GLOBAL GYROKINETIC EQUATIONS: EXTENDED ORDERINGS AND SECOND ORDER TERMS

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GYROKINETIC SIMULATION OF MFE EDGE PLASMAS MUST DEAL WITH LARGE PERTURBATIONS

- e.g. solution of gyrokinetic Poisson or vorticity equation

$$\nabla \cdot \left[ \left( n_0 + \delta n \right) \nabla \phi \right] = S$$

- $\delta n/n_0$ may be significant in the edge

- May have large time dependent ExB flows

X.Q. Xu, BOUT Simulation

Density perturbation at outside midplane

Edge

$\langle N_i \rangle (1 \times 10^{19}/m^3)$

$\delta n_{i,rms}/\langle N_i \rangle (R)$

$\delta n_{i,rms}/N_i(0)$

DIII-D #107822 L-mode

major radius R(m)
OUTLINE

• Simpler extended and subsidiary orderings
  o EM toroidal theory in DDL`92 ordering
  o subsidiary ordering for EM perturbations → “practical minimal model”
• numerical discretization of second-order terms
• gyrokinetics in the most general ordering for $\phi$
  o results and interpretation for slab
  o electromagnetic toroidal theory in this ordering
• summary
EXTENDED ORDERING GOALS: GYROKINETIC THEORY
VALID WHERE STANDARD ORDERINGS INAPPLICABLE

• Existing large-perturbation gyrokinetic theories
  o Allow $\frac{e\phi}{T} \sim 1$, $\frac{V_{exB}}{v_{th}} \ll 1$ - electromagnetic slab theory: A.M. Dimits, L.L. LoDestro, D.H.E. Dubin, Phys. Fluids B4 274 (1992).
  o 2-scale: short-wavelength perturbations with $\frac{e\phi}{T} \ll 1$ + (static) large long-wavelength component with $\frac{V_{exB}}{v_{th}} \sim 1$.
    ▪ T.S. Hahm, Phys Plasmas 3, 4658 (1996)

• More general theory removes 2-scale and static restrictions
  o Basic ordering and electrostatic slab theory: A.M. Dimits, Phys. Plasmas 17, 055901 (2010)
THE DDL`92 ORDERING IS USEFUL FOR LARGE PERTURBATION AMPLITUDES, SMALL FLOWS

- ordering: $q\psi / T \sim 1$, $\hat{b}_0 \times \nabla \psi / (\Omega_0 V_{th}) \ll 1$ where $\psi = \phi - p_\parallel \delta A_\parallel$.
- $u = 0$; use the standard 2-step approach
- Separation is $\psi = \bar{\psi} + \tilde{\psi}$
- $q\tilde{\psi} / T \ll 1 \Leftrightarrow \hat{b}_0 \times \nabla \psi / (\Omega_0 V_{th}) \ll 1$,

$$
\Gamma = \left[ A_{gc} + p_\parallel \hat{b}_0 \right] \cdot d\mathbf{R} - \mu d\theta - \left[ \frac{1}{2} \left\langle \left( p_\parallel - \delta A_\parallel \right)^2 \right\rangle + \mu \Omega + \langle \phi \rangle \right.
+ \frac{1}{2\Omega_0} \left\langle \nabla \left( \bar{\Psi} / \Omega_0 \right) \times \hat{b}_0 \cdot \nabla \bar{\psi} \right\rangle - \frac{1}{2\Omega_0} \frac{\partial}{\partial \mu} \left\langle \bar{\psi}^2 \right\rangle \right] dt,
$$

where $\tilde{\psi} = \tilde{\phi} - \left( p_\parallel - \delta \tilde{A}_\parallel \right) \delta \tilde{A}_\parallel$, $\bar{\Psi} = \int^\theta \bar{\psi} d\theta$, $\bar{\Psi} = \Psi - \langle \Psi \rangle$

- Essentially the same result as in the standard ordering, but now justified for the DDL ordering
THE FIRST-ORDER EQUATIONS OF MOTION CONTAIN THE \( \nabla B \), CURVATURE, AND EXB DRIFTS

\[ \Gamma_{-1,0,1} = \left[ A_{gc} + p_{||} \hat{b}_0 \right] \cdot dR - \mu d\theta - \left[ \frac{1}{2} \left( \langle p_{||} - \langle \delta A_{||} \rangle \rangle \right)^2 + \mu \Omega_0 + \langle \phi \rangle \right] dt, \]

- \( R \) Euler-Lagrange equations

\[ \dot{R} = \left( p_{||} - \delta A_{||} \right) \hat{b}_0 + \frac{1}{\Omega_0} \hat{b}_0 \times \left( \nabla \langle \phi \rangle + \mu \nabla \Omega_0 + \left( p_{||} - \langle \delta A_{||} \rangle \right)^2 \hat{b}_0 \cdot \nabla \hat{b}_0 \right) \]
SUBSIDIARY ORDERING FOR EM TERMS
RESULTS IN “PRACTICAL MINIMAL MODEL”

• Take $\delta \tilde{A}_|| = O(\varepsilon^2)$; still have $\delta \tilde{A}_|| = O(\varepsilon^0)$,

$$\Gamma = \left[ A_{gc} + p_\parallel \hat{b}_0 \right] \cdot dR - \mu d\theta - \left[ \frac{1}{2} \left( p_\parallel - \delta \tilde{A}_|| \right)^2 + \mu \Omega + \phi \right.$$

$$+ \frac{1}{2\Omega_0} \left\langle \nabla \left( \tilde{\Phi}/\Omega_0 \right) \times \hat{b}_0 \cdot \nabla \tilde{\phi} \right\rangle - \frac{1}{2\Omega_0} \frac{\partial}{\partial \mu} \left\langle \tilde{\phi}^2 \right\rangle \right] dt,$$

• The GK Poisson’s equation again has only electrostatic terms

$$0 \approx -\frac{1}{4\pi e} \nabla^2 \phi(x) = \int d\Lambda \delta \left( R + r - x \right) \left[ F_i + \tilde{\phi}_1 \frac{\partial F_i}{\partial \mu} + \nabla_{R \perp} \tilde{\Phi}_1 \cdot \hat{b}_0 \times \nabla_{R \perp} F_i \right] - n_e$$

• The GK Ampere’s Law has only the identity part of $T_{gy}$

$$\left( \nabla^2 / c^2 \right) A_|| = \left( 4\pi e / m_e \right) \left[ \int dVF_e \left( p_\parallel - \delta \tilde{A}_|| \right) \right.$$

$$- \left( Zm_e / m_i \right) \int d\Lambda \delta \left( R + r - x \right) \left( p_\parallel - \delta \tilde{A}_|| \right) \left( F_i + \tilde{\psi}_1 \frac{\partial F_i}{\partial \mu} + \nabla_{R \perp} \tilde{\Psi}_1 \cdot \hat{b}_0 \times \nabla_{R \perp} F_i \right) \left. \right]$$

• A similar model has been obtained for the symplectic representation.
IMPLEMENTATION OF THE SECOND-ORDER TERMS IN
GYROKINETIC MODELS REQUIRES NEW METHODS

GYROCENTER EQUATIONS OF MOTION

First-order terms now standard

- Given $\psi$, calculate $\langle \psi \rangle (R, \mu, t)$ either at each mesh node or cell center, or at each gyrocenter (in the case of a PIC code).
- Also need derivatives for the equations of motion
- Can calculate directly by averaging on the gyro orbit or in Fourier space
  - Save for use on the 4D ($R, \mu$) mesh for continuum codes.
  - For a PIC code, can do this particle by particle.
SECOND-ORDER TERMS IN GYROKINETIC EQUATIONS OF MOTION NEVER YET IMPLEMENTED

\[ \frac{\partial \langle (\tilde{\psi}_1)^2 \rangle}{\partial \mu} : \text{use} \]

\[ \frac{\partial \langle (\tilde{\psi}_1)^2 \rangle}{\partial \mu} = \frac{\partial \langle \psi_1^2 \rangle}{\partial \mu} - 2 \langle \psi_1 \rangle \frac{\partial \langle \psi_1 \rangle}{\partial \mu} , \]

and

\[ \frac{\partial \langle \psi \rangle}{\partial \mu} = \frac{\partial \rho}{\partial \mu} \frac{1}{2\pi} \int d\theta \hat{\rho}(\theta) \cdot \nabla \psi (R + \rho \hat{\rho}(\theta)) . \]

This can be calculated directly by sampling the components of \( \nabla \psi \) around an instantaneous gyro orbit or by using a Fourier decomposition and Bessel functions. Again, at any given \( t \), this is a function of the 4D \( (R, \mu) \) phase space.
\[
\left\langle \nabla_R \tilde{\Phi} \cdot \mathbf{b}_o \times \nabla_R \tilde{\phi} \right\rangle = \mathbf{b}_o \cdot \int_{0}^{\theta_1} d\theta_1 \int_{0}^{\theta_2} d\theta_2 \nabla_R \tilde{\phi} \left( \bar{R} + \rho \hat{\rho}(\theta_1) \right) \times \nabla_R \tilde{\phi} \left( \bar{R} + \rho \hat{\rho}(\theta_2) \right) \\
= \mathbf{b}_o \cdot \int_{0}^{\theta_1} d\theta_1 \int_{0}^{\theta_2} d\theta_2 \nabla_R \phi \left( \bar{R} + \rho \hat{\rho}(\theta_1) \right) \times \nabla_R \phi \left( \bar{R} + \rho \hat{\rho}(\theta_2) \right) - \frac{1}{2} \nabla_R \tilde{\phi} \left( \bar{R} \right) \times \nabla_R \tilde{\phi} \left( \bar{R} \right)
\]

This can be calculated sampling the components of \( \nabla \phi \) over a double gyro orbit. e.g., \( n \)-point \( \theta_1 \) gyro orbit -> \( n(n+1)/2 \)-point \( \theta_1 - \theta_2 \) double gyro orbit

8-point \( \theta_1 \) gyro orbit -> 36-point \( \theta_1 - \theta_2 \) double gyro orbit

Again, at any given time, this is a function of the 4D \((R, \mu)\) phase space.
THE FULL-f GYROKINETIC POISSON EQUATION CAN BE DIRECTLY DISCRETIZED, e.g., USING FINITE-ELEMENTS

- weak form + Galerkin representation of $\phi$.
- All derivatives and gyroaveraging operations can be recast into derivatives operating on the Galerkin basis functions.
- Begin with gyrokinetic Poisson equation:

$$L\phi = S$$

$$L\phi = \frac{1}{4\pi} \nabla^2 \phi(x) + \int d\Lambda \delta(R + r - x) \left[ \tilde{\phi}_1 \frac{\partial F_i}{\partial \mu} + \nabla_{R \perp} \tilde{\Phi}_1 \cdot \bar{b}_o \times \nabla_{R \perp} F_i \right]$$

$$S = n_e - \int d\Lambda \delta(R + r - x) F_i(R); \quad d\Lambda = dR \, dv_{\parallel} \, B \, d\mu \, d\theta$$
• Use Galerkin discretization of \( \frac{\delta A}{\delta \phi(x)} = 0 \)

\[
A = \int d\mathbf{x} \phi \left( S - \frac{1}{2} L \phi \right) \\
= \int d\mathbf{x} \phi S + \frac{1}{8\pi} \int d\mathbf{x} \left( \nabla \phi \right)^2 + \frac{1}{2} \int d\Lambda F_i \frac{\partial \tilde{\phi}^2}{\partial \mu} \\
- \frac{1}{2} \int d\Lambda F_i \left[ \hat{b} \cdot \nabla_{R,\perp} \tilde{\phi} \times \nabla_{R,\perp} \tilde{\Phi} - \frac{1}{B} \tilde{\phi} \nabla_{R,\perp} \tilde{\Phi} \cdot J_{eqm} \right]
\]

\[
\begin{cases}
\phi(x) = \sum_i \phi_i \psi_i(x) ; l = (i, j)
\end{cases}
\]

• Insert Galerkin representation

\[
\psi_{i,j} = \psi \left( \frac{x - x_{i,j}}{\Delta x_{i,j}}, \frac{y - y_{i,j}}{\Delta y_{i,j}} \right)
\]

→ resolves derivatives onto basis functions
\[ \delta \left( \frac{\text{weak form Galerkin GK Poisson}}{\delta \phi_{i,j}} \right) = 0 \rightarrow \text{Matrix equation to be solved} \]

\[ M^S \cdot \Phi = S \]

\[ M^S = \left( M + M^T \right) / 2 \]

\[ M_{k,l} = \frac{1}{4\pi} \int d x \nabla \psi_k \cdot \nabla \psi_l + \int d \Lambda F_i \frac{\partial \left( \bar{\psi}_k \bar{\psi}_l \right)}{\partial \mu} \]

\[ -\int d \Lambda F_i \left[ \hat{b} \cdot \nabla_R \bar{\psi}_k \times \nabla_R \bar{\Psi}_l - \frac{1}{B} \bar{\psi}_k \nabla_R \bar{\Psi}_l \cdot J_{\text{eqm}} \right] \]

\[ S_k = \int d x S \psi_k \]

- "Deposition" or projection from Z to x is needed to calculate the matrix elements.

- The resulting matrices are still sparse
  - For solution on perpendicular slices with \( N^2 \) cells of size \( \sim \rho_i \), about 100 \( N^2 \) nonzero matrix elements are needed (not \( N^4 \)).
TOWARDS A MINIMAL NECESSARY ORDERING FOR GYROKINETIC THEORY

• Low-frequency GK should be possible if in some frame of reference, the perturbation to the gyro orbit is small

• Consider an electrostatic potential $\phi$

• Absolute value of $\phi$ should not matter
  ▪ Can transform away any long-wavelength $E \times B$ drift
  ▪ Only shear (spatial variation) of $E \times B$ drift matters

• Basic ordering: $q\phi/T = O(\varepsilon^{-1})$, $V_{\text{ExB}}/v_{\text{th}} = O(\varepsilon^{0})$, $V'_{\text{ExB}}/\Omega = O(\varepsilon^{1})$
SECOND-ORDER LAGRANGIAN FOR ELECTROSTATIC SLAB CASE

\[
\Gamma = \left[ A_{gc} + U_{\parallel} \hat{b}_0 + u + \frac{1}{2\Omega} \frac{\partial}{\partial \mu} \left\langle \delta_1 \tilde{\phi} \left( \nabla u \right) \cdot \rho \right\rangle \right] \cdot dR
\]

\[
+ \frac{1}{2\Omega^2} \left[ \left\langle \nabla \tilde{\phi} \times \hat{b}_0 \cdot \nabla \left( \rho \cdot \nabla u \right) \right\rangle + \mu \frac{\partial u}{\partial \mu} \cdot \left\langle \rho \frac{\partial \delta_1 \tilde{\phi}}{\partial \mu} \right\rangle \right] d\mu
\]

\[
- \left[ \mu + \frac{1}{2\Omega^2} \left\langle \nabla \tilde{\phi} \times \hat{b}_0 \cdot \nabla \rho \right\rangle \right] d\theta - \left[ \frac{1}{2} U_{\parallel}^2 + \mu \Omega + \frac{1}{2} u^2 + \bar{\phi} \right]
\]

\[
+ \frac{1}{2\Omega^2} \left\langle \nabla \tilde{\phi} \times \hat{b}_0 \cdot \nabla \delta_1 \tilde{\phi} \right\rangle - \frac{1}{2\Omega} \frac{\partial}{\partial \mu} \left\langle \left( \delta_1 \tilde{\phi} \right)^2 \right\rangle - \frac{1}{2\Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial}{\partial \mu} \left\langle \delta_1 \tilde{\phi} \rho \right\rangle \right] dt,
\]

- here, \( u = \frac{1}{\Omega} \hat{b}_0 \times \nabla \tilde{\phi} \) here has temporal and spatial dependences
- many new (noncanonical) components to Lagrange tensor
- \( \nabla \cdot \left\langle \rho \cdot \nabla u \delta_1 \tilde{\phi} \right\rangle \) terms are absent in 2-scale theories, but are of the same order as \( \nabla \left\langle \left( \rho \cdot \nabla u \right)^2 \right\rangle \sim \nabla \left\langle \delta_1 \tilde{\phi}^2 \right\rangle \).
THE FIRST-ORDER EQUATIONS OF MOTION CONTAIN THE EXB AND POLARIZATION DRIFTS

\[ \Gamma_{-1,0,1} = \left[ A_{gc} + U_{||} \hat{b}_0 + u \right] \cdot dR - \mu \, d\theta - \left[ \frac{1}{2} U_{||}^2 + \mu \Omega + \frac{1}{2} u^2 + \phi \right] dt \]

- Euler-Lagrange equation \( \omega_j \dot{Z}^i = H_{,j} + \gamma_{j,t} \Rightarrow \)

\[ \dot{Z}^i = P^{ij} \left( H_{,j} + \gamma_{j,t} \right) \text{ where } P = \omega^{-1}, \quad \omega^{ij} = \frac{\partial \gamma_j}{\partial Z^i} - \frac{\partial \gamma_i}{\partial Z^j} \]

\[ \dot{R} = u + U_{||} \hat{b}_0 + \frac{1}{\Omega^*_{||}} \hat{b}_0 \times \left( \frac{\partial}{\partial t} + u.\nabla + U_{||} \nabla \right) u, \]

\[ u \equiv \frac{1}{\Omega} \hat{b}_0 \times \nabla \phi, \quad \Omega^*_{||} = \hat{b}_0 \cdot \nabla \times \left( A_{gc} + u \right) \]

- \( \frac{1}{\Omega} \hat{b} \times \left( \frac{\partial u}{\partial t}, \, U_{||} \nabla_{||} u \right) \) terms are new.

- \( u \) includes both long- and short-wavelength components.
THE EQUATIONS OF MOTION CAN BE OBTAINED PERTURBATIVELY TO SECOND ORDER

\[ \dot{Z}_i = P^{ij} \left( H_{,j} + \gamma_{,j,t} \right) \]

\[ P = \omega^{-1}, \quad \omega^{ij} = \frac{\partial \gamma_j}{\partial Z^i} - \frac{\partial \gamma_i}{\partial Z^j} \]

- \( \omega \) has many more nonzero components than in previous theories, so its inverse \( P \) is more complicated.

- Use \( \omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 \) where \( \omega_0 \) is canonical (and therefore easily invertible)

\[ P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots, \]

where \( P_0 = \omega^{-1}_0, \quad P_1 = -P_0 \omega_1 P_0, \quad P_2 = P_0 \left( \omega_1 P_0 \omega_1 - \omega_2 \right) P_0 \)
THE EQUATIONS OF MOTION CAN BE OBTAINED PERTURBATIVELY TO SECOND ORDER

\[
\begin{bmatrix}
\dot{R}_{2,\perp} \\
\dot{z}_2 \\
\dot{U}_{2,\parallel} \\
\dot{\mu}_2 \\
\dot{\theta}_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{\Omega} \hat{b} \times \left\{ \nabla_{\perp} H_2 - \left[ \left( \nabla_{\perp} \gamma_{2,\perp} \right) \cdot u_{\perp} + \left( \nabla_{\perp} \gamma_{2,z} \right) \cdot U_{\parallel} - \Omega \nabla_{\perp} \gamma_{2,\theta} \right] - \left( \nabla_{\perp} \times u_{\perp} \right) \frac{1}{\Omega} \frac{d_0 u_{\perp}}{dt} \right\} \\
0 \\
-H_{2,z} + \left[ \gamma_{2,\perp} \cdot u_{\perp} + \gamma_{2,z} \cdot U_{\parallel} - \Omega \gamma_{2,\theta,z} \right] - \frac{1}{\Omega} \frac{d_0 u_{\perp}}{dt} \\
0 \\
-H_{2,\mu} + \left[ \gamma_{2,\perp,\mu} \cdot u_{\perp} + \gamma_{2,z,\mu} \cdot U_{\parallel} - \Omega \gamma_{2,\theta,\mu} \right]
\end{bmatrix}
\]
THE DIRECTLY CALCULATED EQUATIONS OF MOTION ARE COMPLICATED BECAUSE OF THE

· e.g. first row of Det(w)*w⁻¹

\[
\begin{bmatrix}
0, & 1 - 2w2mt - w2mt^2 & w2xt w2ym - w2mt w2xt w2ym + w2xm w2yt + w2mt w2xm w2yt, \\
Bst_1 & Bst_1 & Bst_1 \\
Bst_1 & Bst_1 & Bst_1 \\
0, & - w2yt w2zm - w2mt w2yt w2zm & w2xt w2ym w2yt w2zm + w2xm w2yt^2 w2zm + w2ym w2zt + w2mt w2ym w2zt \\
Bst_1 & Bst_1 & Bst_1 \\
Bst_1 & Bst_1 & Bst_1 \\
w2mt w2ym w2zt + w2xt w2ym^2 w2zt - w2xm w2ym w2yt w2zt - w2xt w2ym^2 w2zt - w2xm w2ym w2yt w2zt \\
Bst_1 & Bst_1 & Bst_1 \\
w2mt^2 w2yt w2zt - w2xt w2ym w2yt - w2mt w2mt w2ym w2yt w2zt - w2mt w2xm w2yt w2zt \\
Bst_1 & Bst_1 & Bst_1 \\
w2mt w2ym w2yt + w2xt w2ym w2yt - w2xm w2yt^2, - w2ym w2mt w2ym - w2xt w2ym^2 + w2xm w2ym w2yt \\
Bst_1 & Bst_1 & Bst_1 & Bst_1 & Bst_1 & Bst_1 \\
\end{bmatrix}
\]

These may be necessary, e.g., for numerical implementations with proper conservation properties.
THE GYROKINETIC POISSON (FIELD) EQUATION IS NONLINEAR EVEN AT T₁ ORDER

Gyrokinetic Poisson equation - quasineutrality

\[ n_e = n_i = \int d^6Z \delta(R + \rho - x) \Omega \left[ 1 + \frac{1}{\Omega^2} \nabla_{\perp}^2 \phi + \rho \times \hat{b} \cdot \frac{\partial u}{\partial \mu} \right] \]

\[ \times \left[ F_i(R, \mu) + \frac{1}{\Omega} \delta \tilde{\phi} \frac{\partial F}{\partial \mu} + \frac{1}{\Omega^2} \nabla \tilde{\Phi} \times \hat{b}_0 \cdot \nabla F \right] \]

\[ = \int \Omega d\mu \, dv_{\parallel} \, d\theta \left[ F_i(x - \rho, \mu) + \frac{1}{\Omega} \tilde{\phi} \frac{\partial F}{\partial \mu} + \frac{1}{\Omega^2} \nabla \tilde{\Phi} \times \hat{b}_0 \cdot \nabla F \right] \]

- To linear order in \( \phi \), this is the same as the standard form.
  - Note the gyrophase dependent Jacobian term.
  - Integrate one term by parts wrt. \( \mu \) and then wrt. \( \theta \).
THE TOROIDAL DERIVATION PROCEEDS SIMILARLY TO THE SLAB DERIVATION

• Given \( Q_0 = (x, V = \dot{x}) \), where \( x \) is the lab-frame position, and the presence of a large \( E \times B \) drift, it is most convenient to transform to \( Q = (x, v = V - u) \) where

• \( v \) is the velocity in the frame of a gyro-averaged \( E \times B \) drift

\[
u \equiv \frac{1}{\Omega_0} \hat{b} \times \nabla \langle \phi \rangle\]
SUCH A TRANSFORMATION CAN BE DEFINED IMPLICITLY

- Define

\[ \rho = \frac{1}{\Omega} \hat{b}_0 \times \nu, \]

\[ \nu_\perp = \hat{b}_0 \times \nu \times \hat{b}_0, \]

\[ \nu_\perp = |\nu_\perp|, \]

\[ \mu = \frac{\nu_\perp^2}{2\Omega}, \]

\[ \langle \phi \rangle (X, \mu, t) = \frac{1}{2\pi} \oint d\theta \, \phi (X + \rho, t) \]

\[ u(x, v, t) = \frac{1}{\Omega_0} \hat{b} \times \nabla_x \langle \phi \rangle (x - \rho, \mu, t) \]

- Now define \( u \) and \( v \) through the implicit equation

\[ v + u(x, v, t) = V = \dot{x} \]

- solution well defined because our ordering \( \Rightarrow \|\nabla_v u\| \ll 1 \)
THE TRANSFORMATION CAN BE VELOCITY DEPENDENT

- Starting point: $Q_0 = (x, V) \rightarrow Q = (x, v)$

\[
L(Q_0, \dot{Q}_0, t) = \left[ A(x, t) + V \right] \cdot \dot{x} - \left\{ \frac{1}{2} V^2 + \phi(x, t) \right\} \rightarrow
\]

\[
L(Q, \dot{Q}, t) = \left[ A(x, t) + u(x, v, t) + v \right] \cdot \dot{x} - \left\{ \frac{1}{2} [u(x, v, t) + v]^2 + \phi(x, t) \right\}
\]

- $v$ Euler-Lagrange equation: $0 = \left( I + \nabla_v u \right) \cdot [\dot{x} - (u + v)]$

- Our ordering $\Rightarrow \| \nabla_v u \| = 1$, so
  - $v = \dot{x} - u(x, v, t)$
  - but $x$ is still the lab-frame position.
TOROIDAL EDGE ORDERING

• Basic ordering: \( q\phi/T = O(\epsilon^{-1}) \), \( V_{\text{ExB}}/v_{\text{th}} = O(\epsilon^0) \), \( V'_{\text{ExB}}/\Omega = O(\epsilon^1) \)
  
  o will use \( \left( \frac{\partial}{\partial t} + V_{\text{ExB}} \cdot \nabla \right) \ln(S)/\Omega \sim V'_{\text{ExB}}/\Omega \sim \epsilon \ll 1 \)

• low \( \beta \) - \( \delta B/B_0 = O(\epsilon^1) \)
  \( A(x,t) = A_0(x) + \delta A_\parallel(x,t) \hat{b}_0 \)

• Edge ordering \( qA_0/(mcv_{\text{th}}) = O(\epsilon^{-2}) \), \( \rho/L_p \sim L_p/R \sim \epsilon \)
LOWEST ORDER TRANSFORMATION

• Transform to \( R = x - \rho \), \( v \)

\[
L = \left[ A_0 (R + \rho) + \delta A_{||} (R + \rho, t) \hat{b}_0 + u(R + \rho, v, t) + v \right] \cdot (\dot{R} + \dot{\rho})
- \left[ \frac{1}{2} (u^2 + v^2) + u \cdot v + \phi \left( R + \rho, t \right) \right].
\]

• Separate \( \phi(R + \rho) \) as \( \phi(R + \rho) = \phi + \rho \cdot \nabla \phi + \delta^1 \phi \)

\[
\delta^1 \tilde{\phi} \equiv \tilde{\phi} - \rho \cdot \nabla \phi = O(\varepsilon = V_{EB} / \Omega); \quad \bar{\phi} \equiv \phi \left( R + \rho \right) - \bar{\phi} \left( R, \mu \right),
\]

• \( \overline{\phi} \equiv \langle \phi \rangle = O(\varepsilon^{-1}) \) is gyrophase independent

• \( u \cdot v + \rho \cdot \nabla \overline{\phi} = 0 \) if \( u \equiv \hat{b} \times \nabla_{R} \overline{\phi} / \Omega_0 \)

• After this transformation and separation, all gyrophase dependences are at order \( \varepsilon^1 \) or higher
ALL GYROPHASE DEPENDENCES ARE AT FIRST OR HIGHTER ORDER

\[ L = \langle A_0 \rangle \cdot \dot{R} - \langle \phi \rangle_0 \]
\[ + \left[ u + \left( U_{\parallel} + \delta A_{\parallel} \right) \hat{b}_0 \right] \cdot \dot{R} - \mu \dot{\theta} - \left[ \frac{1}{2} \left( U_{\parallel}^2 + u^2 \right) + \mu \Omega \right] \]
\[ + u \cdot \dot{\rho} - \delta_{1\dot{\phi}} \]
\[ - \frac{1}{2} \left[ \rho \cdot \nabla \left( \nabla \langle A_0 \rangle \right) \cdot \rho \right] \cdot \dot{R} \]

\[ L = \langle A_0 \rangle \cdot \dot{R} - \langle \phi \rangle_0 \]
\[ + \left[ u + p_{\parallel} \hat{b}_0 \right] \cdot \dot{R} - \mu \dot{\theta} - \left[ \frac{1}{2} \left( p_{\parallel} - \delta \tilde{A}_{\parallel} \right)^2 + \frac{1}{2} u^2 + \mu \Omega \right] \]
\[ + u \cdot \dot{\rho} - \left[ \delta_{1\dot{\phi}} - \left( p_{\parallel} - \delta \tilde{A}_{\parallel} \right) \delta \tilde{A}_{\parallel} \right] \]
\[ - \frac{1}{2} \left( \delta \tilde{A}_{\parallel} \right)^2 - \frac{1}{2} \left[ \rho \cdot \nabla \left( \nabla \langle A_0 \rangle \right) \cdot \rho \right] \cdot \dot{R} \]
LIE-TRANSFORM PERTURBATION THEORY

\[ Z \rightarrow \bar{Z} = (\bar{R}, \bar{U}, \bar{\mu}, \bar{\theta}) = T(\varepsilon)Z \]
\[ \bar{f}(Z) = T^{-1}f(Z) \]
\[ T = \ldots T_3 T_2 T_1 \]
\[ T_n(\varepsilon) = \exp(\varepsilon L_n) \]
\[ \Gamma = T^{-1}\gamma + dS \]
\[ \left(L_n \gamma\right)_\alpha = g^\beta_n \left( \frac{\partial \gamma_\alpha}{\partial Z^\beta} - \frac{\partial \gamma_\beta}{\partial Z^\alpha} \right) \]

- Due to spatial derivatives, \[ L_n \gamma = \left(L_n \gamma\right)_a + \varepsilon \left(L_n \gamma\right)_b + \varepsilon^2 \left(L_n \gamma\right)_c \]

\[ \Gamma_{-2,-1,0} = \gamma_{-2,-1,0} + dS_{-2,-1,0} \]
\[ \Gamma_1 = \gamma_1 - \left(L_1 \gamma_0\right)_a - \left(L_1 \gamma_{-1}\right)_b - \left(L_1 \gamma_{-2}\right)_c + dS_1 \]

- Main results needed

\[ \Gamma_2 = \langle \gamma_2 \rangle - \frac{1}{2} \langle \left(L_1 \gamma_1\right)_a \rangle \]
THE FIRST ORDER EQUATION YIELDS AND THE GAUGE FUNCTION AND THE GENERATORS

\[ g^\perp_1 = -\frac{1}{\Omega_0} \hat{b}_0 \times \nabla S_1, \]
\[ g^\parallel_1 = -\frac{\partial S_1}{\partial p^\parallel}, \]
\[ g^p_1 = \nabla^\parallel S_1, \]
\[ g^\mu_1 = -\frac{\partial}{\partial \theta} (S_1 + u \cdot \rho), \]
\[ g^\theta_1 = \frac{\partial}{\partial \mu} (S_1 + u \cdot \rho) \]
\[ 0 = \frac{\partial S_1}{\partial t} + g_1 \cdot \nabla \phi + g^p_1 (p^\parallel - \delta \overline{A}^\parallel) + \Omega g^\mu_1 - \delta_1 \tilde{\psi}, \]
\[ \delta_1 \tilde{\psi} \equiv \delta_1 \tilde{\phi} - \left( p^\parallel - \delta \overline{A}^\parallel \right) \delta \overline{A}^\parallel \]
THE FIRST ORDER EQUATION YIELDS AND THE GAUGE FUNCTION AND THE GENERATORS

\[
\left( \frac{dS_1}{dt} \right)_{\text{slow}} - \Omega_0 \frac{\partial}{\partial \theta} (S_1 + \mathbf{u} \cdot \mathbf{\rho}) = \delta_1 \tilde{\psi},
\]
\[
S_1 \approx -\left( \delta_1 \tilde{\Psi}/\Omega_0 \right) - \mathbf{\rho} \cdot \mathbf{u} = -\tilde{\psi}/\Omega_0,
\]
\[
\tilde{\Psi} \equiv \Psi - \tilde{\Psi},
\]
\[
\Psi_i = \int_{\theta_0}^{\theta} d\theta \tilde{\psi},
\]
\[
\tilde{\psi} \equiv \tilde{\phi} - (p_{\parallel} - \delta \vec{A}) \delta \vec{A}
\]
\[
g_1^\perp = \frac{1}{\Omega_0} \hat{b}_0 \times \nabla (\tilde{\Psi}/\Omega_0),
\]
\[
g_1^\parallel = -\nabla (\tilde{\Psi}/\Omega_0),
\]
\[
g_1^{p\parallel} = -\nabla (\tilde{\Psi}/\Omega_0),
\]
\[
g_1^\mu = \left( \delta_1 \tilde{\psi}/\Omega_0 \right),
\]
\[
g_1^\theta = \frac{1}{\Omega_0} \frac{\partial \delta_1 \tilde{\Psi}}{\partial \mu}
\]
\[ \Gamma_2 = \frac{1}{2\Omega_0} \left[ \frac{\partial}{\partial \mu} \left\langle \nabla (u \cdot \rho) \delta_1 \tilde{\psi} \right\rangle - \Omega_0 \left\langle \rho \cdot \nabla \left( \nabla \langle A_0 \rangle \right) \cdot \rho \right\rangle \right], \]

\[ \Gamma_2^\mu = -\frac{1}{2\Omega_0} \left\{ \left\langle \hat{b}_0 \times \nabla \left( \tilde{\psi} / \Omega_0 \right) \cdot \nabla \left( u \cdot \frac{\partial \rho}{\partial \mu} \right) \right\rangle - \left\langle \Delta \tilde{A} \nabla \left( u \cdot \frac{\partial \rho}{\partial \mu} \right) \right\rangle \right\} + \frac{\partial u}{\partial \mu} \left\langle \rho \frac{\partial \delta_1 \tilde{\psi}}{\partial \mu} \right\rangle \right\}, \]

\[ \Gamma_2^\theta = \frac{1}{2\Omega_0} \left\{ \left\langle \hat{b}_0 \times \nabla \left( \delta_1 \tilde{\psi} / \Omega_0 \right) \cdot \nabla \left( u \cdot \rho \right) \right\rangle - \left\langle \delta \tilde{A} \nabla \left( u \cdot \rho \right) \right\rangle \right\} + \frac{\partial u}{\partial \mu} \left\langle \frac{\partial \rho}{\partial \theta} \delta_1 \tilde{\psi} \right\rangle \right\}, \]

\[ H_2 = -\frac{1}{2\Omega_0} \left\{ \left\langle \hat{b}_0 \times \nabla \left( \tilde{\psi} / \Omega_0 \right) \cdot \nabla \delta_1 \tilde{\psi} \right\rangle - \left\langle \Delta \tilde{A} \nabla \left( \delta_1 \tilde{\psi} \right) \right\rangle \right\} + \frac{\partial}{\partial \mu} \left\langle \left( \delta_1 \tilde{\psi} \right)^2 \right\rangle + \frac{\partial u}{\partial t} \cdot \frac{\partial}{\partial \mu} \left\langle \delta_1 \tilde{\psi} \rho \right\rangle \right\}. \]
CORE ORDERING – MORE “STANDARD” BUT HARDER

\[ L = \langle A_0 \rangle \cdot \hat{R} - \langle \phi \rangle_0 \]

\[ + \left[ u + \left( U_\| + \delta A_\| \right) \hat{b}_0 \right] \cdot \hat{R} - \mu \dot{\theta} - \left[ \frac{1}{2} U_\|^2 + u^2 \right] + \mu \Omega \]

\[ + u \cdot \rho - \delta_1 \phi - \frac{1}{2} \left[ \rho \cdot \nabla \left( \nabla \langle A_0 \rangle \right) \right] \cdot \rho \cdot \hat{R} \]

\[ \cdots O(e^{-1}) \]

\[ \cdots O(e^0) \]

\[ \cdots O(e^1) \]

- Brings in
  - FLR corrections to equilibrium magnetic field
  - Higher-order equilibrium motion terms, e.g., Banos drift
  - Couplings of equilibrium magnetic field variation and fluctuations
SUMMARY OF (GYROAVERAGED) GYROKINETIC ORDERINGS AND DERIVATIONS

- \( e\psi / T \ll 1, \quad \psi = \phi - \left( p_{||}/mc \right) A_{||} \) - most Hamiltonian GK derivations – core.

- \( e\psi / T \sim 1, \quad V_{ExB} / v_{th} \ll 1 \) - Dimits, Dubin, Lodestro `92, and extended here; many iterative derivations (e.g., Parra-Catto) – likely applicable to many edge situations without large flows.

- 2-scale, with \( u_{E} / v_{th} \sim 1, \quad \delta V_{ExB} / v_{th} \ll 1 \) - Brizard `94; Hahm `96; Qin et. al., `07; Hahm-Wang-Madsen `09. – addresses large flows.

- Our new ordering \( V'_{ExB} / \Omega \ll 1 \) for any perturbations – allows large perturbations and large flows. Captures new cross terms.

- Differences between new ordering/derivation and 2-scale
  - our \( \bar{u} \) not quite the same as 2-scale \( u_{E} \)
  - \( \nabla \cdot \left\langle \rho \cdot \nabla u_{E} \ delta \bar{\phi} \right\rangle \sim \nabla \left\langle \left( \rho \cdot \nabla u_{E} \right)^{2} \right\rangle \sim \nabla \left\langle \delta \bar{\phi}^{2} \right\rangle \)
GYROKINETIC EQUATIONS HAVE BEEN DERIVED IN A NEW MORE GENERAL ORDERING

- Allows for
  - large flow velocities
  - large perturbation amplitudes

- Toroidal electromagneti gyrokinetic theory has been developed in a low-$\beta$ ordering appropriate for MFE edge plasma conditions.

- Useful reduced and subsidiary orderings were found.
  - The “standard” toroidal Hamiltonian theory is valid for the more general DDL ordering, which allows for large relative perturbation amplitudes
  - Now have justification of the reduced “practical minimal model” as valid under the DDL ordering for $\phi$ and a subsidiary ordering for $\delta A_\parallel$

- Numerical algorithms for the discretization of the second-order terms in the gyrokinetic equations of motion and Poisson’s equation were presented.