The “monoenergetic approximation”

A connection between tokamak pedestals and stellarators?

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• DKES (Drift Kinetic Equation Solver) is the standard code for calculating neoclassical transport in stellarators. (PENTA is based on DKES.)

• In DKES, ad-hoc changes are made to the $E_r$ terms in the kinetic equation to expedite computation. These changes are sometimes called the monoenergetic approximation.
  – Kinetic energy instead of total energy is conserved. Magnetic moment is not conserved.
  – The pitch-angle scattering operator is used to model collisions, so speed $\nu$ becomes just a parameter.

• To rigorously assess whether this approximation is justified, the kinetic equation must be solved including collisions. (Hard)

• Simpler test to avoid dealing with collisions: compare DKES trajectories to true trajectories.

• Result: DKES systematically under-predicts the fraction of trapped particles.
Characteristics of the drift-kinetic equation give effective particle guiding-center trajectories.

\[
\left( \frac{d\psi}{dt} \right) \frac{\partial f}{\partial \psi} + \left( \frac{d\theta}{dt} \right) \frac{\partial f}{\partial \theta} + \left( \frac{d\zeta}{dt} \right) \frac{\partial f}{\partial \zeta} + \left( \frac{d\nu}{dt} \right) \frac{\partial f}{\partial \nu} + \left( \frac{d\xi}{dt} \right) \frac{\partial f}{\partial \xi} = C\{f\}
\]

3 spatial coordinates

2 velocity coordinates, e.g. speed & pitch angle
Equations of motion

True equations:
\[
\begin{align*}
\frac{d\theta}{dt} &= \left(\nu b + \frac{c}{B^2} \frac{d\Phi}{d\psi} B \times \nabla \psi + v_m\right) \cdot \nabla \theta \\
\frac{d\xi}{dt} &= \left(\nu b + \frac{c}{B^2} \frac{d\Phi}{d\psi} B \times \nabla \psi + v_m\right) \cdot \nabla \xi \\
\frac{d\psi}{dt} &= v_m \cdot \nabla \psi
\end{align*}
\]

Equivalent true equations:
\[
\begin{align*}
\frac{d\theta}{dt} &= \left(\nu b + \frac{c}{B^2} \frac{d\Phi}{d\psi} B \times \nabla \psi + v_m\right) \cdot \nabla \theta \\
\frac{d\xi}{dt} &= \left(\nu b + \frac{c}{B^2} \frac{d\Phi}{d\psi} B \times \nabla \psi + v_m\right) \cdot \nabla \xi \\
\frac{d\psi}{dt} &= v_m \cdot \nabla \psi
\end{align*}
\]

DKES equations:
\[
\begin{align*}
\frac{d\theta}{dt} &= \left(\nu b + \frac{c}{\langle B^2 \rangle} \frac{d\Phi}{d\psi} B \times \nabla \psi \right) \cdot \nabla \theta \\
\frac{d\xi}{dt} &= \left(\nu b + \frac{c}{\langle B^2 \rangle} \frac{d\Phi}{d\psi} B \times \nabla \psi \right) \cdot \nabla \xi \\
\frac{d\psi}{dt} &= v_m \cdot \nabla \psi
\end{align*}
\]

\[
\begin{align*}
\frac{d\mu}{dt} &= 0 \\
\frac{dH}{dt} &= 0 \text{ where } H = \frac{mv^2}{2} + Ze\Phi \\
\frac{d\xi}{dt} &= -\frac{v}{2B^2} \left(1 - \xi^2\right) B \cdot \nabla B + \frac{c}{2B^3} \frac{d\Phi}{d\psi} \xi \left(1 - \xi^2\right) B \times \nabla \psi \cdot \nabla B \\
+ \frac{\xi^2 - 1}{2\xi B} v_m \cdot \nabla B \text{ where } \xi = \nu_\parallel / \nu \\
\frac{dv}{dt} &= \frac{cv}{2B^3} \frac{d\Phi}{d\psi} \left(1 + \xi^2\right) B \times \nabla \psi \cdot \nabla B \\
\frac{d\xi}{dt} &= -\frac{v}{2B^2} \left(1 - \xi^2\right) B \cdot \nabla B \\
\frac{dv}{dt} &= 0
\end{align*}
\]
Monte-Carlo codes give noisy results for the flow and bootstrap current due to $\pm \nu_||$ cancellation.
**Trapped trajectories**

For axisymmetric or quasisymmetric \( \mathbf{B} \) field with constant \( d\Phi/d\psi \):

\[
E_r = 0
\]

\[
U = \left. -\frac{\mathbf{v}_E \times \mathbf{B} \cdot \nabla B}{\nu_{th} \mathbf{b} \cdot \nabla B} \right| = -\frac{B_t}{B_p} \frac{\left| \mathbf{v}_E \times \mathbf{B} \right|}{\nu_{th}}
\]

\( U \) can be \( \sim 1 \) even if \( |\mathbf{v}_E \times \mathbf{B}| \ll |\nu_{th}| \).

\( U \sim 1 \) in HSX?
Trapped trajectories

For axisymmetric or quasisymmetric $\mathbf{B}$ field with constant $d\Phi/d\psi$:

\[ U = - \frac{\mathbf{v}_{E\times B} \cdot \nabla B}{\nu_{th} \mathbf{b} \cdot \nabla B} = - \frac{B_t}{B_p} \frac{\mathbf{v}_{E\times B}}{\nu_{th}} \]
The true fraction of trapped particles (last page) is not the same as the “effective trapped fraction”

\[ f_{t}^{\text{eff}} = 1 - \frac{3}{4} \left\langle B^2 \right\rangle \int_0^{1/B_{\text{max}}} \frac{\lambda \, d\lambda}{\left\langle \sqrt{1 - \lambda B} \right\rangle} \approx 1.46\sqrt{\varepsilon} \]

which appears in banana-regime neoclassical quantities.

The conventional heat flux, viscosities, and bootstrap current are all \( \propto f_{t}^{\text{eff}} \).
Drift-kinetic equation:

\[ \nu_{||} \mathbf{b} \cdot \nabla f_1 + \mathbf{v}_{\mathbf{E} \times \mathbf{B}} \cdot \nabla f_1 + \ldots = C \{ f_1 \} \]

In either axisymmetry or quasisymmetry, the ratio of these terms is

\[ \frac{\mathbf{v}_{\mathbf{E} \times \mathbf{B}} \cdot \nabla f_1}{\nu_{||} \mathbf{b} \cdot \nabla f_1} \sim \frac{\mathbf{v}_{\mathbf{E} \times \mathbf{B}} \cdot \nabla B}{\nu_{th} \mathbf{b} \cdot \nabla B} = \frac{\mathbf{v}_{\mathbf{E} \times \mathbf{B}} \cdot \nabla B}{\nu_{th} \mathbf{b} \cdot \nabla B} = U = \frac{B_t}{B_p} \frac{\mathbf{v}_{\mathbf{E} \times \mathbf{B}}}{\nu_{th}} \]

Tokamak: \[ V_{||} = -\frac{T_i R B_i}{m_i \Omega_i} \left( \frac{Z e}{T_i} \frac{d \Phi}{d \psi} + \frac{1}{p_i} \frac{d p_i}{d \psi} - \frac{1.17}{T_i} \frac{d T_i}{d \psi} \right) \]

\[ \nu_{th,i} U \quad \nu_{th,i} \rho_p / L_p \quad \nu_{th,i} \rho_p / L_T \]

So in a tokamak, you can simultaneously have \( U \sim 1 \) and \( \mathbf{v}_{\mathbf{E} \times \mathbf{B}} \ll \nu_{th,i} \) only if radial gradient scale length is \( \rho_p \).

This is precisely the ordering we used to analyze tokamak pedestals.
Radial particle diffusion coefficient (normalized)

$U = \text{normalized } E_r$

- ○ DKES
- □ MOCA (Monte-Carlo) with monoenergetic trajectories
- △ MOCA with correct trajectories

From CD Beidler et al., *Proc. 17th Int’l Toki Conf. and 16th Int’l Stellarator/Heliotron Workshop* (2007)
• Kagan-Catto calculations for the tokamak pedestal ordering can be generalized to a quasisymmetric stellarator.

• We use generalized pitch-angle scattering model collision operator, to follow the shift in trapped-passing boundary.

For $B = B(\psi, M\theta - N\zeta)$,

$$\langle q_i \cdot \nabla \psi \rangle = -1.35 Q(U) \sqrt{\varepsilon} \frac{(MG + NI)^2 v_i n_i T_i}{(\mu M - N)^2 m_i \left\langle \Omega^2 \right\rangle} dT_i$$
New results for quasisymmetry

For $B = B(\psi, M\theta - N\zeta)$,

$$\langle V_{ii}B \rangle = \frac{cp_i(MG + NI)}{n_ie(N + \Phi)} \left[ \frac{1}{p_i} \frac{dp_i}{d\psi} + \frac{Ze}{T_i} \frac{d\Phi}{d\psi} - \frac{1.17 A(U)}{T_i} \frac{dT_i}{d\psi} \right]$$

$$\langle j_{ii}B \rangle = 2.42\sqrt{\epsilon} \frac{c(MG + NI)}{N + \Phi M} \left( \frac{dp_e}{d\psi} + \frac{dp_i}{d\psi} - 0.75n \frac{dT_e}{d\psi} - 1.17 A(U) \frac{dT_i}{d\psi} \right)$$

Electric field causes enhancement of bootstrap current.
Both in stellarators and tokamak pedestals, the simultaneous ordering $\nu_{E \times B} \ll \nu_{th,i}$ and $v_{E \times B} \cdot \nabla B \sim \nu || b \cdot \nabla B$ is of interest.

The conventional approach for calculating neoclassical transport in stellarators involves ad-hoc changes to the kinetic equation.

These changes lead to $O(1)$ errors in the trapped fraction when $E_r$ is large.