Sub-Larmor dual cascade in 2D gyrokinetic turbulence

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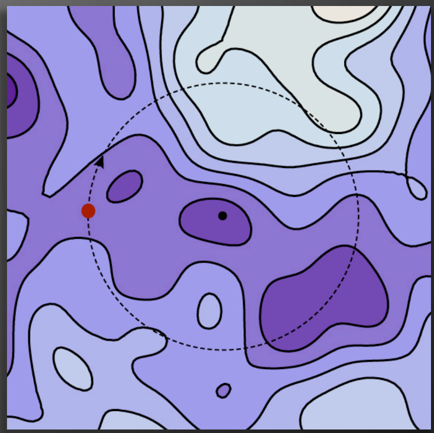

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M. Barnes, S. Cowley, W. Dorland, G. Howes, R. Numata, A. Schekochihin

Outline

- Two-dimensional gyrokinetics: Dual cascade by nonlinear phase-mixing
  - Equations + cartoon physics
  - Phase-space spectrum
  - Generalized Fjørtoft argument for dual cascade
- Dual cascade simulations
  - Setup and initial conditions
  - Decay laws
  - Shell-to-shell spectral transfer function

The Equations

Two-dimensional electrostatic gyrokinetic system:

"Gyro-averaged" ExB velocity

$$\frac{\partial g}{\partial t} + \mathbf{v}_E \cdot \boldsymbol{\nabla} g = \langle C[h] \rangle_{\mathbf{R}} \qquad (1) \qquad \mathbf{v}_E = \hat{\mathbf{z}} \times \boldsymbol{\nabla} \langle \varphi \rangle_{\mathbf{R}}$$

$$2\pi \int v dv \, \langle g \rangle_{\mathbf{r}} = (1+\tau)\varphi - \Gamma_0 \varphi \qquad (2)$$

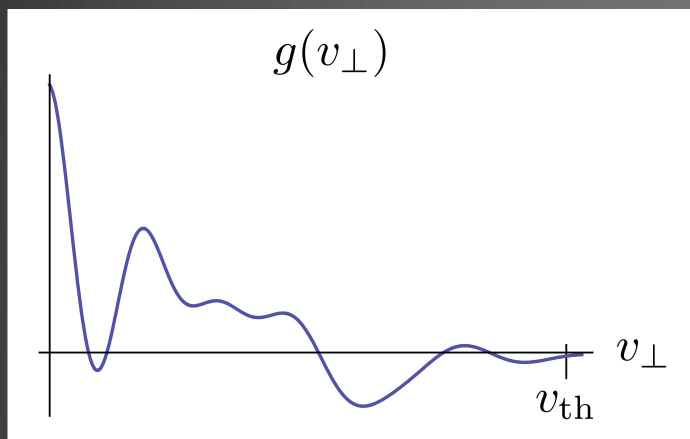
Quasi-neutrality

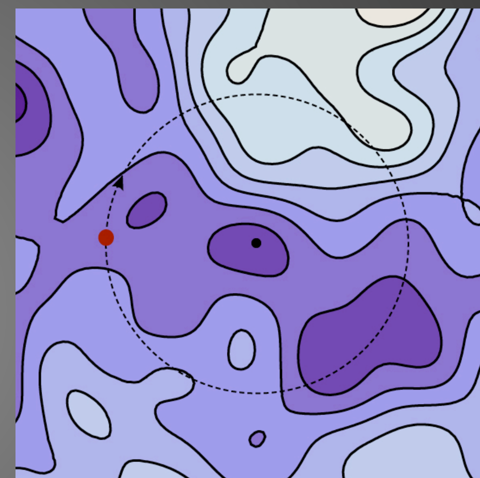
$g(\mathbf{R}_\perp,\ v_\perp,\ t)$ is the phase-space density of ion gyrocenters.

Intrinsic scales

velocity scale: $v_{\rm th}$

spatial scale: $\rho_{\rm th} = v_{\rm th}/\Omega_c$



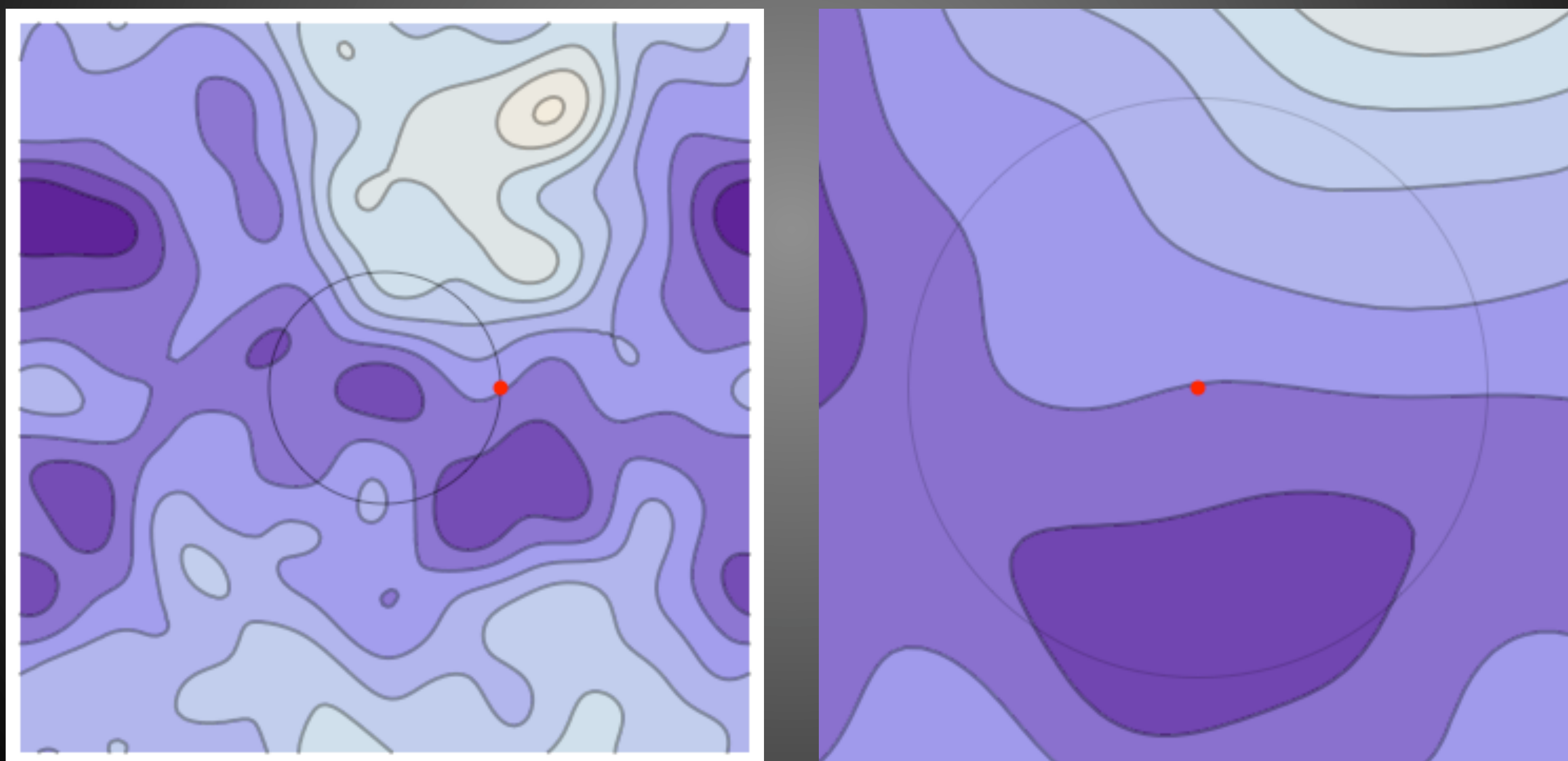


We will focus on fluctuations at **Sub-Larmor** scales:

$$k > \rho_{\rm th}^{-1}$$

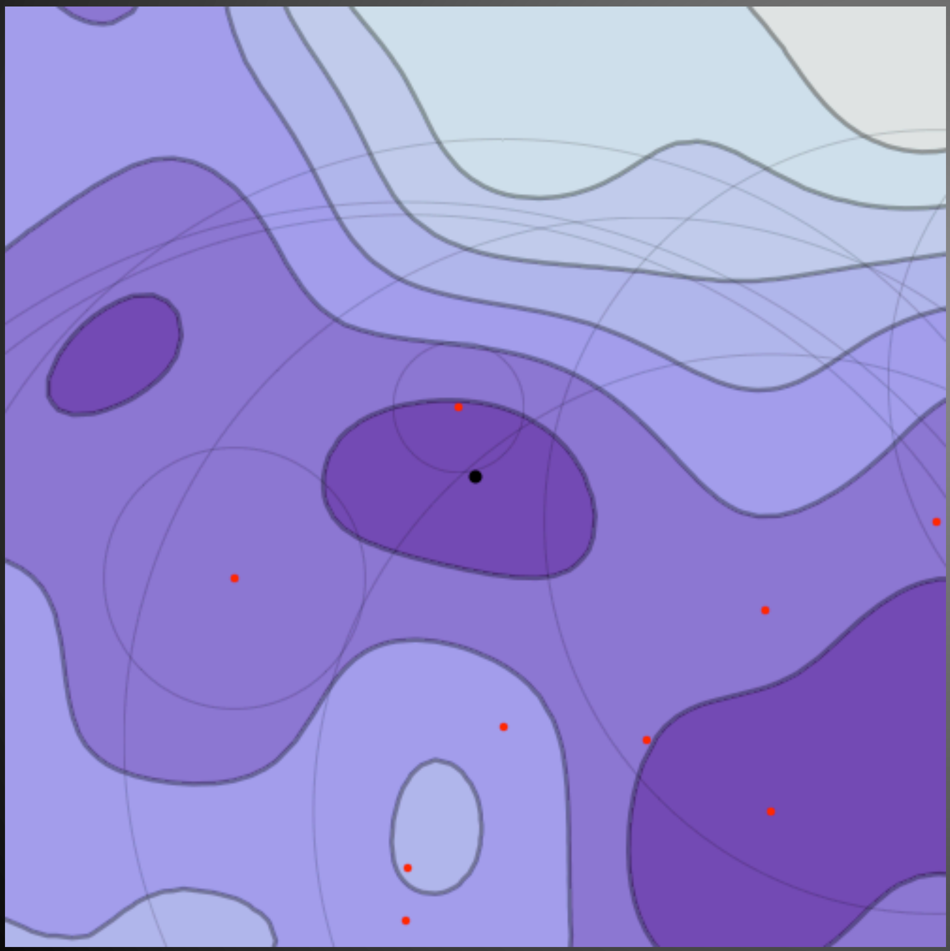
Particle motion

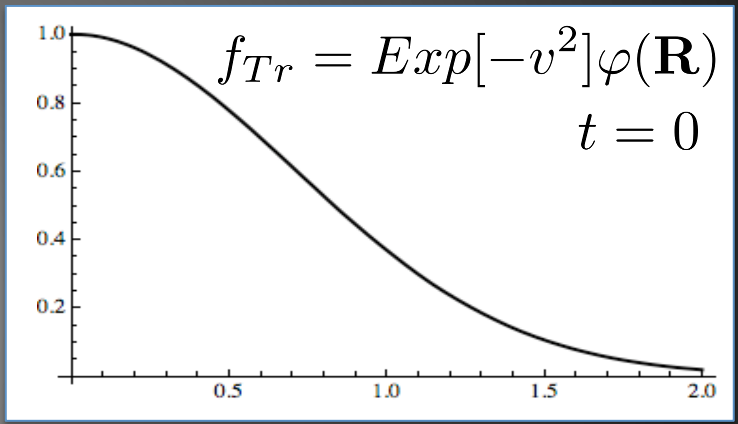
φ $\langle \varphi \rangle_{\mathbf{R}}$

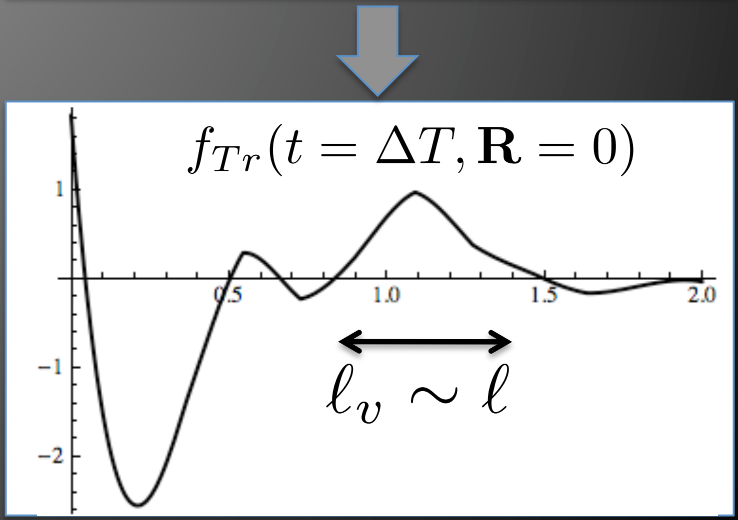


Cartoon of Phase Mixing

(Passive advection of *tracer* particle distribution f_{Tr})







Collisionless Invariants of 2D Gyrokinetics

"Generalized free energy"

$$W_g = 2\pi \int v dv \int \frac{d^2\mathbf{r}}{V} \frac{\langle g^2 \rangle_{\mathbf{r}}}{2F}$$

$$\left(G = \int \frac{d^2\mathbf{R}}{V} g^2 \right)$$

Electrostatic invariant (only in 2D!)

$$E = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} \left[\alpha\varphi^2 - \varphi\Gamma_0\varphi\right]$$

(Actual free energy (electrostatic):

$$W = T_0 \delta S = -\int \frac{d^2\mathbf{r}}{V} \frac{\delta f^2}{F_0} = -\int \frac{d^2\mathbf{r}}{V} \left[2\pi \int v dv \frac{\langle h^2 \rangle_{\mathbf{r}}}{F_0} - \alpha\varphi^2\right]$$)

Phase-space spectrum

Hankel-Fourier transform

$$\hat{g}(\mathbf{k},p) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} d^2\mathbf{R}\int_0^\infty v\,dv\,J_0(pv)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}}g(\mathbf{R},v)$$

Wavenumber spectrum for position and velocity space:

$$W_g = \int dk\,dp\,W_g(k,p)$$

Thus the spectra are defined:

$$W_g(k,p) = \pi\;pk\;\overline{|\hat{g}(k,p)|^2}$$

$$E(k) = 2\pi\beta(k)k^{-1}W_g(k,k)$$

Navier—Stokes:
Simple mechanism for dual cascade

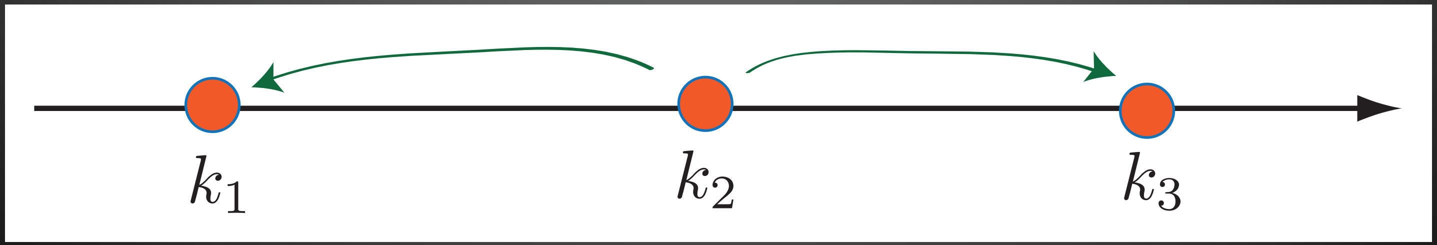
[Fjørtoft 1953]

$$\Delta E = 0 \longrightarrow \Delta E_1 + \Delta E_2 + \Delta E_3 = 0$$

$$\Delta Z = 0 \longrightarrow k_1^2 \Delta E_1 + k_2^2 \Delta E_2 + k_3^2 \Delta E_3 = 0$$

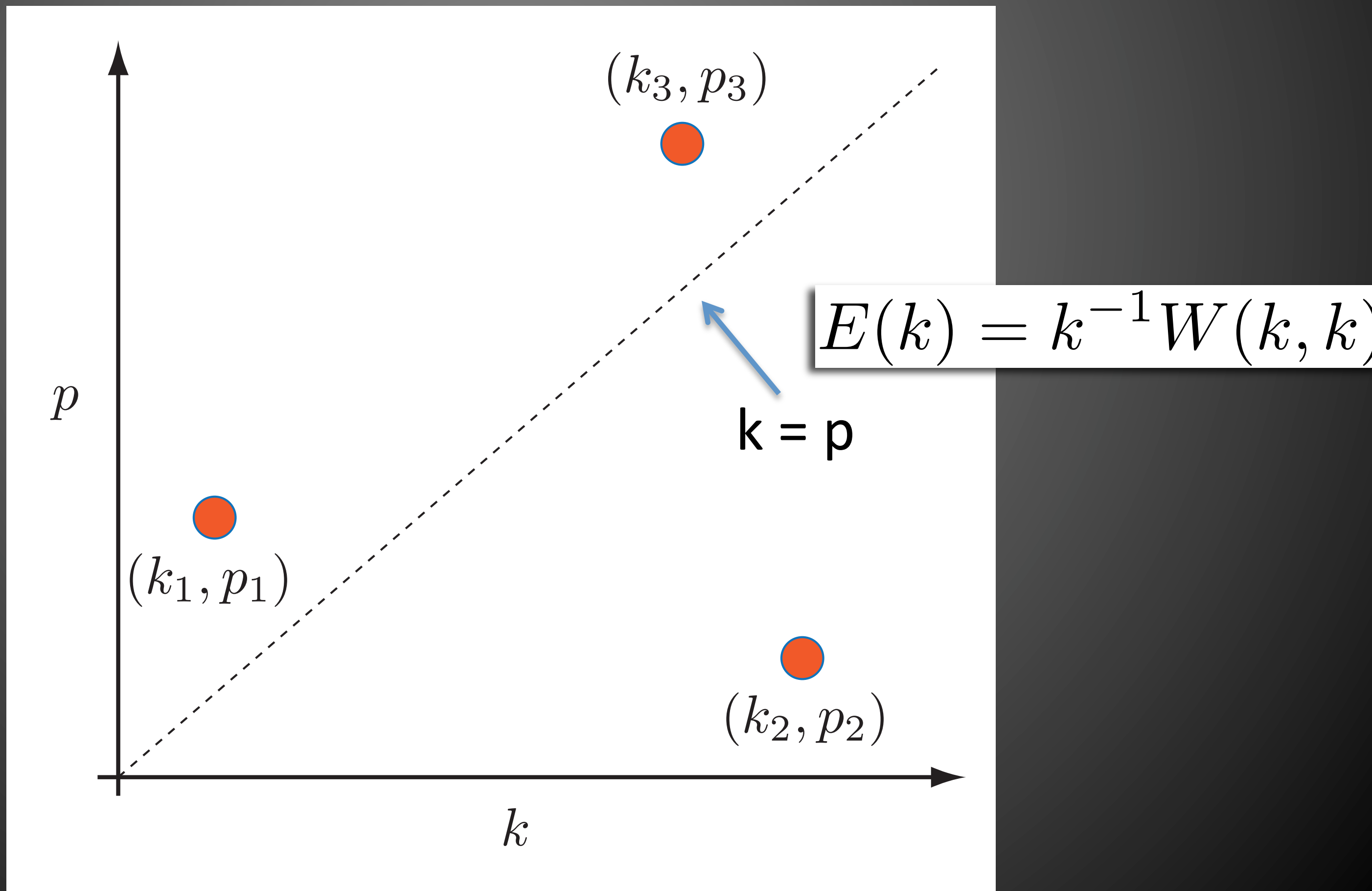
$$\Delta E_1 = -\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2} \Delta E_2 \qquad \Delta E_3 = -\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2} \Delta E_2$$

Inspection of the coefficients reveals that the intermediate wavenumber is either only a source or only a sink.



Three-scale interactions in phase-space: (k_1, p_1), (k_2, p_2) and (k_3, p_3)

$\Delta E_1 + \Delta E_2 + \Delta E_3 = 0$ and $\Delta W_1 + \Delta W_2 + \Delta W_3 = 0$



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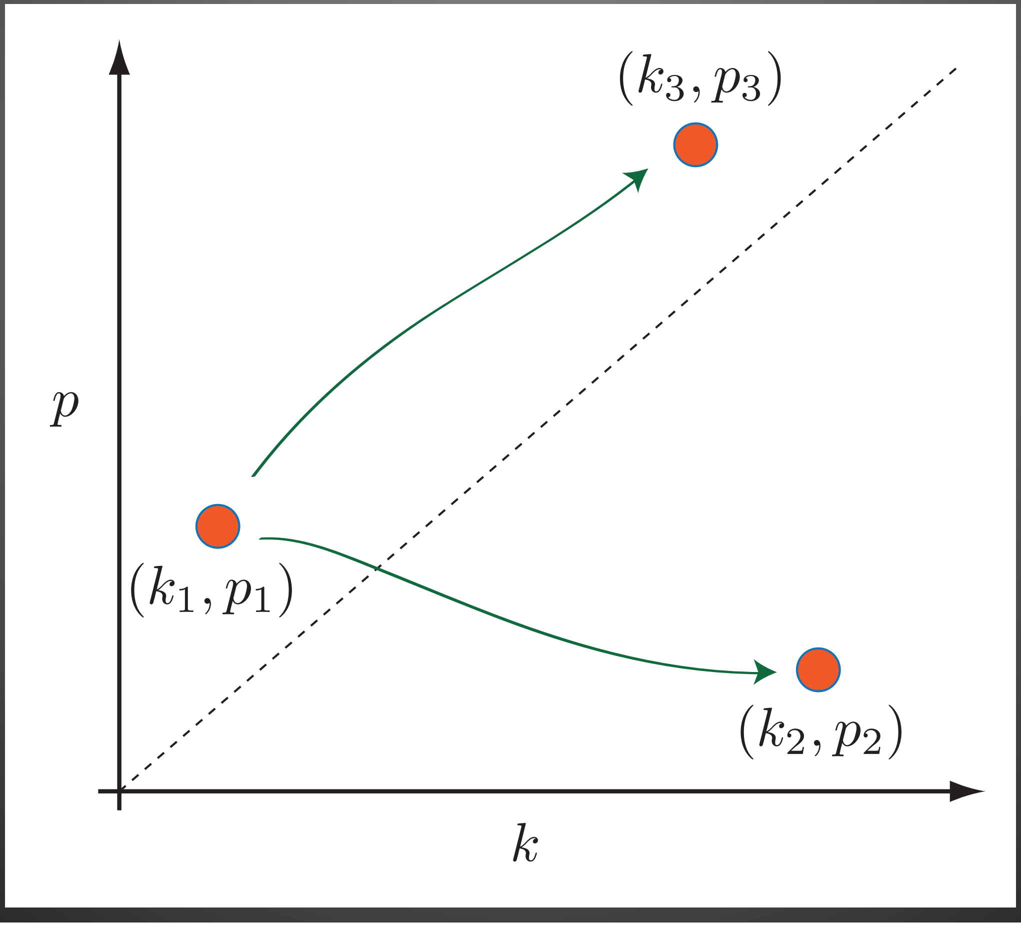
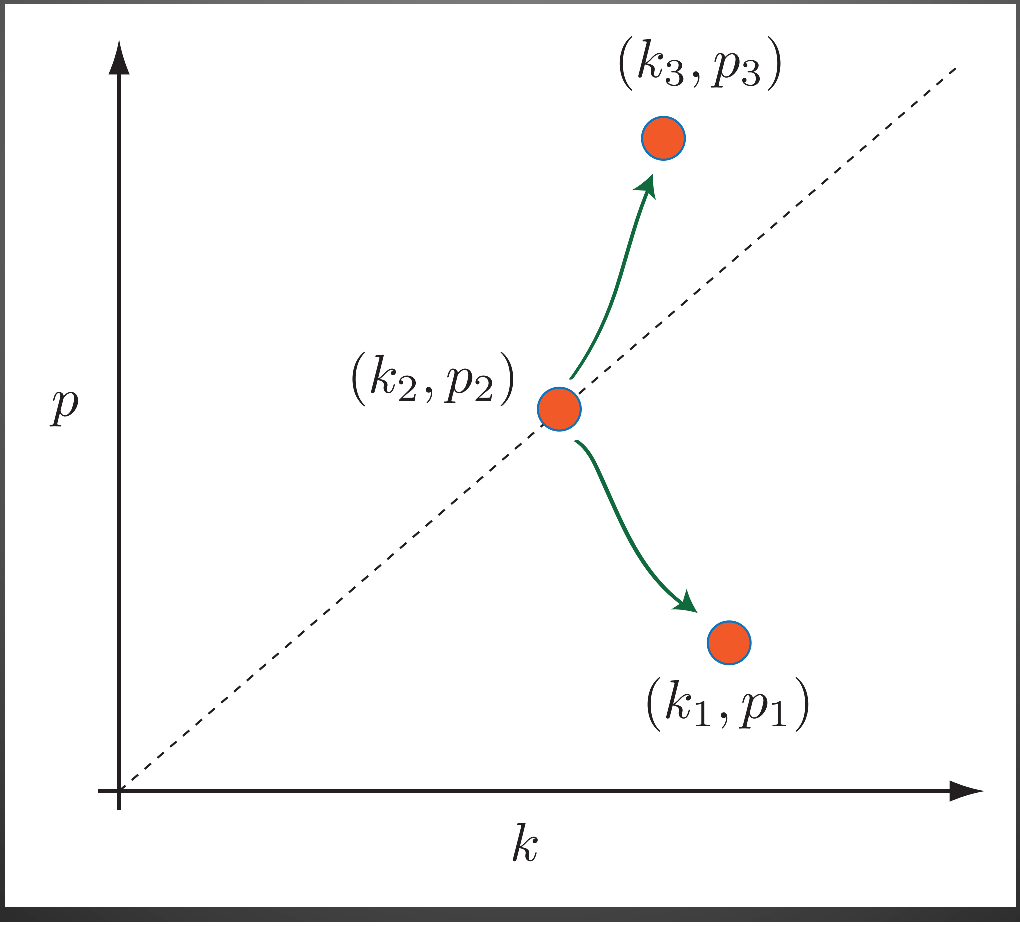


Three-scale interactions in phase-space: (k_1, p_1), (k_2, p_2) and (k_3, p_3)

$\boxed{\Delta W_1 + \Delta W_2 + \Delta W_3 = 0}$ but $\boxed{\Sigma\, \Delta E_i = \Delta E_2 \neq 0 \text{ !!}}$



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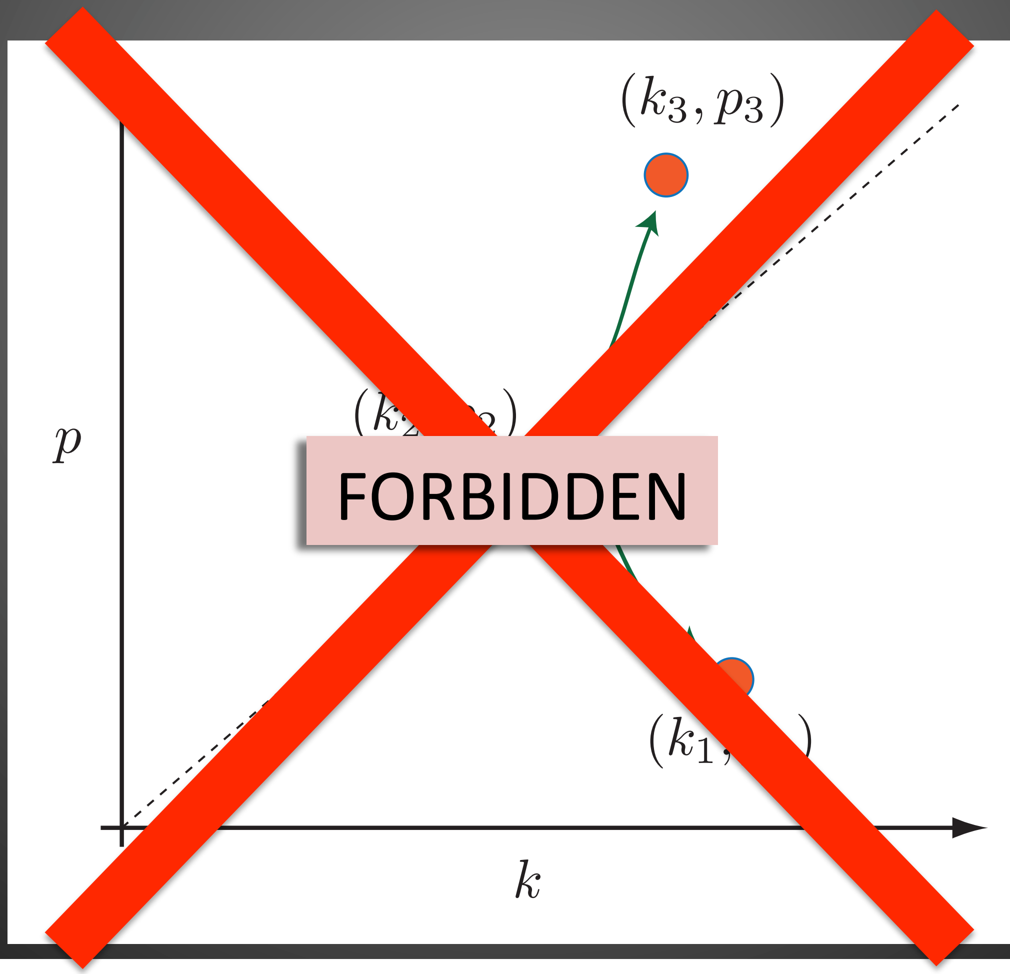
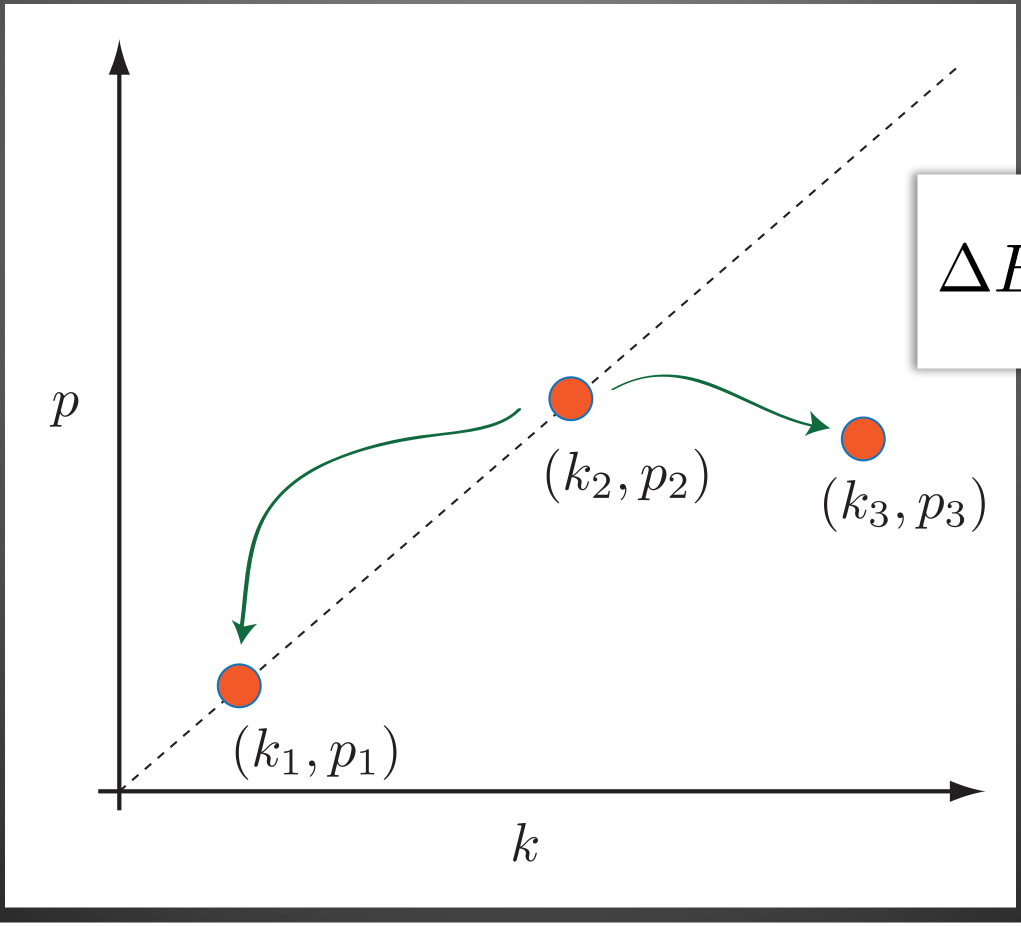


Three-scale interactions in phase-space: (k_1, p_1), (k_2, p_2) and (k_3, p_3)

$$k_1 \Delta E_1 + k_2 \Delta E_2 = -\Delta W_3$$ and $$\Delta E_1 + \Delta E_2 = 0$$

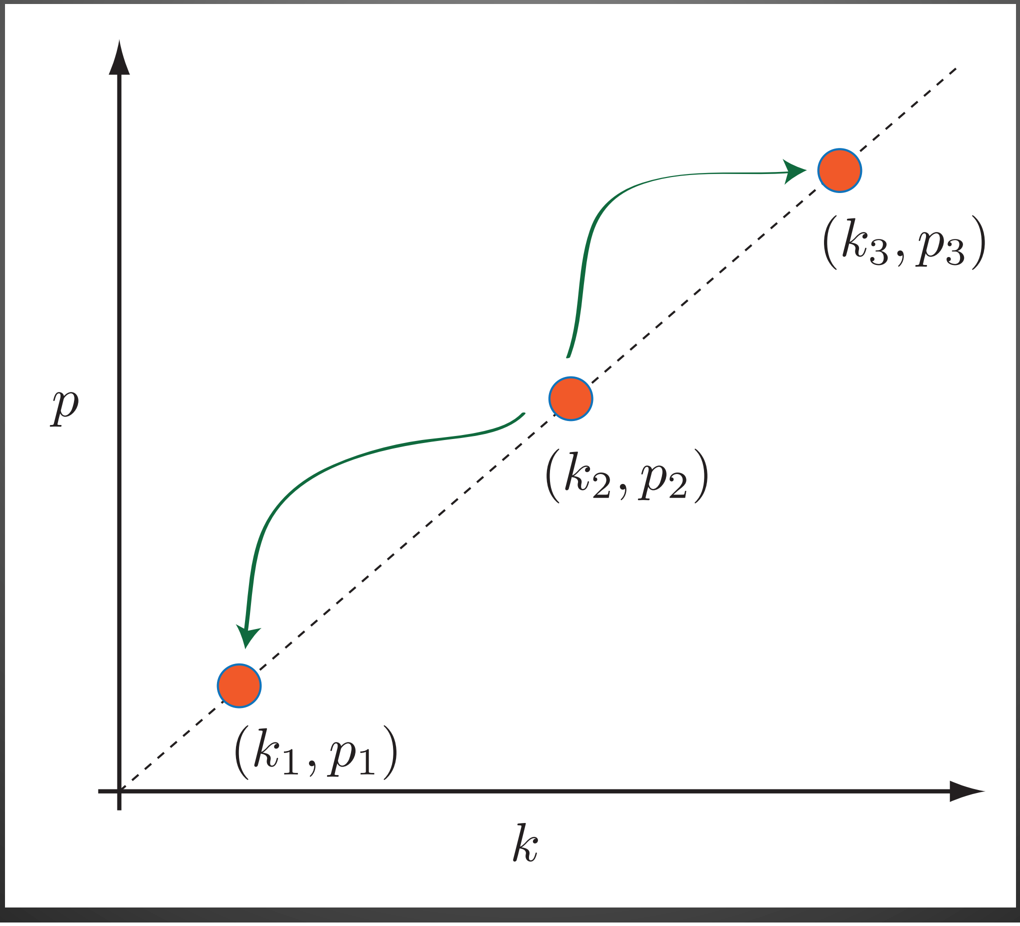
$$\Delta E_1 = \frac{\Delta W_3}{k_2 - k_1}$$



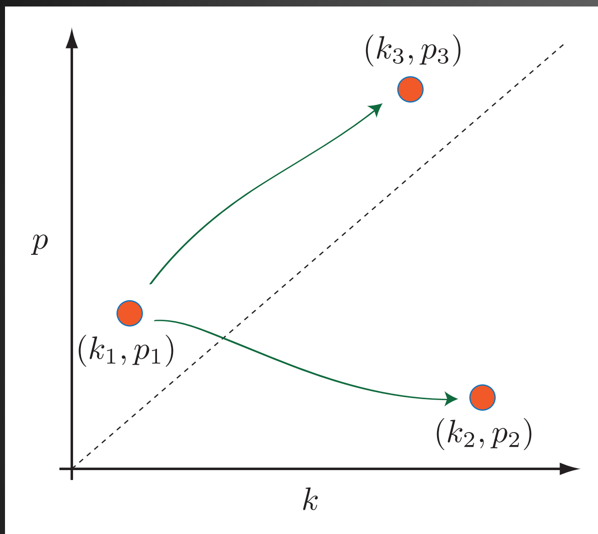
Three-scale interactions in phase-space: (k_1, p_1), (k_2, p_2) and (k_3, p_3)

$$k_1 \Delta E_1 + k_2 \Delta E_2 + k_3 \Delta E_3 = 0$$

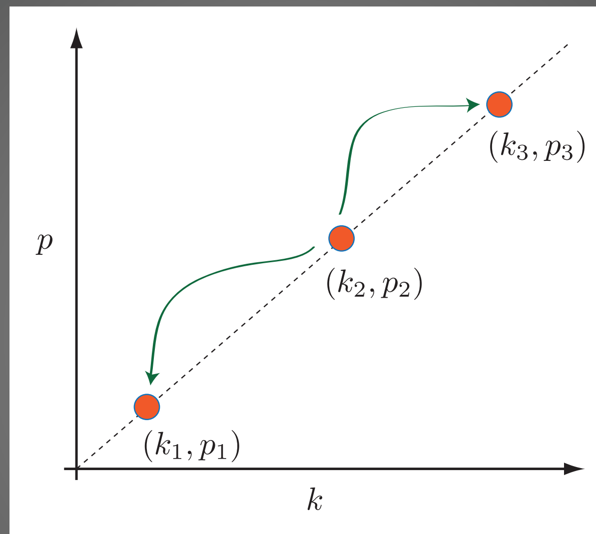
$$\Delta E_1 + \Delta E_2 + \Delta E_3 = 0$$



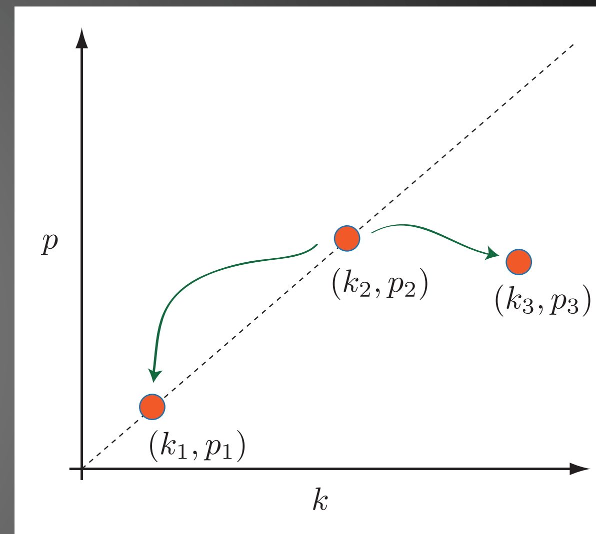
Three-scale interactions in phase-space: (k_1, p_1), (k_2, p_2) and (k_3, p_3)



Unconstrained



Fjørtoft-type



Enhanced

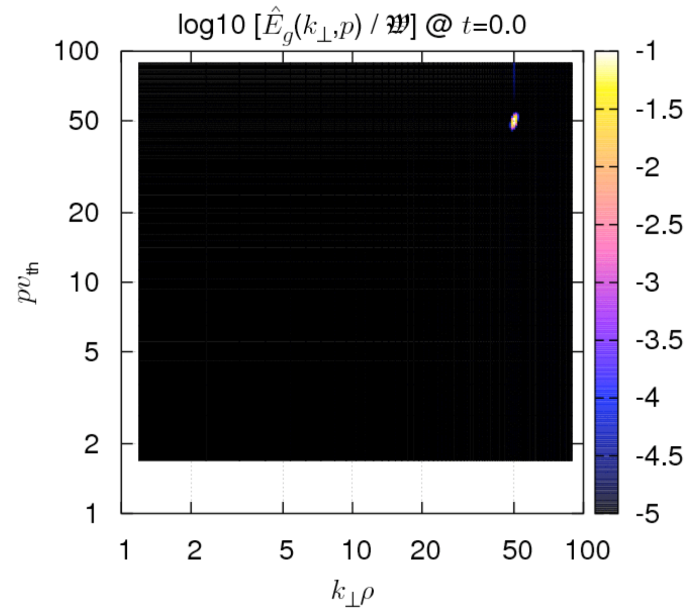
- Relaxation of free energy among three wavenumber pairs forces inverse transfer of electrostatic energy.
- This corresponds to an inverse transfer along the diagonal in phase-space spectrum

Dual cascade simulation — coherent dist. —

Straight homogeneous slab: $L_x = L_y = 2\pi\rho$.
Coherent initial condition

$$g_{\boldsymbol{k}} = C \frac{k_\perp^2}{k_0^2} \exp\left[-\left(\frac{k_\perp - k_0}{k_{\mathrm{w}}} \right)^2 \right] J_0\left(\frac{k_\perp v_\perp}{\Omega} \right) e^{-v^2}$$

run	ν	$k_0\rho$	resolution
A	10^{-2}	15	$64^2 \times 32^2$
B	4×10^{-3}	25	$128^2 \times 64^2$
C	5×10^{-3}	25	$128^2 \times 64^2$
D	2×10^{-3}	50	$256^2 \times 128^2$



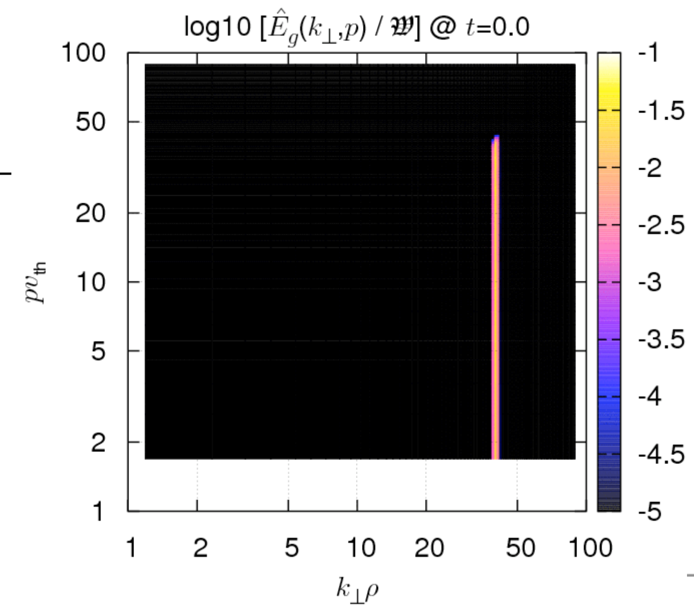
Dual cascade simulation — random dist. —

Random initial condition

$$g_{\boldsymbol{k}} = C\frac{k_\perp^2}{k_0^2}\exp\left[-\left(\frac{k_\perp - k_0}{k_\mathrm{w}}\right)^2\right]\frac{1}{N}\sum_{j=1}^{N}(2\delta_j - 1)\sqrt{p_j}J_0(p_j v_\perp)e^{-v^2}$$

where $p_j = (k_\perp + k_\mathrm{w})\eta_j$; δ_j & η_j: random numbers

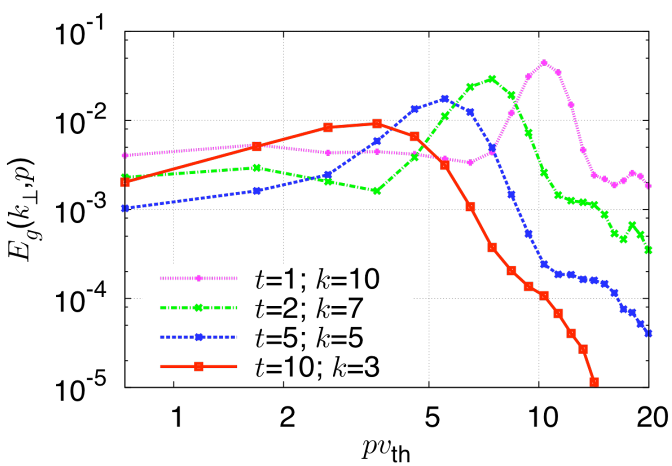
run	ν	$k_0\rho$	resolution
E	0.01	25	$128^2 \times 64^2$
F	5×10^{-3}	30	$256^2 \times 128^2$
G	5×10^{-3}	40	$256^2 \times 128^2$

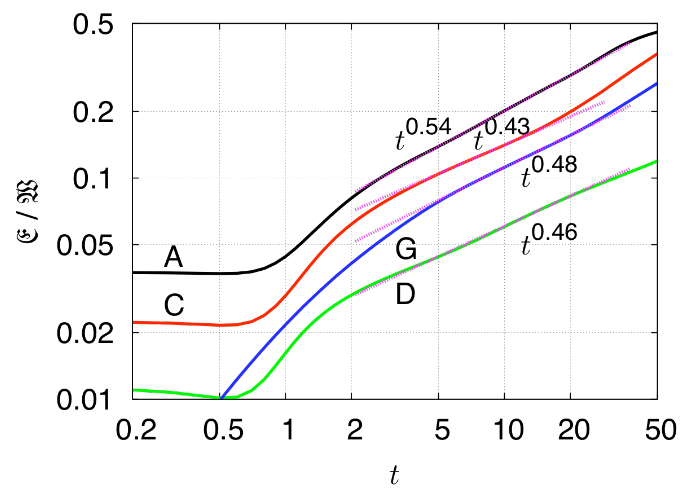


Dual cascade simulation — random dist. —

- Slices of 2D spectrum show peak at diagonal (left).
- Decay shows similar power law for both initial conditions (right):

$$\frac{\mathfrak{E}}{\mathfrak{W}} \sim t^{0.5}.$$





Dual cascade — decay law —

From simulations:

- nearly monochromatic spectrum of scale l_*

$$\mathfrak{W} \propto g_*^2, \qquad \mathfrak{E} \propto \varphi_*^2.$$

Dual cascade — decay law —

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- nearly monochromatic spectrum of scale l_*

$$\mathfrak{W} \propto g_*^2, \qquad \mathfrak{E} \propto \varphi_*^2.$$

- entropy concentrated at the diagonal ($k_\perp \rho \simeq p v_{\rm th}$)
  - $g_k \simeq J_0(k_\perp v_\perp) e^{-v^2/v_{\rm th}^2}$ [NB: $g_k \simeq R(k_\perp v_\perp) e^{-v^2/v_{\rm th}^2}$ in direct cascade]
  - $\mathfrak{W} \propto k_* \mathfrak{E}$ [NB: $\mathfrak{W} \propto k_*^2 \mathfrak{E}$ in direct case]

Dual cascade — decay law —

From simulations:

- nearly monochromatic spectrum of scale l_*

$$\mathfrak{W} \propto g_*^2, \qquad \mathfrak{E} \propto \varphi_*^2.$$

- entropy concentration along $k_\perp \rho \simeq p v_{\mathrm{th}}$: $\mathfrak{W} \propto k_* \mathfrak{E}$
- second invariant decays collisionally

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From simulations:

- nearly monochromatic spectrum of scale l_*

$$\mathfrak{W} \propto g_*^2, \qquad \mathfrak{E} \propto \varphi_*^2.$$

- entropy concentration along $k_\perp \rho \simeq p v_{\rm th}$: $\mathfrak{W} \propto k_* \mathfrak{E}$
- second invariant decays collisionally

$$\frac{d\mathfrak{E}}{dt} \propto -\nu k_*^2 \mathfrak{E}$$

$$\Downarrow$$

$$k_* \propto t^{-1/2}, \qquad \frac{\mathfrak{E}}{\mathfrak{W}} \propto t^{1/2}.$$

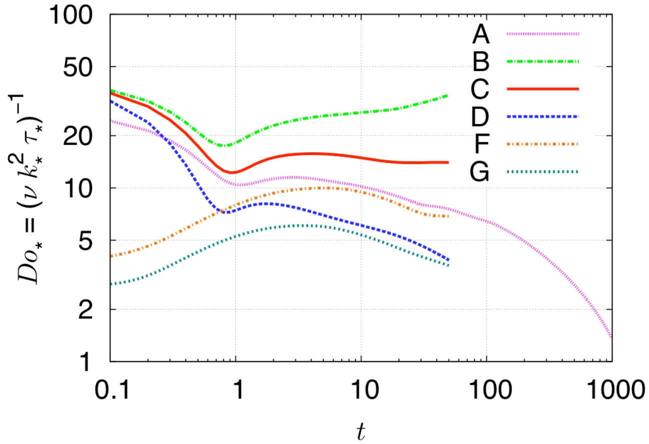
Dual cascade — critical Do_* —

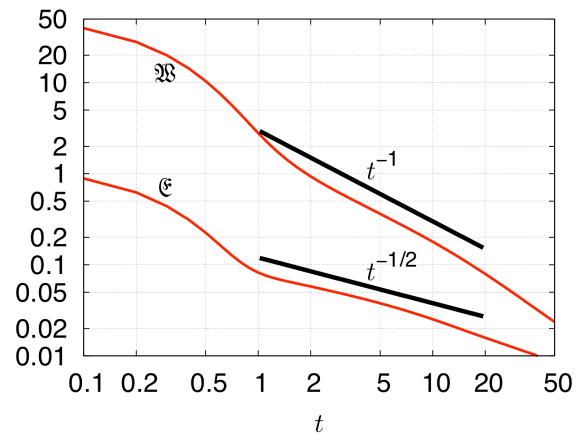
What happens to change of $\mathfrak{W}$?

$$\frac{d\mathfrak{W}}{dt} \propto -\frac{\mathfrak{W}}{\tau_*} - \nu(k_*\rho)^2\mathfrak{W} \quad \Rightarrow \quad Do_* = \frac{1}{\nu(k_*\rho)^2\tau_*} \simeq \frac{\mathfrak{E}}{\nu\mathfrak{W}^{1/2}}$$

In the critical case these two terms behave similarly.

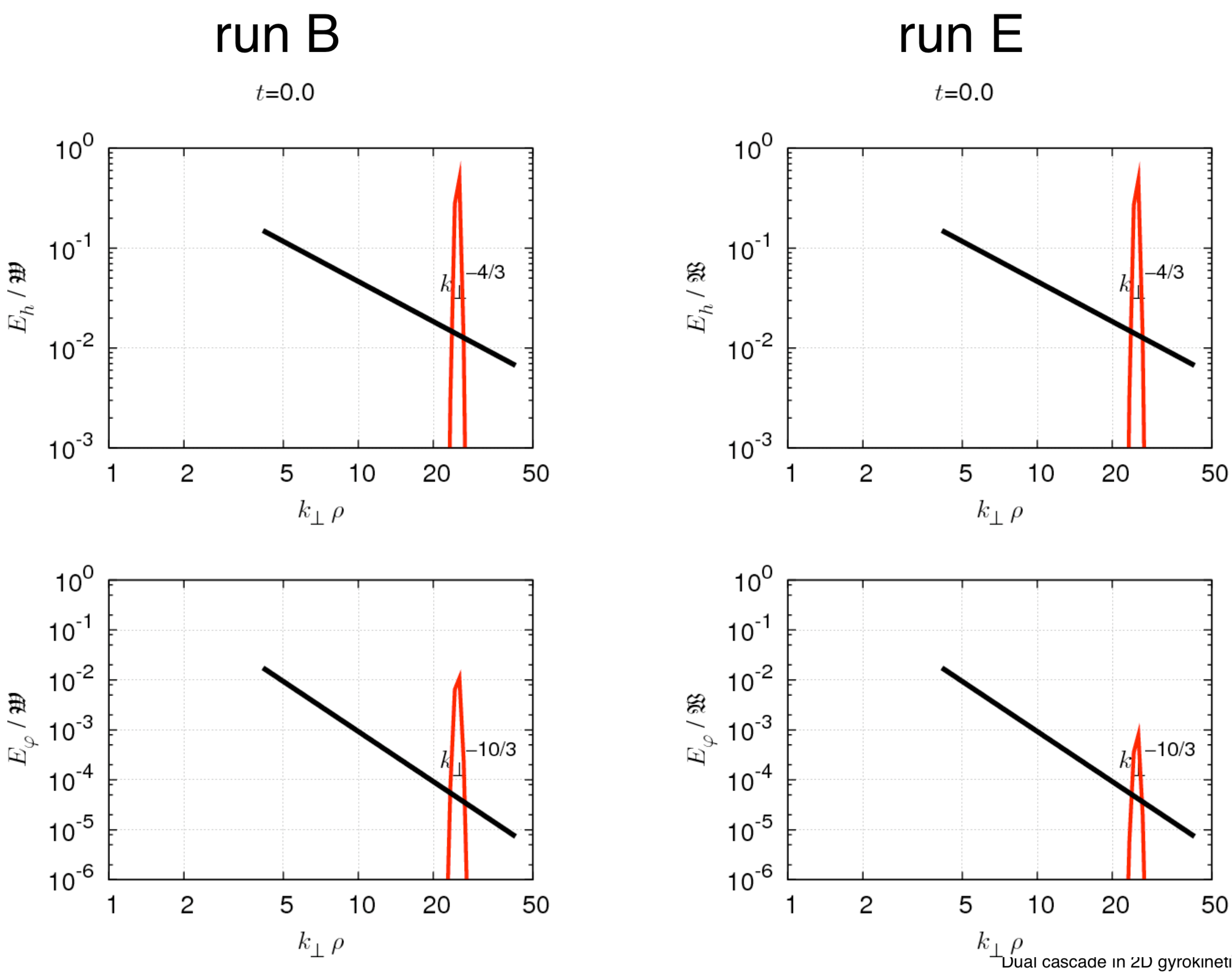
$$\tau_* \sim t \qquad \Rightarrow \qquad \mathfrak{E} \propto t^{-1/2}, \quad \mathfrak{W} \propto t^{-1}.$$





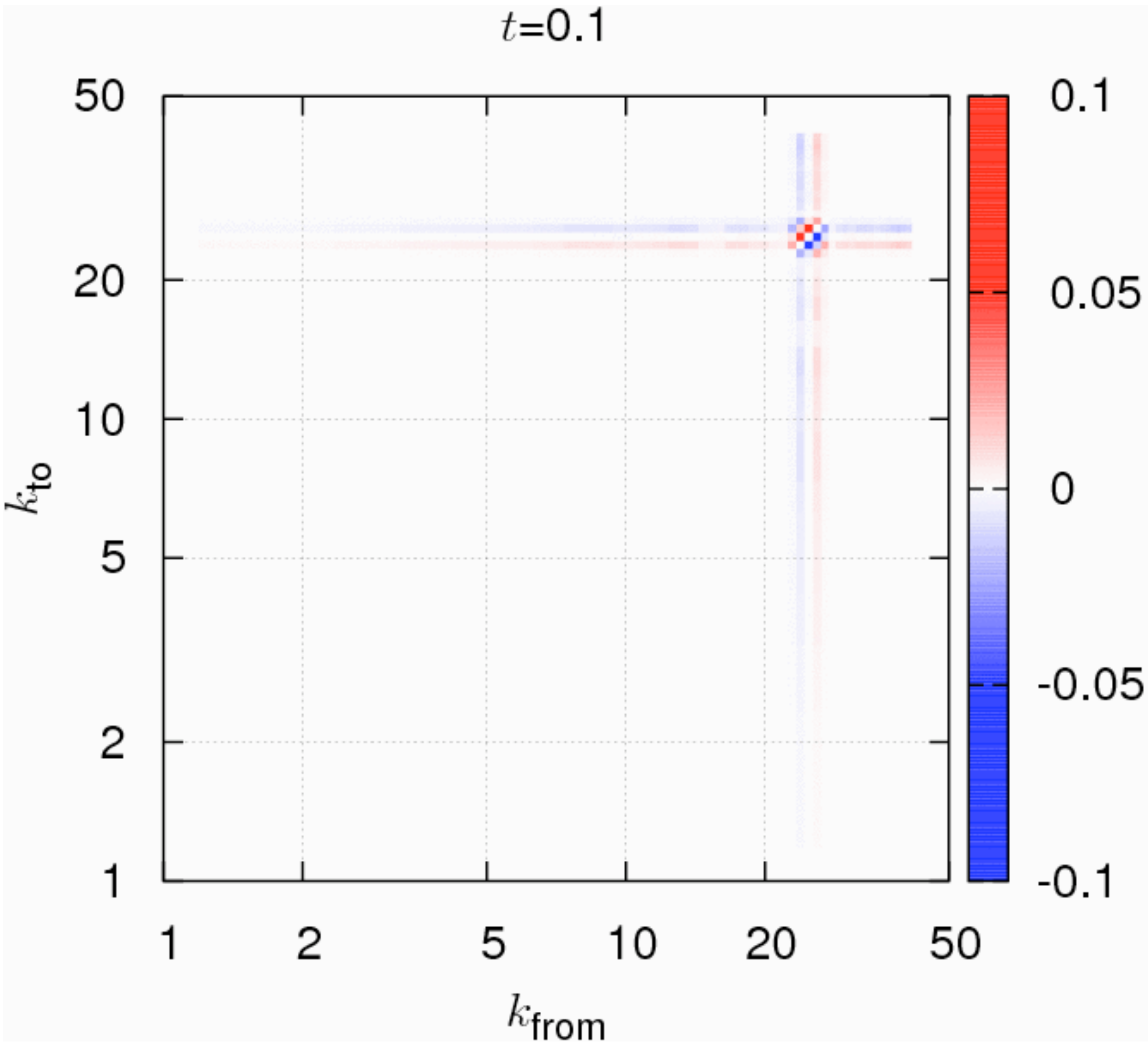
J. R. Chasnov, Phys. Fluids 9, 171 (1997).

Time evolution of wave number spectra

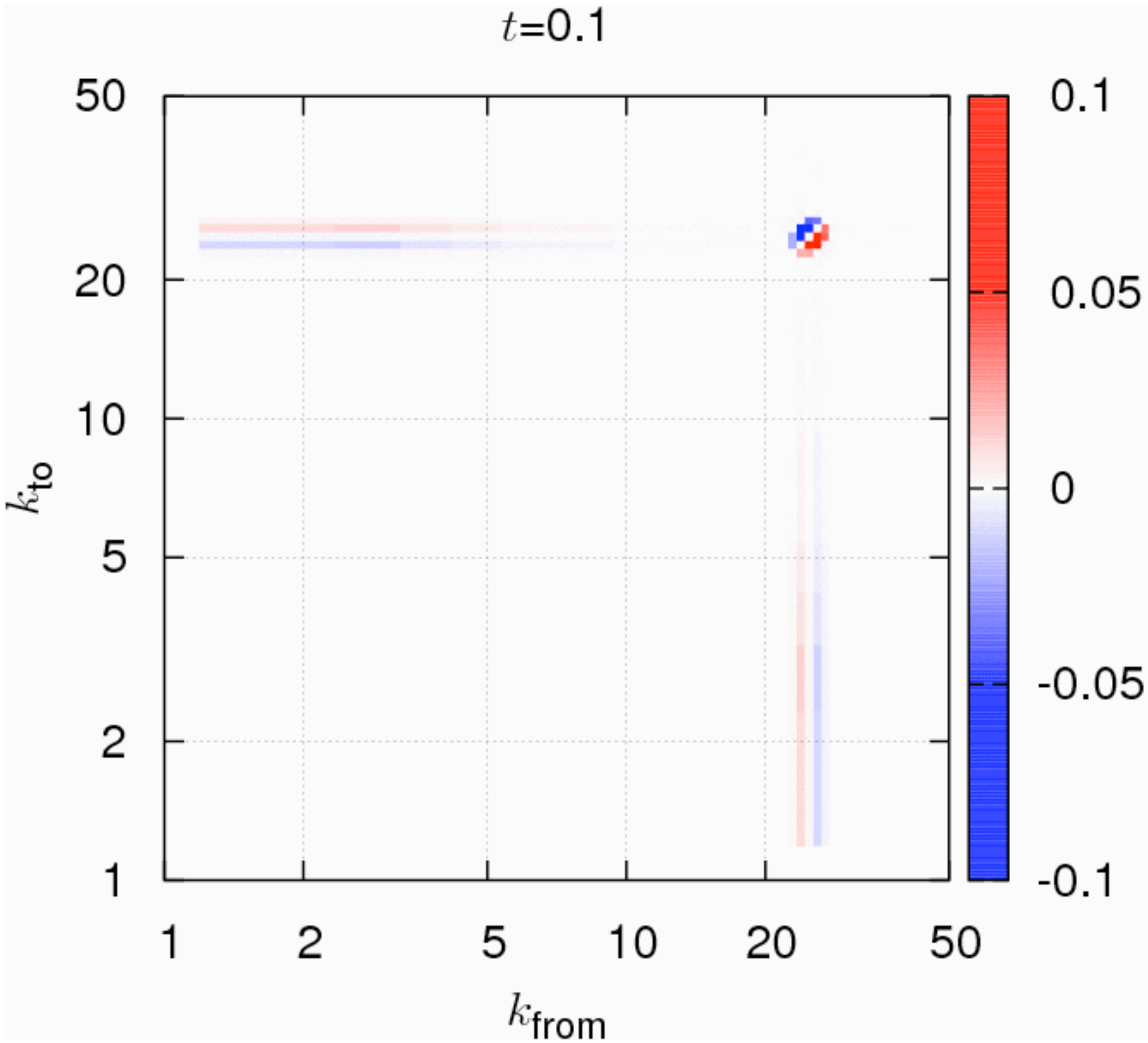


Shell-to-shell transfer

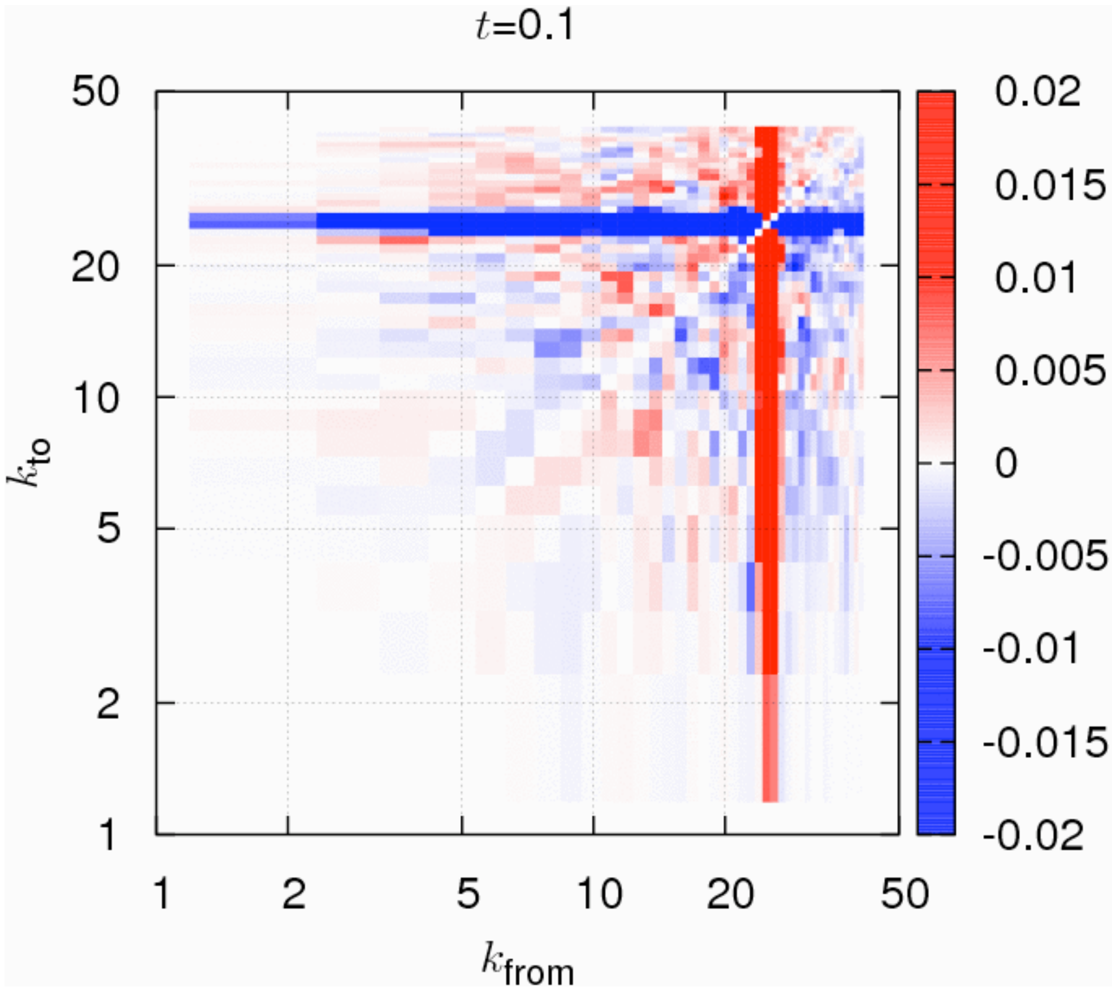
Shell-to-shell transfer of W



Shell-to-shell transfer of E



Shell-to-shell transfer E
(random initial condition)



References

[G. Plunk, Cowley, Schekochihin, Tatsuno, *submitted* JFM (2009) arXiv:0904.0243]
[T. Tatsuno, W. Dorland, A. A. Schekochihin, G. G. Plunk, M. Barnes, S. C. Cowley, and G. G. Howes, PRL (2009) arXiv:0811.2538]
[T. Tatsuno, M. Barnes, S. C. Cowley, W. Dorland, G. G. Howes, R. Numata, G. Plunk and A. A. Schekochihin J. Plasma Fusion Res. (2010), arxiv:1003.3933]