Radial electric field in full f simulations

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Radial electric field in gyrokinetics

- First, show requirements on accuracy of charge density conservation for tokamaks
  - Intrinsic ambipolarity
- Next, show what we know for slab gyrokinetics
- Finally, brief discussion on the tokamak problem

Intrinsic ambipolarity (I)

- In axisymmetric systems, at $k_\perp L \sim 1$

$\langle n_i - n_e \rangle_\psi$ (or $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$) $\cong 0$ for any radial electric field

- Related to conservation of toroidal angular momentum

$$\frac{\partial}{\partial t}\left\langle R n_i M \mathbf{V}_i \cdot \hat{\zeta} \right\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi}\left(V'\Pi\right)$$

- Possible to show that to relevant order

$$\frac{\partial}{\partial t}\left\langle n_i - n_e \right\rangle_\psi = 0$$

Intrinsic ambipolarity (II)

- Consider equation for toroidal angular momentum

$$R\hat{\zeta}\cdot\left[\frac{\partial}{\partial t}\left(n_i M\mathbf{V}_i\right)+\nabla\cdot\left(\overleftrightarrow{\mathbf{P}}_i+\overleftrightarrow{\mathbf{P}}_e\right)=e\left(n_i-n_e\right)\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}\right]$$

- Using $\nabla\left(R\hat{\zeta}\right)=\left(\nabla R\right)\hat{\zeta}-\hat{\zeta}\left(\nabla R\right)=$ antisymmetic

$$R\left(\mathbf{J}\times\mathbf{B}\right)\cdot\hat{\zeta}=\mathbf{J}\cdot\left(R\mathbf{B}\times\hat{\zeta}\right)=\mathbf{J}\cdot\nabla\psi$$

we obtain

$$\frac{\partial}{\partial t}\left(Rn_i M\mathbf{V}_i\cdot\hat{\zeta}\right)+\nabla\cdot\left[R\hat{\zeta}\cdot\left(\overleftrightarrow{\mathbf{P}}_i+\overleftrightarrow{\mathbf{P}}_e\right)\right]$$

$$=e\left(n_i-n_e\right)\left(R\hat{\zeta}\cdot\mathbf{E}\right)+\frac{1}{c}\mathbf{J}\cdot\nabla\psi$$

Intrinsic ambipolarity (III)

- Flux surface averaged toroidal angular momentum

$$\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi = -\left\langle \left(cR\hat{\boldsymbol{\zeta}}\cdot\mathbf{E}\right)e\left(n_i - n_e\right)\right\rangle_\psi + c\frac{\partial}{\partial t}\left\langle Rn_i M\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\right\rangle_\psi$$

$$+\frac{c}{V'}\frac{\partial}{\partial\psi}V'\left\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi\right\rangle_\psi$$

- Neglect $\left\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_e\cdot\nabla\psi\right\rangle_\psi << \left\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi\right\rangle_\psi$ because it is off-diagonal

- Finally, using $\frac{\partial}{\partial t}\left\langle e\left(n_i - n_e\right)\right\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial\psi}V'\langle\mathbf{J}\cdot\nabla\psi\rangle_\psi$

Intrinsic ambipolarity (IV)

- Flux surface averaged toroidal angular momentum

$$\frac{\partial}{\partial t}\left\langle e(n_i - n_e)\right\rangle_\psi - \frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle \left(cR\hat{\zeta}\cdot\mathbf{E}\right)e(n_i - n_e)\right\rangle_\psi =$$

$$-\frac{c}{V'}\frac{\partial}{\partial \psi}V'\left\{\frac{\partial}{\partial t}\left\langle Rn_i M\mathbf{V}_i\cdot\hat{\zeta}\right\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle R\hat{\zeta}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi\right\rangle_\psi\right\}$$

- For $n_i - n_e = 0$, conservation of toroidal angular momentum

$$\frac{\partial}{\partial t}\left\langle Rn_i M\mathbf{V}_i\cdot\hat{\zeta}\right\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle R\hat{\zeta}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi\right\rangle_\psi$$

Intrinsic ambipolarity (V)

- BUT for gyroBohm transport, $D_{gB} = (\rho_i / a)\rho_i \upsilon_i$

$$\frac{\partial}{\partial t}\left\langle R n_i M \mathbf{V}_i \cdot \hat{\zeta} \right\rangle_\psi \sim \frac{\partial}{\partial r}\left[D_{gB} \frac{\partial}{\partial r}\left(R n_i M V_i \right) \right] \sim \left(\frac{\rho_i}{a} \right)^2 \frac{V_i}{\upsilon_i} \frac{R p_i}{a}$$

Then need

$$\frac{\partial}{\partial t}\left\langle e\left(n_i - n_e \right) \right\rangle_\psi \sim \frac{B}{B_p}\left(\frac{\rho_i}{a} \right)^3 \frac{V_i}{\upsilon_i} e n_e \frac{\upsilon_i}{a}$$

- Usual models have $\dfrac{\partial \left\langle n_i \right\rangle_\psi}{\partial t} \sim \left(\dfrac{\rho_i}{a} \right)^2 n_e \dfrac{\upsilon_i}{a}$

$\langle n_i - n_e \rangle_\psi = 0$ to relevant order $\Rightarrow$ intrinsic ambipolarity

Bottom line

- Axisymmetric devices are intrinsically ambipolar $\Leftrightarrow$ toroidal angular momentum is conserved

- To recover conservation of toroidal angular momentum, need to work hard on

$$\frac{\partial}{\partial t}\left\langle n_i - n_e \right\rangle_\psi$$

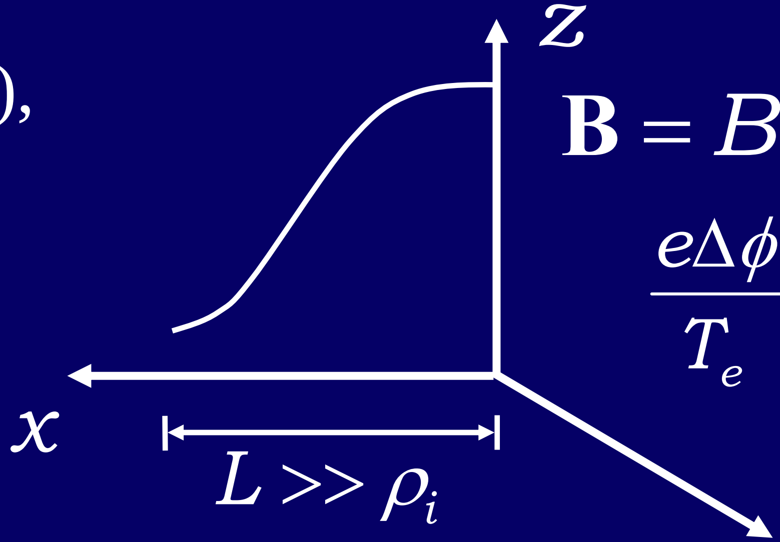
  Working with quasineutrality is a BAD IDEA

- Better to use conservation of angular momentum

Electrostatic gyrokinetics in a slab

$$n_e(x,t), T_{i,e}(x,t),$$

$$V_{iy}(x,t) \sim \frac{\rho_i}{L} c_s$$



$$\mathbf{B} = B\hat{\mathbf{z}} = \text{const}$$

$$\frac{e\Delta\phi}{T_e} \sim \frac{\Delta f}{f} \sim \frac{\rho_i}{L}$$

$$\frac{\partial f_i^{GK}}{\partial t} + \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_i^{GK} + \dot{E} \frac{\partial f_i^{GK}}{\partial E} = 0$$

$$\frac{\partial f_e}{\partial t} + \left(v_\parallel \hat{\mathbf{b}} - \frac{c}{B} \nabla\phi \times \hat{\mathbf{b}} \right) \cdot \nabla f_e + \frac{e}{m} v_\parallel \hat{\mathbf{b}} \cdot \nabla\phi \frac{\partial f_e}{\partial E_0} = 0$$

$$n_i^{GK} + n_{ip} \equiv \int d^3v\, f_i^{GK} + \int d^3v\, f_{ip} = \int d^3v\, f_e \equiv n_e$$

$$f_i^{GK}(\mathbf{R}, E, \mu, t)$$

$$f_e(\mathbf{r}, E_0, \mu_0, t)$$

$$\phi(\mathbf{r}, t)$$

Transport of y-momentum in a slab

- Full Vlasov $\Rightarrow$ momentum conservation

$$\frac{\partial}{\partial t}\left\langle n_i M V_{iy} \right\rangle_T = -\frac{\partial}{\partial x}\left\langle \pi_{i,xy} \right\rangle_T$$

  - $\langle \ldots \rangle_T \equiv \int dt\, dx\, dy\, dz \equiv$ coarse grain average

- At $k_\perp L \sim 1,\; n_i V_{iy} = \dfrac{c}{eB}\left(\dfrac{\partial p_i}{\partial x} + e n_i \dfrac{\partial \phi}{\partial x} \right)$

  - Relation valid for both gyrokinetics and full Vlasov

- Full Vlasov $\Rightarrow$ momentum transport to $(\rho_i / L)^3\, p_i$

$$\left\langle \pi_{i,xy} \right\rangle = -\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y} \int d^3 v\, f_i M (\mathbf{v}\cdot\hat{\mathbf{y}}) \right\rangle_T - \frac{1}{2\Omega_i}\frac{\partial p_i}{\partial t} + \frac{M}{2\Omega_i}\frac{\partial}{\partial x}\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y} \int d^3 v\, f_i (\mathbf{v}\cdot\hat{\mathbf{y}})^2 \right\rangle_T$$

Ordering for f_i and ϕ

- First term in momentum transport

$$-\left\langle \underbrace{\frac{c}{B}\frac{\partial\phi}{\partial y}}_{\delta_i v_i} \underbrace{\int d^3v\, f_i M(\mathbf{v}\cdot\hat{\mathbf{y}})}_{n_i M \delta_i v_i} \right\rangle_T \sim \delta_i^2 p_i >> \text{gyroBohm}$$

- In reality, f_i and ϕ are decorrelated to first order
  - Shown in δf flux tube simulations

- Need to obtain f_i and ϕ to second order in δ_i

Gyrokinetic variables

- Expansion in $\delta_i = \dfrac{\rho_i}{L} \sim \dfrac{\omega}{\Omega_i} << 1$
  - Potential fluctuations with $k_\perp \rho_i \sim 1$ small by δ_i
- New variables $\mathbf{R} = \mathbf{r} + \ldots, E = \dfrac{v^2}{2} + \ldots, \mu = \dfrac{v_\perp^2}{2B} + \ldots$

  Dubin *et al.* PoP 1983

$$\dot{\mathbf{R}} = u\hat{\mathbf{b}} - \frac{c}{B}\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}} + O(\delta_i^3 v_i), \; \dot{E} = -\frac{e}{M}u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\Psi + O\left(\delta_i^3 \frac{v_i^3}{L}\right)$$

with $$\Psi = \langle \phi \rangle - \frac{e}{2MB}\frac{\partial}{\partial \mu}\left\langle \tilde{\phi}^2 \right\rangle - \frac{c}{2B\Omega_i}\left\langle \left(\nabla_{\mathbf{R}}\tilde{\phi} \times \hat{\mathbf{b}}\right)\cdot \nabla_{\mathbf{R}}\tilde{\Phi} \right\rangle$$

  - $\langle \ldots \rangle \equiv$ average in a gyromotion

Polarization density

- Two cases considered
  - Case 1: polarization density to second order

$$f_{ip} = f_{ip,1} + f_{ip,2}$$

Obtain $f_i^{GK} = f_i^{(2)} = f_{i0} + f_{i1} + f_{i2} + \ldots,\ \phi^{(2)} = \phi_0 + \phi_1 + \phi_2 + \ldots$

  - Case 2: polarization density from variational principle

$$f_{ip} = f_{ip,1}$$

Obtain $f_i^{GK} = f_i^{(1)} = f_{i0} + f_{i1} + G_{i2} + \ldots,\ \phi^{(2)} = \phi_0 + \phi_1 + \Phi_2 + \ldots$

Time evolution of $n_i^{GK} - n_e$

- Evolution of f_i^{GK} in conservative form

$$\frac{\partial f_i^{GK}}{\partial t} + \nabla \cdot \left[f_i^{GK} \underbrace{\left(\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} \mathbf{r} + \dot{E} \frac{\partial \mathbf{r}}{\partial E} \right)}_{\neq \dot{\mathbf{r}} = \mathbf{v}} \right] + \nabla_v \cdot \left[f_i^{GK} \underbrace{\left(\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} \mathbf{v} + \dot{E} \frac{\partial \mathbf{v}}{\partial E} \right)}_{\neq \dot{\mathbf{v}}} \right] = 0$$

- Using this expression and the drift kinetic electrons

$$\frac{\partial}{\partial t}(n_i^{GK} - n_e) = -\nabla \cdot \left\{ \int d^3v \left[f_i^{GK} \left(\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} \mathbf{r} + \dot{E} \frac{\partial \mathbf{r}}{\partial E} \right) - f_e \left(v_\parallel \hat{\mathbf{b}} - \frac{c}{B} \nabla \phi \times \hat{\mathbf{b}} \right) \right] \right\}$$

- In each time step, $n_i^{GK} - n_e$ is set equal to n_{ip} by choosing the electric field

Calculation of electric field at $k_\perp L \sim 1$

- Electric field in x direction at $k_\perp L \sim 1$ obtained such that $\langle n_i^{GK} - n_e \rangle_T = 0$

- With steady state turbulence and at $k_\perp L \sim 1$

$$\left\langle \frac{D}{Dt}\left(n_i^{GK} - n_e\right)\right\rangle_T = \frac{c}{eB}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left\langle n_i M V_{iy}\right\rangle_T + \frac{\partial}{\partial x}\left\langle \Pi \right\rangle_T - F_y\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\left\langle n_i M V_{iy}\right\rangle_T = -\frac{\partial}{\partial x}\left\langle \Pi \right\rangle_T + F_y$$

with $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} - \dfrac{c}{B}\left(\nabla \phi \times \hat{\mathbf{b}}\right)\cdot \nabla$

Case 1

- Obtain $f_i^{GK} = f_i^{(2)},\ \phi^{(2)}$
- UNPHYSICAL force but physical transport

$$F_y = \frac{e^3}{2M^2B^2}\left\langle \int d^3v\, f_i \frac{\partial^2}{\partial y \partial \mu}\left(\frac{1}{3}\frac{\partial}{\partial \mu}\left\langle \tilde{\phi}^3 \right\rangle + \left\langle \tilde{\phi}^2 \right\rangle \frac{\partial \langle \phi \rangle}{\partial \mu}\right)\right\rangle_x \sim \delta_i^3 \frac{p_i}{L}$$

$$\langle \Pi \rangle_T \{f_i, \phi\} = -\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_i M(\mathbf{v}\cdot\hat{\mathbf{y}})\right\rangle_T - \frac{1}{2\Omega_i}\frac{\partial p_i}{\partial t}$$

$$+ \frac{M}{2\Omega_i}\frac{\partial}{\partial x}\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_i (\mathbf{v}\cdot\hat{\mathbf{y}})^2\right\rangle_T$$

- With $f_i = f_i^{GK} + f_{ip,1} + f_{ip,2} = f_i^{(2)} + f_{ip,1} + f_{ip,2},\ \phi = \phi^{(2)}$

Case 2

- Obtain $f_i^{GK} = f_i^{(1)},\ \phi^{(1)}$
- No force but UNPHYSICAL transport

$$F_y = 0$$

$$\langle \Pi \rangle_T \{f_i, \phi\} = -\left\langle \frac{c}{B} \frac{\partial \phi}{\partial y} \int d^3 v\, f_i M (\mathbf{v} \cdot \hat{\mathbf{y}}) \right\rangle_T - \frac{1}{2\Omega_i} \frac{\partial p_i}{\partial t}$$

$$+ \frac{M}{2\Omega_i} \frac{\partial}{\partial x} \left\langle \frac{c}{B} \frac{\partial \phi}{\partial y} \int d^3 v\, f_i (\mathbf{v} \cdot \hat{\mathbf{y}})^2 \right\rangle_T$$

- With $f_i = f_i^{GK} + f_{ip,1} = f_i^{(1)} + f_{ip,1},\ \phi = \phi^{(1)} \Rightarrow \Pi \neq \pi_{i,xy}$
- Not obvious that we should recover this form of Π!

Consequences

- Employing the gyrokinetic quasineutrality gives the wrong velocity profile
  - Non-physical terms comparable to gyroBohm transport of momentum

- Lower order gyrokinetic equation with $\Psi \simeq \langle \phi \rangle$ gives a stronger force $F_y \sim \delta_i^2 p_i / L$

  Makes $V_{iy} \sim v_i$ in a confinement time $t_E \sim \delta_i^{-2} L / v_i$

- Need higher order drifts, in particular to order $\delta_i^3 v_i$

Transport of momentum at long times

- Possible to obtain expression for

$$\left\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\right\rangle_T = O\left(\delta_i^3 p_i R\left|\nabla\psi\right|\right)$$

  with f_i and ϕ only good to second order
- Work in progress
  - Expressions are far more complicated!
- Work in the slab and preliminary work in tokamaks seems to indicate that drifts are needed to $\delta_i^3 v_i$
- Possible to reduce requirements for $B_p/B << 1$?
  - Then neglect $\mathbf{v}_\perp$ contributions, the difficult ones!