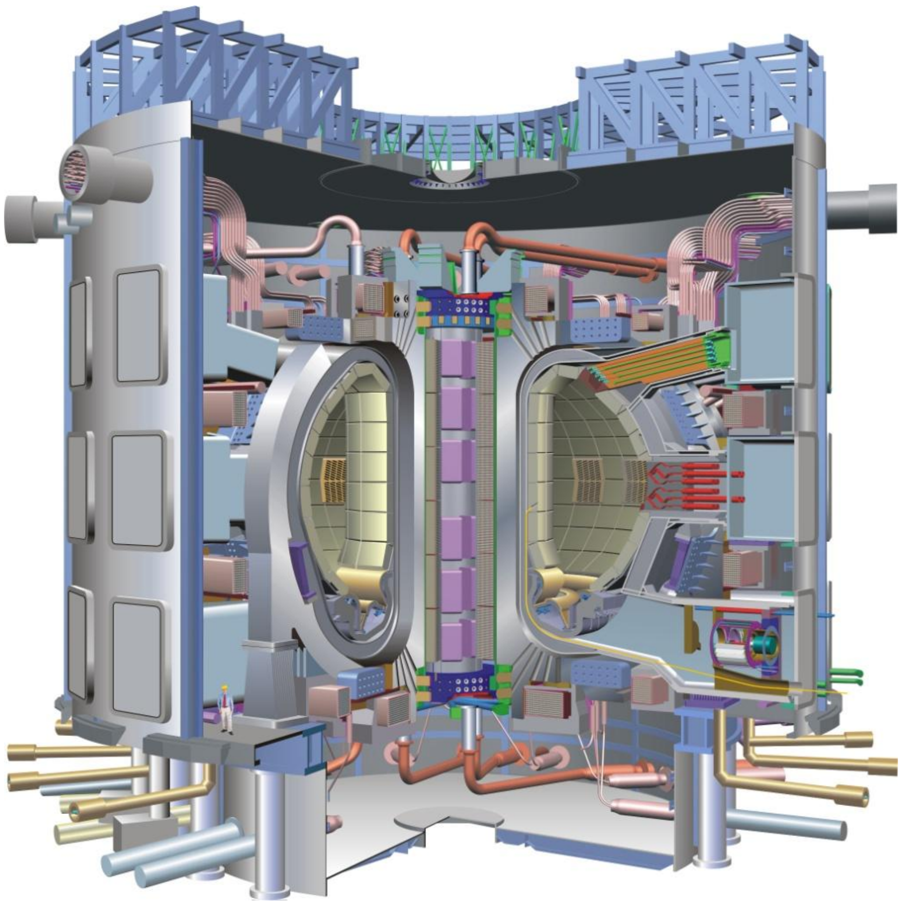
Inverse cascades, zonal jets and turbulence/transport suppression in CHM model

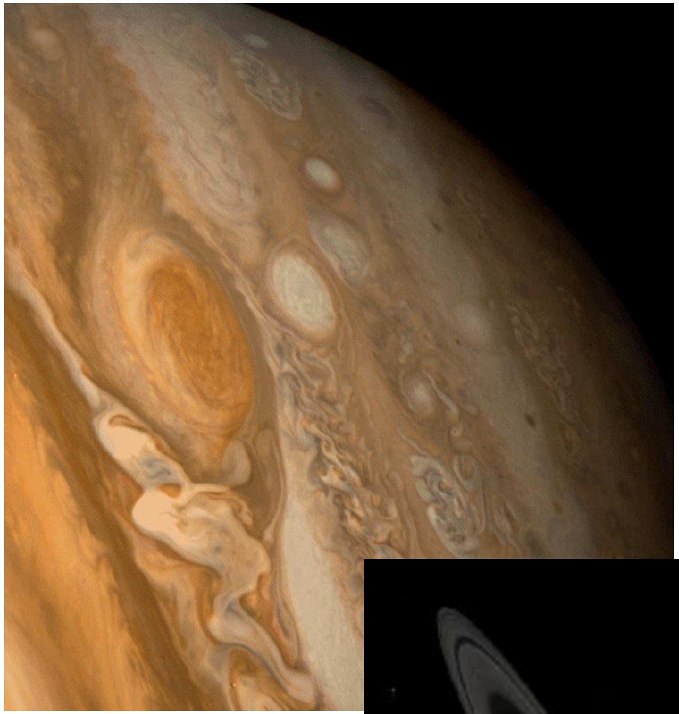
Sergey Nazarenko, Warwick, UK

Balk, Connaughton, Dyachenko, Manin, Nadiga, Quinn, Zakharov, 1988-2010

Drift waves in fusion devices



Rossby waves in atmospheres of rotating planets

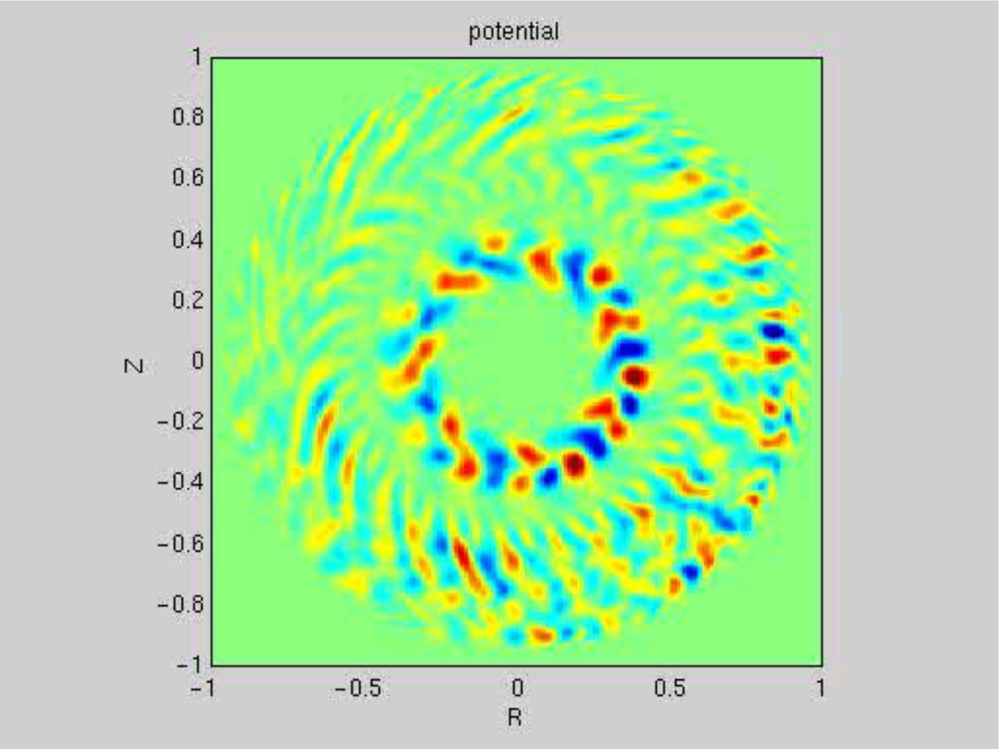


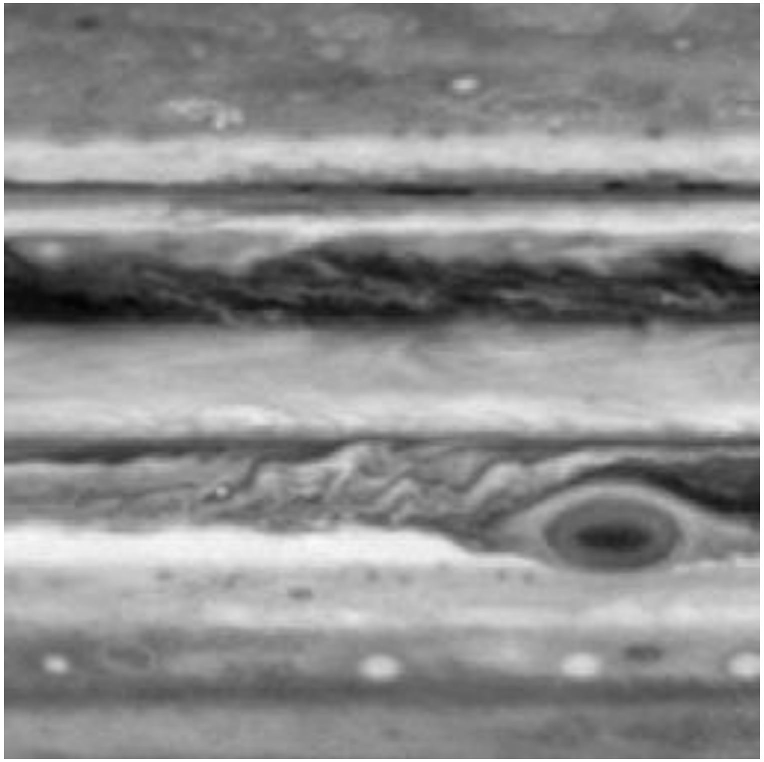
Charney-Hasegawa-Mima equation

$$\frac{\partial}{\partial t}\left(\rho^2\nabla^2\psi - \psi\right) - \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0$$

- Ψ – streamfunction (electrostatic potential).
- ρ – Deformation radius (ion Larmor radius).
- β – PV gradient (diamagnetic drift).
- x – east-west (poloidal arc-length)
- y – south-north (radial length).

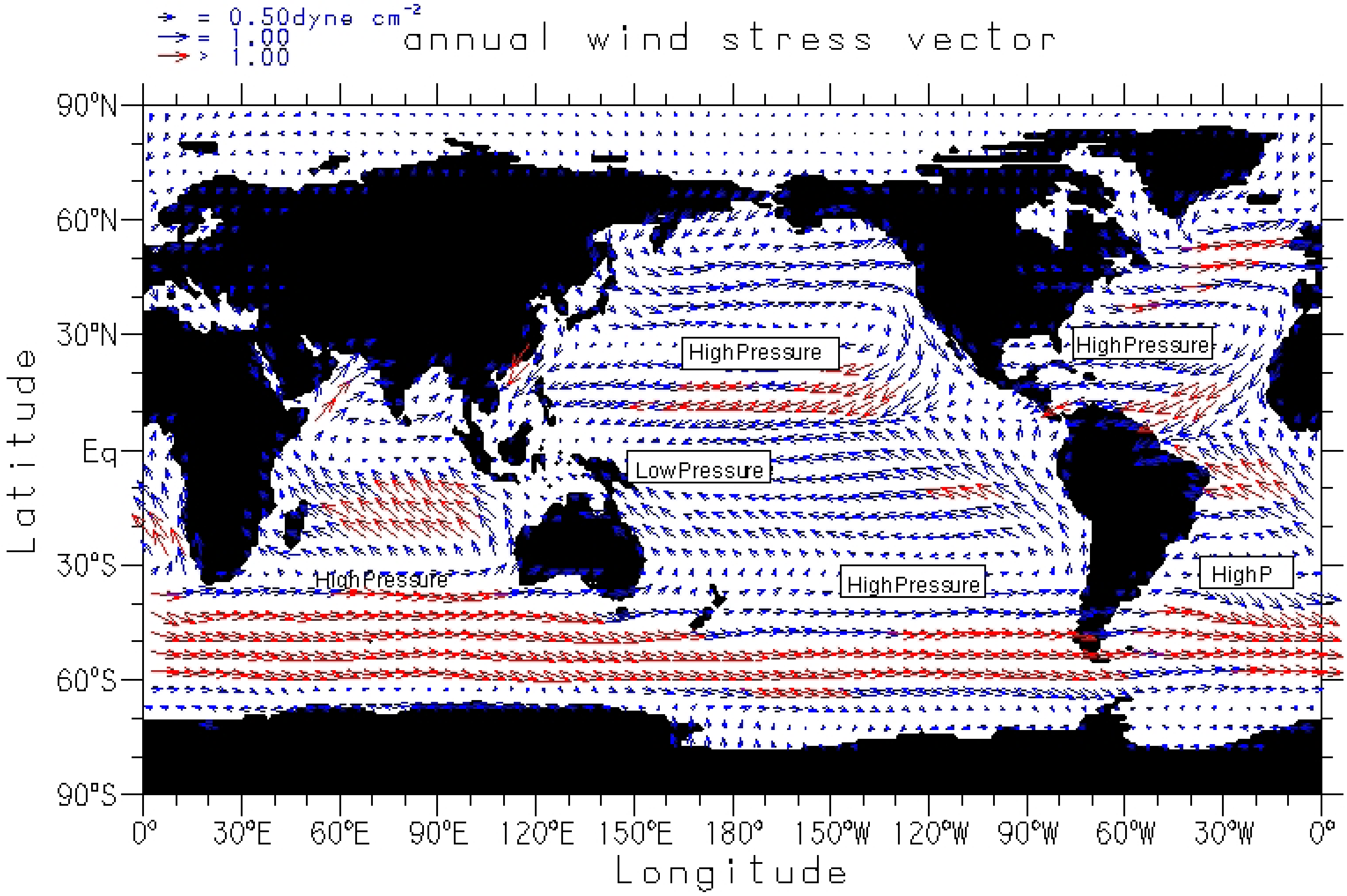
Turbulence-ZF system. LH-transition





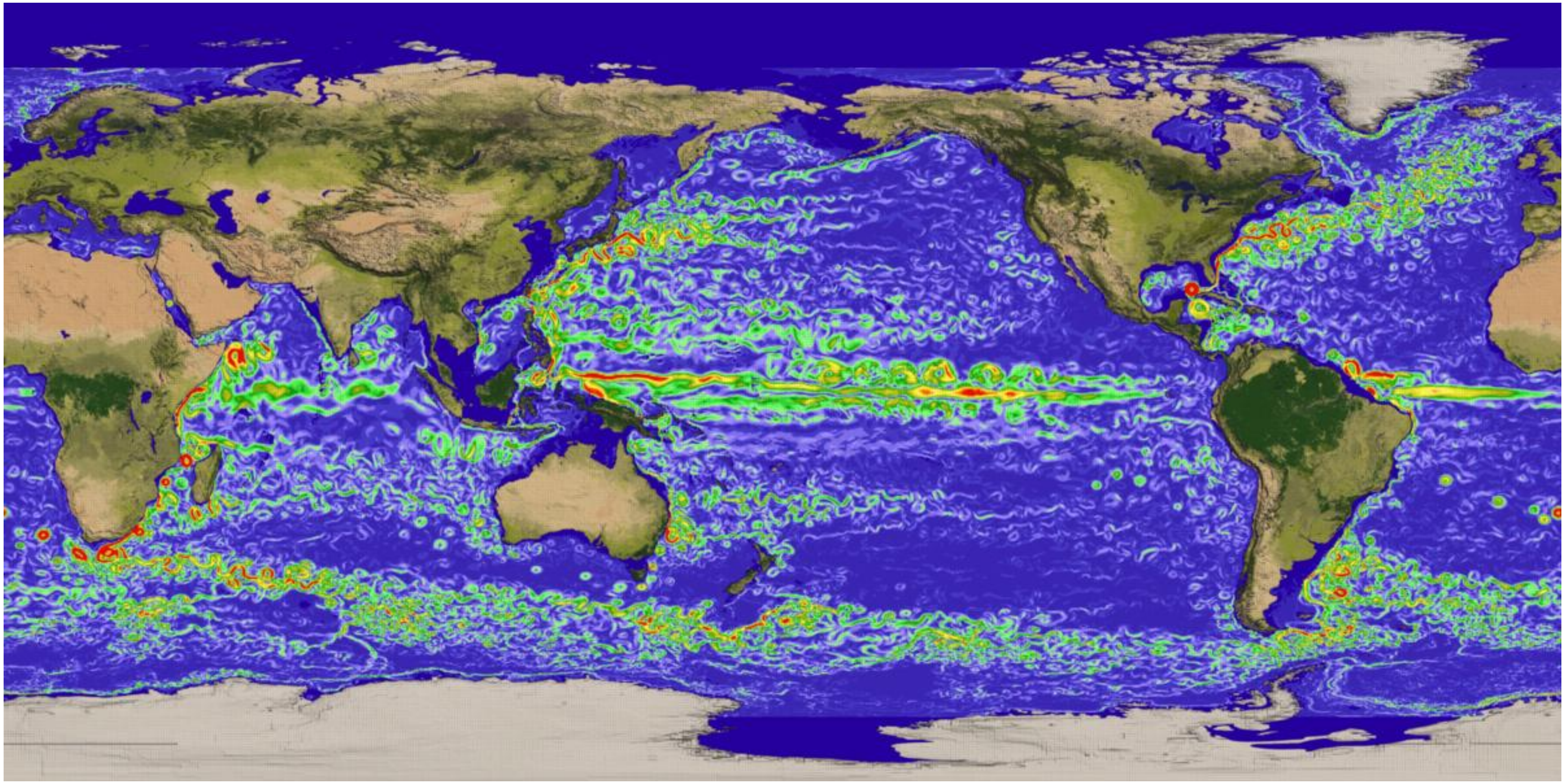
- Small-scale turbulence generates zonal flows
- ZF’s suppress waves
- Hence transport barriers, LH transition

Zonal Jets in Earth's Atmosphere



Average oceanic winds on Earth (QSCAT)

Zonal Jets in Earth's Oceans



Eddy-resolving simulation of Earths oceans (Earth Simulator Center/JAMSTEC)

Barotropic governor in GFD

- James and Gray' 1986

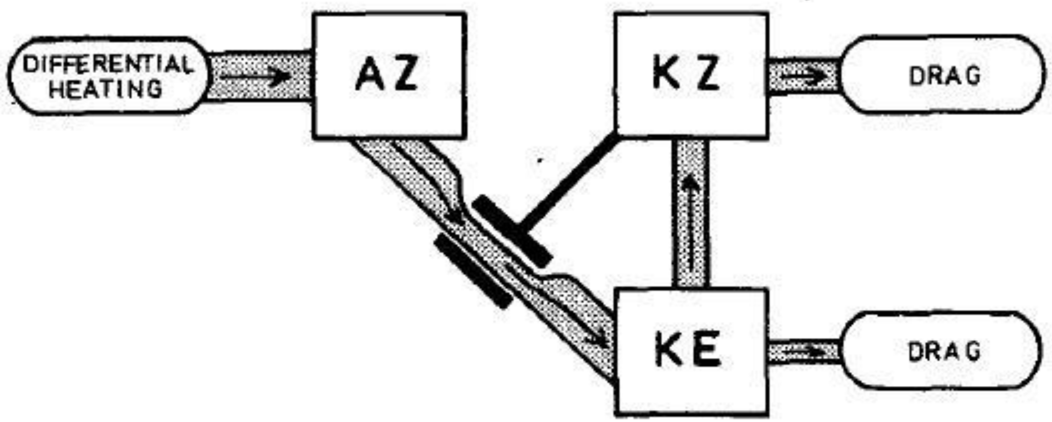


FIG. 1. Schematic illustration of the "barotropic governor," summarizing the effect of horizontal shears on baroclinic instability as postulated in JG. Energy conversion from available potential energy (AZ) to eddy kinetic energy (KE) results in momentum fluxes which increase the barotropic contribution to the zonal kinetic energy (KZ). As barotropic shears build up in the zonal flow, the baroclinic conversions are inhibited.

Mechanisms of zonal flow generation:

- (part 1) Anizotropic inverse cascade
- (part 2) Modulational instability

(part 3) *Feedback of ZF onto turbulence: turbulence suppression, LH transition*

Part 1

ZF generation by anisotropic inverse cascades

2D Euler equation

$$\frac{\partial}{\partial t}\left(\rho^2\nabla^2\psi - \cancel{\psi}\right) - \beta\cancel{\frac{\partial\psi}{\partial x}} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0$$

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Conservation laws for Euler

$$E(\vec{k}) = \int \langle \vec{u}(\vec{x}+\vec{r}) \cdot \vec{u}(\vec{x}) \rangle e^{-i\vec{k}\cdot\vec{x}} d\vec{r}$$

- energy spectrum

$$\langle u^2 \rangle = \int E(\vec{k}) d\vec{k}$$

- energy

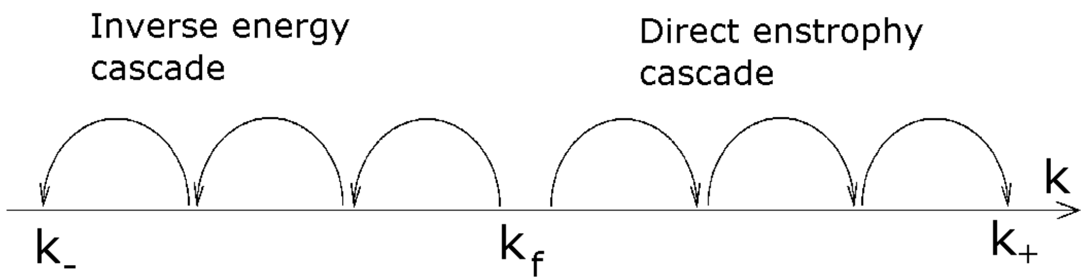
$$\langle (\nabla \times \vec{u})^2 \rangle = \int k^2 E(\vec{k}) d\vec{k}$$

-enstrophy

Fjørtoft'53 argument for 2D turbulence.

$$\langle u^2 \rangle = \int E(\vec{k})d\vec{k}$$

$$\langle (\nabla \times \vec{u})^2 \rangle = \int k^2 E(\vec{k})d\vec{k}$$

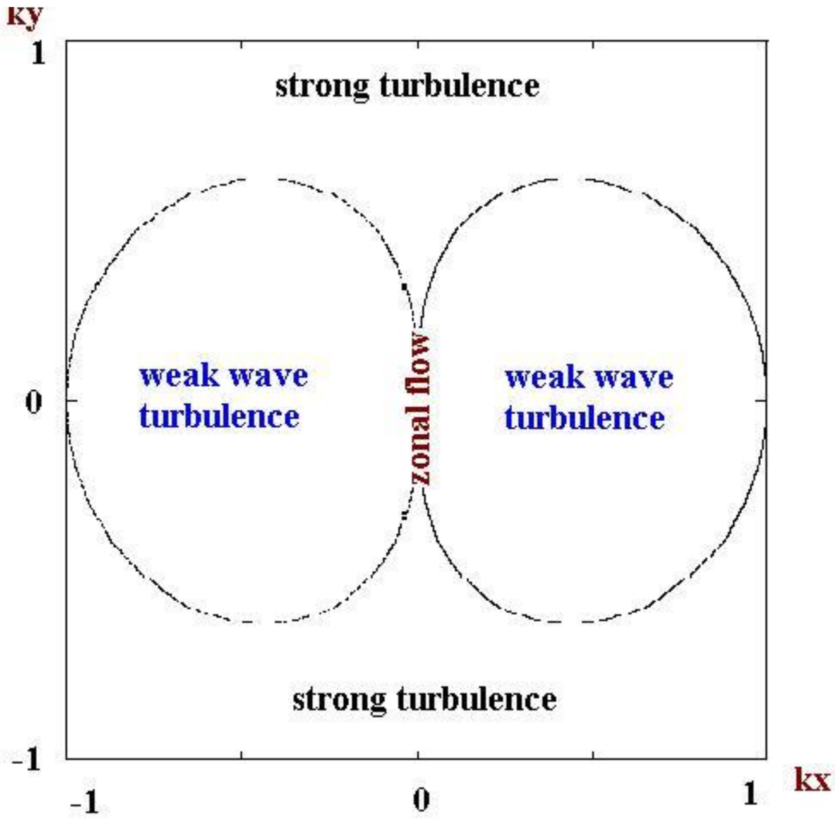


- *Produce turbulence at k_f and have two dissipation regions at k_- and k_+ separated by large inertial ranges.*
- *Production rates for energy and enstrophy are related as $\eta \approx k_f^2\, \varepsilon$.*
- *If energy is dissipated at k_+ at rate ~ ε then enstrophy is dissipated at a rate $k_+^2\, \varepsilon >> k_f^2\, \varepsilon \approx \eta$ which is a contradiction. Therefore, energy must be dissipated at k_- – inverse energy cascade.*
- *Similar ad absurdum argument is used to show that enstrophy cannot be dissipated at k_-, and therefore cascades forward in k.*

Rhines scale crossover

$$\frac{\partial}{\partial t}\left(\rho^2\nabla^2\psi-\psi\right)-\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}=0$$

$$k_t(\phi)=k_\beta\cos^{3/5}\phi, \qquad k_\beta=(\beta^3/\epsilon)^{1/5}.$$



- *Nonlinear=linear → Rhines scale.*
- *"Lazy 8" separates vortex-dominated and wave-dominated scales (Rhines'75, Holloway'84)*
- *Outside of lazy-8: Kraichnan's isotropic inverse cascade.*
- *Inside lazy-8 the cascade is anisotropic and dominated by triad wave resonances.*

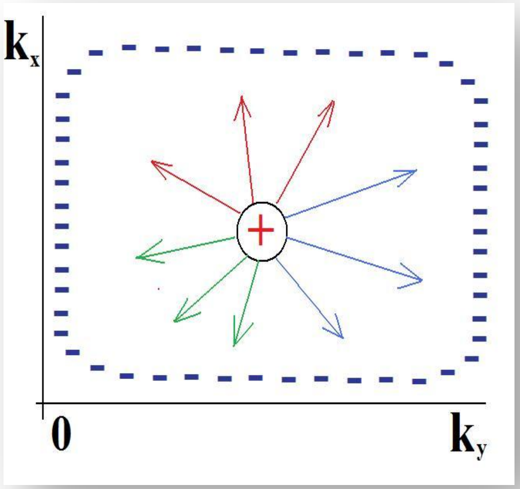
Extra quadratic invariant on β-plane

- *Balk, Nazarenko & Zakharov (1990)*
- *Adiabatic for the original β-plane equation: requires small nonlinearity and possibly random phases.*
- *For case kρ >>1:*

$$\Phi = \int \frac{k_x^2}{k^6}(k_x^2 + 5k_y^2)|\hat{\psi}_k|^2 d\mathbf{k}, \quad \text{- Zonostrophy invariant.}$$

Anisotropic cascades in β-plane turbulence

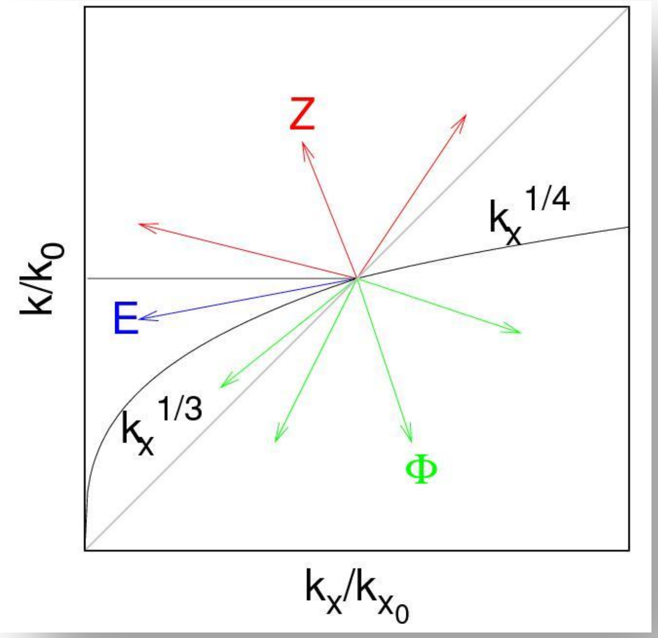
- *3 cascades cannot be isotropic.*
- *Let us produce turbulence near some k_0, surround it by a large non-dissipative area and dissipate at large k and at small k_x and small k_y.*



$$E = \int k^2 |\hat{\psi}_k|^2 \, d\mathbf{k},$$

$$Z = \int k^4 |\hat{\psi}_k|^2 \, d\mathbf{k},$$

$$\Phi = \int \frac{k_x^2}{k^6} (k_x^2 + 5k_y^2) |\hat{\psi}_k|^2 \, d\mathbf{k},$$



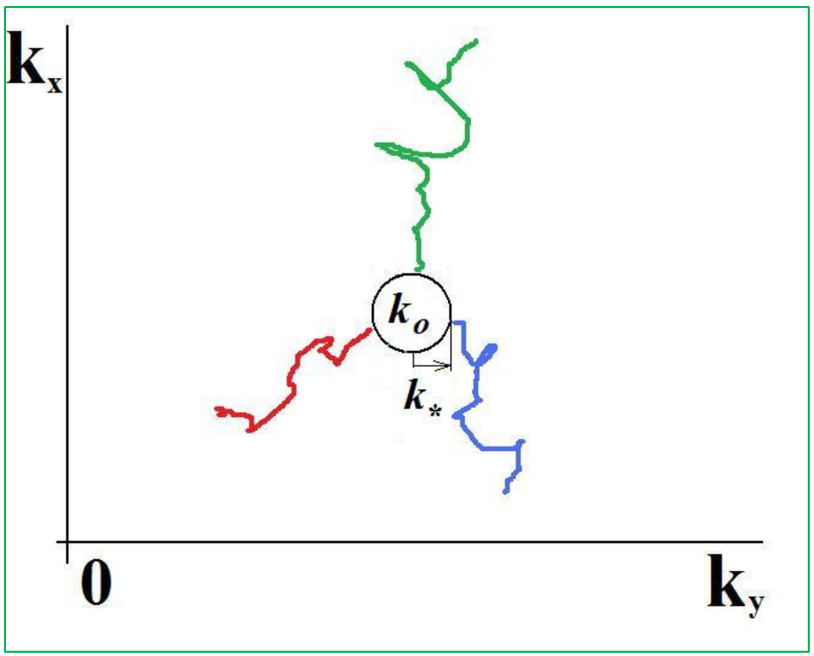
•*Fjortoft's argument separates the k-space into three non-intersecting sectors to which the energy, enstrophy and zonostrophy can cascade.*

•*Zonostrophy Φ forces energy E to the scales corresponding to zonal flows.*

Numerics

- Pseudo-spectral, no dissipation.
- Initial condition:

$$\hat{\psi}_k = A\, e^{(|\mathbf{k}-\mathbf{k}_0|^2/k_*^2 + i\phi_k)},$$ with random independent phases.



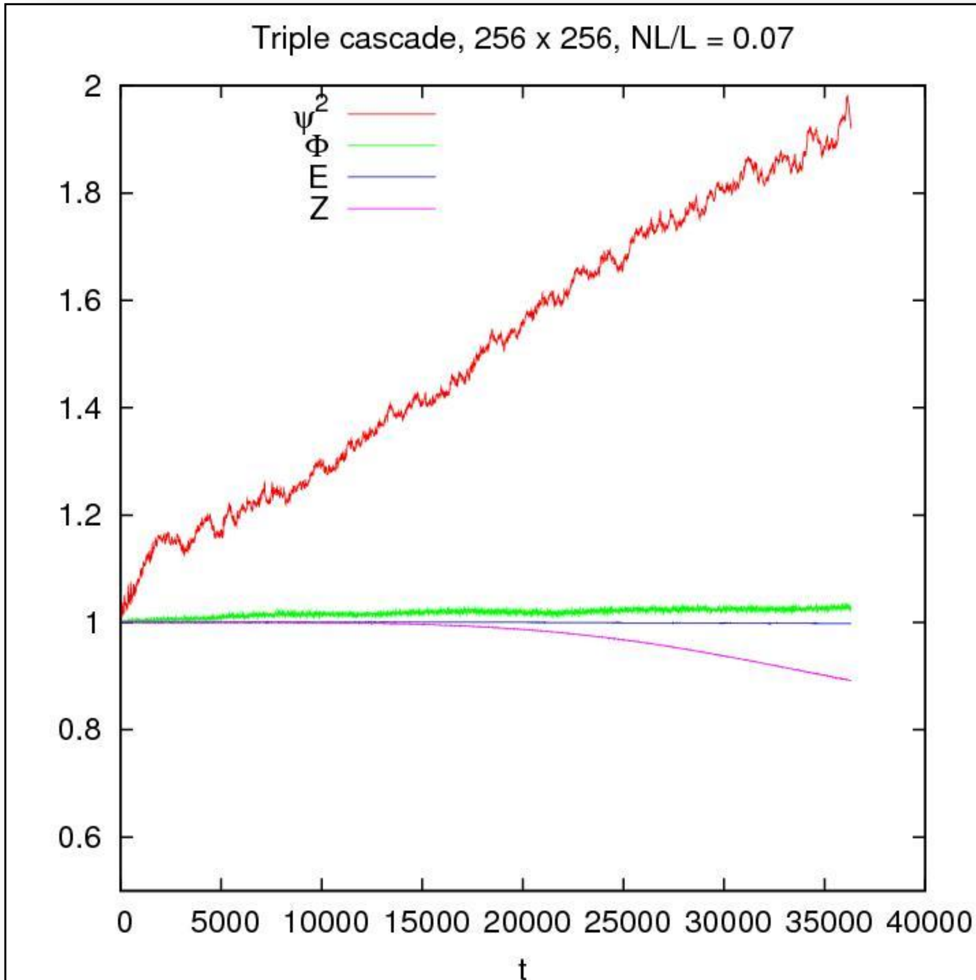
Quantify cascades via trajectory
Of centroids for E, Z and Φ:

$$\mathbf{k}_E(t) = \int \mathbf{k}\, k^2 |\hat{\psi}_k|^2 \, d\mathbf{k} \;/E \; , \quad \text{E-centroid (integral scale)}$$

$$\mathbf{k}_Z(t) = \int \mathbf{k}\, k^4 |\hat{\psi}_k|^2 \, d\mathbf{k} \;/Z \; , \quad \text{Z-centroid}$$

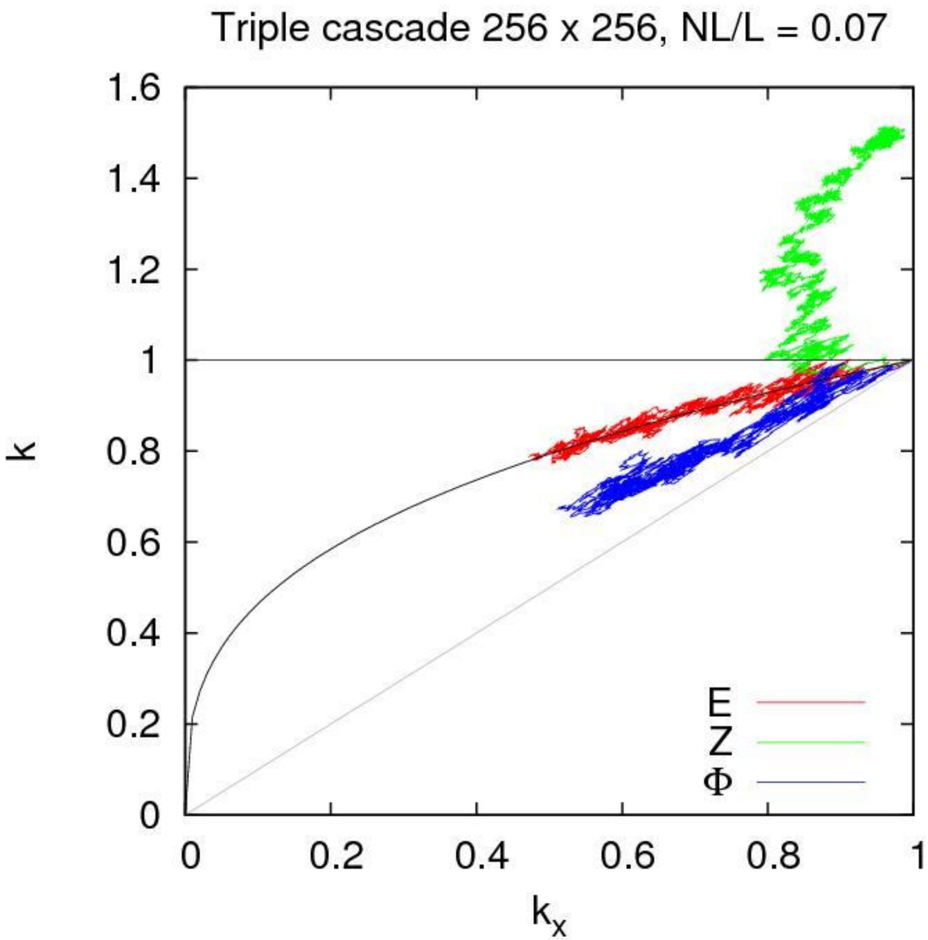
$$\mathbf{k}_\Phi(t) = \int \mathbf{k}\, \frac{k_x^4}{k^6}(k_x^2 + 5k_y^2) |\hat{\psi}_k|^2 \, d\mathbf{k} \;/\,\Phi \; , \quad \Phi\text{-centroid}$$

Weak nonlinearity run, NL/L $=k_0^2/k_\beta^2=0.07$



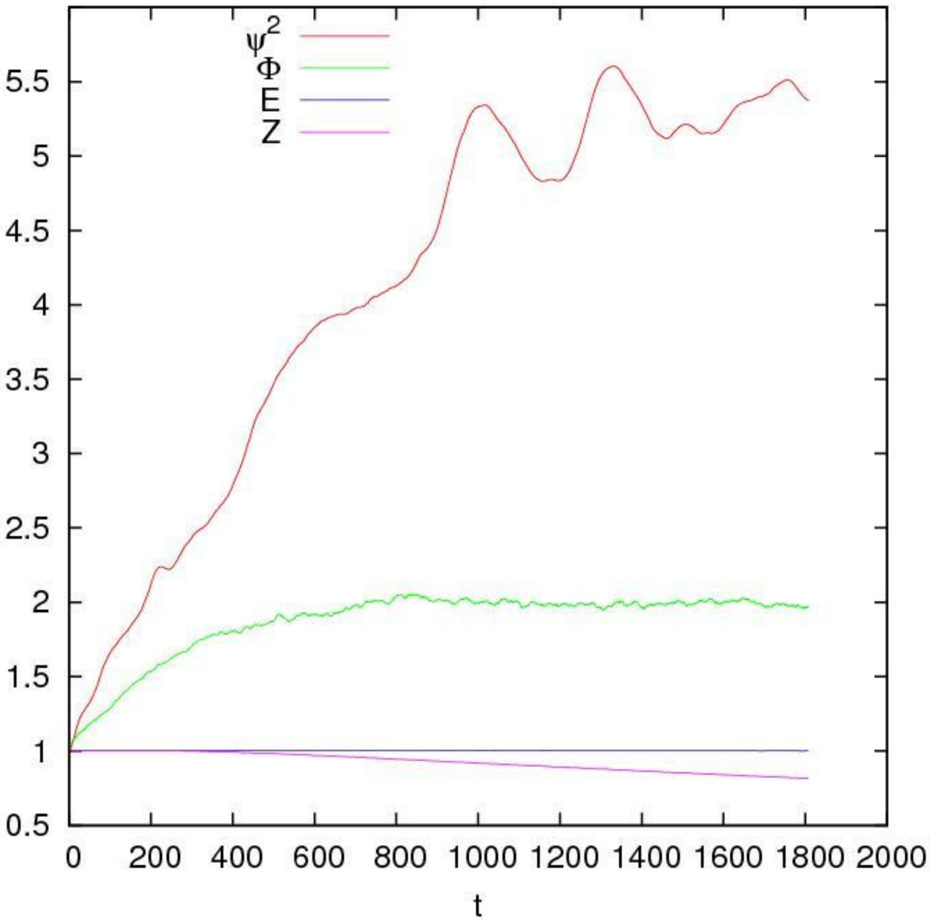
- *Initial turbulence is well within the dumbbell.*
- *Because of slow weakly nonlinear evolution, we compare with a non-conserved quantity (red).*
- *Energy and zonostrophy are well conserved, enstrophy less well.*

Weak nonlinearity run, NL/L $=k_0^2/k_\beta^2=0.07$



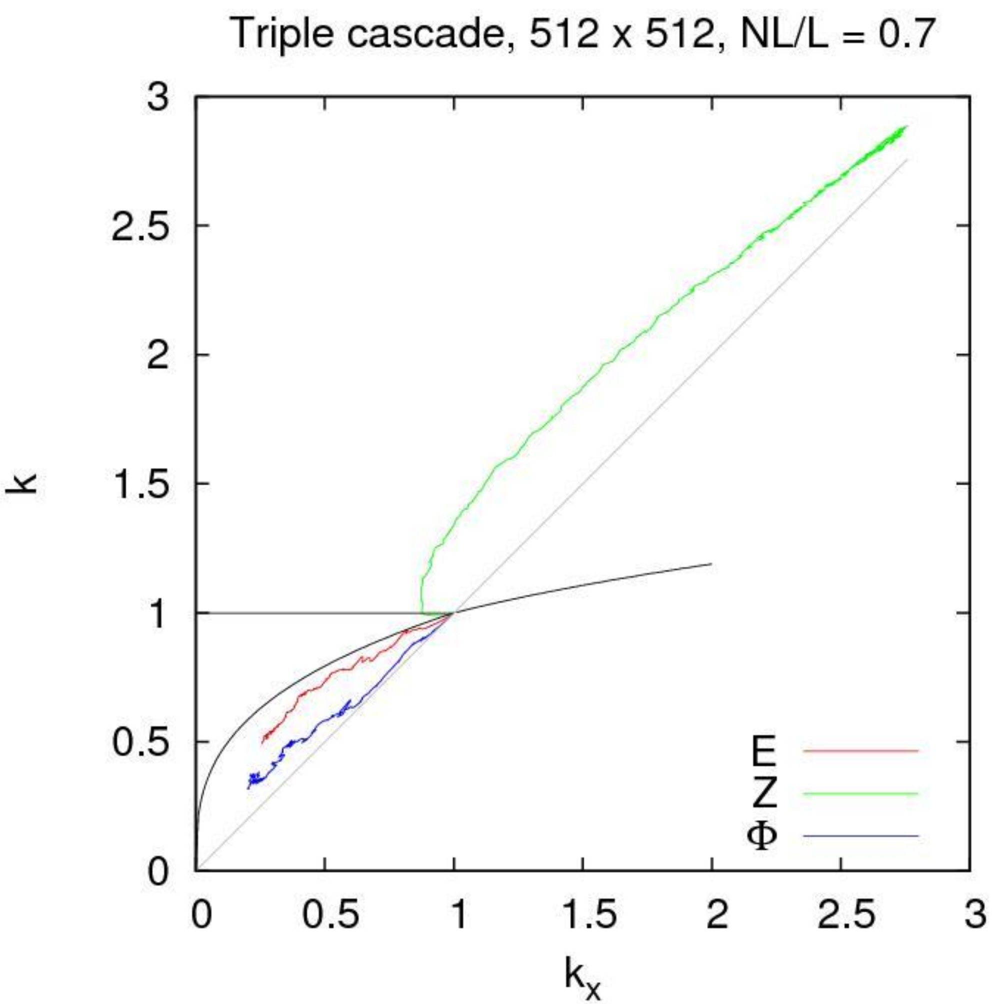
- *All three invariants cascade as predicted.*
- *Energy cascades to zonal scales along the boundary of its sector.*
- *Zonostrophy cascade is slightly anisotropic.*

Strong nonlinearity run, NL/L $=k_0^2/k_\beta^2=0.7$



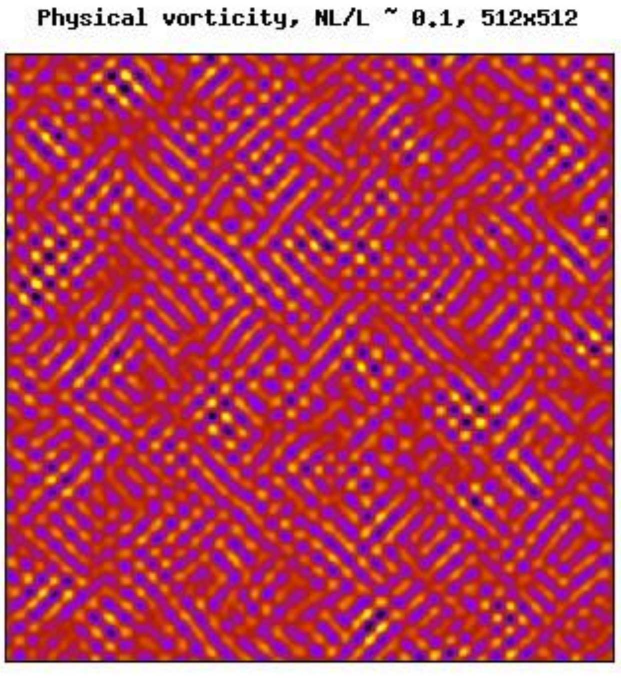
- *Initial turbulence is at the border of the dumbbell.*
- *Zonostropy is not conserved initially, but is conserved later.*
- *This is because the nonlinearity weakens as the inverse cascade enters into the dumbbell.*

Strong nonlinearity run, NL/L $=k_0^2/k_\beta^2=0.7$

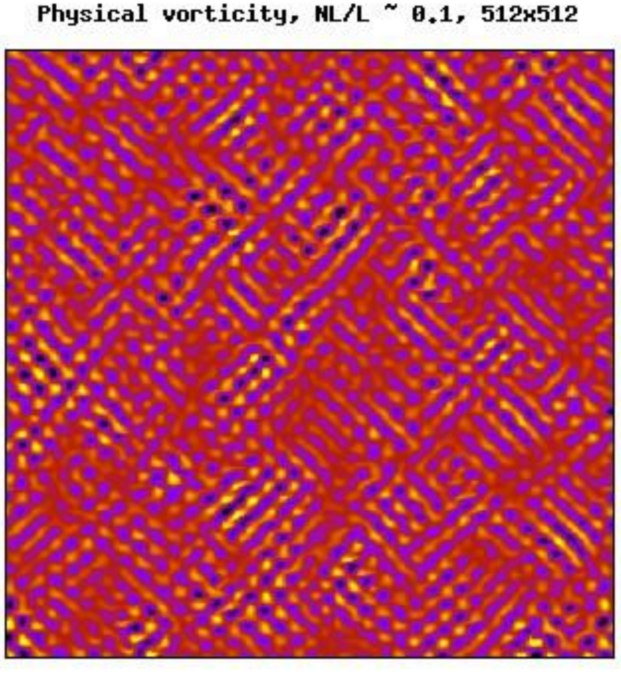


- E, Z and Φ cascade similar to the weakly nonlinear case.
- Faster and less chaotic trajectories.
- Enstrophy and Zonostrophy cascades are almost isotropic.

Vorticity field



Vorticity field



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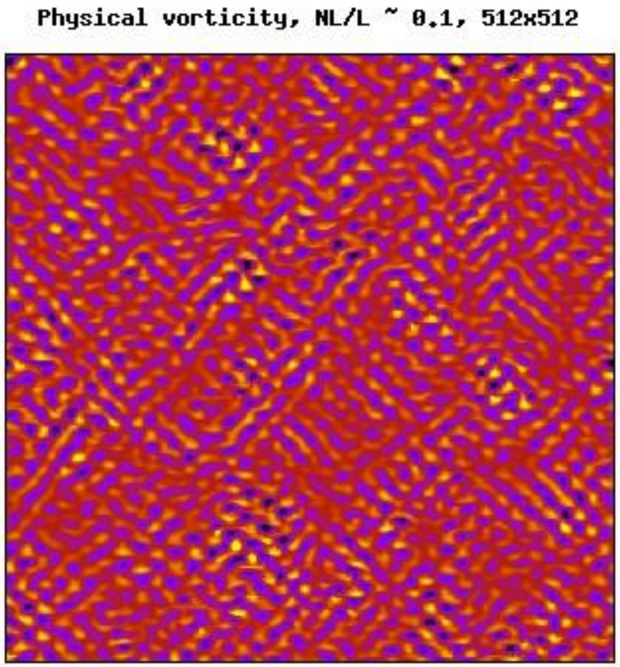

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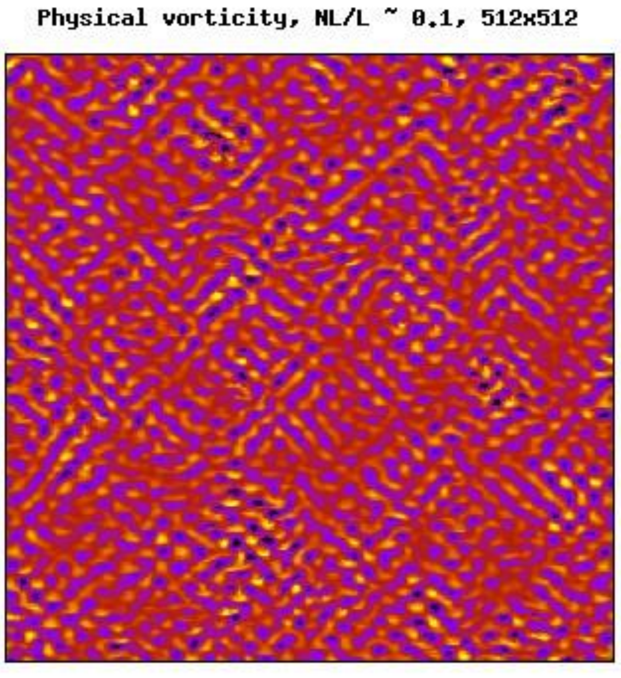

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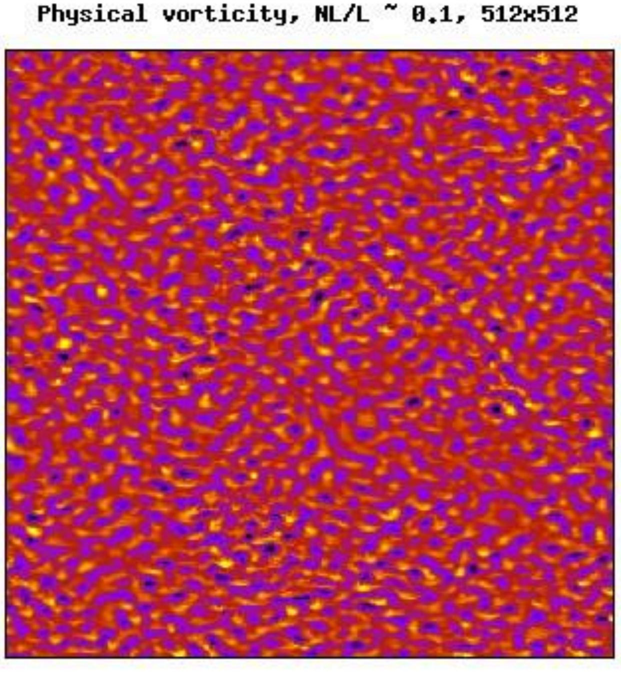

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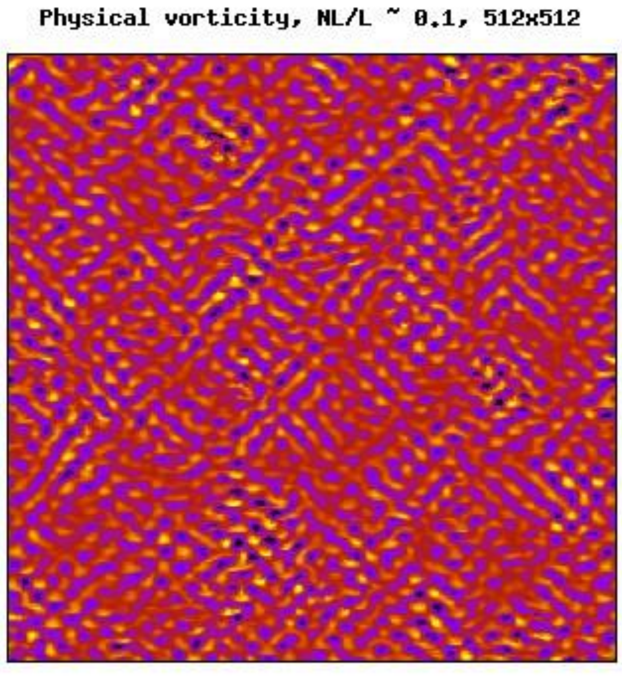
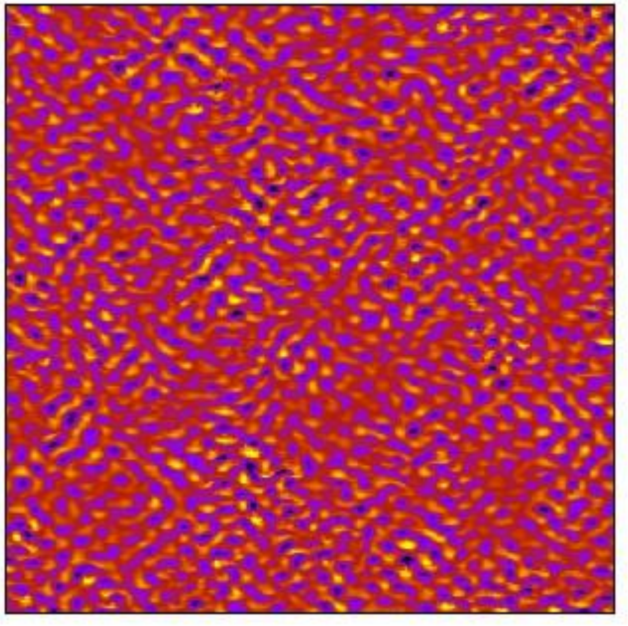


Vorticity field

Physical vorticity, NL/L ~ 0.1, 512x512



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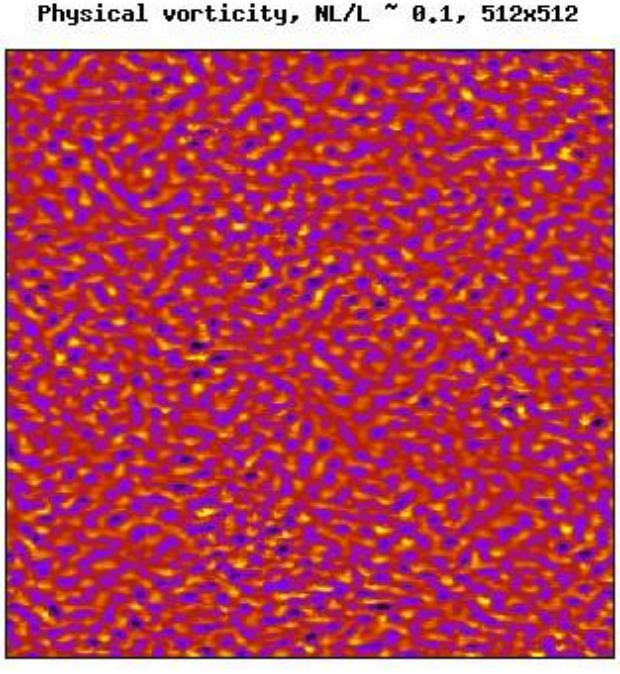

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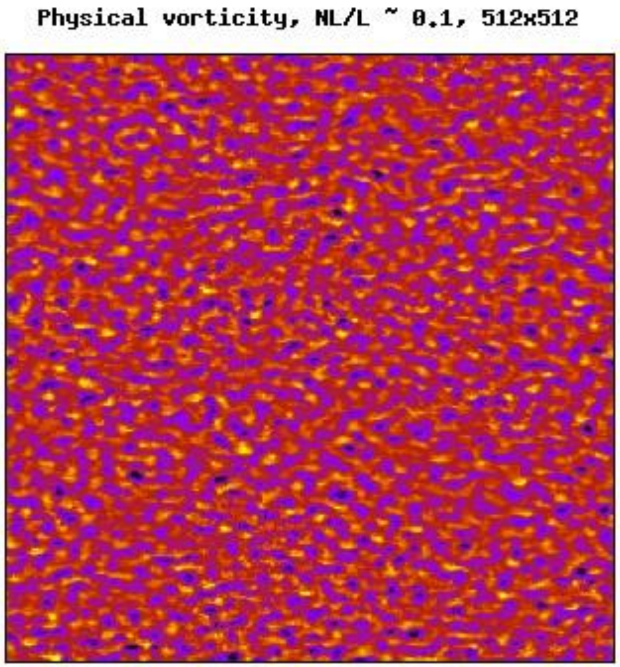

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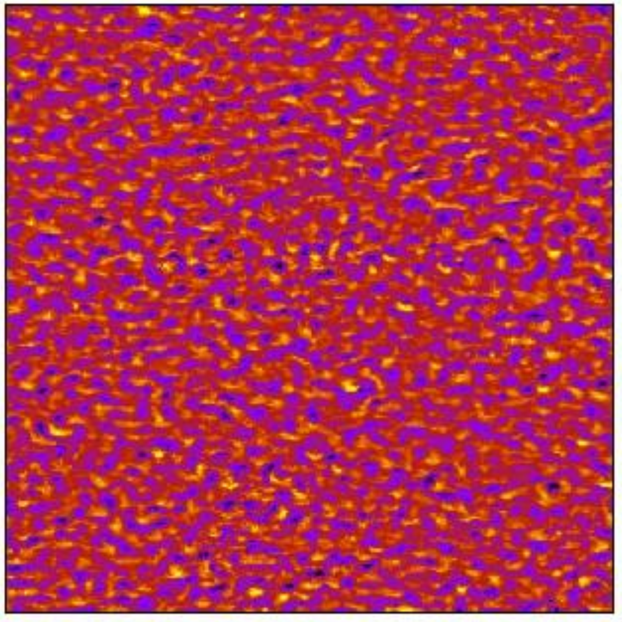

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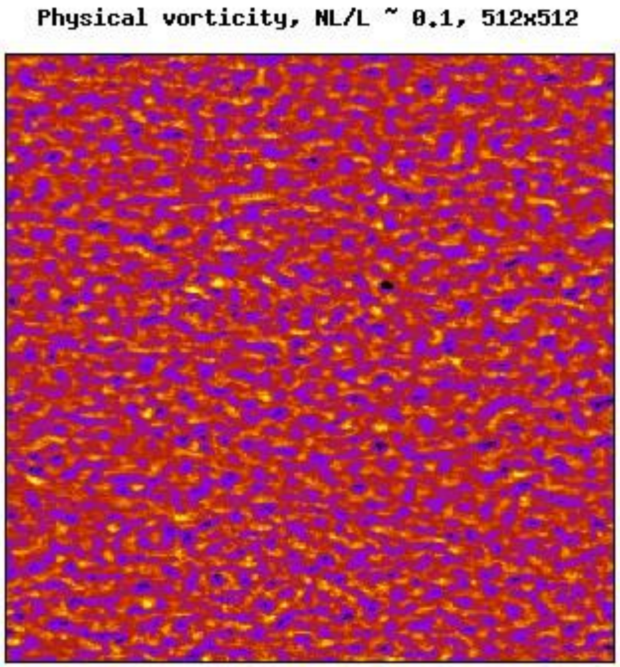

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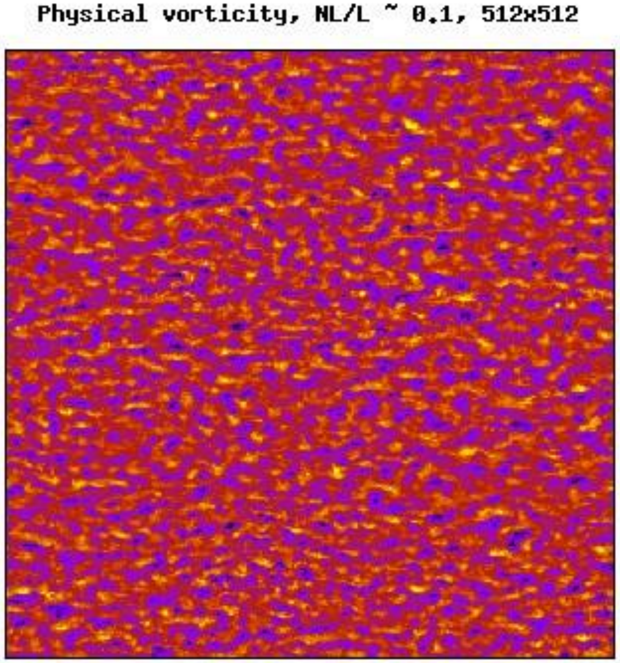

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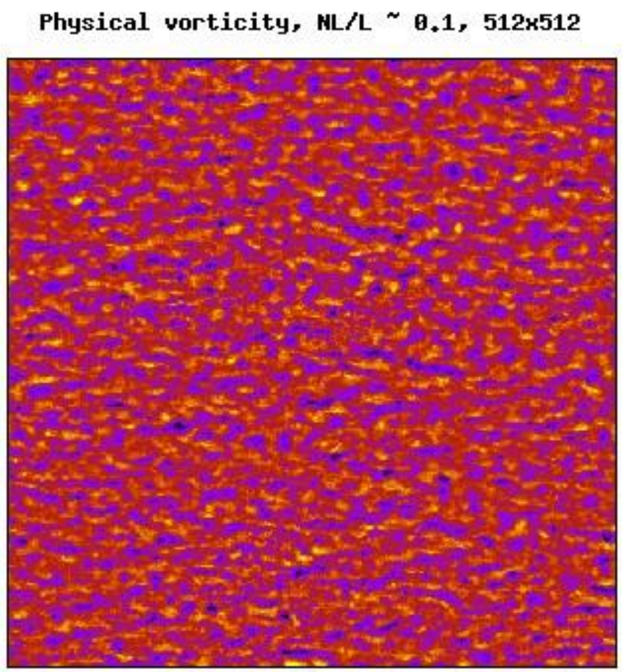

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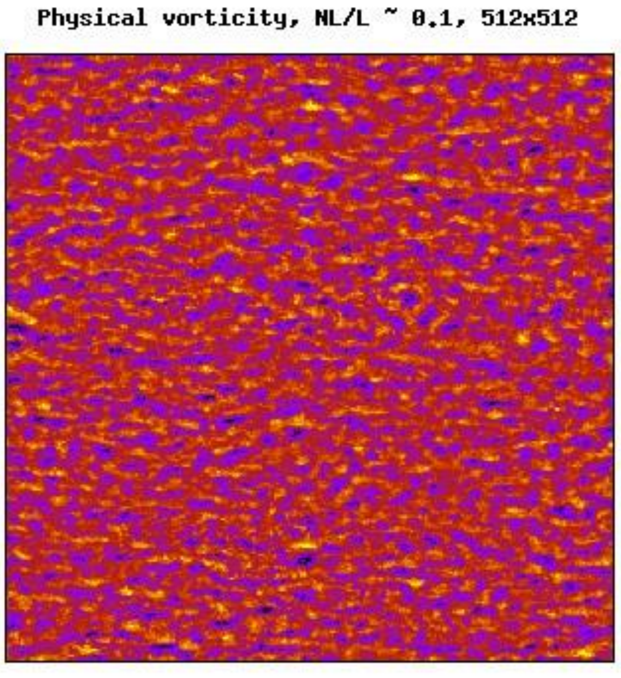

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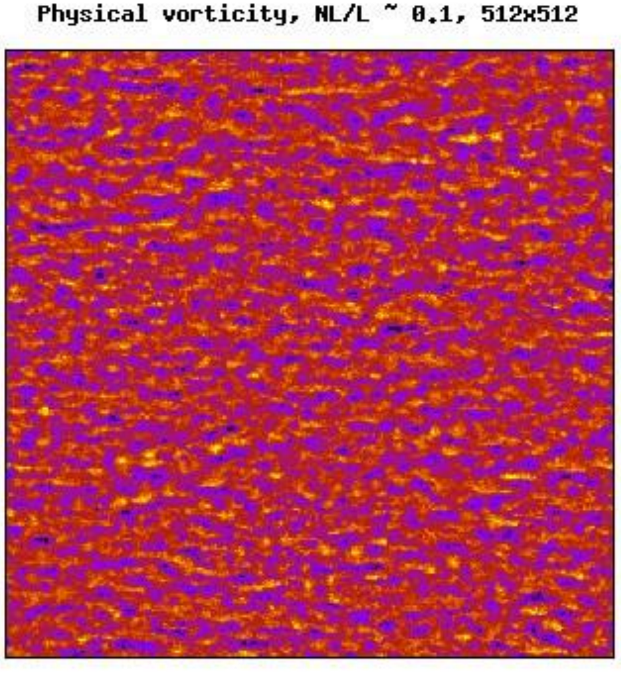

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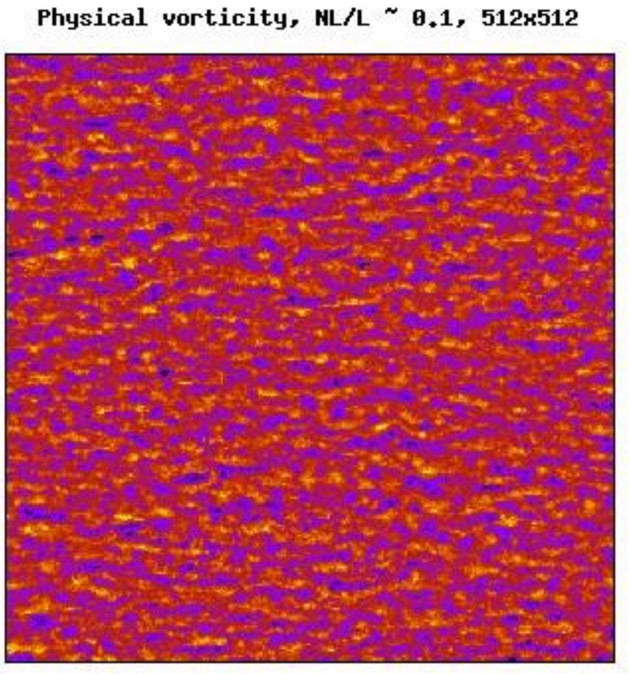

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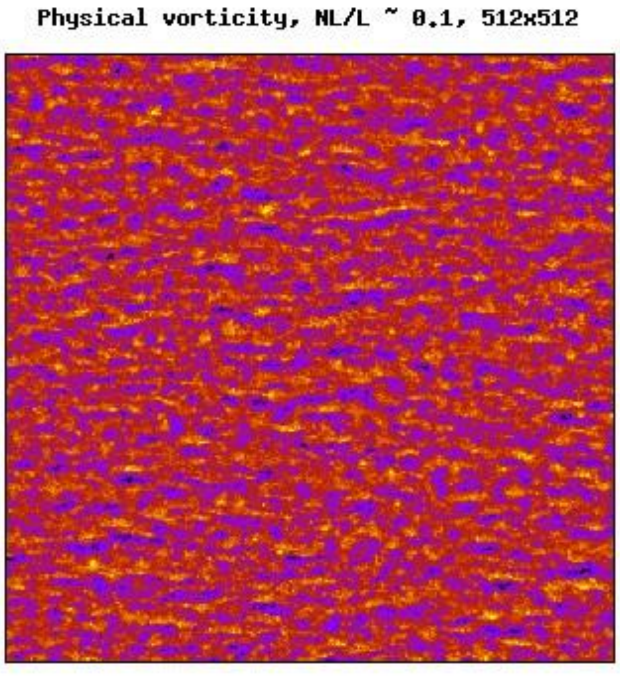

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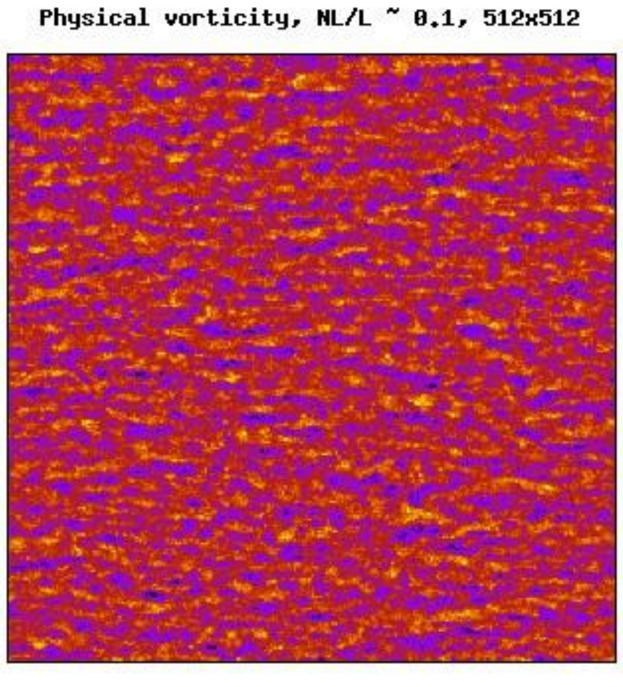

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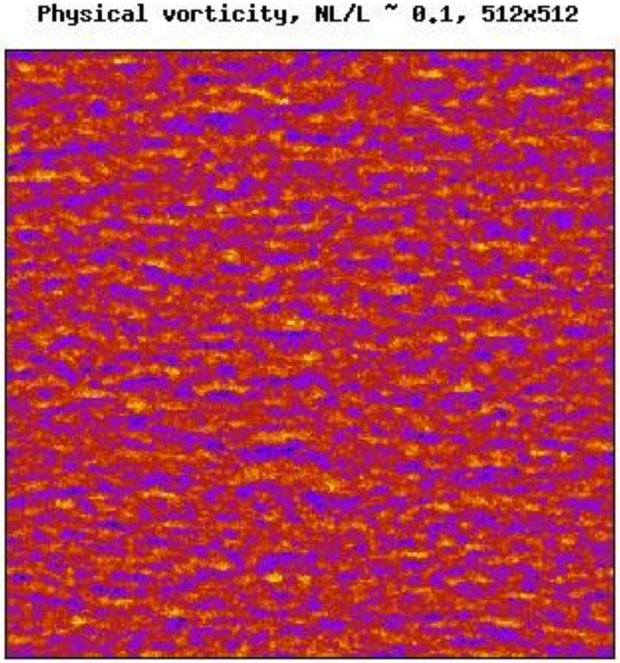
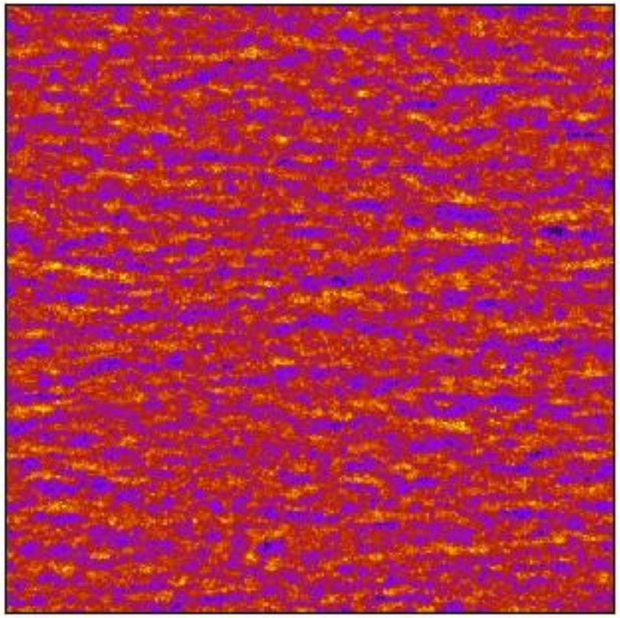


Vorticity field

Physical vorticity, NL/L ~ 0.1, 512x512



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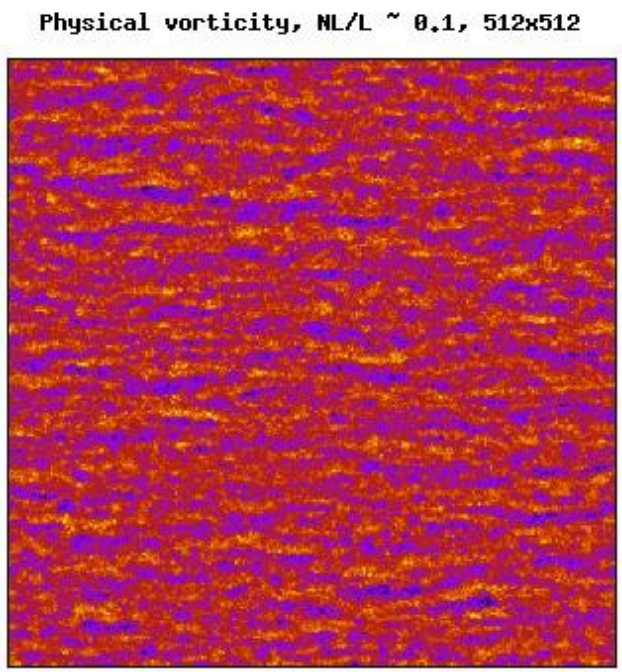

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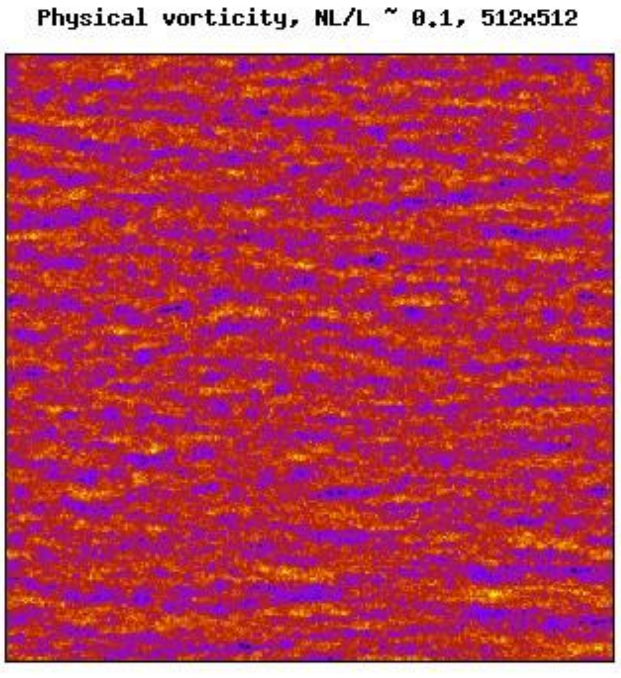
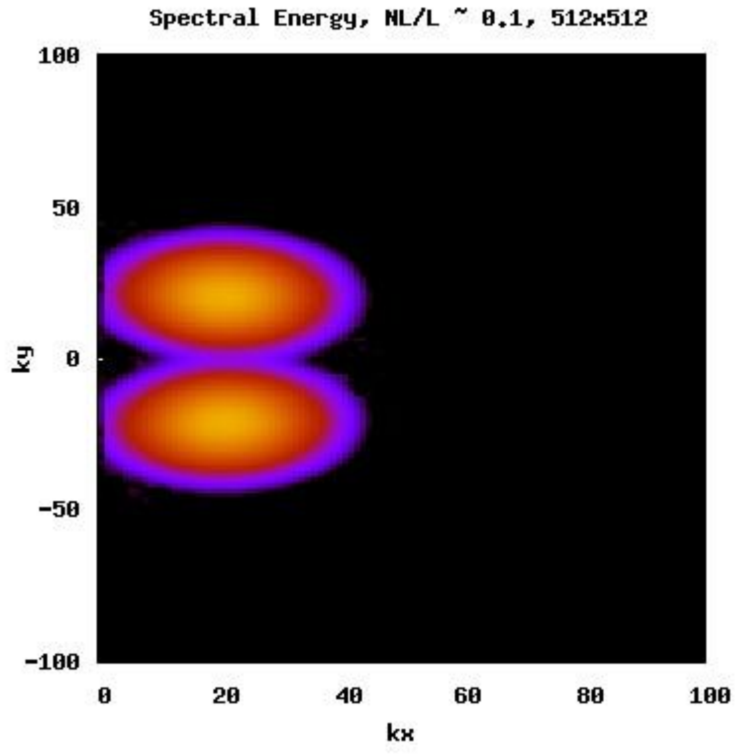
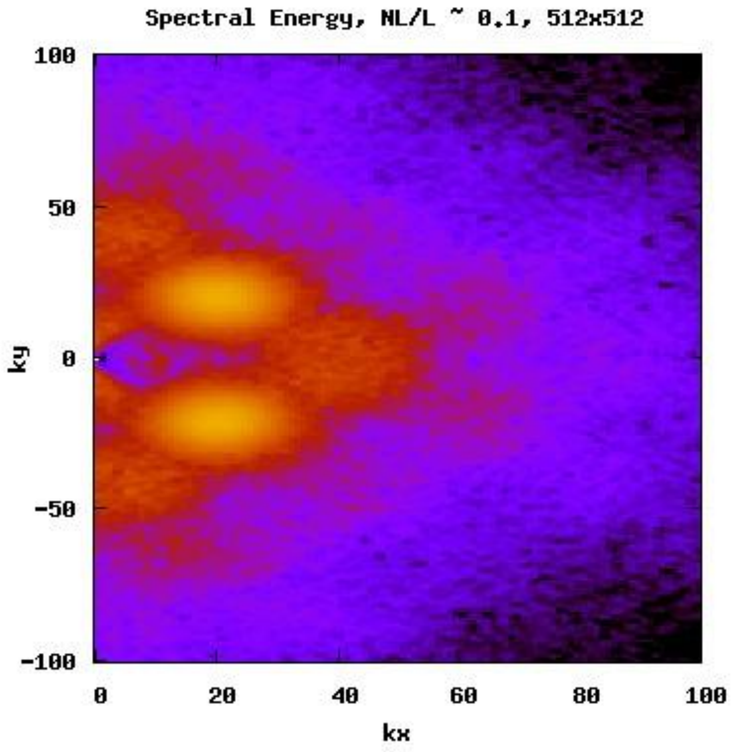


2D spectrum


Spectral Energy, NL/L ~ 0.1, 512x512
100
50
0
-50
-100
ky
0
20
40
60
80
100
kx

2D spectrum



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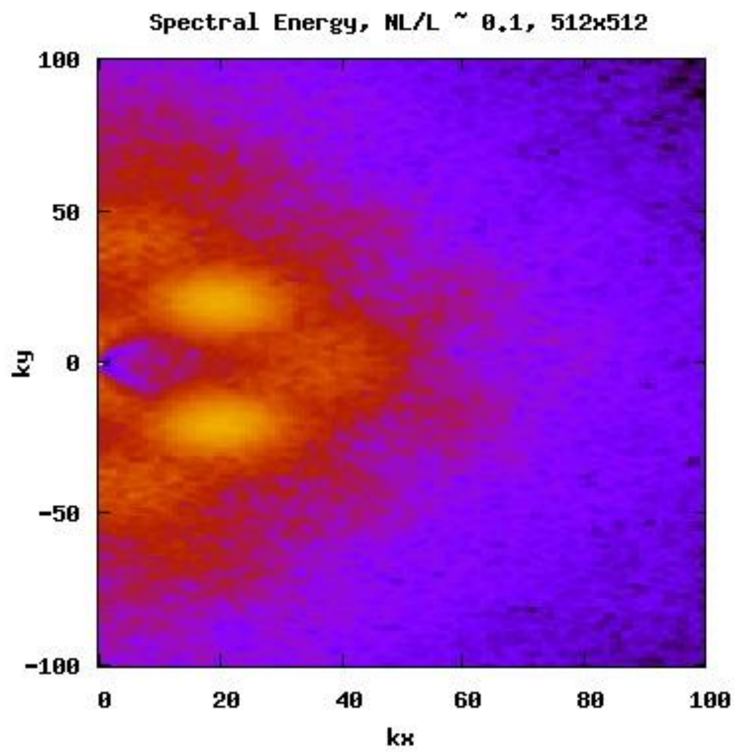

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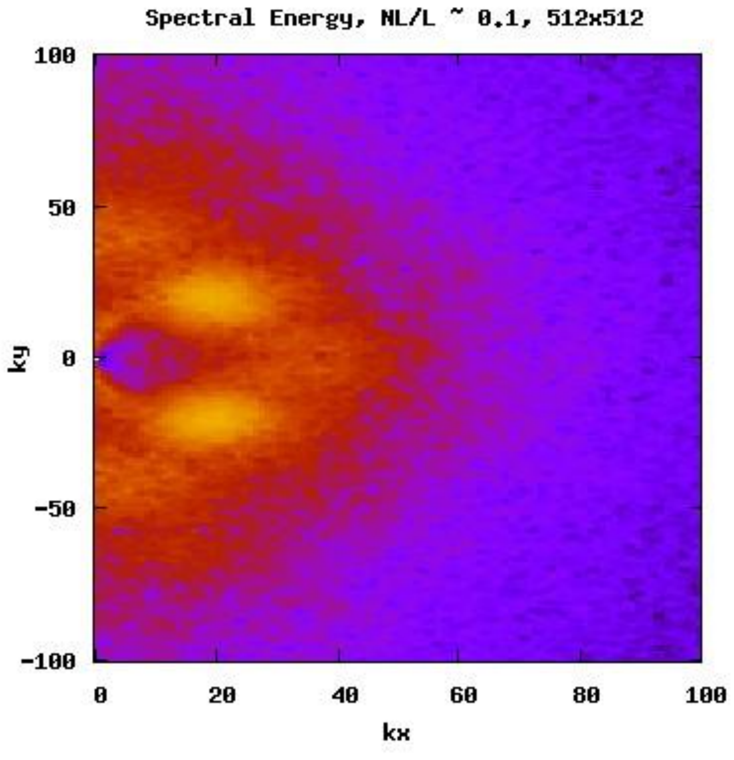

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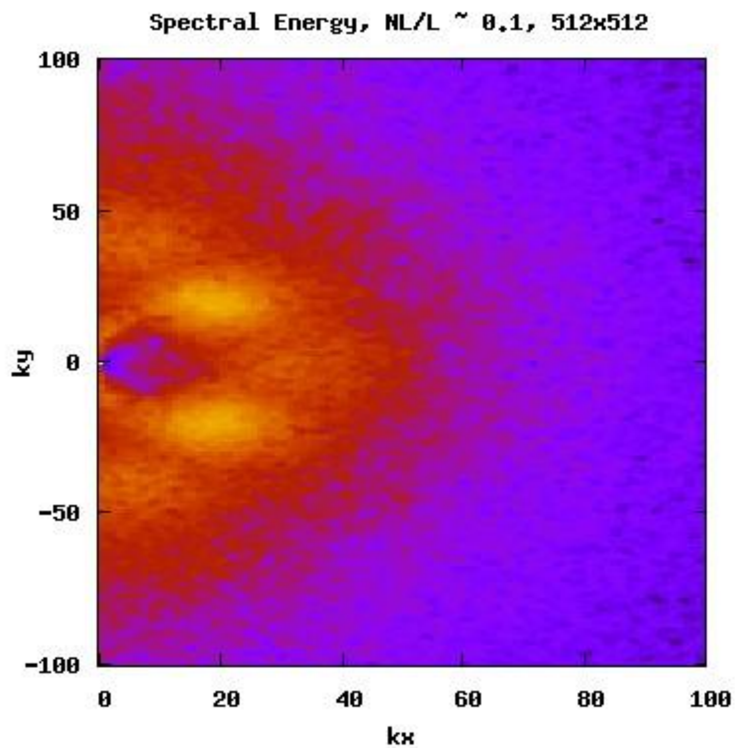

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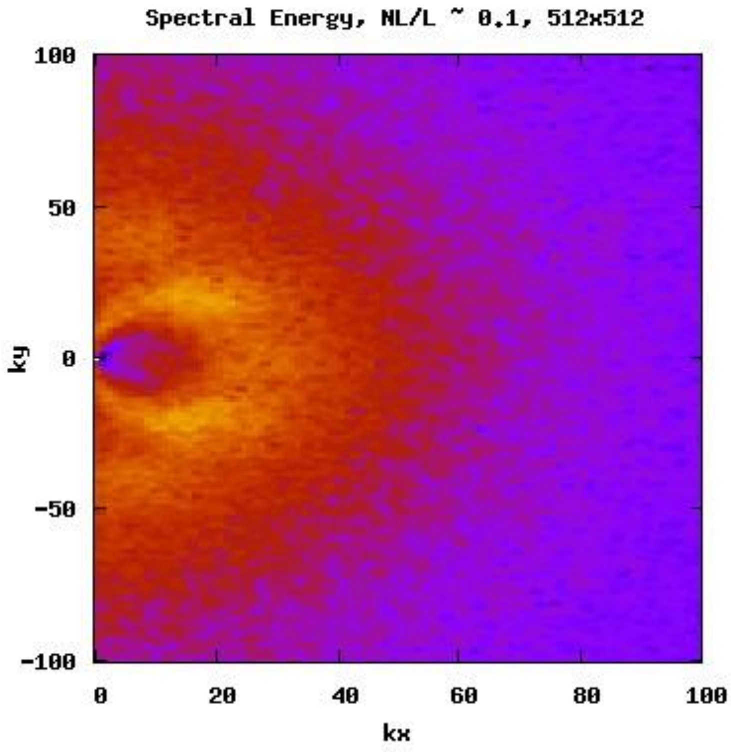

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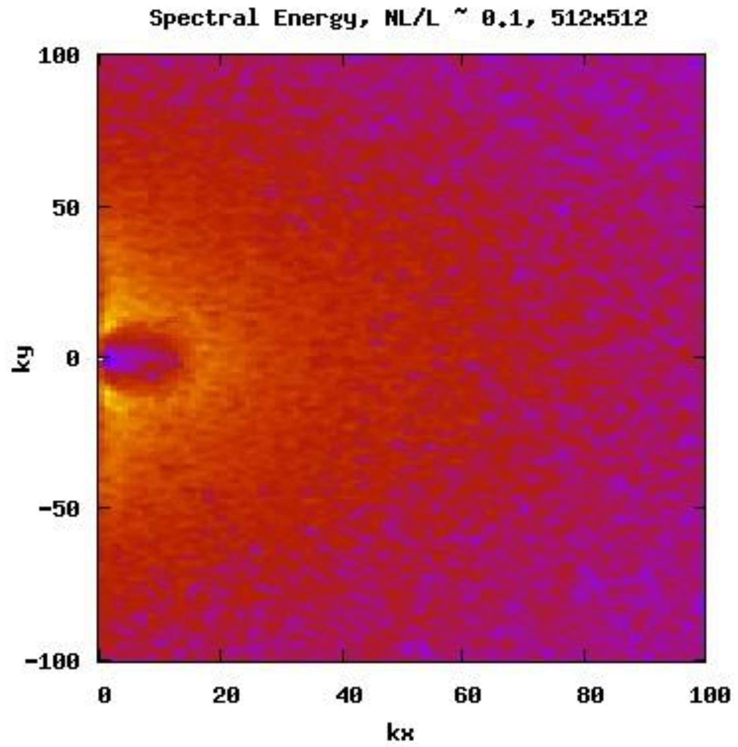

Spectral Energy, NL/L ~ 0.1, 512x512
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40
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100
kx

Part 2: Modulational Instability

$$\psi_0(\mathbf{x},t) = \Psi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + \bar{\Psi}_0 e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}$$

$$\boxed{\omega(\mathbf{k}) = -\frac{\beta k_x}{k^2+F} \quad - \text{frequency of linear waves.}}$$

- These waves are solutions of CHM equation for any amplitude. Are they stable? (Lorentz 1972, Gill 1973).

$$\psi(\mathbf{x},0) = \psi_0(\mathbf{x}) + \epsilon\,\psi_1(\mathbf{x}),$$

$$\psi_1(\mathbf{x}) = \psi_Z(\mathbf{x}) + \psi^+(\mathbf{x}) + \psi^-(\mathbf{x}) \quad - \text{perturbation.}$$

$$\psi_Z(\mathbf{x}) = a e^{i\mathbf{q}\cdot\mathbf{x}} + \bar{a} e^{-i\mathbf{q}\cdot\mathbf{x}} \quad - \text{zonal part} \quad \mathbf{q} = (0,q),$$

$$\psi^+(\mathbf{x}) = b^+ e^{i\mathbf{p}_+\cdot\mathbf{x}} + \bar{b}^+ e^{-i\mathbf{p}_+\cdot\mathbf{x}} \quad - \quad {}^+\text{satellite} \quad \mathbf{p}_+ = \mathbf{k}+\mathbf{q},$$

$$\psi^-(\mathbf{x}) = b^- e^{i\mathbf{p}_-\cdot\mathbf{x}} + \bar{b}^- e^{-i\mathbf{p}_-\cdot\mathbf{x}} \quad - \quad {}^-\text{satellite} \quad \mathbf{p}_- = \mathbf{k}-\mathbf{q}.$$

Instability dispersion relation

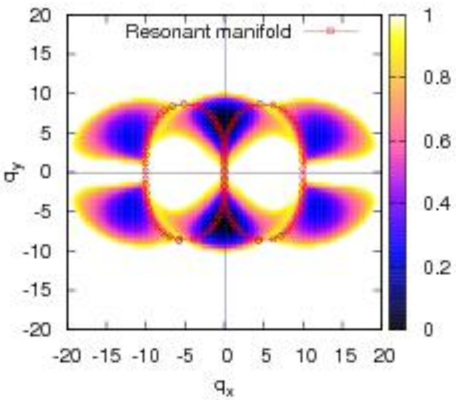
$$(q^2+F)\Omega+\beta q_x+\left|\Psi_0\right|^2\left|\mathbf{k}\times\mathbf{q}\right|^2(k^2-q^2)\left[\frac{p_+{}^2-k^2}{(p_+{}^2+F)(\Omega+\omega)+\beta p_{+x}}-\frac{p_-{}^2-k^2}{(p_-{}^2+F)(\Omega-\omega)+\beta p_{-x}}\right]=0$$

$$\boxed{M=\frac{\Psi_0 k^3}{\beta}\ -\text{nonlinearity parameter.}}$$

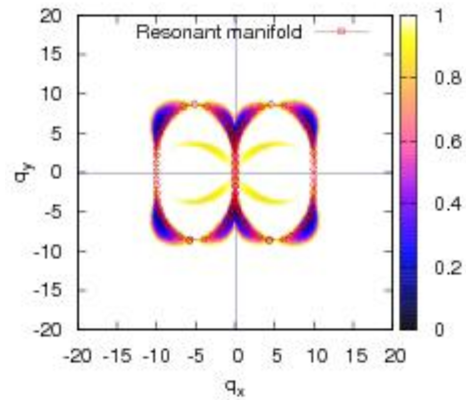
$M\to\infty$ – Euler limit (Rayleigh instabilty);

$M\to 0$ – weak monlinearity: resonant wave inetraction.

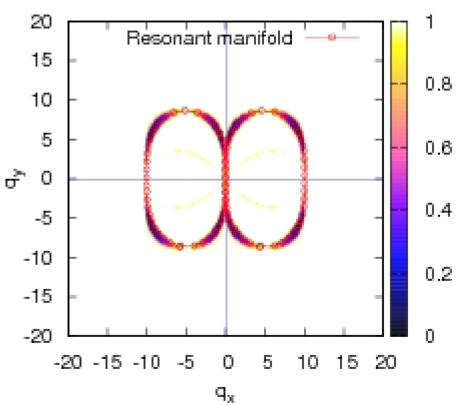
Structure of instability as a function of M



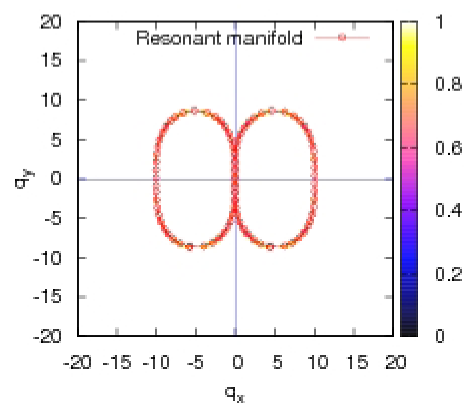
M=10



M=1



M=0.5

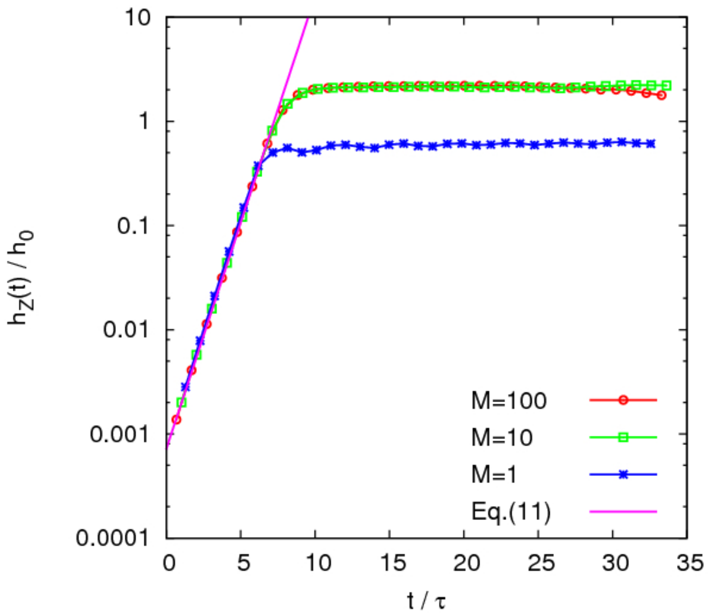


M=0.1

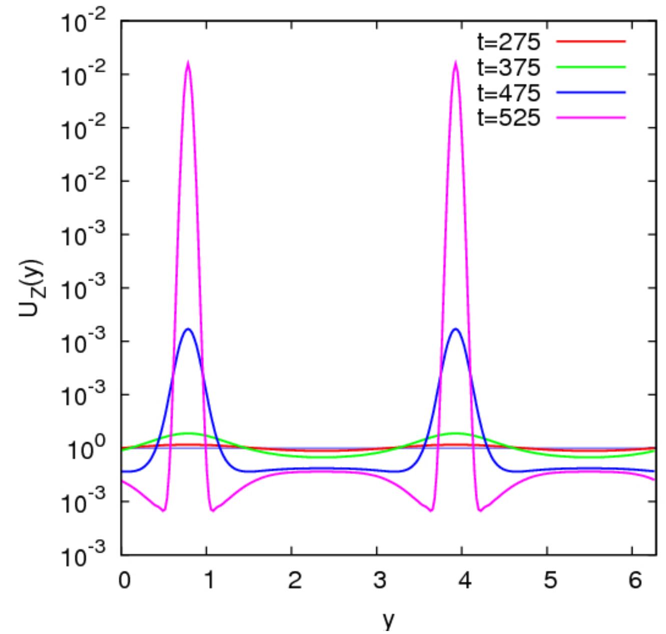
$$\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2,$$
$$\omega(\mathbf{k})+\omega(\mathbf{k}_1)=\omega(\mathbf{k}_2)$$

Unstable region collapses onto the resonant curve. For small M the most unstable disturbance is not zonal.

Nonlinear stage of modulational instability



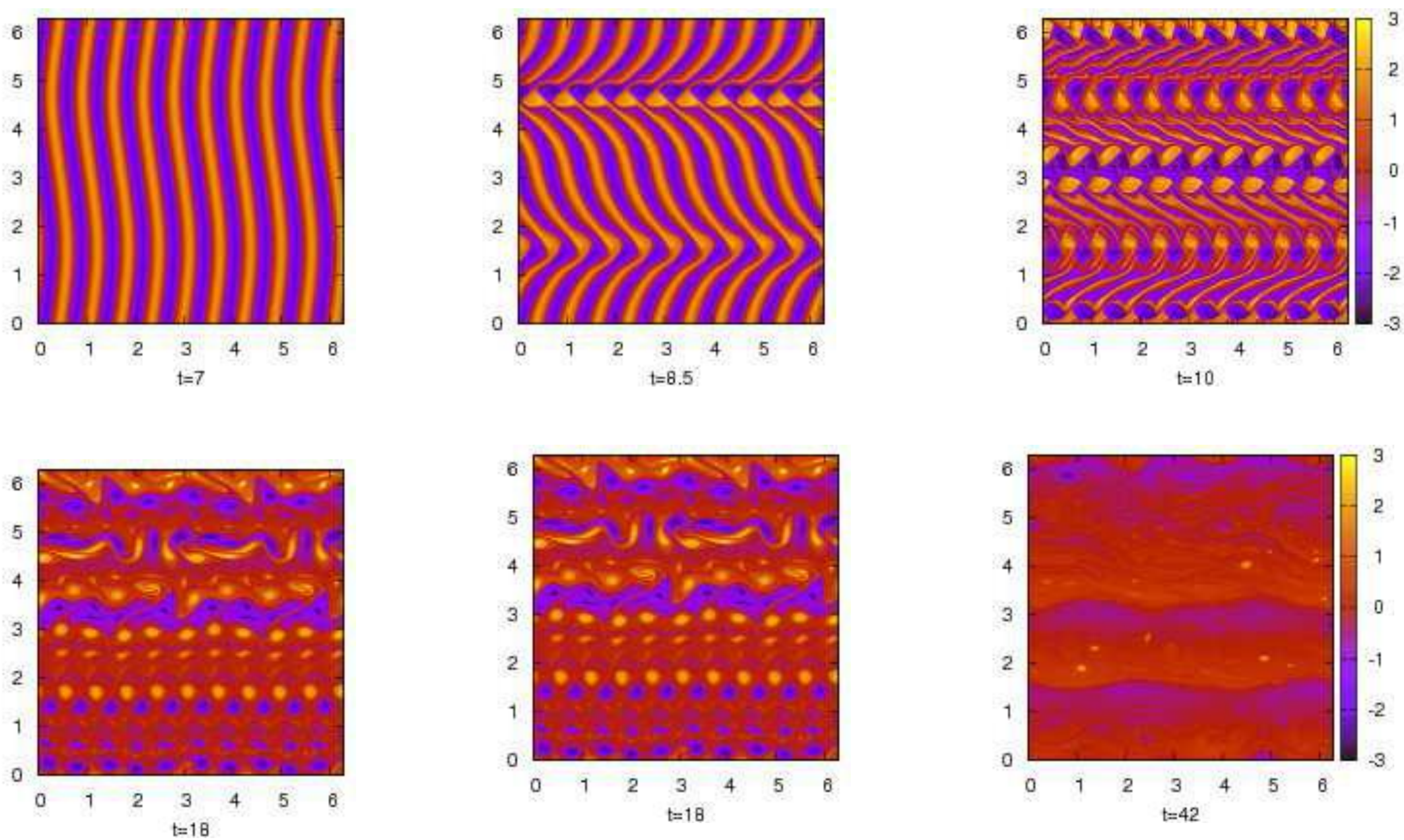
Growth of the q-mode compared to predictions of linear stability.



Zonal velocity profile (averaged over x).

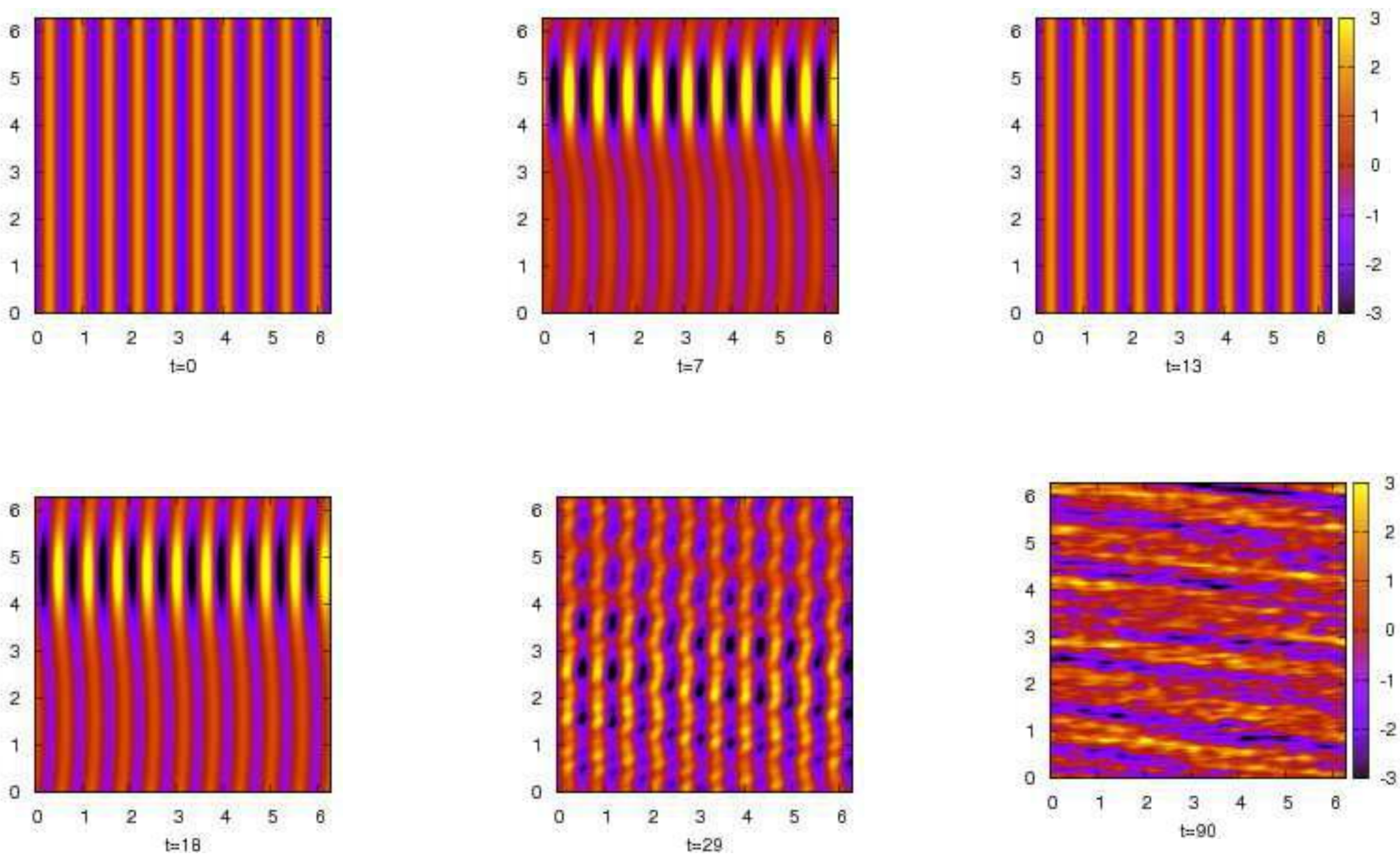
Pinching of jets predicted by Manin & SN, 1994. Transport barriers.

Strong wave case (M = 10)



- Jet pinch, roll-up into a double vortex street. After long time, the street breaks via a vortex pairing instability, leading to turbulence with a PV staircase structure.

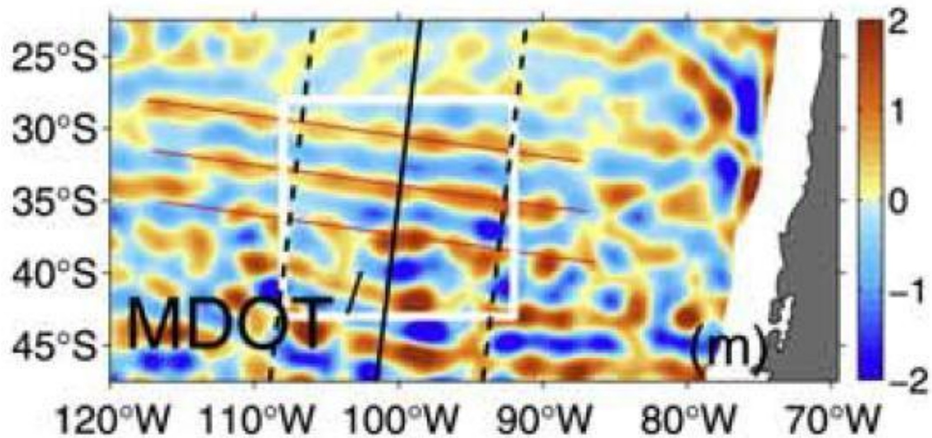
Weak wave case (M = 0.1)

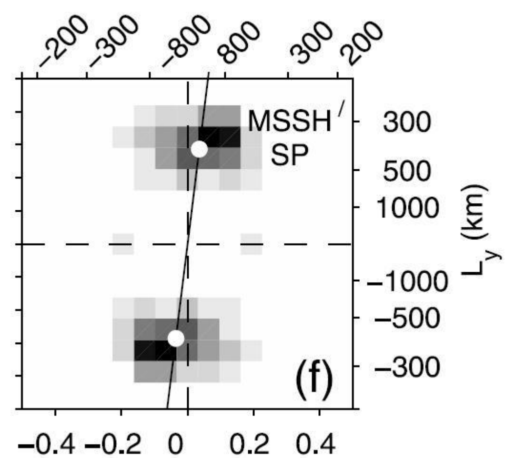


Original drift wave experiences self-focusing, but jets do not roll into vortices. Energy oscillates between 4-modes as predicted by the 4-mode truncation. At long time: transition to turbulence with inclined jets.

Ocean jets

- From Maximenko et al 2008.
- Slightly off-zonal jets.

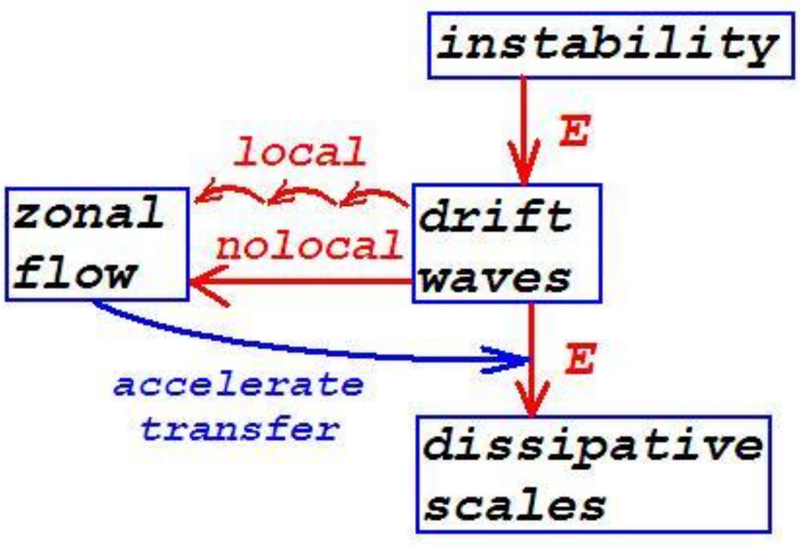




Summary for part 2.

- MI of a travelling drift wave exists for any nonlinearity M. Two limits : Euler limit for M>>1 vs weak resonance interaction for M<<1.
- Most unstable disturbance is zonal for large M’s and an inclined wave for small M. Inclined jets are seen of small M for long-time nonlinear stage.
- ZF’s are mostly eastward due to the beta-effect.
- Nonlinear pinching of ZF’s (for any M). Simplest model for the transport barriers.
- Role of MI for broad initial wave spectra?

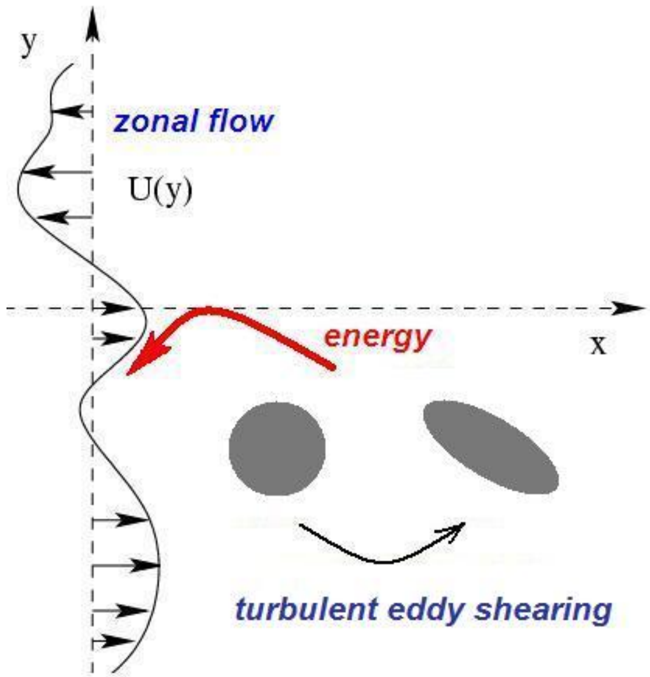
Part 3: LH transition



Balk, SN and Zakharov 1990

- Small-scale turbulence causes anomalous transport, hence L-mode.
- Negative feedback loop.
- Suppressed turbulence →no transport →improved confinement & H-mode.

Cartoon of nonlocal interaction



Victor P. Starr, *Physics of Negative Viscosity Phenomena* (McGraw Hill Book Co., New York 1968).

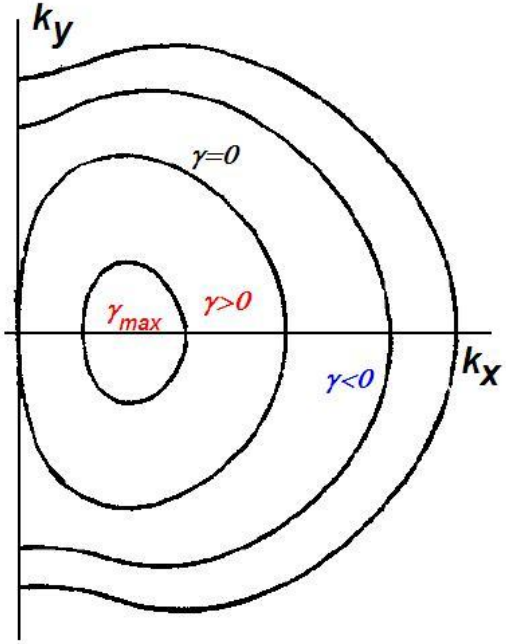
- Eddy scale ***L*** decreases via shearing by ZF
- Potential enstrophy ***Z*** is conserved.
- => Eddy energy $E = Z L^2$ is decreasing
- Total ***E*** is conserved, => ***E*** is transferred from the eddy to ZF
- Wrong! Both smaller and larger ***L's*** are produced. The energy of the eddy is unchanged. (Kraichnan 1976).

Small-scale energy conservation

- Energy in SS eddies is conserved if they are initially isotropic (Kraichnan 1976)
- 1. Dissipation: ellipse cannot get too thin.
- 2. Anisotropic initial eddies

Wave instabilities

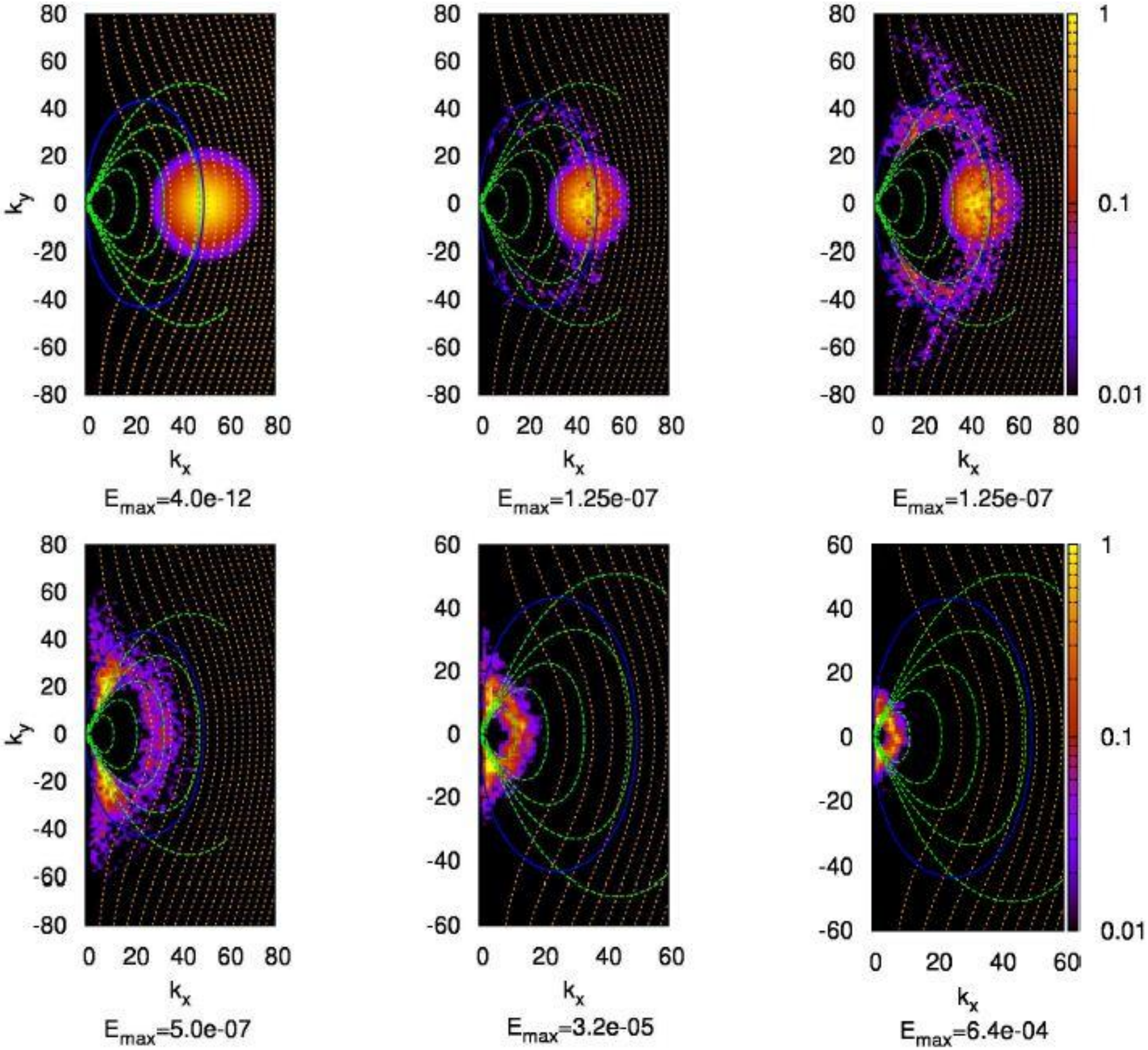
$$\frac{\partial}{\partial t}\left(\rho^2\nabla^2\psi-\psi\right)-\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}=\hat{\gamma}\,\psi$$



Access to stored free energy:
GFD: Baroclinic instability.
In plasmas: ITG, ETG instablities.

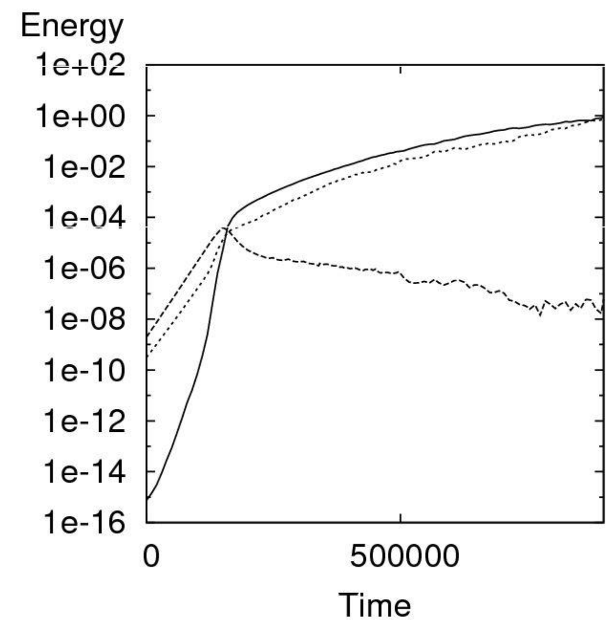
- Maximum on the k_x-axis at $k\rho \sim 1$.
- $\gamma=0$ line crosses $k=0$ point.

Forced CHM simulation



- Generation of ZF and suppression of small-scale turbulence
- Diffusion on k-space curves (as predicted in Balk et al 1991)
- ZF saturation:

$$U_{ZF} = \frac{\sqrt{v_* L \gamma_{max}}}{\rho k_f}$$



Summary for all parts.

- ZF's can be generated via modulational instability (for narrowband initial data) and by anisotropic inverse cascade (for broadband initial data)
- ZF's suppress turbulence thereby causing transport barriers.
- All effects present in forced/dissipated CHM model.
- Examine 2-potential models which includes instabilities.