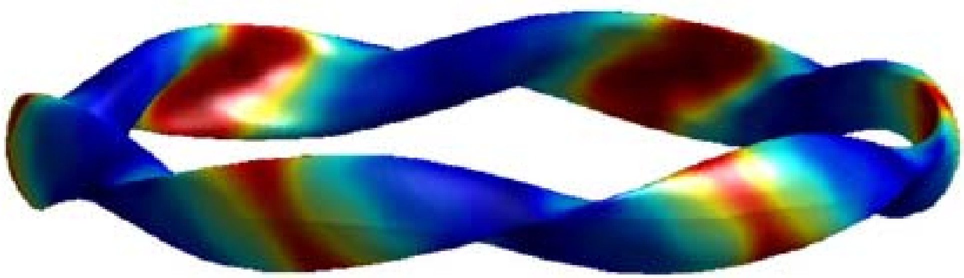
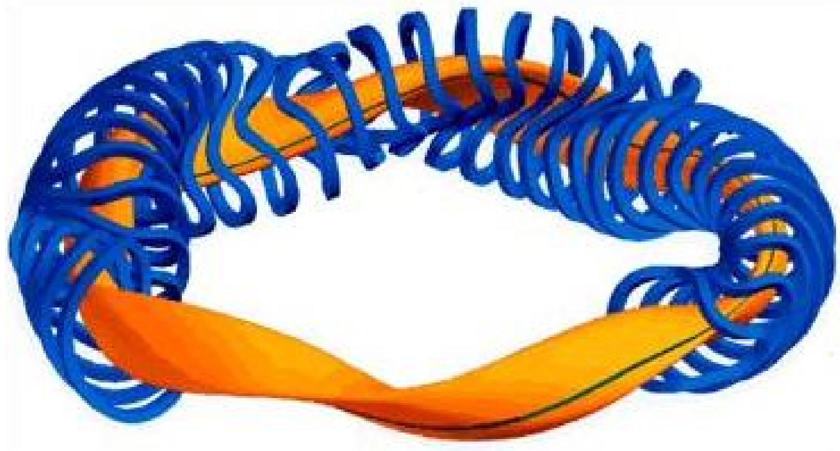
The “monoenergetic approximation”

A connection between tokamak pedestals and stellarators?

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The monoenergetic approximation

- DKES (Drift Kinetic Equation Solver) is the standard code for calculating neoclassical transport in stellarators. (PENTA is based on DKES.)

- In DKES, ad-hoc changes are made to the E_r terms in the kinetic equation to expedite computation. These changes are sometimes called the monoenergetic approximation.

  - Kinetic energy instead of total energy is conserved. Magnetic moment is not conserved.

  - The pitch-angle scattering operator is used to model collisions, so speed υ becomes just a parameter.

- To rigorously assess whether this approximation is justified, the kinetic equation must be solved including collisions. (Hard)

- Simpler test to avoid dealing with collisions: compare DKES trajectories to true trajectories.

- Result: DKES systematically under-predicts the fraction of trapped particles.

Drift-kinetic equation

Characteristics of the drift-kinetic equation give effective particle guiding-center trajectories.

$$\underbrace{\left(\frac{d\psi}{dt}\right)\frac{\partial f}{\partial \psi}+\left(\frac{d\theta}{dt}\right)\frac{\partial f}{\partial \theta}+\left(\frac{d\zeta}{dt}\right)\frac{\partial f}{\partial \zeta}}_{\text{3 spatial coordinates}}+\underbrace{\left(\frac{d\upsilon}{dt}\right)\frac{\partial f}{\partial \upsilon}+\left(\frac{d\xi}{dt}\right)\frac{\partial f}{\partial \xi}}_{\text{2 velocity coordinates, e.g. speed \& pitch angle}}=C\{f\}$$

Equations of motion

True equations:

$$\frac{d\theta}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi + \mathbf{v}_m \right)\cdot\nabla\theta$$

$$\frac{d\zeta}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi + \mathbf{v}_m \right)\cdot\nabla\zeta$$

$$\frac{d\psi}{dt} = \mathbf{v}_m\cdot\nabla\psi$$

$$\frac{d\mu}{dt} = 0$$

$$\frac{dH}{dt} = 0 \quad \text{where} \quad H = \frac{m\upsilon^2}{2} + Ze\Phi$$

Equivalent true equations:

$$\frac{d\theta}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi + \mathbf{v}_m \right)\cdot\nabla\theta$$

$$\frac{d\zeta}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi + \mathbf{v}_m \right)\cdot\nabla\zeta$$

$$\frac{d\psi}{dt} = \mathbf{v}_m\cdot\nabla\psi$$

$$\frac{d\xi}{dt} = -\frac{\upsilon}{2B^2}\left(1-\xi^2\right)\mathbf{B}\cdot\nabla B + \frac{c}{2B^3}\frac{d\Phi}{d\psi}\xi\left(1-\xi^2\right)\mathbf{B}\times\nabla\psi\cdot\nabla B + \frac{\xi^2-1}{2\xi B}\mathbf{v}_m\cdot\nabla B \quad \text{where} \quad \xi = \upsilon_{\parallel}/\upsilon$$

$$\frac{d\upsilon}{dt} = \frac{c\upsilon}{2B^3}\frac{d\Phi}{d\psi}\left(1+\xi^2\right)\mathbf{B}\times\nabla\psi\cdot\nabla B$$

DKES equations:

$$\frac{d\theta}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{\langle B^2\rangle}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi \right)\cdot\nabla\theta$$

$$\frac{d\xi}{dt} = \left(\upsilon_{\parallel}\mathbf{b} + \frac{c}{\langle B^2\rangle}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi \right)\cdot\nabla\zeta$$

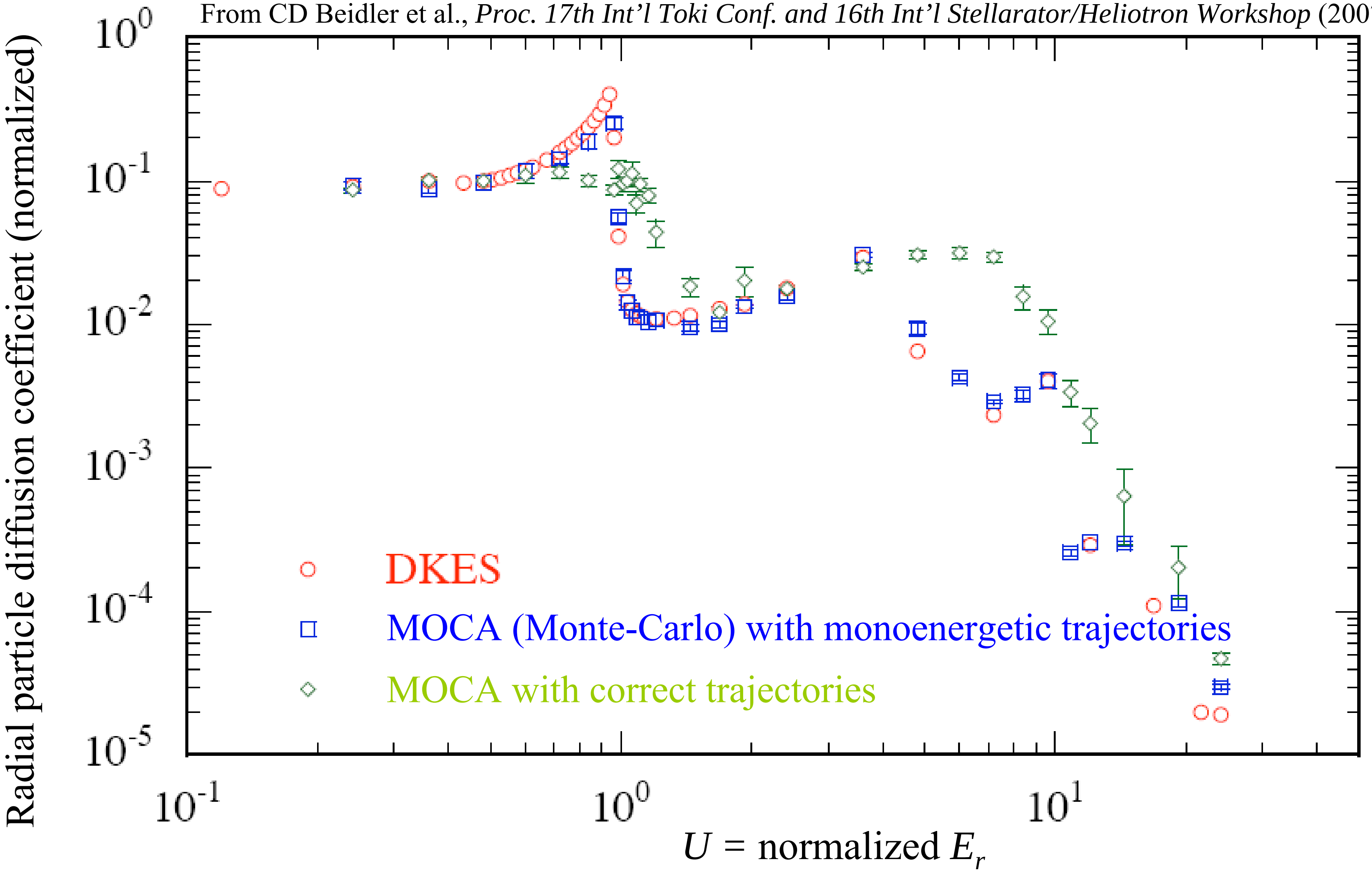
$$\frac{d\psi}{dt} = \mathbf{v}_m\cdot\nabla\psi$$

$$\frac{d\xi}{dt} = -\frac{\upsilon}{2B^2}\left(1-\xi^2\right)\mathbf{B}\cdot\nabla B$$

$$\frac{d\upsilon}{dt} = 0$$

W7-X

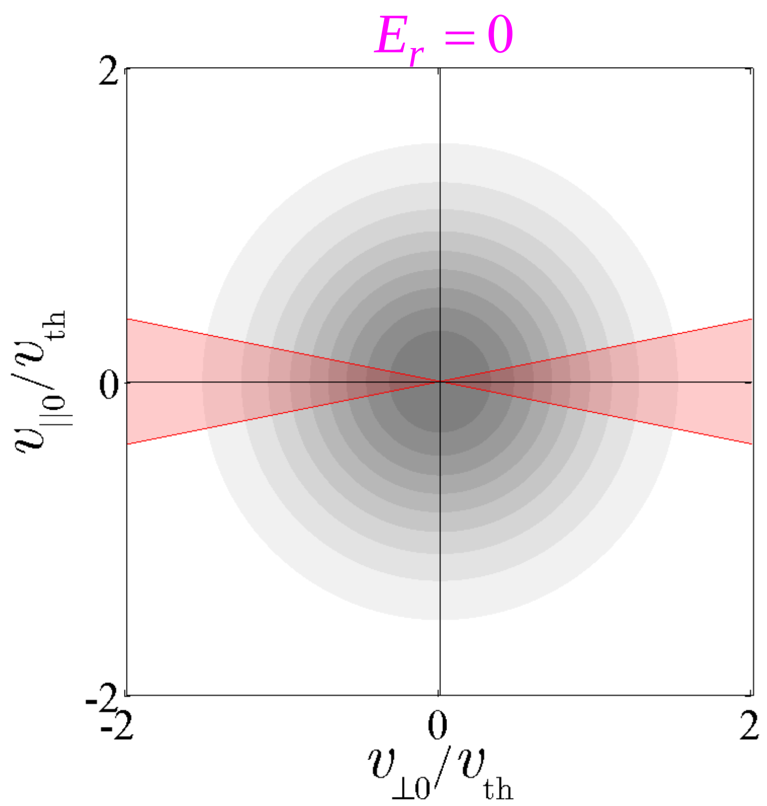
From CD Beidler et al., *Proc. 17th Int'l Toki Conf. and 16th Int'l Stellarator/Heliotron Workshop* (2007)

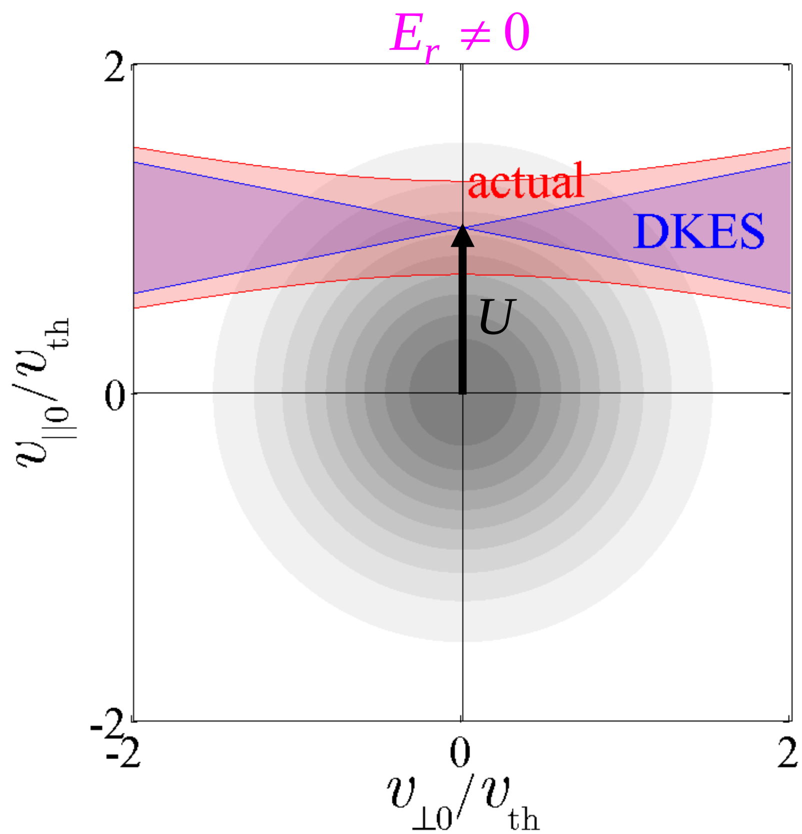


Monte-Carlo codes give noisy results for the flow and bootstrap current due to $\pm\upsilon_\parallel$ cancellation.

Trapped trajectories

For axisymmetric or quasisymmetric $\mathbf{B}$ field with constant $\mathrm{d}\Phi/\mathrm{d}\psi$:





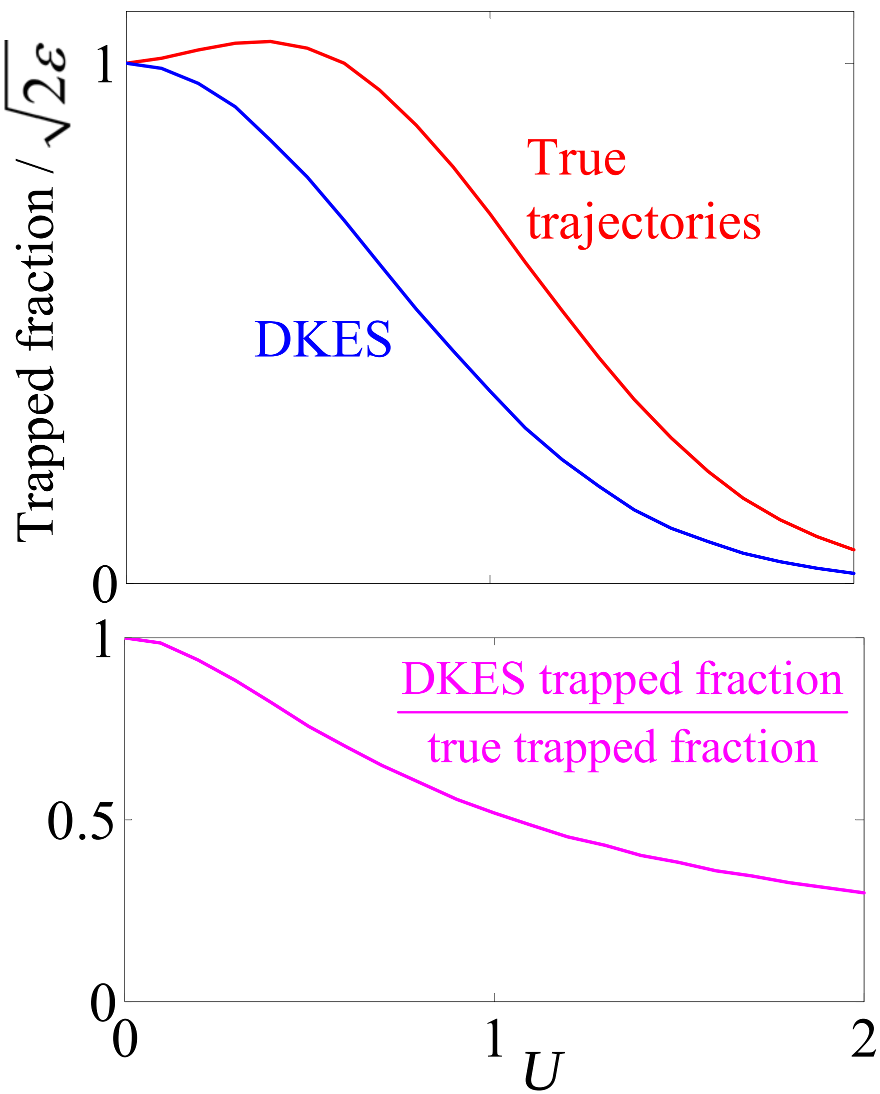
U can be ~ 1 even if $|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}| \ll |v_{th}|$.

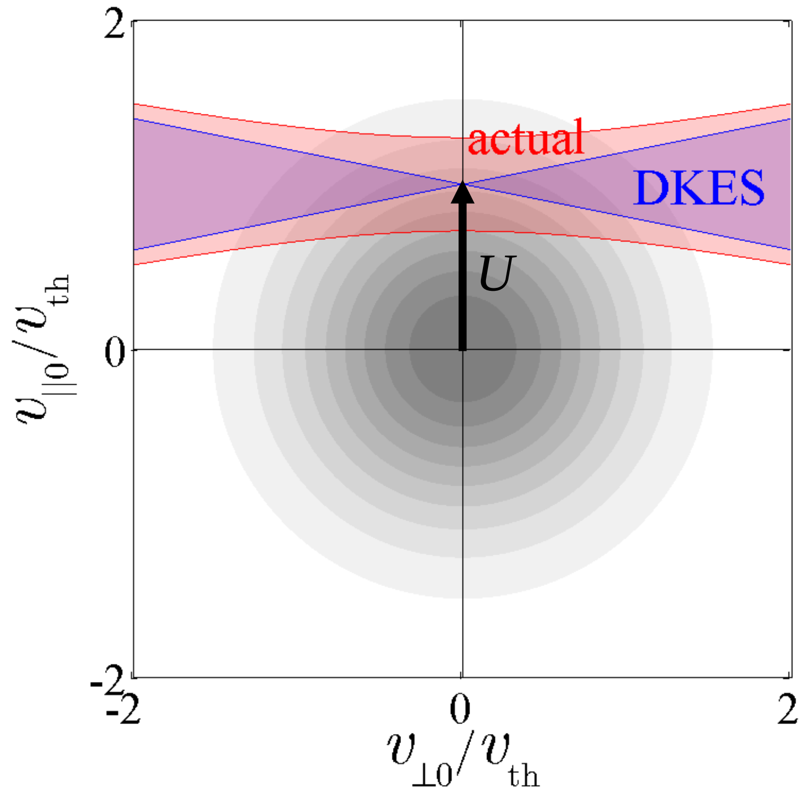
$U \sim 1$ in HSX?

$$U = -\frac{\mathbf{v}_{\mathbf{E}\times\mathbf{B}} \cdot \nabla B}{v_{th}\mathbf{b} \cdot \nabla B} = -\frac{B_t}{B_p} \frac{|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}|}{v_{th}}$$

Trapped trajectories

For axisymmetric or quasisymmetric **B** field with constant $\mathrm{d}\Phi/\mathrm{d}\psi$:





$$U = -\frac{\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla B}{\upsilon_{th}\mathbf{b}\cdot\nabla B} = -\frac{B_t}{B_p}\frac{\left|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right|}{\upsilon_{th}}$$

Trapped fraction

The true fraction of trapped particles (last page) is not the same as the "effective trapped fraction"

$$f_t^{eff} = 1 - \frac{3}{4}\left\langle B^2 \right\rangle \int_0^{1/B\max} \frac{\lambda \, d\lambda}{\left\langle \sqrt{1-\lambda B} \right\rangle} \approx 1.46\sqrt{\varepsilon}$$

which appears in banana-regime neoclassical quantities.

The conventional heat flux, viscosities, and bootstrap current are all $\propto f_t^{eff}$.

Connection to tokamak pedestals

Drift-kinetic equation:

$$\upsilon_{\|}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla f_1 + ... = C\{f_1\}$$

In either axisymmetry or quasisymmetry, the ratio of these terms is

$$\frac{\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla f_1}{\upsilon_{\|}\mathbf{b}\cdot\nabla f_1} \sim \frac{\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla f_1}{\upsilon_{th}\mathbf{b}\cdot\nabla f_1} = \frac{\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla B}{\upsilon_{th}\mathbf{b}\cdot\nabla B} = U = \frac{B_t}{B_p}\frac{\left|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}\right|}{\upsilon_{th}}$$

Tokamak: $$V_{i\|} = -\frac{T_i R B_t}{m_i \Omega_i}\left(\frac{Ze}{T_i}\frac{d\Phi}{d\psi} + \frac{1}{p_i}\frac{dp_i}{d\psi} - \frac{1.17}{T_i}\frac{dT_i}{d\psi}\right)$$

$\upsilon_{th,i}U$ $\quad$ $\upsilon_{th,i}\rho_p/L_p$ $\quad$ $\upsilon_{th,i}\rho_p\,/\,L_T$

So in a tokamak, you can simultaneously have $U\sim 1$ and $\upsilon_{\mathrm{E\times B}} \ll \upsilon_{\mathrm{th,i}}$ only if radial gradient scale length is ρ_p.

This is precisely the ordering we used to analyze tokamak pedestals.

Tokamak

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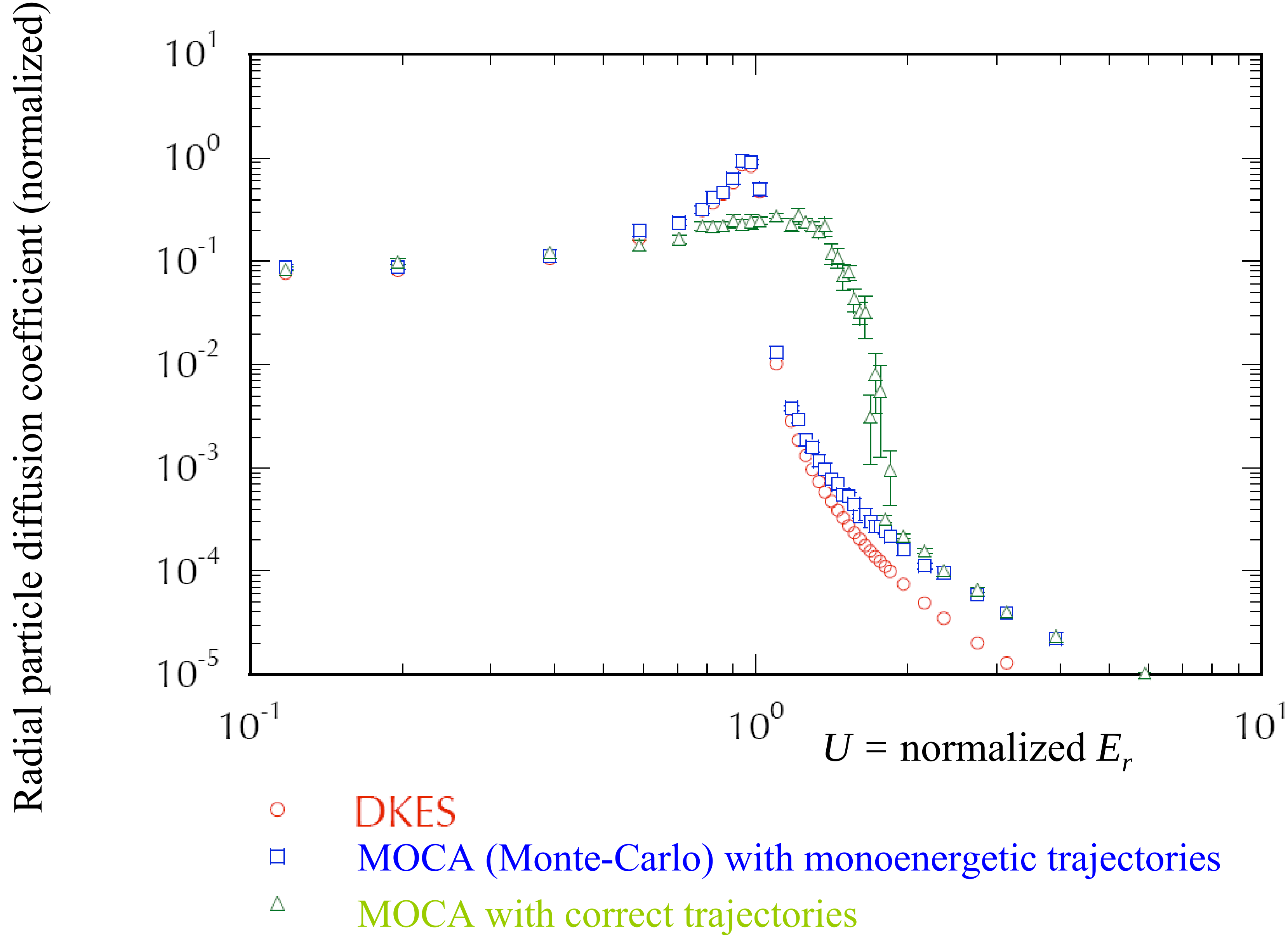
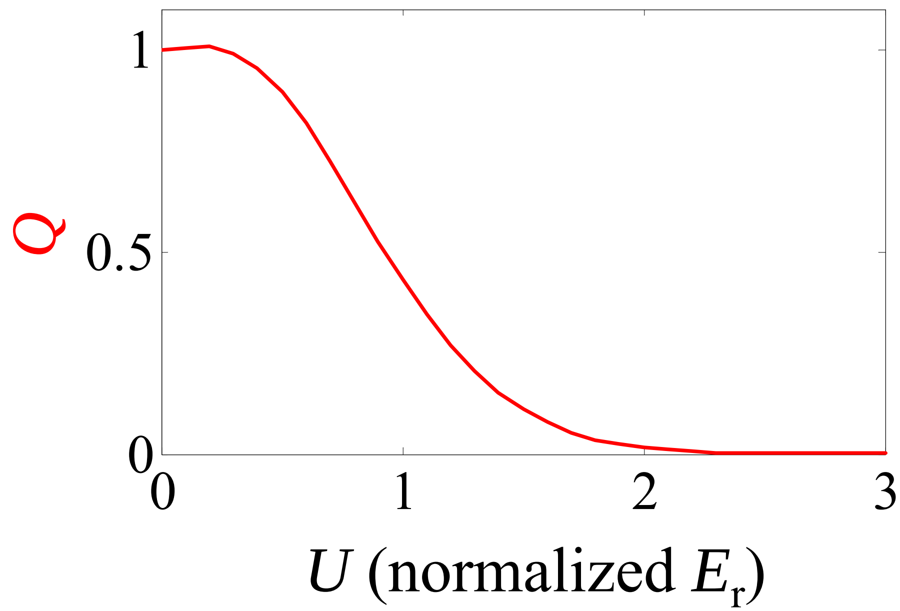


New results for quasisymmetry

- Kagan-Catto calculations for the tokamak pedestal ordering can be generalized to a quasisymmetric stellarator.
- We use generalized pitch-angle scattering model collision operator, to follow the shift in trapped-passing boundary.

For $B = B(\psi,\ M\theta - N\zeta)$,

$$\langle \mathbf{q}_\mathrm{i} \cdot \nabla\psi \rangle = -1.35\, Q(U) \sqrt{\varepsilon}\, \frac{(MG + NI)^2 \nu_\mathrm{i} n_\mathrm{i} T_\mathrm{i}}{(\iota M - N)^2 m_\mathrm{i} \langle \Omega^2 \rangle} \frac{dT_\mathrm{i}}{d\psi}$$

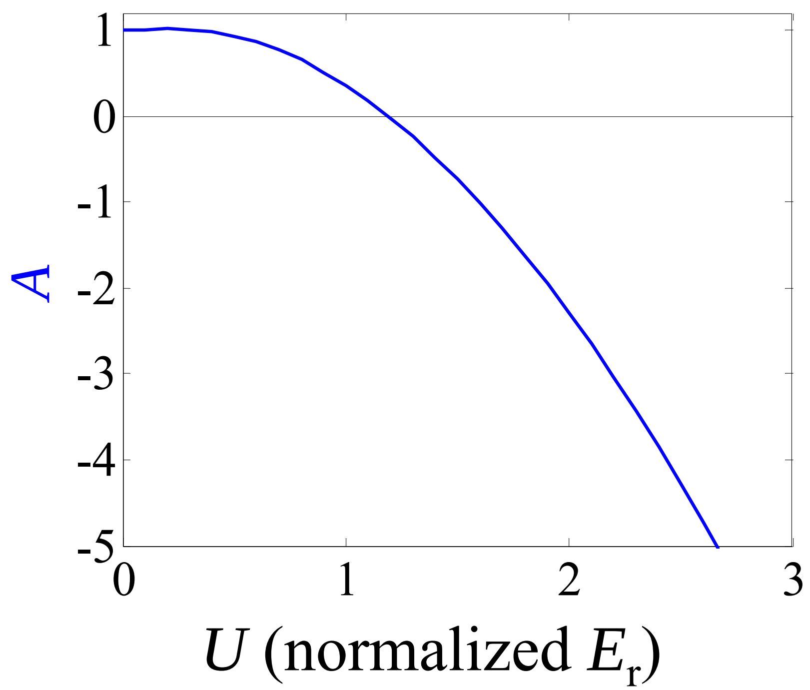


New results for quasisymmetry

For $B = B(\psi,\ M\theta - N\zeta)$,

$$\left\langle V_{\mathrm{i}\|} B \right\rangle = \frac{c p_{\mathrm{i}} \left(MG + NI \right)}{n_{\mathrm{i}} e \left(N - \iota M \right)} \left[\frac{1}{p_{\mathrm{i}}} \frac{dp_{\mathrm{i}}}{d\psi} + \frac{Ze}{T_{\mathrm{i}}} \frac{d\Phi}{d\psi} - \frac{1.17 A(U)}{T_{\mathrm{i}}} \frac{dT_{\mathrm{i}}}{d\psi} \right]$$

$$\left\langle j_{\|} B \right\rangle = 2.42 \sqrt{\varepsilon} \frac{c \left(MG + NI \right)}{N - \iota M} \left(\frac{dp_{\mathrm{e}}}{d\psi} + \frac{dp_{\mathrm{i}}}{d\psi} - 0.75 n \frac{dT_{\mathrm{e}}}{d\psi} - 1.17\, A(U)\, n \frac{dT_{\mathrm{i}}}{d\psi} \right)$$



Electric field causes enhancement of bootstrap current.

Recap

- Both in stellarators and tokamak pedestals, the simultaneous ordering $\upsilon_{\mathrm{E\times B}} \ll \upsilon_{\mathrm{th,i}}$ and $\mathbf{v}_{\mathrm{E\times B}} \cdot \nabla B \sim \upsilon_{\|} \mathbf{b} \cdot \nabla B$ is of interest.

- The conventional approach for calculating neoclassical transport in stellarators involves ad-hoc changes to the kinetic equation.

- These changes lead to $O(1)$ errors in the trapped fraction when E_r is large.