(30) and (31) can be rewritten in terms of A and β in an even more concise form:

$$\boxed{\begin{aligned}&\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp A\cdot\overrightarrow{\nabla}_\perp\right)\beta = 0\\ &\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp A\cdot\overrightarrow{\nabla}_\perp\right)\overrightarrow{\nabla}^2_\perp A = \overrightarrow{e_y}\cdot\overrightarrow{\nabla}_\perp\beta\end{aligned}} \tag{33}$$

Once the coupled equations in eq.(33) are solved for A and β, we immediately know A_1 and β_1. The magnetic field and current density are then readily calculated from eq.(17), which we repeat here:

$$\begin{aligned}\frac{\overrightarrow{B}}{B_0} &= \left(1-\epsilon x-\frac{\beta_1}{2}\right)\overrightarrow{e_\phi} + \overrightarrow{\nabla}_\perp A_1\times\overrightarrow{e_\phi}\\ \frac{\mu_0 a\overrightarrow{J_1}}{B_0} &= \frac{1}{2}\overrightarrow{e_\phi}\times\overrightarrow{\nabla}_\perp\beta_1 - \overrightarrow{e_\phi}\overrightarrow{\nabla}^2_\perp A_1\end{aligned} \tag{34}$$

The stellarator equilibrium is then fully determined.

The numerical methods which we use to solve eq.(33) for either the fixed boundary problem or the free boundary problem will be the topic of another writeup.