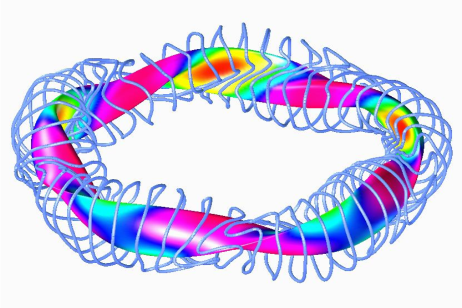
MHD equilibrium calculations for stellarators



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MAGNETIC FIELD LINE HAMILTONIAN

- Consider a general toroidal coordinate system (r, θ, ϕ)
  θ poloidal angle, ϕ toroidal angle

- Using the gauge freedom for the vector potential $\overrightarrow{A}$, it is always possible to construct Ψ and χ such that

$$\overrightarrow{B} = \overrightarrow{\nabla}\phi \times \overrightarrow{\nabla}\Psi + \overrightarrow{\nabla}\chi \times \overrightarrow{\nabla}\theta$$

  Ψ and χ not necessarily poloidal and toroidal flux

- In (χ, θ, ϕ) coordinates, the field line trajectories are given by

$$\frac{d\chi}{d\phi} = -\frac{\partial \Psi}{\partial \theta}$$

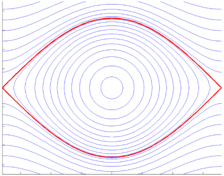
$$\frac{d\theta}{d\phi} = \frac{\partial \Psi}{\partial \chi}$$

Hamilton's equations with $H \leftrightarrow \Psi$, $x \leftrightarrow \theta$ and $p \leftrightarrow \chi$

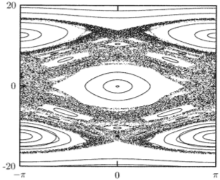
The "time" ϕ is periodic, so this is a "1.5 degree of freedom Hamiltonian"

FUNDAMENTAL COMPLICATIONS IN 3D

Pendulum

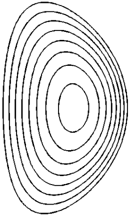


Energy conserved

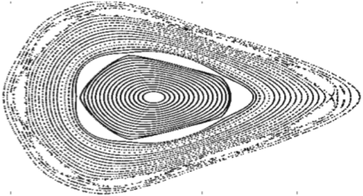


Energy not conserved

Magnetic field surfaces



Axisymmetry



No Symmetry

THE WORK HORSE: VMEC (1)

- Based on an idea by Kruskal & Kulsrud; Numerical scheme first proposed by F. Bauer, O. Betancourt, and P. Garabedian; Implemented and developed as VMEC by S. Hirschman et al.
- Minimization of the plasma potential energy

$$E = \int_{\Omega} \left(\frac{B^2}{2\mu_0} + \frac{\rho^\gamma}{\gamma - 1} \right) dV = E_B + E_{int}$$

- Only allowed virtual displacements $\overrightarrow{\xi}$ satisfy

$$\delta\rho = -\overrightarrow{\nabla} \cdot \rho \overrightarrow{\xi} \qquad \delta \overrightarrow{B} = \overrightarrow{\nabla} \times (\overrightarrow{\xi} \times \overrightarrow{B})$$

$\Rightarrow$ Assuming there is a nested family of flux surfaces $s = cst$, define

$$\iint_{s \leq s_0} \overrightarrow{B} \cdot \overrightarrow{dS} = F_T(s_0) \quad \iint_{s \leq s_0} \overrightarrow{B} \cdot \overrightarrow{dS} = F_P(s_0) \quad \iiint_{s \leq s_0} \rho d^3x = M(s_0)$$

F_T, F_P and M are given functions of s, held fixed during variation

The work horse: VMEC (2)

Turn constrained minimization into unconstrained minimization:

- Flux and solenoidal constraints incorporated by writing

$$\vec{B} = \vec{\nabla} s \times \vec{\nabla} G \quad \text{with} \quad G = -F'_T(s)u + F'_P(s)v + \lambda(s,u,v)$$

and with $\lambda(s,u,v)$ a *periodic* function of u and v

- Mass constraint obtained from Hölder's inequality

$$M(s) = \iiint \rho \frac{dsdvdu}{\mathcal{J}} \ , \ \ M'(s) \equiv m(s) = \iint \frac{\rho}{\mathcal{J}} dvdu \ , \ \mathcal{J} = \frac{\partial(s,u,v)}{\partial(x,y,z)}$$

$$m(s) = \iint \frac{\rho}{\mathcal{J}^{\frac{1}{\gamma}}} \frac{1}{\mathcal{J}^{\frac{\gamma-1}{\gamma}}} dvdu \leq \left(\iint \frac{\rho^\gamma}{\mathcal{J}} dvdu \right)^{\frac{1}{\gamma}} \left(\iint \frac{dvdu}{\mathcal{J}} \right)^{\frac{\gamma-1}{\gamma}}$$

Minimum of E_{int} when $\rho = \rho(s) = m(s)/(\iint dudv/\mathcal{J}) \ \Rightarrow p = p(s)$

The work horse: VMEC (3)

Choose v s.t. $v = \phi$, ϕ usual toroidal angle

$$E = \iiint \frac{D_1^2 + D_2^2 + D_3^2}{2\mu_0} \frac{dsdvdu}{D} + \frac{1}{\gamma - 1} \int \frac{m(s)^\gamma}{\left(\iint Ddvdu\right)^{\gamma-1}} ds$$

$$D \equiv \frac{\partial(R,Z)}{\partial(s,u)} \ , \quad D_1 \equiv \frac{\partial(\psi,R)}{\partial(u,v)} \ , \quad D_2 \equiv R\psi_u \ , \quad D_3 \equiv \frac{\partial(\psi,Z)}{\partial(u,v)}$$

Write

$$R = \sum R_{mn}(s)\cos(mu + nv)$$

$$Z = \sum Z_{mn}(s)\cos(mu + nv)$$

Find R_{mn} and Z_{mn} corresponding to minimum of E using the Steepest Descent Method

SIESTA can be used in conjunction with VMEC to compute equilibria with islands

A DIRECT SOLVER WITH ISLANDS: PIES

- Solves the MHD equilibrium equations iteratively

$$1.\ \vec{B} \cdot \vec{\nabla} p = 0$$

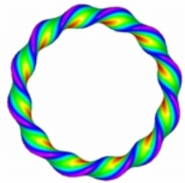
$$2.\ \vec{J_\perp} = \frac{\vec{B} \times \vec{\nabla} p}{B^2}$$

$$3.\ \vec{B} \cdot \vec{\nabla} \left(\frac{J_\parallel}{B} \right) = -\vec{\nabla} \cdot \vec{J_\perp}$$

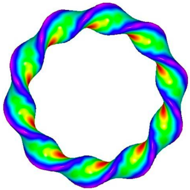
$$4.\ \vec{\nabla} \times \vec{B} = \vec{J}$$

- Eq.(1) is used to compute magnetic coordinates
  This step is where most of the action is – complications with rational surfaces, stochasticity…
- Eq.(2)-(3) are then solved analytically, mode by mode, in magnetic coordinates
- Eq.(4) is then solved for the updated magnetic field, in lab (toroidal) coordinates

ASYMPTOTIC APPROACHES: GREENE-JOHNSON (1961)



ATF



LHD

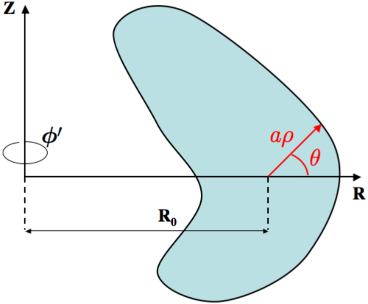
Assumes strong toroidal guide field B_0

Expansion based on the small parameter $\delta \equiv B_{hel}/B_0 \ll 1$

Parameter	Symbol	Scaling
Beta	$\beta \sim \beta_t$	δ^2
Inverse aspect ratio	$a/R_0 \equiv \epsilon$	δ^2
Number of helical periods	N_0	$1/\delta^2$
Poloidal helicity	l	1
Rotational transform	$\iota/2\pi$	1

NOTE: $\beta \sim \epsilon$

Consequences of the G-J ordering



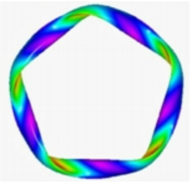
$$R' = R_0 + a\rho\cos\theta$$
$$Z' = a\rho\sin\theta$$
$$\phi' = -\frac{\phi}{N_0}$$

$$a\vec{\nabla}' \equiv \vec{\nabla} = \vec{\nabla}_\perp + \vec{\nabla}_\parallel$$
$$\vec{\nabla}_\perp = \frac{\partial}{\partial\rho}\vec{e_\rho} + \vec{e_\theta}\frac{1}{\rho}\frac{\partial}{\partial\theta} \sim 1$$
$$\vec{\nabla}_\parallel = \vec{e_\phi}\frac{\epsilon N_0}{1+\epsilon\rho cos\theta}\frac{\partial}{\partial\phi} \sim 1$$

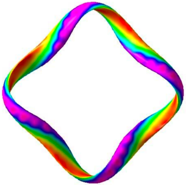
To lowest order, $\vec{B}\cdot\vec{\nabla}\psi = 0 \quad \Rightarrow \quad \epsilon N_0 B_0 \frac{\partial\psi}{\partial\phi} = 0 \quad \Rightarrow \quad \psi = \psi(\rho,\theta)$

Greene-Johnson stellarators are perturbed tokamaks!

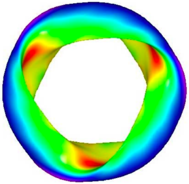
A new ordering



W7-X



HSX



NCSX

Same expansion parameter as Greene-Johnson: $\delta \equiv B_{hel}/B_0 \ll 1$

Parameter	Symbol	G-J Scaling	New scaling
Beta	$\beta \sim \beta_t$	δ^2	δ
Inverse aspect ratio	ϵ	δ^2	δ
Number of helical periods	N_0	$1/\delta^2$	1
Poloidal helicity	l	1	1
Rotational transform	$\iota/2\pi$	1	1

NOTE: In both expansions, $\beta \sim \epsilon$

Consequence of the new ordering

To lowest order, $\overrightarrow{B} \cdot \overrightarrow{\nabla} \psi = 0 \Rightarrow \left(\epsilon N_0 B_0 \frac{\partial}{\partial \phi} + \overrightarrow{B_p} \cdot \overrightarrow{\nabla}_\perp \right) \psi = 0$

Equilibria can have a non-planar magnetic axis

Our expansion:

$$
\begin{array}{rlr}
 & O(1) & O(\delta) \\
\overrightarrow{B} = & B_0 \overrightarrow{e_\phi} & + \left(B_{\phi 1} - \epsilon x B_0 \right) \overrightarrow{e_\phi} + \overrightarrow{B_{p1}} \\
\overrightarrow{J} = & & J_{\phi 1} \overrightarrow{e_\phi} + \overrightarrow{J_{p1}} \\
p = & & p_1(\rho, \theta, \phi)
\end{array}
$$

Greene-Johnson expansion:

$$
\begin{array}{rllll}
 & O(1) & O(\delta) & O(\delta^2) & O(\delta^3) \\
\overrightarrow{B} = & B_0 \overrightarrow{e_\phi} & + \overrightarrow{B_1}(\rho, \theta, \phi) & + \left(B_{\phi 2} - \epsilon x B_0 \right) \overrightarrow{e_\phi} + \overrightarrow{B_{p2}} & + \overrightarrow{B_3}(\rho, \theta, \phi) \\
\overrightarrow{J} = & & & \overrightarrow{J_2}(\rho, \theta) & + \overrightarrow{J_3}(\rho, \theta, \phi) \\
p = & & & p_2(\rho, \theta) & + p_3(\rho, \theta, \phi)
\end{array}
$$

The new expansion (1)

- **The $\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$ equation**
  To lowest order, this is $\overrightarrow{\nabla}_\perp \cdot \overrightarrow{B_{p1}} = 0$. Introduce $A_1(\rho, \theta, \phi)$ s.t.

$$\frac{\overrightarrow{B}}{B_0} = \left(1 - \epsilon x + \frac{B_{\phi 1}}{B_0}\right) \overrightarrow{e_\phi} + \overrightarrow{\nabla}_\perp A_1 \times \overrightarrow{e_\phi}$$

- **The $\mu_0 \overrightarrow{J} = \overrightarrow{\nabla} \times \overrightarrow{B}$ equation**

$$\frac{\mu_0 a \overrightarrow{J_1}}{B_0} = -\overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp \frac{B_{\phi 1}}{B_0} - \overrightarrow{e_\phi} \overrightarrow{\nabla}_\perp^2 A_1$$

- **The $\overrightarrow{J_\perp} = (\overrightarrow{B} \times \overrightarrow{\nabla} p)/B^2$ equation**

$$\overrightarrow{\nabla}_\perp \left(\frac{\beta_1}{2} + \frac{B_{\phi 1}}{B_0}\right) = 0 \Rightarrow \frac{B_{\phi 1}}{B_0} = -\frac{\beta_1}{2} \qquad \beta_1 = \frac{2\mu_0 p_1}{B_0^2}$$

  θ-pinch pressure balance relation

THE NEW EXPANSION (2)

- **The $\overrightarrow{B} \cdot \overrightarrow{\nabla} p = 0$ equation**

$$\left(\epsilon N_0 \frac{\partial}{\partial \phi} - \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A_1 \cdot \overrightarrow{\nabla}_\perp \right) \beta_1 = 0$$

- **The $\overrightarrow{\nabla} \cdot \overrightarrow{J} = 0$ equation**

$$\left(\epsilon N_0 \frac{\partial}{\partial \phi} - \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A_1 \cdot \overrightarrow{\nabla}_\perp \right) \overrightarrow{\nabla}^2_\perp A_1 = -\epsilon \overrightarrow{e_Z} \cdot \overrightarrow{\nabla} \beta_1$$

$$\left(\frac{\partial}{\partial \phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp \right) \beta = 0$$

$$\left(\frac{\partial}{\partial \phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp \right) \overrightarrow{\nabla}^2_\perp A = \overrightarrow{e_Z} \cdot \overrightarrow{\nabla} \beta$$

$$A = -\frac{A_1}{\epsilon N_0} \qquad \beta = \frac{1}{\epsilon N_0^2} \frac{2\mu_0 p_1}{B_0^2} = \frac{\beta_1}{\epsilon N_0^2}$$

The new expansion: Results

$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp\right)\beta = 0$$

$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp\right)J_\phi = \overrightarrow{e_Z} \cdot \overrightarrow{\nabla}\beta$$

$$\overrightarrow{\nabla}_\perp^2 A = J_\phi$$

$$A = -\frac{A_1}{\epsilon N_0} \qquad \beta = \frac{1}{\epsilon N_0^2}\frac{2\mu_0 p_1}{B_0^2} = \frac{\beta_1}{\epsilon N_0^2} \qquad J_\phi = \frac{\mu_0 a J_{1\phi}}{\epsilon N_0 B_0}$$

$$\frac{\overrightarrow{B}}{B_0} = \left(1 - \epsilon x - \frac{\beta_1}{2}\right)\overrightarrow{e_\phi} + \overrightarrow{\nabla}_\perp A_1 \times \overrightarrow{e_\phi} \qquad \frac{\mu_0 a \overrightarrow{J_1}}{B_0} = \frac{1}{2}\overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp \beta_1 - \overrightarrow{e_\phi}\overrightarrow{\nabla}_\perp^2 A_1$$

- Equations agree with the Greene-Johnson expansion in the right limit
- Equations are in a convenient form for iterations

ITERATION STEPS

Start with given $A^{(0)}$

for $\mathbf{k = 1, 2, \ldots}$

$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A^{(k-1)} \cdot \overrightarrow{\nabla}_\perp\right)\beta^{(k)} = 0$$

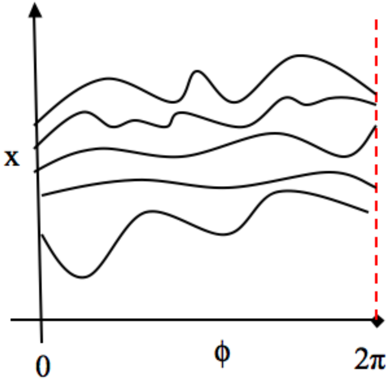
$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A^{(k-1)} \cdot \overrightarrow{\nabla}_\perp\right)J_\phi^{(k)} = \overrightarrow{e_Z} \cdot \overrightarrow{\nabla}\beta^{(k)}$$

$$\overrightarrow{\nabla}_\perp^2 A^{(k)} = J_\phi^{(k)}$$

until a stopping criterion holds

Hyperbolic equations and Initial conditions

$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp\right)\beta = 0$$



- How to choose the I.C. at $\phi = 0$ such that β is periodic in ϕ?

Dick and Jane calculate 3D equilibria

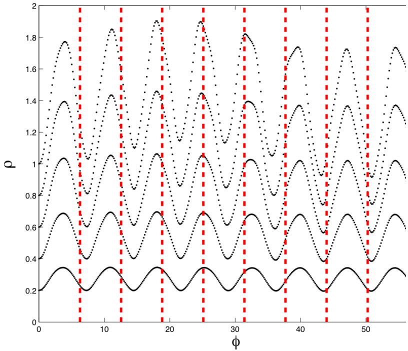
$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp\right)\beta = 0$$

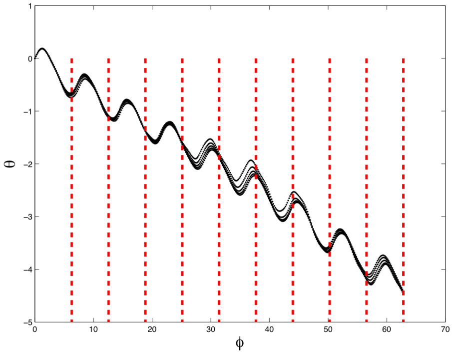
- Given A, the field line trajectories are given by the characteristics:

$$\begin{cases} \frac{d\rho}{d\phi} = -\frac{1}{\rho}\frac{\partial A}{\partial\theta} \\ \\ \frac{d\theta}{d\phi} = \frac{1}{\rho}\frac{\partial A}{\partial\rho} \end{cases}$$

- Given a starting point $(\rho_0, \theta_0, \phi_0)$, we can easily integrate these equations $\Rightarrow \rho(\phi), \theta(\phi)$
- Integrate for many turns (very large ϕ) to sample the whole magnetic surface
- If starting point is always of the form $(\rho_0, 0, 0)$, ρ_0 is a unique label for each magnetic surface

Extracting information from Characteristics

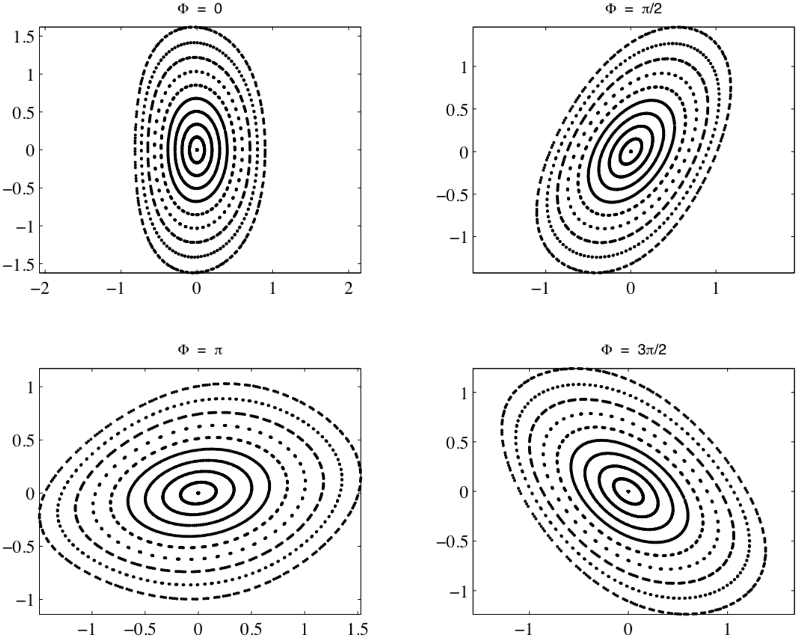


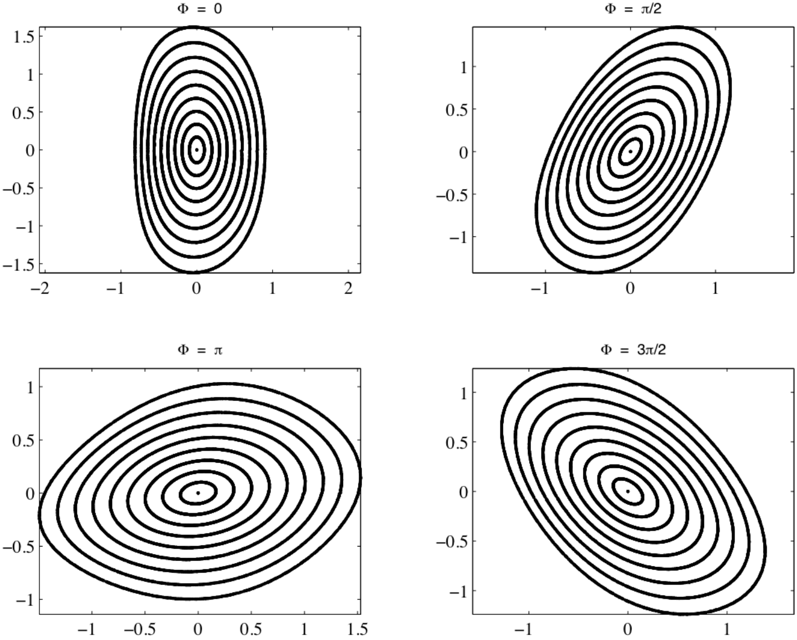


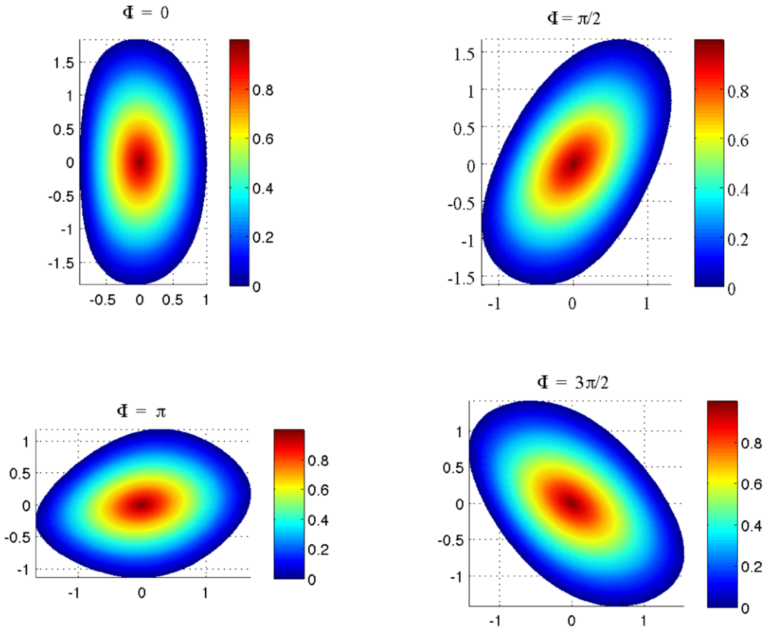
Through this process, we have

$$\rho = \rho(\rho_0, \theta, \phi)$$

$$\vec{e_Z} \cdot \vec{\nabla}_{\perp}\beta = \frac{d\beta}{d\rho_0}\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\left[\sin\theta - \frac{\cos\theta}{\rho}\frac{\partial\rho}{\partial\theta}\right]$$


Φ = 0
Φ = π/2
Φ = π
Φ = 3π/2


Φ = 0
Φ = π/2
Φ = π
Φ = 3π/2


Φ = 0
1.5
1
0.5
0
-0.5
-1
-1.5
-0.5 0 0.5 1
0.8
0.6
0.4
0.2
0
Φ = π/2
1.5
1
0.5
0
-0.5
-1
-1.5
-1 0 1
0.8
0.6
0.4
0.2
0
Φ = π
1
0.5
0
-0.5
-1
-1 0 1
0.8
0.6
0.4
0.2
0
Φ = 3π/2
1
0.5
0
-0.5
-1
-1 0 1
0.8
0.6
0.4
0.2
0

Closing the iteration loop

$$\left(\frac{\partial}{\partial\phi} + \overrightarrow{e_\phi} \times \overrightarrow{\nabla}_\perp A \cdot \overrightarrow{\nabla}_\perp\right) J_\phi = \overrightarrow{e_Z} \cdot \overrightarrow{\nabla}\beta$$

- We already know the characteristics for this equation, and the RHS of this equation on the characteristics
  $\Rightarrow$ Solving for J_ϕ is a straightforward 1D (Lagrangian) integral
- We can then evaluate $\partial A/\partial\rho$ and $\partial A/\partial\theta$ required for the next iteration:

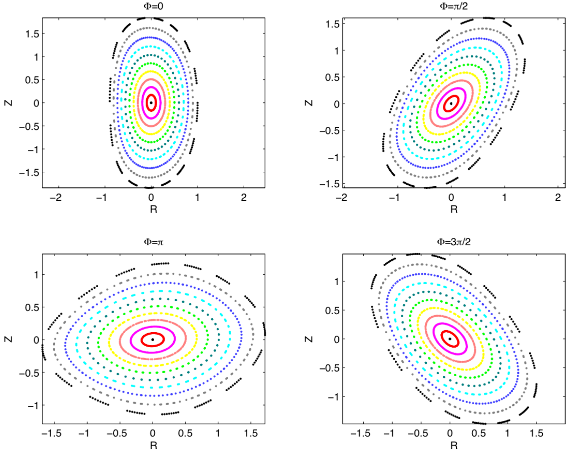
$$\frac{\partial A}{\partial\rho}(\rho,\theta,\phi) = \frac{1}{2\pi}\iint d\rho' d\theta' \rho' \left[\frac{\rho - \rho'\cos(\theta-\theta')}{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}\right] J_\phi(\rho',\theta',\phi)$$

$$\frac{\partial A}{\partial\theta}(\rho,\theta,\phi) = \frac{1}{2\pi}\iint d\rho' d\theta' \frac{\rho\rho'^2\cos(\theta-\theta')}{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')} J_\phi(\rho',\theta',\phi)$$

PRELIMINARY RESULTS – LAST ISSUE?

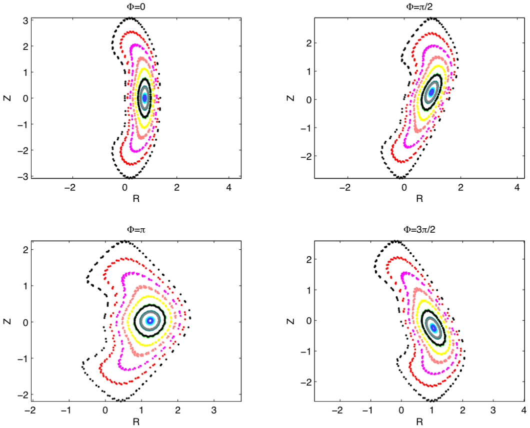
Start with vacuum field (in this case, dominant $l = 2$ component, some $l = 3$ and $l = 4$)

$$\beta(\rho_0) = 0.01(1 - \rho_0^2)^2$$



Preliminary results – Last Issue?

Flux surfaces after 1st iteration



Need to calculate the new location of the magnetic axis to evaluate new β profile; is there a fast numerical scheme to do that?

References

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