Phase-space turbulence in 2D magnetized plasma using gyrokinetics

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Collaborators:

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G. G. Howes	University of Iowa

Contents

Tatsuno, Dorland, Schekochihin *et al*., Phys. Rev. Lett. **103**, 015003 (2009);
Tatsuno, Barnes, Cowley *et al*., J. Plasma & Fusion Res. SER. **9**, 509 (2010);
Numata, Howes, Tatsuno *et al*., J. Comput. Phys. **229**, 9347 (2010).

Navier-Stokes turbulence

- Navier-Stokes eqn. ($\boldsymbol{u}$: velocity, p: pressure, ν: viscosity [small])

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

- Assuming periodic box, energy evolution yields

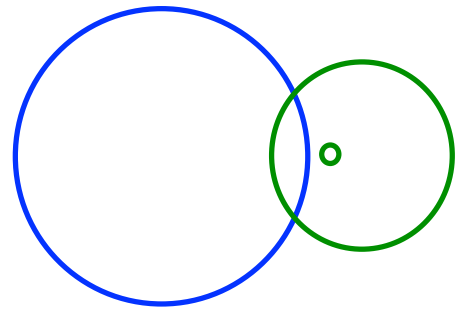
$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{r} = -\nu \int |\nabla \times \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{r}$$

- Inviscid invariant = energy: $E = \frac{1}{2} \int |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{r}$

- Due to higher-order derivative, energy can dissipate efficiently if it is transferred to small scales.

Kolmogorov's scaling law

- Assumptions
  - Isotropy
  - Local-in-scale interactions
  - Existence of inertial range (no dissipation)

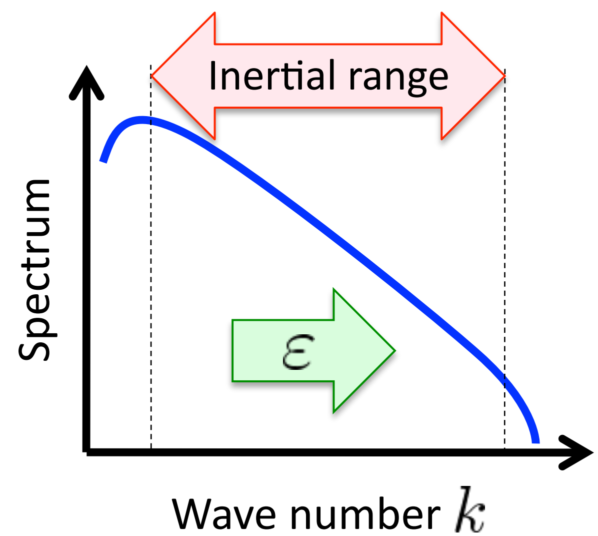


Similar-size eddies interact

- Dimensional analysis
  - $\varepsilon \sim \frac{u_\ell^2}{\tau_\ell} \sim \text{const}$ (Constant energy flux)
  - $\tau_\ell \sim \frac{\ell}{u_\ell}, \quad u_\ell^2 \sim \int^{\ell^{-1}} E(k)\,\mathrm{d}k$

$$\Rightarrow \quad E(k) \propto k^{-5/3}$$

$(k \sim \ell^{-1})$

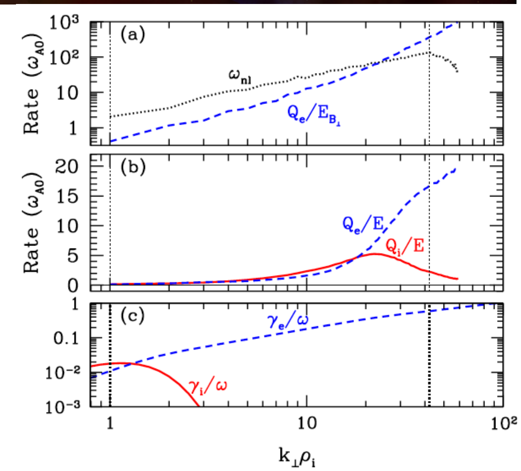


Kinetic turbulence in magnetized plasma

- Examples in nature
  - Magnetic fusion
  - Solar wind
  - Great Red Spot?
  - Accretion disks?

	Fusion	Solar wind
Particle energy	1 keV	30 eV
Field strength	1 T	10 nT
Larmor radius (typical small scale)	5 mm	80 km

- How dissipation takes place in small scales?
  - Turbulent spectra connects to small scales
  - Irreversible dissipation achieved by collisions
  - Collision operator is diffusion in velocity space
  - Need to understand velocity space structure



Prelim.: Wave-number dependence of dissipation

FT-2 tokamak: Gurchenko *et al.*, Plasma Phys. Control. Fusion **53**, 035010 (2010).
Tokamak simulation: Navarro *et al.*, Phys. Rev. Lett. **106**, 055001 (2011).
Solar wind: Howes, Dorland, Cowley *et al.*, Phys. Rev. Lett. **100**, 065004 (2008).

Gyrokinetics (GK)

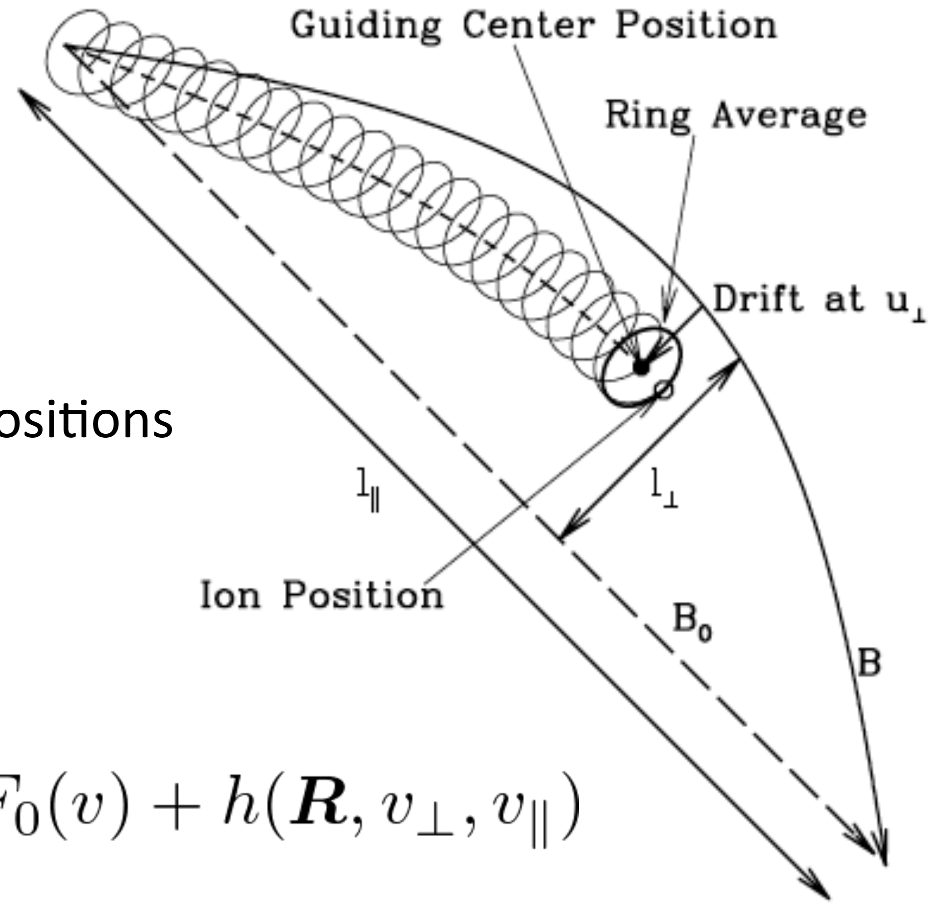
- Anisotropic magnetized plasma
- Average Larmor motion
- GK ordering

$$\frac{\omega}{\Omega} \sim \frac{\ell_\perp}{\ell_\parallel} \sim \frac{q\varphi}{T} \sim \epsilon$$

- Particle ($\boldsymbol{r}$) and ring center ($\boldsymbol{R}$) positions

$$\boldsymbol{R} = \boldsymbol{r} + \frac{\boldsymbol{v} \times \hat{\boldsymbol{b}}}{\Omega}$$

- Decompose distribution function



$$f(\boldsymbol{r}, \boldsymbol{v}) = F_0(v) - \frac{q\varphi(\boldsymbol{r})}{T} F_0(v) + h(\boldsymbol{R}, v_\perp, v_\parallel)$$

Catto, Plasma Phys. **20**, 719 (1978); Frieman & Chen, Phys. Fluids **25**, 502 (1982); Howes *et al.*, Astrophys. J. **651**, 590 (2006).

Gyrokinetics

Particle ($\boldsymbol{r}$) & ring center ($\boldsymbol{R}$) positions

$$\boldsymbol{R} = \boldsymbol{r} + \frac{\boldsymbol{v} \times \hat{\boldsymbol{b}}}{\Omega}$$

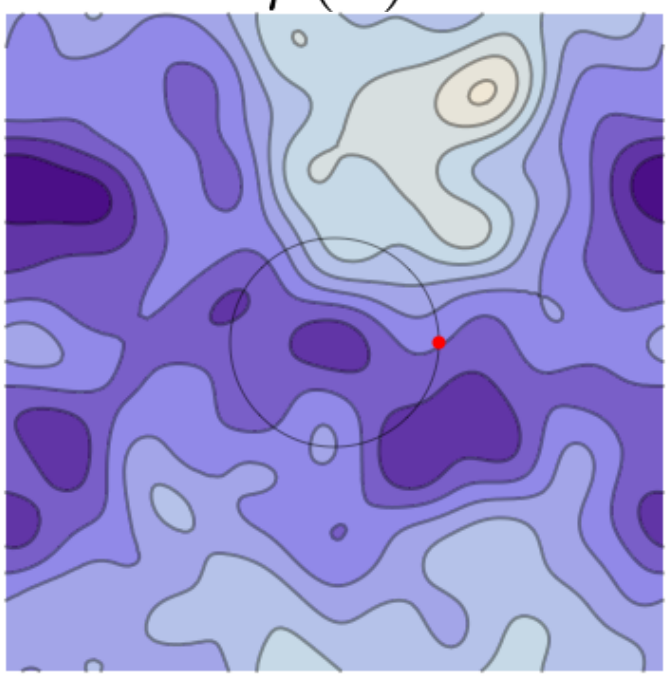
$$\boldsymbol{E} = -\nabla \varphi$$

$\hat{\boldsymbol{b}} := \boldsymbol{B}/|\boldsymbol{B}|$

Ω: Larmor frequency

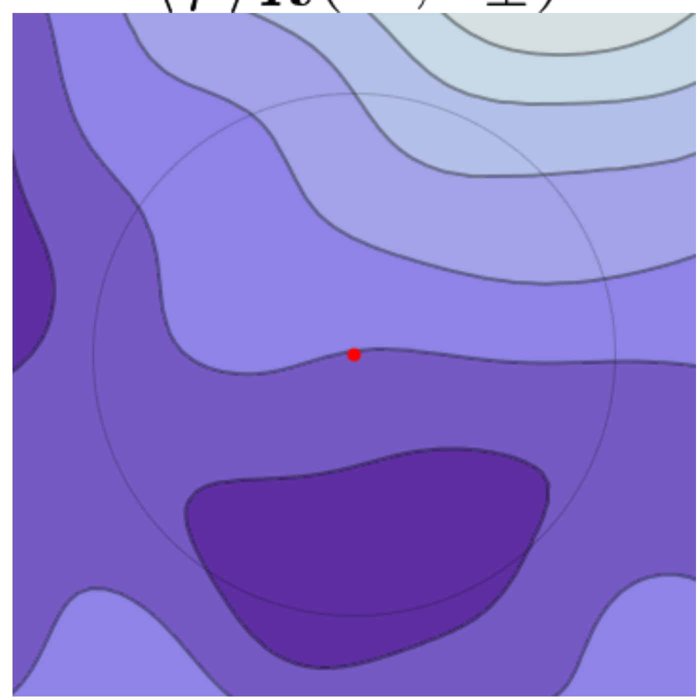
$\langle \cdot \rangle_{\boldsymbol{R}}$: Gyroaverage

$\varphi(\boldsymbol{r})$

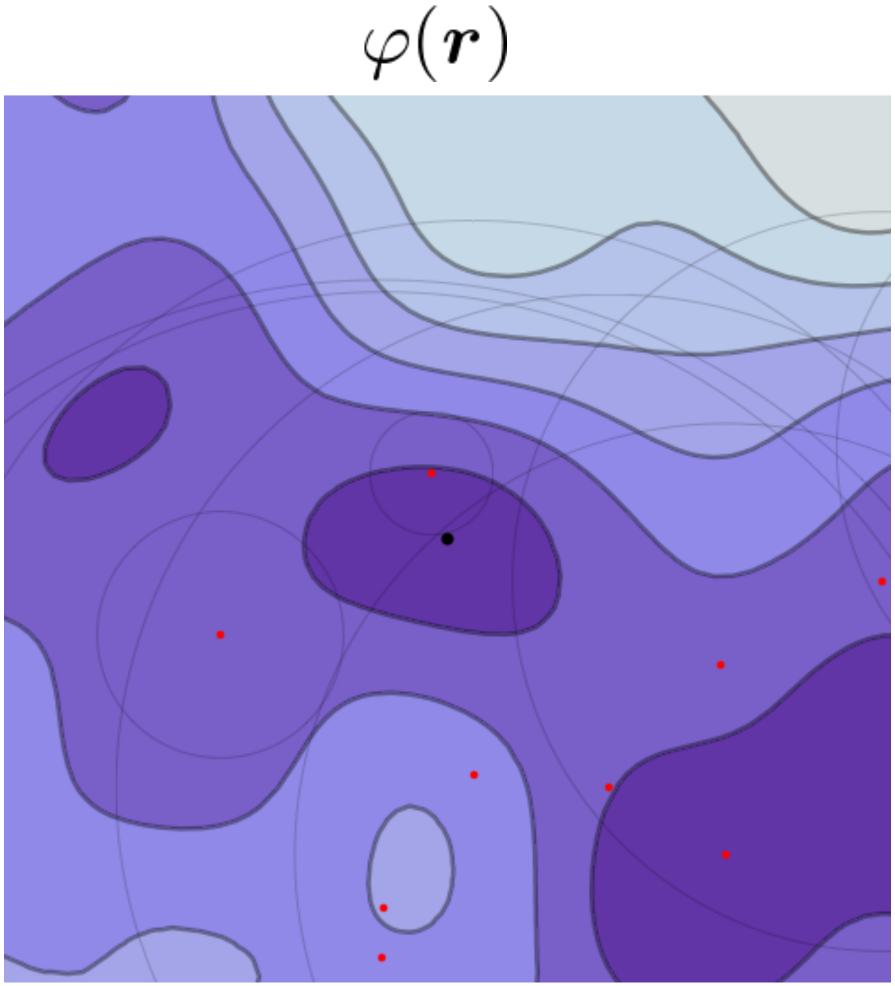


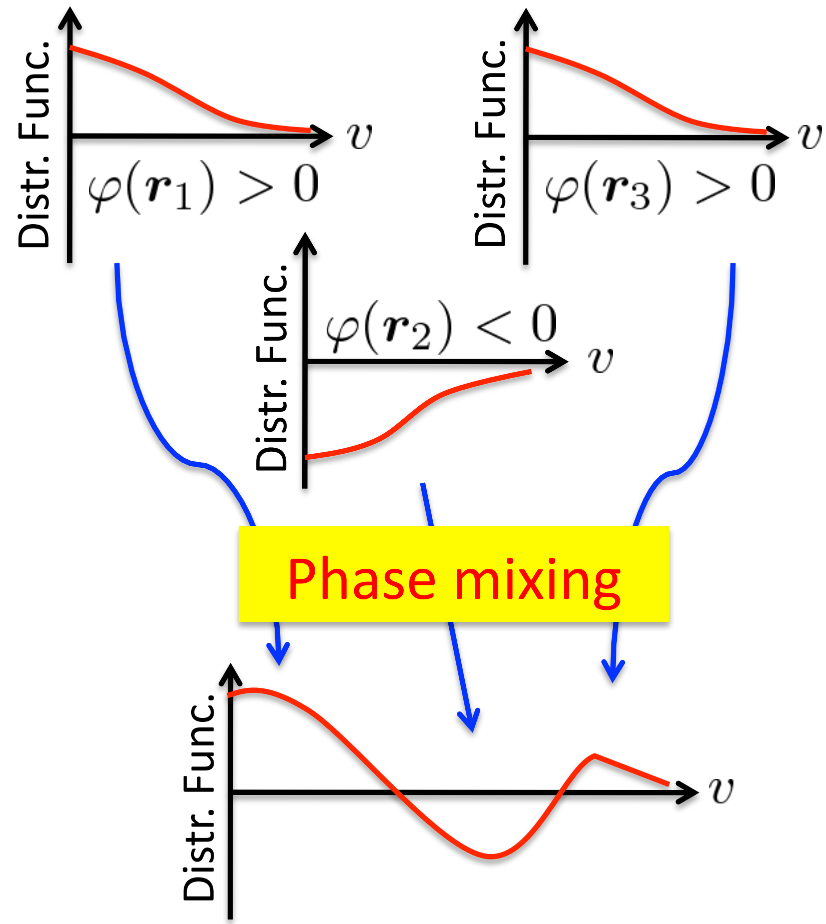
$\boldsymbol{B} \odot$

$\langle \varphi \rangle_{\boldsymbol{R}}(\boldsymbol{R}, v_\perp)$



Nonlinear phase mixing





Dorland & Hammett, Phys. Fluids B **5**, 812 (1993).
Movie by G. G. Plunk

Gyrokinetic (GK) Equation

2D $(k_{||} = 0)$ electrostatic GK Eqn.

$$\frac{\partial g}{\partial t} + \frac{\hat{\boldsymbol{b}}}{B_0} \times \nabla \langle \varphi \rangle_{\boldsymbol{R}} \cdot \nabla g = \langle \mathcal{C}(h) \rangle_{\boldsymbol{R}}$$

δf	: Particle distr. func.
g, h	: Ring distr. func.
$\mathcal{C}$	: Collision operator
W_g	: Perturbed entropy
E	: Perturbed energy

Quasi-neutrality [no-resp. elec.: $Q = \frac{q^2 n_0}{T_0}$]

$$Q\varphi = q \int \langle h \rangle_{\boldsymbol{r}} \, \mathrm{d}\boldsymbol{v} \quad \Longleftrightarrow \quad \int \delta f \, \mathrm{d}\boldsymbol{v} = 0$$

Collisionless invariants $(\Gamma_0 = I_0(b) e^{-b}, \; b = k_\perp^2 \rho^2 / 2)$

$$W_g = \iint \frac{T_0 g^2}{2F_0} \, \mathrm{d}\boldsymbol{R} \, \mathrm{d}\boldsymbol{v}, \quad E = \frac{Q}{2} \sum (1 - \Gamma_0) |\varphi_{\boldsymbol{k}}|^2$$

Schekochihin *et al.*, Astrophys. J. Suppl. Ser. **182**, 310 (2009); Tatsuno *et al.*, Phys. Rev. Lett. **103**, 015003 (2009); Plunk *et al.*, J. Fluid Mech. **664**, 407 (2010).

Scaling theory of forward cascade

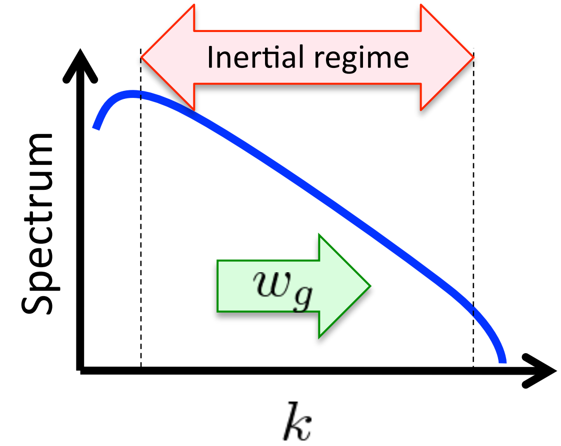
- Turbulent cascade of entropy

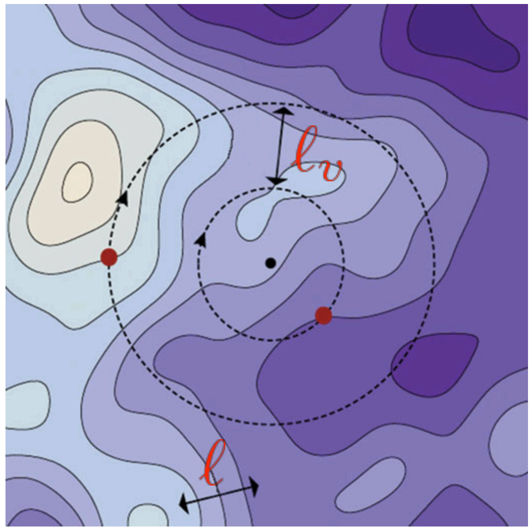
$$w_g \sim \frac{v_{\rm th}^2}{\tau_\ell}\left(\frac{g_\ell v_{\rm th}^3}{n_0}\right)^2 \sim {\rm const}$$

- Nonln. correl. time (ρ : Larmor radius)

$$\tau_\ell \sim \left(\frac{\rho}{\ell}\right)^{1/2} \frac{\ell^2 B_0}{\varphi_\ell}$$

- Correlation betw. v- & x-scales: $\ell \sim \ell_v$
- QN cond.: $\frac{\varphi_\ell}{B_0} \sim \frac{v_{\rm th}^4}{n_0} g_\ell \ell$
- Turbulent spectra $(k \sim \ell^{-1})$





$$W_g(k) \propto k^{-4/3},\ E(k) \propto k^{-10/3}$$

AstroGK

Developed by M. Barnes, W. Dorland, G. Howes, R. Numata and T. Tatsuno based on GS2.

- Spatial discretization
  - Fourier spectral perpendicular to field line (xy plane)
  - 2^{nd} order compact finite difference along field line (z direction)
  - Legendre, Laguerre spectral integration in velocity space
- Temporal discretization
  - Linear streaming: Implicit 2^{nd} order trapezoidal
  - Nonlinear term: 3^{rd} order Adams-Bashforth
  - Collisions: Implicit Euler

    (P-a scattering + Energy diffusion + Moments conservation)

F95, MPI parallel open source code: http://sourceforge.net/projects/gyrokinetics/

Numata, Howes, Tatsuno *et al*., J. Comput. Phys. **229**, 9347 (2010).

Simulation parameters

Setup

- Straight homogeneous slab: $L_x = L_y = 2\pi\rho$
- Initial condition (decaying turbulence)

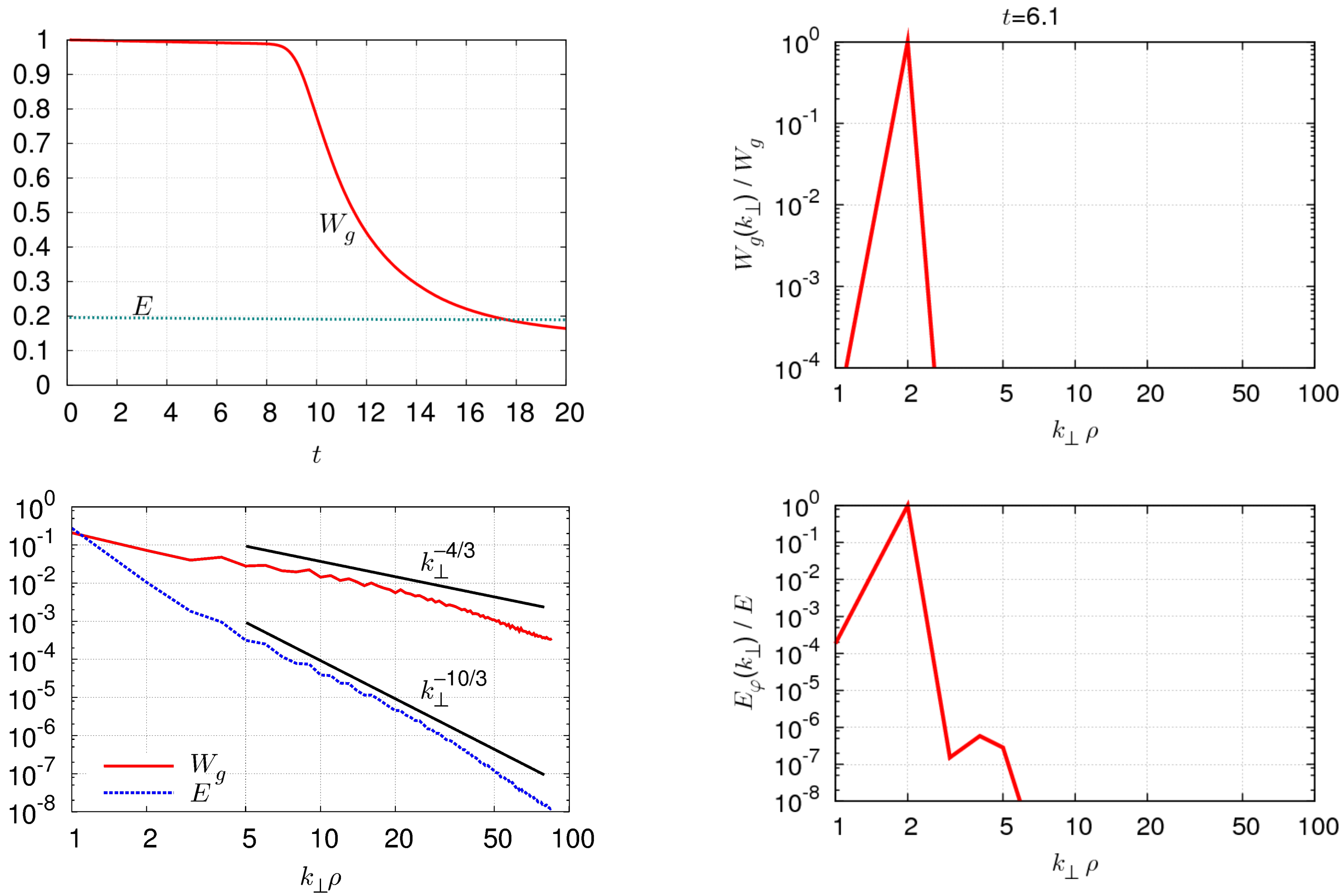
$$g_{\mathrm{init}} = g_0 \left(\cos\frac{2x}{\rho} + \cos\frac{2y}{\rho} + \mathrm{noise} \right) F_0$$

where F_0 is a Maxwellian.

Run table

case	$\nu_{ii}\,\tau_{init}$	*Do*	k_c	N_x X N_y	N_E X N_ξ
(i)	5.2×10^{-3}	32	16	64^2	32^2
(ii)	3.0×10^{-3}	48	21	64^2	32^2
(iii)	1.0×10^{-3}	118	42	128^2	64^2
(iv)	4.2×10^{-4}	440	72	256^2	128^2

Wave-number spectra



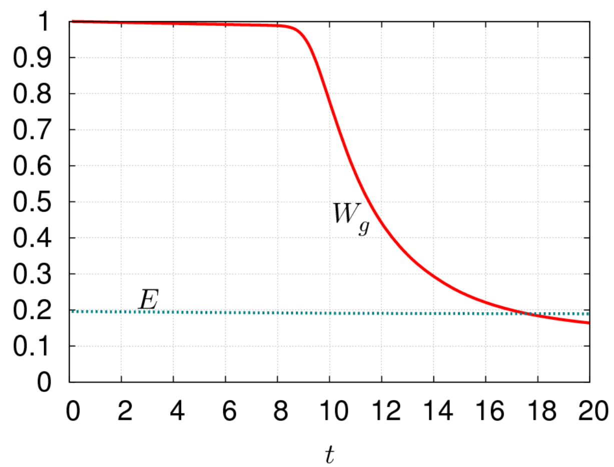
Phase-space cascade

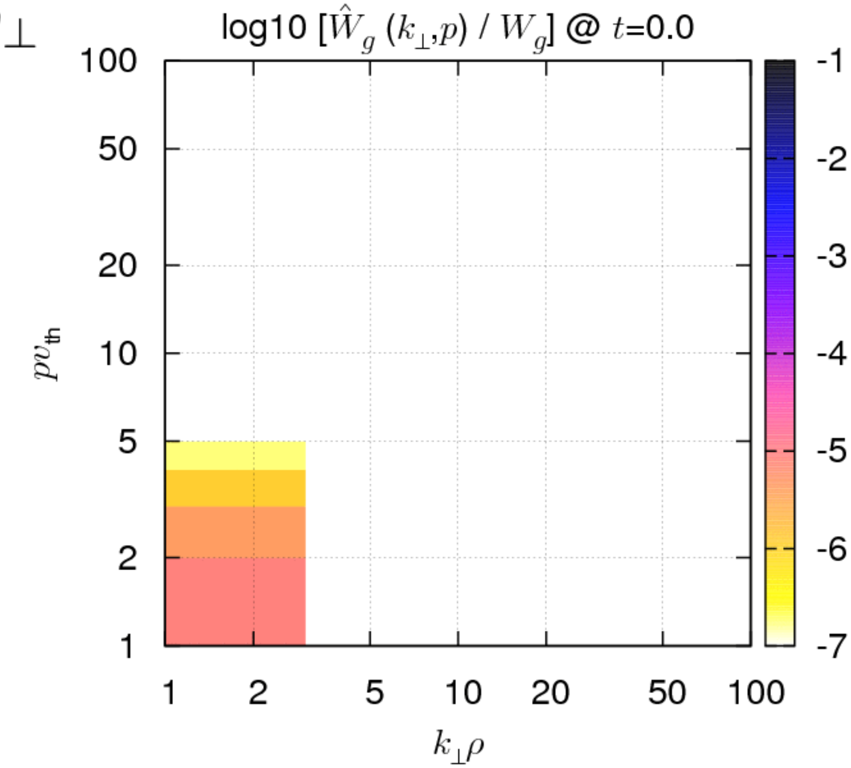
Hankel trans. & phase-space spectra ($\mathcal{K} = \{\boldsymbol{k} : K\rho - 1/2 \le |\boldsymbol{k}|\rho < K\rho + 1/2\}$)

$$\hat{g}_{\boldsymbol{k}}(p) = \int v_\perp J_0(p v_\perp) g_{\boldsymbol{k}}(v_\perp)\, \mathrm{d}v_\perp$$

$$\hat{W}_g(k_\perp, p) = \sum_{\boldsymbol{k}\in\mathcal{K}} |\hat{g}_{\boldsymbol{k}}(p)|^2$$

Time evol. collisionless invariants





Tatsuno *et al.*, Phys. Rev. Lett. **103**, 015003 (2009); J. Plasma & Fusion Res. SER. **9**, 509 (2010).

Entropy transfer function

Shell filtering ($\mathcal{K} = \{\boldsymbol{k} : K\rho - 1/2 \le |\boldsymbol{k}|\rho < K\rho + 1/2\}$)

$$g_K(\boldsymbol{R}, v_\perp, v_\parallel) := \sum_{\boldsymbol{k}\in\mathcal{K}} g_{\boldsymbol{k}}(v_\perp, v_\parallel) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}$$

Entropy transfer function

$$T^{(W)}(K,Q) := -\iint \frac{1}{F_0} g_K \{\langle\varphi\rangle_{\boldsymbol{R}}, g_Q\}\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{v}$$

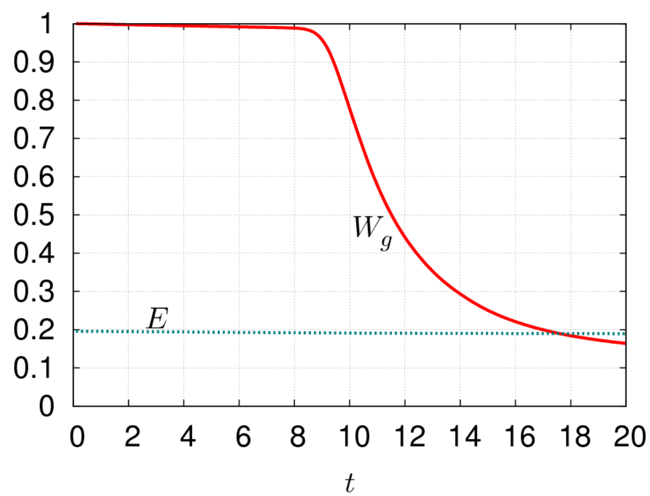
Time evolution of entropy

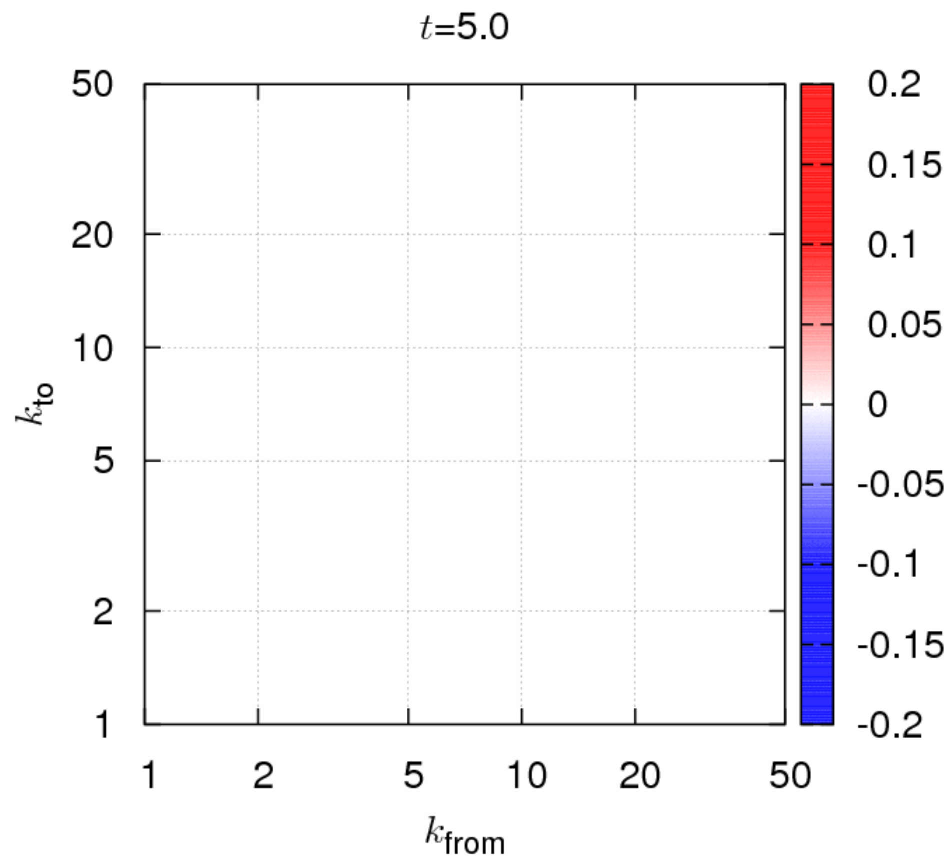
$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|g_K|^2}{2F_0}\,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{v} = \sum_Q T^{(W)}(K,Q) - \text{collisions}$$

Dar, Verma & Eswaran, Physica D **157**, 207 (2001);
Alexakis, Mininni & Pouquet, Phys. Rev. E **72**, 046301 (2005) (MHD);
Tatsuno *et al.*, J. Plasma & Fusion Res. SER. **9**, 509 (2010) [arXiv:1003.3933];
Navarro *et al.*, Phys. Rev. Lett. **106**, 055001 (2011).

Entropy transfer diagnosed

$$T^{(W)}(k_{\mathrm{to}}, k_{\mathrm{from}}) = -\iint \frac{1}{F_0} g_{k_{\mathrm{to}}} \{\langle\varphi\rangle_{\boldsymbol{R}}, g_{k_{\mathrm{from}}}\} \,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{v}$$





Contents

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  - Principle of dual cascade
  - Nonlinear transfer of energy & inverse cascade
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  - Spectral collapse
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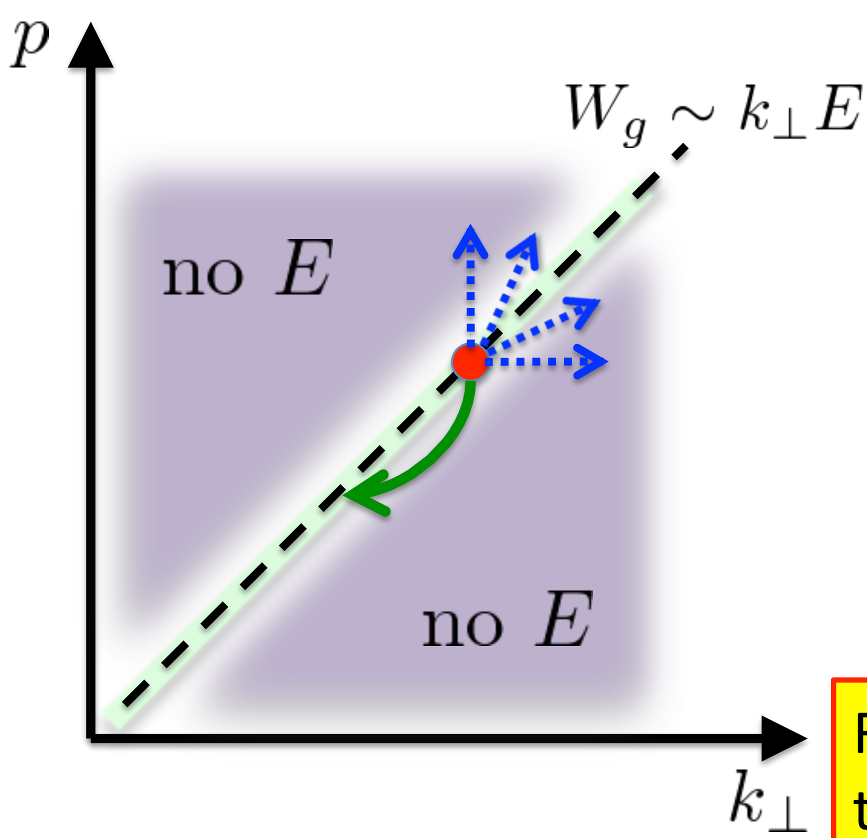
- Summary

Tatsuno, Dorland, Schekochihin *et al*., Phys. Rev. Lett. **103**, 015003 (2009);
Tatsuno, Barnes, Cowley *et al*., J. Plasma & Fusion Res. SER. **9**, 509 (2010);
Numata, Howes, Tatsuno *et al*., J. Comput. Phys. **229**, 9347 (2010).

Dual cascade

- What if small scale initial condition?



Invariants

$$W_g \sim \sum_{\boldsymbol{k}} \int \frac{|g_{\boldsymbol{k}}|^2}{F_0} \mathrm{d}\boldsymbol{v}$$

$$E \sim \sum_{\boldsymbol{k}} |\varphi_{\boldsymbol{k}}|^2$$

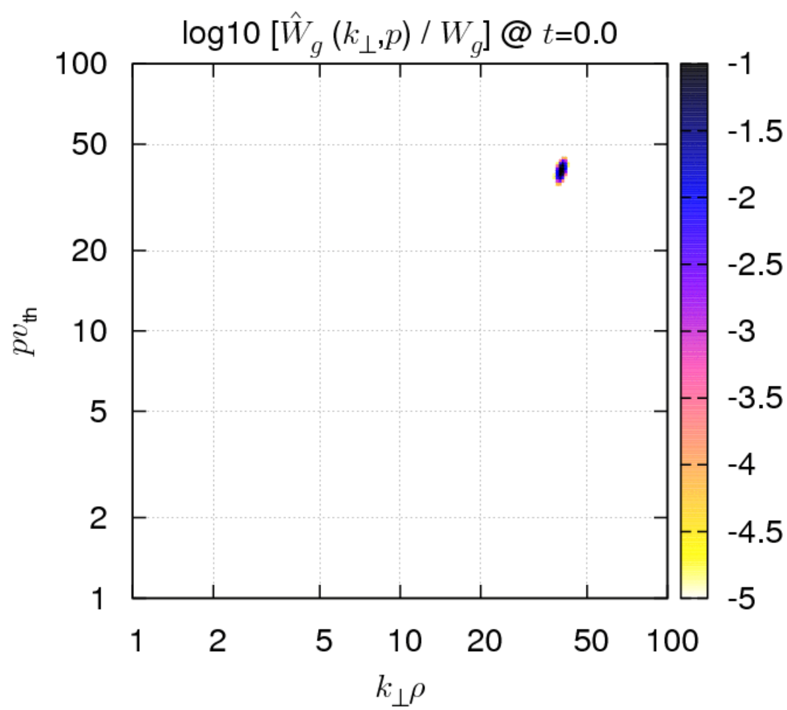
Quasi-neutrality

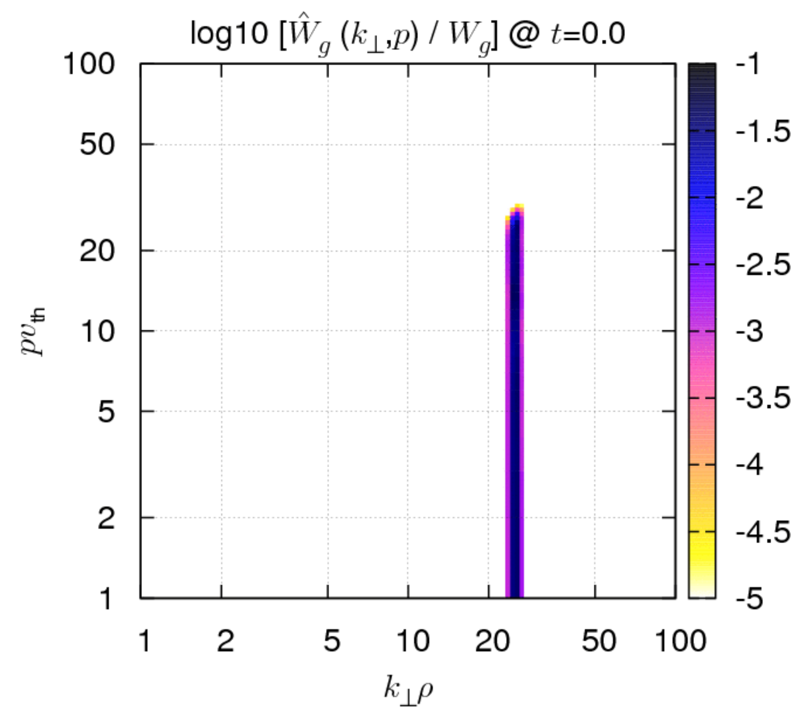
$$\varphi_{\boldsymbol{k}} \sim \int J_0(k_\perp v_\perp) g_{\boldsymbol{k}} \, \mathrm{d}\boldsymbol{v}$$

For W_g to cascade forward, E needs to be transferred to large scale.

Freely decaying turbulence

- Small scale initial condition
- Coherent v-dist. & random v-dist.





Example of small-scale instability: Ricci *et al.*, Phys. Plasmas **13**, 062102 (2006).

Energy transfer function

Shell filtering ($\mathcal{K} = \{\boldsymbol{k} : K\rho - 1/2 \le |\boldsymbol{k}|\rho < K\rho + 1/2\}$)

$$\varphi_K(\boldsymbol{r}) := \sum_{\boldsymbol{k}\in\mathcal{K}} \varphi_{\boldsymbol{k}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

Energy transfer function

$$T^{(E)}(K,Q) := -\iint \frac{1}{F_0} \langle\varphi_K\rangle_{\boldsymbol{R}} \{\langle\varphi_Q\rangle_{\boldsymbol{R}}, g\} \,\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{v}$$

Time evolution of energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\boldsymbol{k}\in\mathcal{K}} (1-\Gamma_0)|\varphi_{\boldsymbol{k}}|^2 = \sum_Q T^{(E)}(K,Q) - \text{collisions}$$

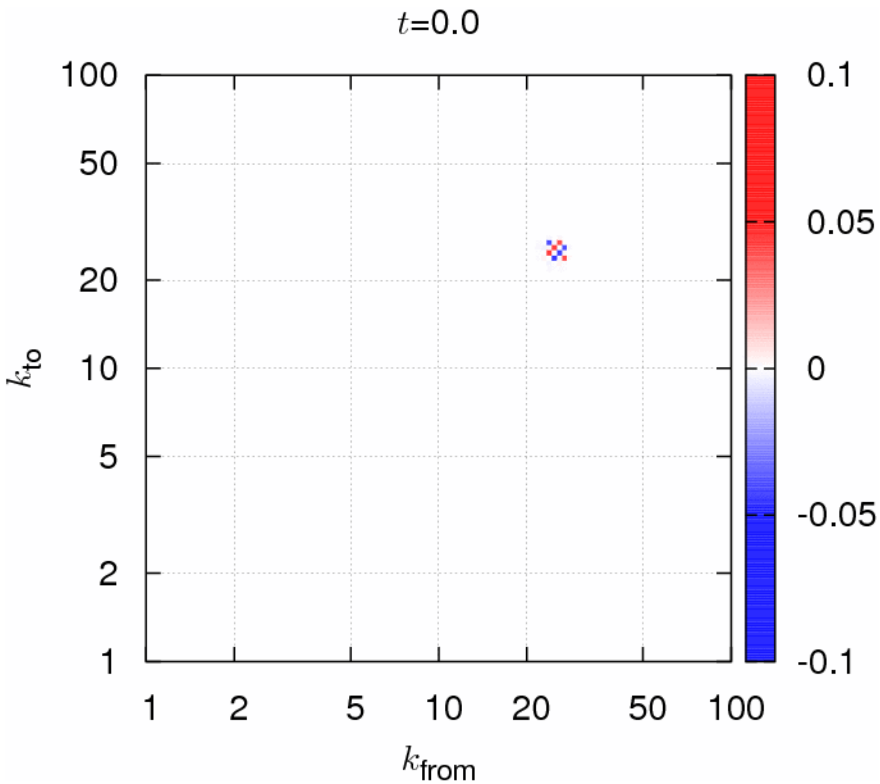
Tatsuno, Plunk *et al.*, Phys. Plasmas, *in preparation*.

Inverse cascade & nonlinear transfer

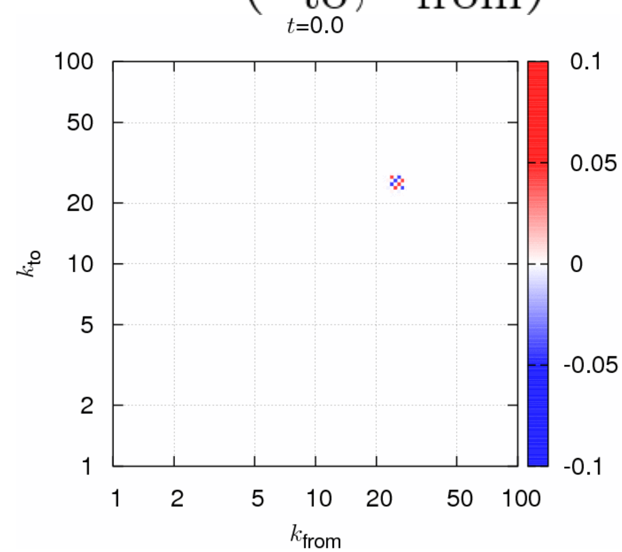
$$E = \frac{Q}{2} \sum_{\boldsymbol{k}} (1 - \Gamma_0) |\varphi_{\boldsymbol{k}}|^2$$

$$W_g = \iint \frac{T_0 g^2}{2F_0} \, \mathrm{d}\boldsymbol{R} \, \mathrm{d}\boldsymbol{v}$$

$$T^{(E)}(k_{\mathrm{to}}, k_{\mathrm{from}})$$



$$T^{(W)}(k_{\mathrm{to}}, k_{\mathrm{from}})$$



Scaling law of freely decaying turbulence

- Monochromatic spectrum of scale length l_*

$$W_g \propto g_*^2, \quad E \propto \varphi_*^2$$

- Invariants relation [Diagonals ($k_\perp \rho \sim p v_{\mathrm{th}}$) dominate]

$$W_g \propto k_* E$$

- Constant energy & nonlinear decay of entropy

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim 0, \quad \frac{\mathrm{d}W_g}{\mathrm{d}t} \sim -\frac{W_g}{\tau_*}$$

Based on these assumptions, in the small collision frequency limit,

$$E \sim \text{const.}, \quad W_g \propto k_* \propto t^{-2/3}$$

Fluid analogy: Chasnov, Phys. Fluids **9**, 171 (1997).

How small is small?

- Define a dimensionless number from change of entropy

$$\frac{\mathrm{d}W_g}{\mathrm{d}t} \sim \underbrace{-\frac{W_g}{\tau_*}}_{\text{nonlinear}} \underbrace{-\nu(k_*\rho)^2 W_g}_{\text{collision}}$$

- When two terms balance (in time), $\tau_* \sim t$ thus (marginal collisionality)

$$E \propto k_* \propto t^{-1/2}, \quad W_g \propto t^{-1}$$

- Dorland number

$$D_* := \frac{1}{\nu\tau_*(k_*\rho)^2}$$

Cf. In forward cascade runs measured at Larmor scale

$$D = \frac{1}{\nu\tau_\rho}$$

Collisional case

- Both energy and entropy decay by collisions

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim \nu (k_* \rho)^2 E, \quad \frac{\mathrm{d}W_g}{\mathrm{d}t} \sim \nu (k_* \rho)^2 W_g$$

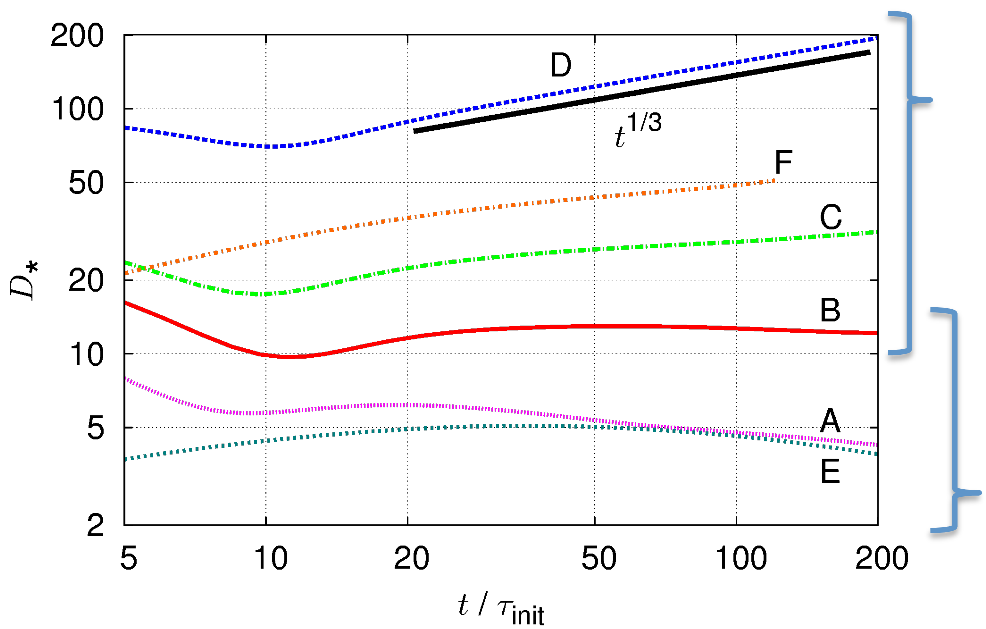
$$k_* \propto \frac{W_g}{E} \propto t^{-1/2}$$

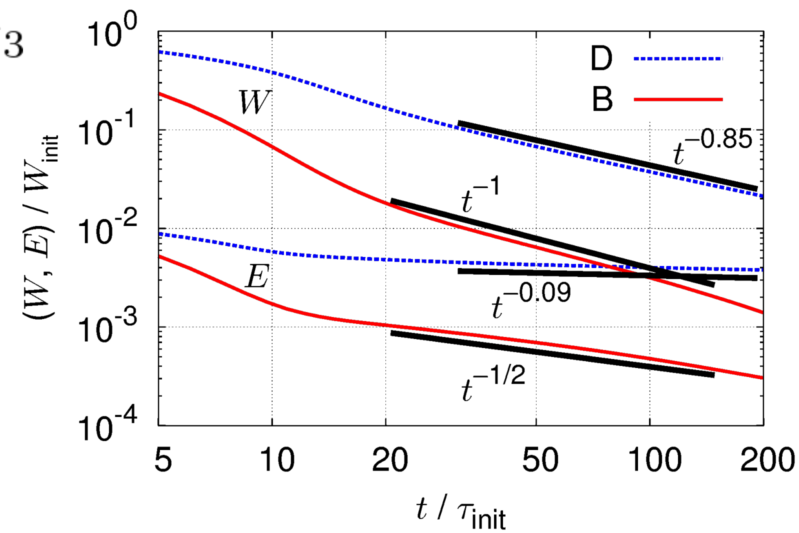
- This was also true for marginal case

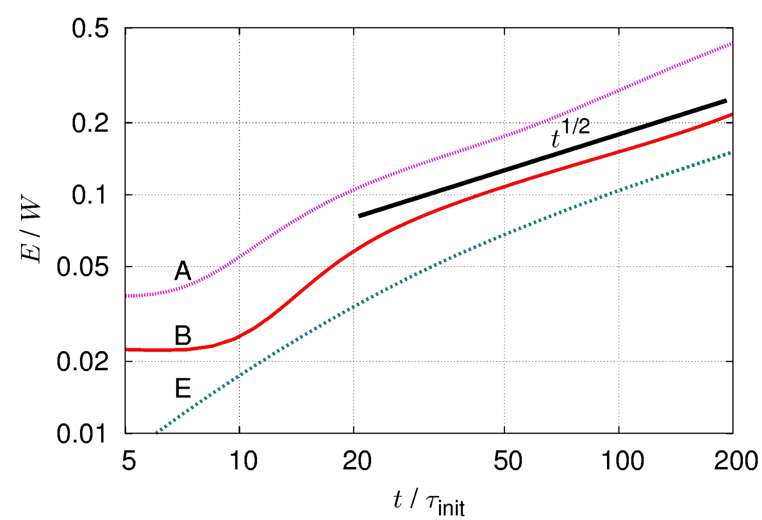
Classify runs with D_*

Collisionless: $E \sim \text{const.}, \quad W_g \propto k_* \propto t^{-2/3}$

Marginal: $E \propto k_* \propto t^{-1/2}, \quad W_g \propto t^{-1}$







Collisional: $k_* \propto \dfrac{W_g}{E} \propto t^{-1/2}$

Dorland number: $D_* := \dfrac{1}{\nu \tau_* (k_* \rho)^2}$

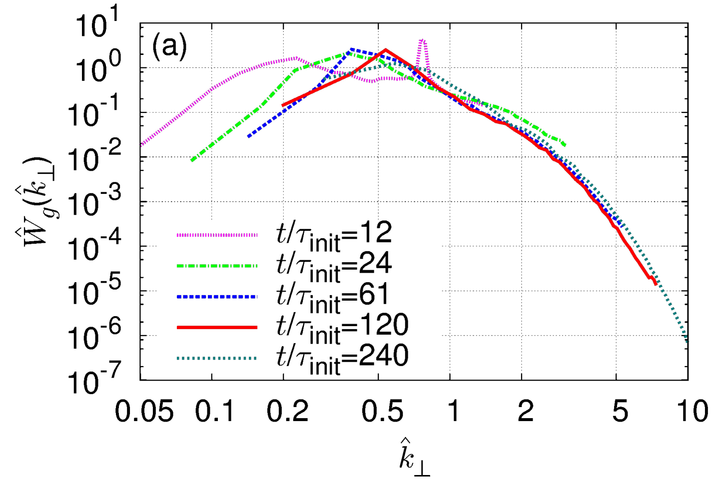
Spectral collapse

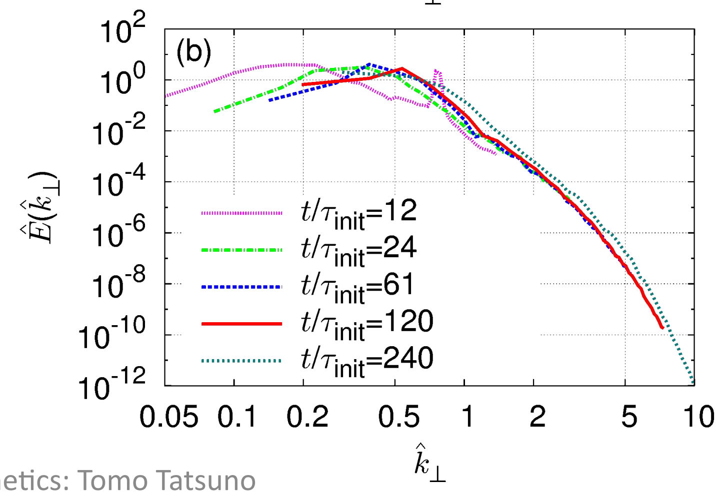
- In marginal case we can collapse wave-number spectra as D_* is constant in time -> self-similar

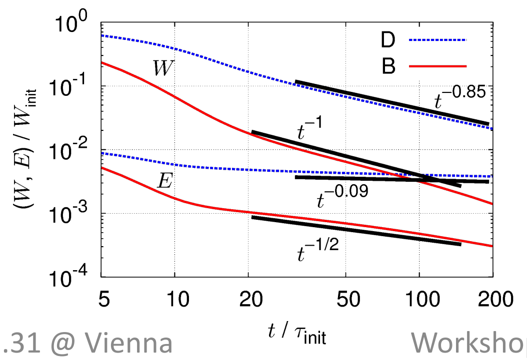
$$\hat{k}_\perp = \frac{k_\perp}{k_*} \quad \left(k_* := \frac{W_g}{\rho E}\right)$$

$$\hat{W}_g(\hat{k}_\perp) = \frac{k_* W_g(k_\perp, t)}{\sum_{k_\perp} W_g(k_\perp, t)}$$

$$\hat{E}(\hat{k}_\perp) = \frac{k_* E(k_\perp, t)}{\sum_{k_\perp} E(k_\perp, t)}$$

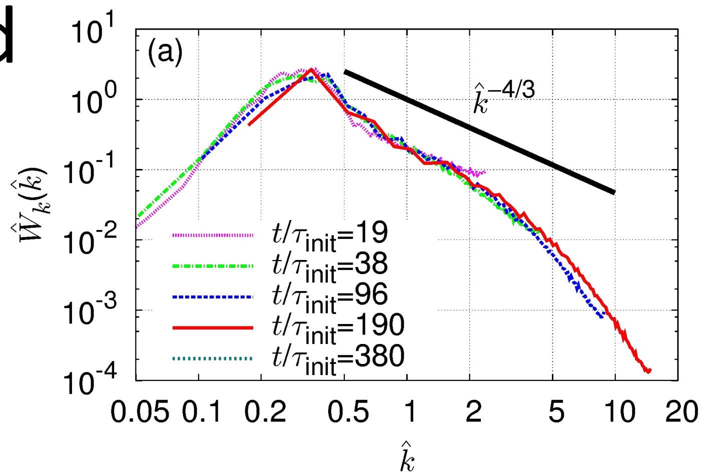


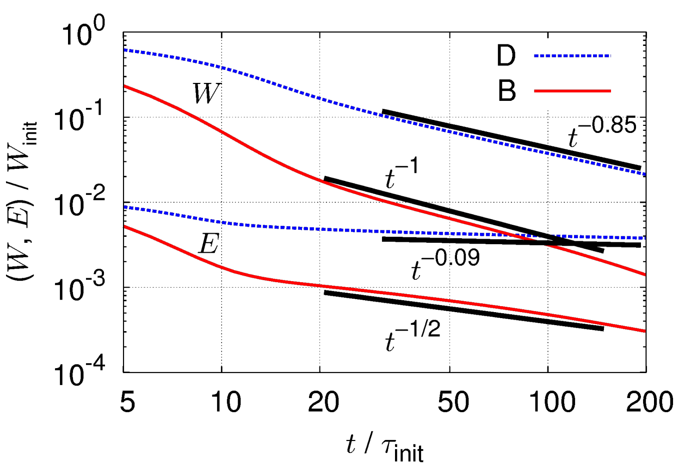


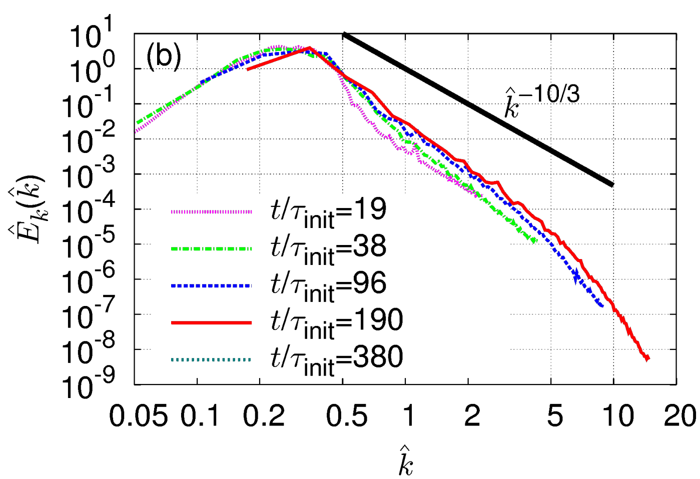


Spectral collapse 2

- Collisionless case --- not bad
- Spectral slope of forward cascade reproduced
- Slope extends in time as D_* increases

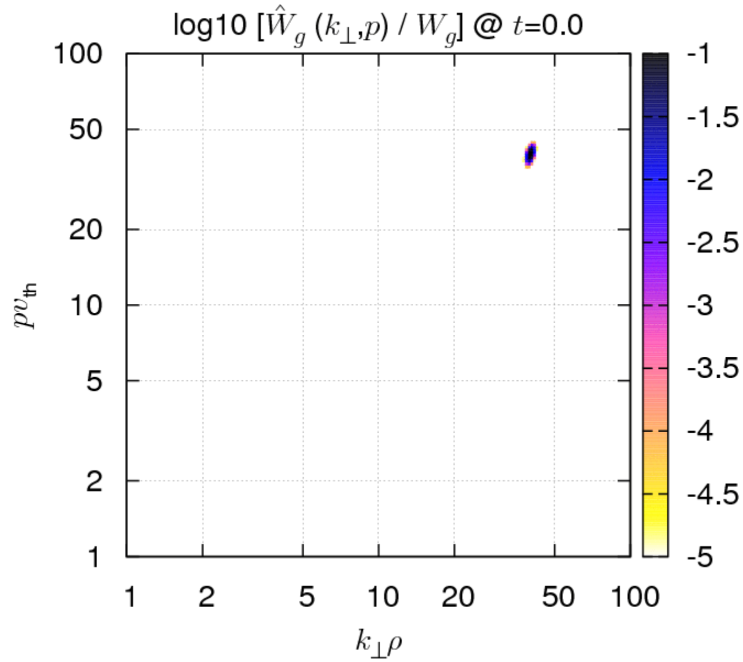




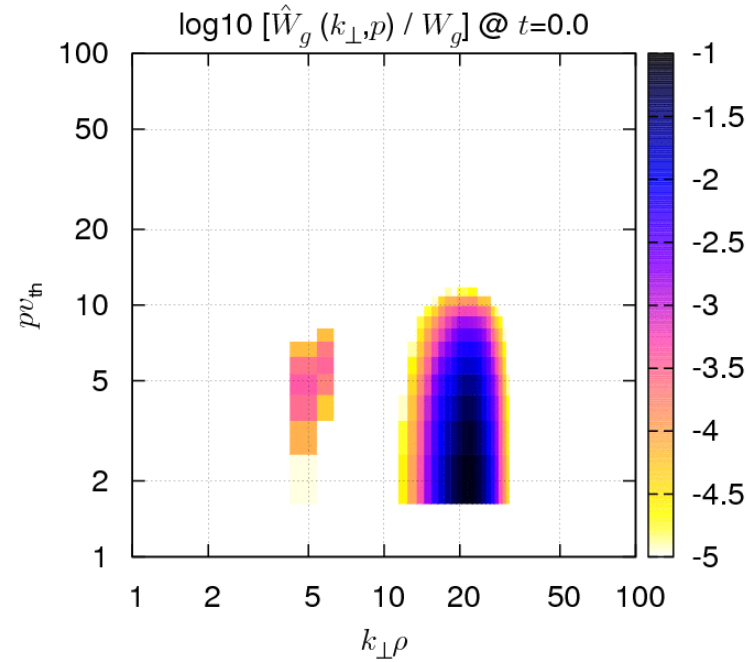


Variety of interactions

- Phase-space cascade has two degs. freedom
- Depending on ratio of invariants W/E, variety of transient interactions is possible



Small W/E



Large W/E

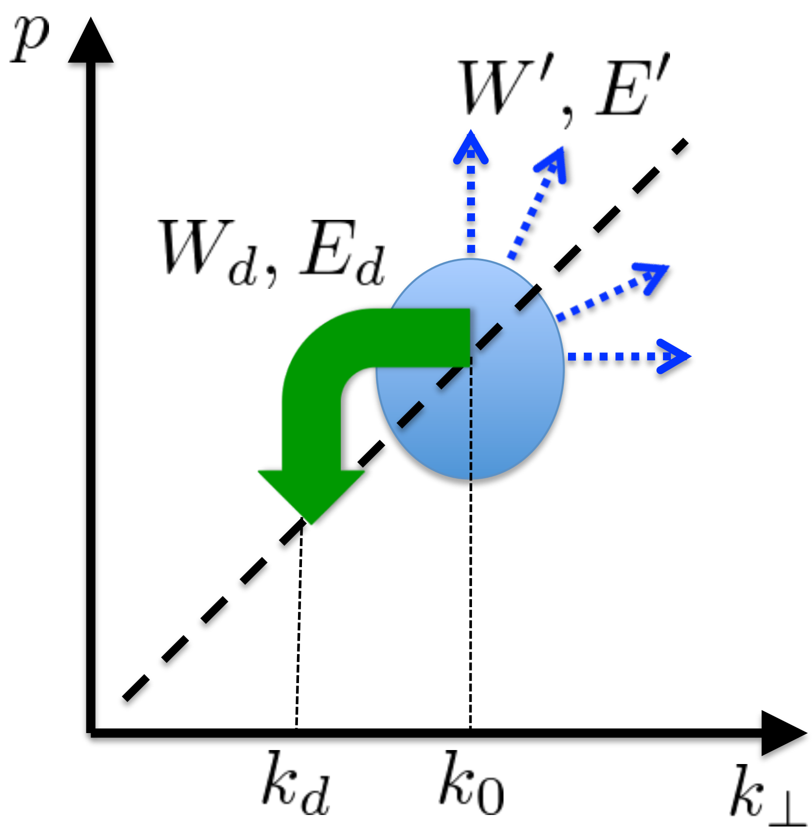
Plunk & Tatsuno, Phys. Rev. Lett., accepted (2011).

Classify in terms of W/E

- Energy partition ratio

$$R := E'/E_d$$

Subscript d: diagonal



- Initial condition

$$W_0/E_0 =: \kappa$$

- Partition of invariants

$$W_0 = W' + W_d$$

$$E_0 = E' + E_d$$

- Law of transfer

$$W' \sim k_0^2 E'$$

$$E' \sim k_d E_d$$

- Diagonal wave number

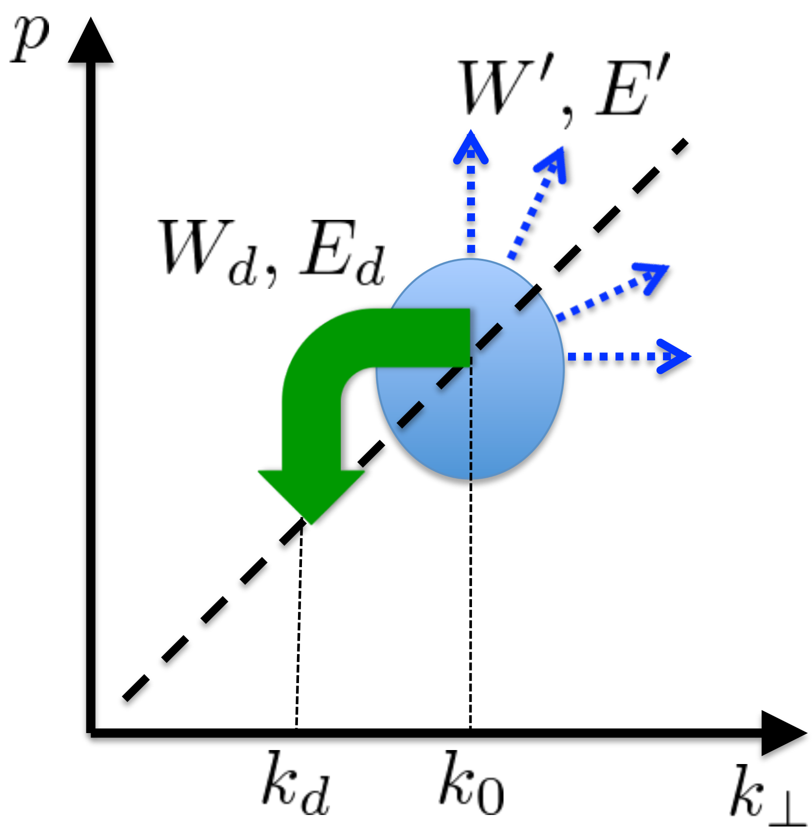
$$k_d = \kappa(R+1) - k_0^2 R$$

Classify in terms of W/E

- Energy partition ratio

$$R := E'/E_d$$

Subscript d: diagonal



- Initial condition

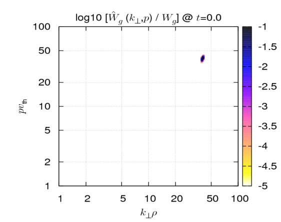
$$W_0/E_0 =: \kappa$$

- Diagonal wave number

$$k_d = \kappa(R+1) - k_0^2 R$$

- Small κ => small k_d

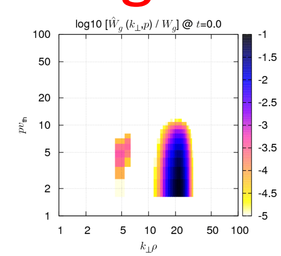
Diag. => strong inverse



- Large κ => large k_d

Off-diag. => strong fwd.

$R < 0$



Summary

- We analyzed small-scale electrostatic turbulence of 2D magnetized plasmas using gyrokinetics
- Direct cascade
  - Coupling of position and velocity space scales
  - Entropy transfers locally to small scales both in position and velocity spaces
- Freely decaying turbulence (dual cascade)
  - Entropy cascades to small scales, energy to large scales
  - Decay laws are identified by means of simple argument on collisionless invariants
  - Classification in terms of dimensionless number is given
  - Spectral collapse supports self-similarity
  - Variety of energy transfer observed