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GLOBAL GYROKINETIC EQUATIONS: EXTENDED ORDERINGS AND SECOND ORDER TERMS

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GYROKINETIC SIMULATION OF MFE EDGE PLASMAS MUST DEAL WITH LARGE PERTURBATIONS

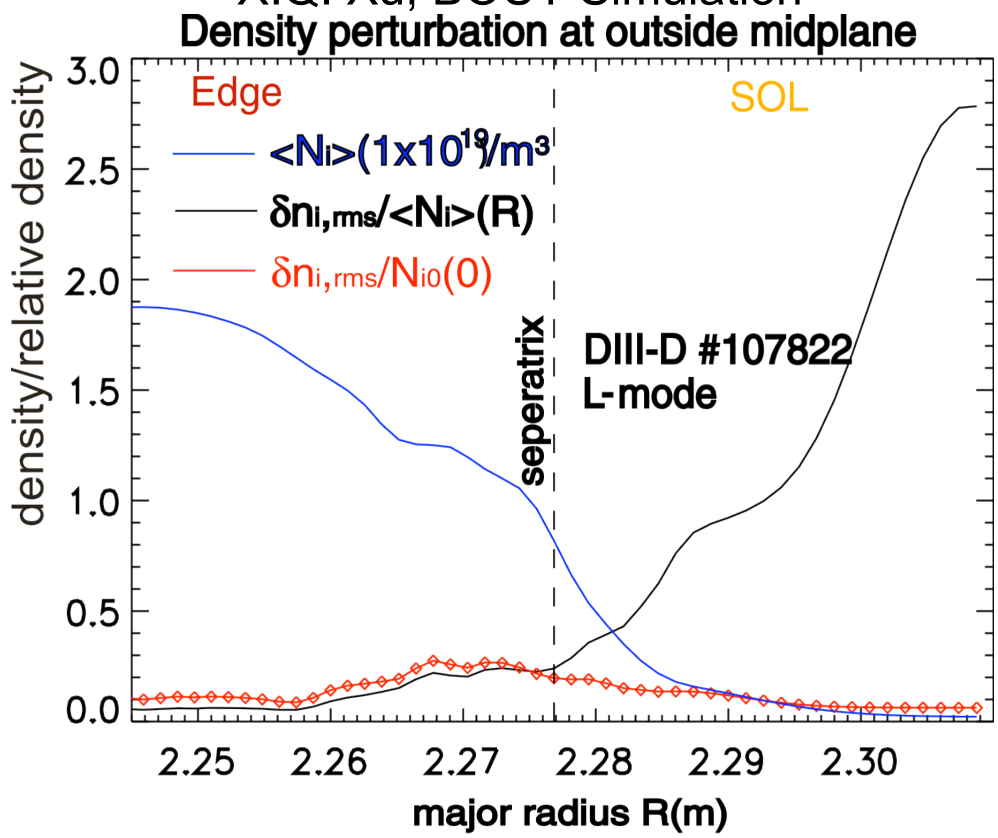
•e.g. solution of gyrokinetic Poisson or vorticity equation

$$\nabla \cdot \left[\left(n_0 + \delta n \right) \nabla \phi \right] = S$$

– $\delta n / n_0$ may be significant in the edge

–May have large time dependent ExB flows

X.Q. Xu, BOUT Simulation



OUTLINE

- Simpler extended and subsidiary orderings
  - EM toroidal theory in DDL`92 ordering
  - subsidiary ordering for EM perturbations $\rightarrow$ "practical minimal model"
- numerical discretization of second-order terms
- gyrokinetics in the most general ordering for ϕ
  - results and interpretation for slab
  - electromagnetic toroidal theory in this ordering
- summary

EXTENDED ORDERING GOALS: GYROKINETIC THEORY VALID WHERE STANDARD ORDERINGS INAPPLICABLE

- **Existing large-perturbation gyrokinetic theories**
  - Allow $e\phi / T \sim 1, \quad V_{ExB} / v_{th} \ll 1$ - electromagnetic slab theory: A.M. Dimits, L.L. LoDestro, D.H.E. Dubin, Phys. Fluids B4 274 (1992).
  - 2-scale: short-wavelength perturbations with $e\phi / T \ll 1$ + (static) large long-wavelength component with $V_{ExB} / v_{th} \sim 1$.
    - M. Artun and W.M. Tang, Phys. Plasmas, 1, 2682 (1994)
    - A.J. Brizard, Phys. Plasmas 2, 459 (1995)
    - T.S. Hahm, Phys Plasmas 3, 4658 (1996)
    - T.S. Hahm L. Wang, J. Madsen, Phys. Plasmas 16, 022305 (2009)
    - H. Qin, et. al., Phys. Plasmas 14, 056110 (2007)
- **More general theory removes 2-scale and static restrictions**
  - Basic ordering and electrostatic slab theory: A.M. Dimits, Phys. Plasmas 17, 055901 (2010)

THE DDL`92 ORDERING IS USEFUL FOR LARGE PERTURBATION AMPLITUDES, SMALL FLOWS

- ordering: $q\psi/T \sim 1$, $\hat{\boldsymbol{b}}_0 \times \nabla\psi / (\Omega_0 V_{\mathrm{th}}) \ll 1$ where $\psi = \phi - p_\parallel \delta A_\parallel$.
- $\boldsymbol{u} = 0$; use the standard 2-step approach
- Separation is $\psi = \bar{\psi} + \tilde{\psi}$
- $q\tilde{\psi}/T \ll 1 \Leftrightarrow \hat{\boldsymbol{b}}_0 \times \nabla\psi / (\Omega_0 V_{\mathrm{th}}) \ll 1$,

$$\Gamma = \left[\boldsymbol{A}_{gc} + p_\parallel \hat{\boldsymbol{b}}_0\right] \cdot d\boldsymbol{R} - \mu d\theta - \left[\frac{1}{2}\left\langle \left(p_\parallel - \delta A_\parallel\right)^2 \right\rangle + \mu\Omega + \langle\phi\rangle \right.$$

$$\left. + \frac{1}{2\Omega_0}\left\langle \boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega_0\right) \times \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{\nabla}\tilde{\psi} \right\rangle - \frac{1}{2\Omega_0}\frac{\partial}{\partial\mu}\left\langle \tilde{\psi}^2 \right\rangle \right] dt,$$

where $\tilde{\psi} = \tilde{\phi} - \left(p_\parallel - \delta\bar{A}_\parallel\right)\delta\tilde{A}_\parallel$, $\Psi = \int^\theta \tilde{\psi}\, d\theta$, $\tilde{\Psi} = \Psi - \langle\Psi\rangle$

- Essentially the same result as in the standard ordering, but now justified for the DDL ordering

THE FIRST-ORDER EQUATIONS OF MOTION CONTAIN THE ∇B, CURVATURE, AND EXB DRIFTS

- $\Gamma_{-1,0,1} = \left[\boldsymbol{A}_{gc} + p_{\parallel}\hat{\boldsymbol{b}}_0 \right] \cdot d\boldsymbol{R} - \mu d\theta - \left[\frac{1}{2}\left(p_{\parallel} - \left\langle \delta A_{\parallel} \right\rangle \right)^2 + \mu\Omega_0 + \left\langle \phi \right\rangle \right] dt,$

- $\boldsymbol{R}$ Euler-Lagrange equations

$$\dot{\boldsymbol{R}} = \left(p_{\parallel} - \delta A_{\parallel} \right) \hat{\boldsymbol{b}}_0 + \frac{1}{\Omega_0} \hat{\boldsymbol{b}}_0 \times \left[\boldsymbol{\nabla} \left\langle \phi \right\rangle + \mu \boldsymbol{\nabla} \Omega_0 + \left(p_{\parallel} - \left\langle \delta A_{\parallel} \right\rangle \right)^2 \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{\nabla} \hat{\boldsymbol{b}}_0 \right]$$

SUBSIDIARY ORDERING FOR EM TERMS RESULTS IN “PRACTICAL MINIMAL MODEL”

- Take $\delta\tilde{A}_\parallel = O\left(\varepsilon^2\right)$; still have $\delta\bar{A}_\parallel = O\left(\varepsilon^0\right)$,

$$\Gamma = \left[\boldsymbol{A}_{gc} + p_\parallel \hat{\boldsymbol{b}}_0\right]\cdot d\boldsymbol{R} - \mu d\theta - \left[\frac{1}{2}\left(p_\parallel - \delta\bar{A}_\parallel\right)^2 + \mu\Omega + \bar{\phi}\right.$$

$$\left. + \frac{1}{2\Omega_0}\left\langle \boldsymbol{\nabla}\left(\tilde{\Phi}/\Omega_0\right)\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\tilde{\phi}\right\rangle - \frac{1}{2\Omega_0}\frac{\partial}{\partial\mu}\left\langle\tilde{\phi}^2\right\rangle\right]dt,$$

- The GK Poisson’s equation again has only electrostatic terms

$$0 \approx -\frac{1}{4\pi e}\nabla^2\phi(x) = \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{r}-\boldsymbol{x}\right)\left[F_i + \tilde{\phi}_1\frac{\partial F_i}{\partial\mu} + \nabla_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\nabla_{R\perp}F_i\right] - n_e$$

- The GK Ampere’s Law has only the identity part of T_{gy}

$$\left(\nabla^2/c^2\right)A_{1\parallel} = (4\pi e/m_e)\left[\int dV F_e\left(p_\parallel - \delta A_\parallel\right)\right.$$

$$\left. -(Zm_e/m_i)\int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{r}-\boldsymbol{x}\right)\left(p_\parallel - \delta\bar{A}_\parallel\right)\left(F_i \cancel{+\tilde{\psi}_1\frac{\partial F_i}{\partial\mu} + \nabla_{R\perp}\tilde{\Psi}_1\cdot\boldsymbol{b_0}\times\nabla_{R\perp}F_i}\right)\right]$$

- A similar model has been obtained for the symplectic representation.

IMPLEMENTATION OF THE SECOND-ORDER TERMS IN GYROKINETIC MODELS REQUIRES NEW METHODS

GYROCENTER EQUATIONS OF MOTION

First-order terms now standard

- Given ψ, calculate $\langle\psi\rangle(\boldsymbol{R},\mu,t)$ either at each mesh node or cell center, or at each gyrocenter (in the case of a PIC code).
- Also need derivatives for the equations of motion
- Can calculate directly by averaging on the gyro orbit or in Fourier space
  - Save for use on the 4D $(\boldsymbol{R},\mu)$ mesh for continuum codes.
  - For a PIC code, can do this particle by particle.

SECOND-ORDER TERMS IN GYROKINETIC EQUATIONS OF MOTION NEVER YET IMPLEMENTED

$\partial \left\langle (\tilde{\psi}_1)^2 \right\rangle / \partial\mu$: use

$$\frac{\partial \left\langle (\tilde{\psi}_1)^2 \right\rangle}{\partial\mu} = \frac{\partial \left\langle \psi_1^2 \right\rangle}{\partial\mu} - 2\left\langle \psi_1 \right\rangle \frac{\partial \left\langle \psi_1 \right\rangle}{\partial\mu},$$

and

$$\frac{\partial \left\langle \psi \right\rangle}{\partial\mu} = \frac{\partial\rho}{\partial\mu}\frac{1}{2\pi}\oint d\theta\, \hat{\rho}(\theta)\cdot\nabla\,\psi\left(R+\rho\hat{\rho}(\theta)\right).$$

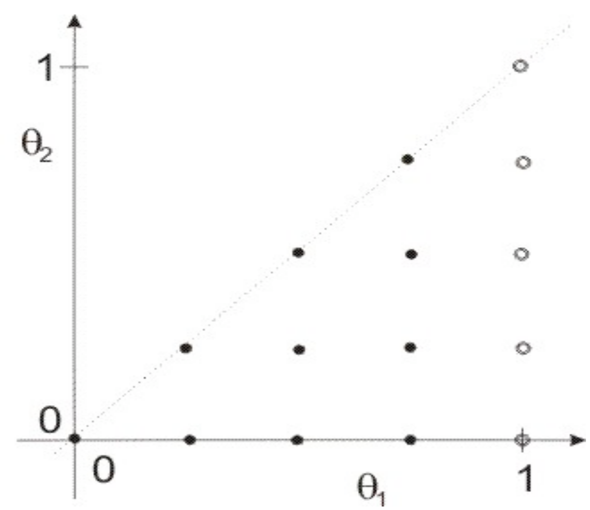
This can be calculated directly by sampling the components of $\nabla\psi$ around an instantaneous gyro orbit or by using a Fourier decomposition and Bessel functions. Again, at any given *t*, this is a function of the 4D $(\boldsymbol{R},\mu)$ phase space.

$$\left\langle \boldsymbol{\nabla}_{R}\tilde{\Phi}\cdot \boldsymbol{b_0}\times \boldsymbol{\nabla}_{R}\tilde{\phi}\right\rangle = \boldsymbol{b_0}\cdot \oint d\theta_1 \int_0^{\theta_1} d\theta_2\, \boldsymbol{\nabla}_{R}\tilde{\phi}\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_1)\right)\times \boldsymbol{\nabla}_{R}\tilde{\phi}\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_2)\right)$$

$$= \boldsymbol{b_0}\cdot \oint d\theta_1 \int_0^{\theta_1} d\theta_2\, \boldsymbol{\nabla}_{R}\phi\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_1)\right)\times \boldsymbol{\nabla}_{R}\phi\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_2)\right) - \frac{1}{2}\boldsymbol{\nabla}_{R}\bar{\phi}\left(\vec{R}\right)\times \boldsymbol{\nabla}_{R}\bar{\phi}\left(\vec{R}\right)$$

This can be calculated sampling the components of $\nabla\phi$ over a double gyro orbit.
e.g., n-point θ_1 gyro orbit -> n(n+1)/2-point θ_1 - θ_2 double gyro orbit
8-point θ_1 gyro orbit -> 36-point θ_1 - θ_2 double gyro orbit

Again, at any given time, this is a function of the 4D $(\boldsymbol{R},\mu)$ phase space.



THE FULL-f GYROKINETIC POISSON EQUATION CAN BE DIRECTLY DISCRETIZED, e.g., USING FINITE-ELEMENTS

- weak form + Galerkin representation of ϕ.
- All derivatives and gyroaveraging operations can be recast into derivatives operating on the Galerkin basis functions.
- Begin with gyrokinetic Poisson equation:

$$L\phi = S$$

$$L\phi = \frac{1}{4\pi}\nabla^2\phi(x) + \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{r}-\boldsymbol{x}\right)\left[\tilde{\phi}_1\frac{\partial F_i}{\partial\mu} + \nabla_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\nabla_{R\perp}F_i\right]$$

$$S = n_e - \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{r}-\boldsymbol{x}\right)F_i\left(\boldsymbol{R}\right);\; d\Lambda = d\boldsymbol{R}\,dv_{\|}\,B\,d\mu\,d\theta$$

- Use Galerkin discretization of $\dfrac{\delta \mathcal{A}}{\delta \phi(\boldsymbol{x})} = 0$

$$
\begin{aligned}
\mathcal{A} &= \int d\boldsymbol{x} \phi \left(S - \tfrac{1}{2} L\phi \right) \\
&= \int d\boldsymbol{x} \phi S + \frac{1}{8\pi} \int d\boldsymbol{x} \left(\nabla \phi \right)^2 + \frac{1}{2} \int d\Lambda \, F_i \frac{\partial \tilde{\phi}^2}{\partial \mu} \\
&\quad - \frac{1}{2} \int d\Lambda \, F_i \left[\hat{\boldsymbol{b}} \cdot \nabla_{R\perp} \tilde{\phi} \times \nabla_{R\perp} \tilde{\Phi} - \frac{1}{B} \tilde{\phi} \nabla_{R\perp} \tilde{\Phi} \cdot \boldsymbol{J}_{\text{eqm}} \right]
\end{aligned}
$$

- Insert Galerkin representation $\begin{cases} \phi(\boldsymbol{x}) = \sum_{\boldsymbol{l}} \phi_{\boldsymbol{l}} \psi_{\boldsymbol{l}}(\boldsymbol{x}) \, ; \boldsymbol{l} = (i, j) \\ \psi_{i,j} = \psi\left(\dfrac{x - x_{i,j}}{\Delta_{x;i,j}}, \dfrac{y - y_{i,j}}{\Delta_{y;i,j}} \right) \end{cases}$

$\rightarrow$ resolves derivatives onto basis functions

$$\frac{\delta\left(\text{weak form Galerkin GK Poisson }\right)}{\delta\phi_{i,j}} = 0 \rightarrow$$ Matrix equation to be solved

$$\boldsymbol{M}^{\mathrm{S}} \cdot \boldsymbol{\Phi} = \boldsymbol{S}$$

$$\boldsymbol{M}^{\mathrm{S}} \quad = \left(\boldsymbol{M} + \boldsymbol{M}^{\mathrm{T}}\right)\Big/2$$

$$M_{\boldsymbol{k},\boldsymbol{l}} \quad = \frac{1}{4\pi}\int d\boldsymbol{x} \nabla\psi_{\boldsymbol{k}} \cdot \nabla\psi_{\boldsymbol{l}} + \int d\Lambda\, F_i \frac{\partial\left(\tilde{\psi}_{\boldsymbol{k}}\tilde{\psi}_{\boldsymbol{l}}\right)}{\partial\mu}$$

$$-\int d\Lambda\, F_i \left[\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}_{\mathbb{R}}\tilde{\psi}_{\boldsymbol{k}} \times \boldsymbol{\nabla}_{\mathbb{R}}\tilde{\Psi}_{\boldsymbol{l}} - \frac{1}{B}\tilde{\psi}_{\boldsymbol{k}}\boldsymbol{\nabla}_{\mathbb{R}}\tilde{\Psi}_{\boldsymbol{l}} \cdot \boldsymbol{J}_{\mathrm{eqm}}\right]$$

$$S_{\boldsymbol{k}} \quad = \int d\boldsymbol{x}\, S\psi_{\boldsymbol{k}}$$

- “Deposition” or projection from Z to x is needed to calculate the matrix elements.

- **The resulting matrices are still sparse**
  - For solution on perpendicular slices with N^2 cells of size ~ρ_i, about 100 N^2 nonzero matrix elements are needed (not N^4).

TOWARDS A MINIMAL NECESSARY ORDERING FOR GYROKINETIC THEORY

- Low-frequency GK should be possible if in some frame of reference, the perturbation to the gyro orbit is small
- Consider an electrostatic potential ϕ
- Absolute value of ϕ should not matter
  - Can transform away any long-wavelength $E \times B$ drift
  - Only shear (spatial variation) of $E \times B$ drift matters
- Basic ordering: $q\phi / T = O(\varepsilon^{-1})$, $V_{\mathrm{ExB}} / v_{\mathrm{th}} = O(\varepsilon^{0})$, $V'_{ExB} / \Omega = O(\varepsilon^{1})$

SECOND-ORDER LAGRANGIAN FOR ELECTROSTATIC SLAB CASE

$$\begin{aligned}\Gamma = &\left[\boldsymbol{A}_{gc} + U_{\|}\hat{\boldsymbol{b}}_0 + \boldsymbol{u} + \frac{1}{2\Omega}\frac{\partial}{\partial\mu}\left\langle \delta_1\tilde{\phi}\left(\boldsymbol{\nabla}\boldsymbol{u}\right)\cdot\boldsymbol{\rho}\right\rangle\right]\cdot d\boldsymbol{R} \\ &+ \frac{1}{2\Omega^2}\left[\left\langle \boldsymbol{\nabla}\tilde{\Phi}\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\rho}}{\partial\mu}\right)\right\rangle + \Omega\frac{\partial\boldsymbol{u}}{\partial\mu}\cdot\left\langle\boldsymbol{\rho}\frac{\partial\delta_1\tilde{\phi}}{\partial\mu}\right\rangle\right]d\mu \\ &- \left[\mu + \frac{1}{2\Omega^2}\left\langle\boldsymbol{\nabla}\tilde{\phi}\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\left(\boldsymbol{\rho}\cdot\boldsymbol{u}\right)\right\rangle\right]d\theta - \left[\frac{1}{2}U_{\|}^2 + \mu\Omega + \frac{1}{2}\boldsymbol{u}^2 + \bar{\phi}\right. \\ &\left. + \frac{1}{2\Omega^2}\left\langle\boldsymbol{\nabla}\tilde{\Phi}\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}\delta_1\tilde{\phi}\right\rangle - \frac{1}{2\Omega}\frac{\partial}{\partial\mu}\left\langle\left(\delta_1\tilde{\phi}\right)^2\right\rangle - \frac{1}{2\Omega}\frac{\partial\boldsymbol{u}}{\partial t}\cdot\frac{\partial}{\partial\mu}\left\langle\delta_1\tilde{\phi}\boldsymbol{\rho}\right\rangle\right]dt,\end{aligned}$$

- here, $\boldsymbol{u} = \frac{1}{\Omega}\hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\bar{\phi}$ here has temporal and spatial dependences
- many new (noncanonical) components to Lagrange tensor
- $\nabla\cdot\left\langle\boldsymbol{\rho}\cdot\nabla\boldsymbol{u}\ \delta_1\tilde{\phi}\right\rangle$ terms are absent in 2-scale theories, but are of the same order as $\nabla\left\langle\left(\boldsymbol{\rho}\cdot\nabla\boldsymbol{u}\right)^2\right\rangle \sim \nabla\left\langle\delta_1\tilde{\phi}^2\right\rangle$.

THE FIRST-ORDER EQUATIONS OF MOTION CONTAIN THE EXB AND POLARIZATION DRIFTS

- $\Gamma_{-1,0,1} = \left[\boldsymbol{A}_{gc} + U_{\parallel}\hat{\boldsymbol{b}}_0 + \boldsymbol{u} \right] \cdot d\boldsymbol{R} - \mu\, d\theta - \left[\frac{1}{2}U_{\parallel}^2 + \mu\Omega + \frac{1}{2}\boldsymbol{u}^2 + \bar{\phi} \right] dt$

- Euler-Lagrange equation $\omega_{ji}\dot{Z}^i = H_{,j} + \gamma_{j,t} \quad \Rightarrow$

$\dot{Z}^i = P^{ij}\left(H_{,j} + \gamma_{j,t}\right)$ where $\boldsymbol{P} = \boldsymbol{\omega}^{-1}$, $\boldsymbol{\omega}^{ij} = \frac{\partial \gamma_j}{\partial Z^i} - \frac{\partial \gamma_i}{\partial Z^j}$

$$\dot{\boldsymbol{R}} = \boldsymbol{u} + U_{\parallel}\hat{\boldsymbol{b}}_0 + \frac{1}{\Omega_{\parallel}^*}\hat{\boldsymbol{b}}_0 \times \left(\frac{\partial}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} + U_{\parallel}\nabla_{\parallel} \right)\boldsymbol{u},$$

$$\boldsymbol{u} \equiv \frac{1}{\Omega}\hat{\boldsymbol{b}}_0 \times \boldsymbol{\nabla}\bar{\phi}, \quad \Omega_{\parallel}^* = \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{\nabla} \times \left(\boldsymbol{A}_{gc} + \boldsymbol{u} \right)$$

- $\frac{1}{\Omega}\hat{\boldsymbol{b}} \times \left(\frac{\partial \boldsymbol{u}}{\partial t},\ U_{\parallel}\nabla_{\parallel}\boldsymbol{u} \right)$ terms are new.

- $\boldsymbol{u}$ includes both long- and short-wavelength components.

THE EQUATIONS OF MOTION CAN BE OBTAINED PERTURBATIVELY TO SECOND ORDER

- $\dot{Z}_i = P^{ij}\left(H_{,j} + \gamma_{j,t}\right)$

- $\boldsymbol{P} = \boldsymbol{\omega}^{-1}$, $\omega^{ij} = \dfrac{\partial \gamma_j}{\partial Z^i} - \dfrac{\partial \gamma_i}{\partial Z^j}$

- $\boldsymbol{\omega}$ has many more nonzero components than in previous theories, so its inverse $\boldsymbol{P}$ is more complicated.

- Use $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \varepsilon \boldsymbol{\omega}_1 + \varepsilon^2 \boldsymbol{\omega}_2$ where $\boldsymbol{\omega}_0$ is canonical (and therefore easily invertible)

- $\boldsymbol{P} = \boldsymbol{P}_0 + \varepsilon \boldsymbol{P}_1 + \varepsilon^2 \boldsymbol{P}_2 + ...$,

  where $\boldsymbol{P}_0 = \boldsymbol{\omega}_0^{-1}$, $\boldsymbol{P}_1 = -\boldsymbol{P}_0 \omega_1 \boldsymbol{P}_0$ $\quad \boldsymbol{P}_2 = \boldsymbol{P}_0\left(\omega_1 \boldsymbol{P}_0 \omega_1 - \omega_2\right)\boldsymbol{P}_0$

THE EQUATIONS OF MOTION CAN BE OBTAINED PERTURBATIVELY TO SECOND ORDER

$$\begin{bmatrix} \dot{\boldsymbol{R}}_{2\perp} \\ \dot{z}_2 \\ \dot{U}_{2\parallel} \\ \dot{\mu}_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\Omega}\hat{\boldsymbol{b}} \times \left\{ \boldsymbol{\nabla}_\perp H_2 - \left[\left(\boldsymbol{\nabla}_\perp \gamma_{2\perp} \right) \cdot \boldsymbol{u}_\perp + \left(\boldsymbol{\nabla}_\perp \gamma_{2z} \right) \cdot U_\parallel - \Omega \boldsymbol{\nabla}_\perp \gamma_{2\theta} \right] - \left(\boldsymbol{\nabla}_\perp \times \boldsymbol{u}_\perp \right)_z \dfrac{1}{\Omega} \dfrac{d_0 \boldsymbol{u}_\perp}{dt} \right\} \\ 0 \\ -H_{2,z} + \left[\gamma_{2\perp,z} \cdot \boldsymbol{u}_\perp + \gamma_{2z,z} U_\parallel - \Omega \gamma_{2\theta,z} \right] - \boldsymbol{u}_{\perp,z} \cdot \dfrac{1}{\Omega} \dfrac{d_0 \boldsymbol{u}_\perp}{dt} \\ 0 \\ -H_{2,\mu} + \left[\gamma_{2\perp,\mu} \cdot \boldsymbol{u}_\perp + \gamma_{2z,\mu} U_\parallel - \Omega \gamma_{2\theta,\mu} \right] \end{bmatrix}$$

THE DIRECTLY CALCULATED EQUATIONS OF MOTION ARE COMPLICATED BECAUSE OF THE

- **e.g. first row of Det(w)*w^{-1}**

$$
\begin{aligned}
&\Big\{\Big\{0,\ -\frac{1}{\mathrm{Bsti}}-\frac{2\,\mathrm{w2mt}}{\mathrm{Bsti}}-\frac{\mathrm{w2mt}^2}{\mathrm{Bsti}}-\mathrm{w2xt}\,\mathrm{w2ym}-\mathrm{w2mt}\,\mathrm{w2xt}\,\mathrm{w2ym}+\mathrm{w2xm}\,\mathrm{w2yt}+\mathrm{w2mt}\,\mathrm{w2xm}\,\mathrm{w2yt},\\
&\quad 0,\ -\frac{\mathrm{w2yt}\,\mathrm{w2zm}}{\mathrm{Bsti}}-\frac{\mathrm{w2mt}\,\mathrm{w2yt}\,\mathrm{w2zm}}{\mathrm{Bsti}}-\mathrm{w2xt}\,\mathrm{w2ym}\,\mathrm{w2yt}\,\mathrm{w2zm}+\mathrm{w2xm}\,\mathrm{w2yt}^2\,\mathrm{w2zm}+\frac{\mathrm{w2ym}\,\mathrm{w2zt}}{\mathrm{Bsti}}+\\
&\quad \frac{\mathrm{w2mt}\,\mathrm{w2ym}\,\mathrm{w2zt}}{\mathrm{Bsti}}+\mathrm{w2xt}\,\mathrm{w2ym}^2\,\mathrm{w2zt}-\mathrm{w2xm}\,\mathrm{w2ym}\,\mathrm{w2yt}\,\mathrm{w2zt}-\frac{\mathrm{wyz}}{\mathrm{Bsti}}-\frac{2\,\mathrm{w2mt}\,\mathrm{wyz}}{\mathrm{Bsti}}-\\
&\quad \frac{\mathrm{w2mt}^2\,\mathrm{wyz}}{\mathrm{Bsti}}-\mathrm{w2xt}\,\mathrm{w2ym}\,\mathrm{wyz}-\mathrm{w2mt}\,\mathrm{w2xt}\,\mathrm{w2ym}\,\mathrm{wyz}+\mathrm{w2xm}\,\mathrm{w2yt}\,\mathrm{wyz}+\mathrm{w2mt}\,\mathrm{w2xm}\,\mathrm{w2yt}\,\mathrm{wyz},\\
&\quad \frac{\mathrm{w2yt}}{\mathrm{Bsti}}+\frac{\mathrm{w2mt}\,\mathrm{w2yt}}{\mathrm{Bsti}}+\mathrm{w2xt}\,\mathrm{w2ym}\,\mathrm{w2yt}-\mathrm{w2xm}\,\mathrm{w2yt}^2,\ -\frac{\mathrm{w2ym}}{\mathrm{Bsti}}-\frac{\mathrm{w2mt}\,\mathrm{w2ym}}{\mathrm{Bsti}}-\mathrm{w2xt}\,\mathrm{w2ym}^2+\mathrm{w2xm}\,\mathrm{w2ym}\,\mathrm{w2yt}\Big\},
\end{aligned}
$$

…………

These may be necessary, e.g., for numerical implementations with proper conservation properties.

THE GYROKINETIC POISSON (FIELD) EQUATION IS NONLINEAR EVEN AT T_1 ORDER

Gyrokinetic Poisson equation - quasineutrality

$$n_e = n_i = \int d^6 Z\, \delta\left(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x}\right)\Omega\left[1 + \frac{1}{\Omega^2}\boldsymbol{\nabla}_\perp^2\overline{\phi} + \boldsymbol{\rho}\times\hat{\boldsymbol{b}}\cdot\frac{\partial \boldsymbol{u}}{\partial\mu}\right]\times$$

$$\left[F_i\left(\boldsymbol{R},\mu\right) + \frac{1}{\Omega}\delta_1\tilde{\phi}\frac{\partial F}{\partial\mu} + \frac{1}{\Omega^2}\boldsymbol{\nabla}\tilde{\Phi}\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}F\right]$$

$$= \int \Omega d\mu\, dv_\parallel\, d\theta\left[F_i\left(\boldsymbol{x} - \boldsymbol{\rho},\mu\right)_i + \frac{1}{\Omega}\tilde{\phi}\frac{\partial F}{\partial\mu} + \frac{1}{\Omega^2}\boldsymbol{\nabla}\tilde{\Phi}\times\hat{\boldsymbol{b}}_0\cdot\boldsymbol{\nabla}F\right]$$

- To linear order in ϕ, this is the same as the standard form.
    - Note the gyrophase dependent Jacobian term.
    - Integrate one term by parts wrt. μ and then wrt. θ.

THE TOROIDAL DERIVATION PROCEEDS SIMILARLY TO THE SLAB DERIVATION

- Given $Q_0 = (\boldsymbol{x}, \boldsymbol{V} = \dot{\boldsymbol{x}})$, where $\boldsymbol{x}$ is the lab-frame position, and the presence of a large $E \times B$ drift, it is most convenient to transform to $Q = \left(\boldsymbol{x}, \boldsymbol{v} = \boldsymbol{V} - \boldsymbol{u}\right)$ where

- $\boldsymbol{v}$ is the velocity in the frame of a gyro-averaged $E \times B$ drift

$$\boldsymbol{u} \equiv \frac{1}{\Omega_0} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \left\langle \phi \right\rangle$$

SUCH A TRANSFORMATION CAN BE DEFINED IMPLICITLY

- Define

$$\boldsymbol{\rho} = \frac{1}{\Omega}\hat{\boldsymbol{b}}_0 \times \boldsymbol{v},$$
$$\boldsymbol{v}_\perp = \hat{\boldsymbol{b}}_0 \times \boldsymbol{v} \times \hat{\boldsymbol{b}}_0,$$
$$v_\perp = \left|\boldsymbol{v}_\perp\right|,$$
$$\mu = \frac{v_\perp^2}{2\Omega},$$

$$\left\langle\phi\right\rangle\left(\boldsymbol{X},\mu,t\right) = \frac{1}{2\pi}\oint d\theta\; \phi\left(\boldsymbol{X}+\boldsymbol{\rho},t\right)$$

$$\boldsymbol{u}\left(\boldsymbol{x},\boldsymbol{v},t\right) \equiv \frac{1}{\Omega_0}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}_{\boldsymbol{X}}\left\langle\phi\right\rangle\left(\boldsymbol{x}-\boldsymbol{\rho},\mu,t\right)$$

- Now define $\boldsymbol{u}$ and $\boldsymbol{v}$ through the implicit equation

$$\boldsymbol{v} + \boldsymbol{u}\left(\boldsymbol{x},\boldsymbol{v},t\right) = \boldsymbol{V} = \dot{\boldsymbol{x}}$$

- solution well defined because our ordering $\Rightarrow \left\|\boldsymbol{\nabla}_v \boldsymbol{u}\right\| \ll 1$

THE TRANSFORMATION CAN BE VELOCITY DEPENDENT

- Starting point: $Q_0 = (\boldsymbol{x}, \boldsymbol{V}) \to Q = (\boldsymbol{x}, \boldsymbol{v})$

$$L\left(Q_0, \dot{Q}_0, t\right) = \left[\boldsymbol{A}\left(\boldsymbol{x}, t\right) + \boldsymbol{V}\right] \cdot \dot{\boldsymbol{x}} - \left\{\frac{1}{2}\boldsymbol{V}^2 + \phi\left(\boldsymbol{x}, t\right)\right\} \to$$

$$L\left(Q, \dot{Q}, t\right) = \left[\boldsymbol{A}\left(\boldsymbol{x}, t\right) + \boldsymbol{u}\left(\boldsymbol{x}, \boldsymbol{v}, t\right) + \boldsymbol{v}\right] \cdot \dot{\boldsymbol{x}} - \left\{\frac{1}{2}\left[\boldsymbol{u}\left(\boldsymbol{x}, \boldsymbol{v}, t\right) + \boldsymbol{v}\right]^2 + \phi\left(\boldsymbol{x}, t\right)\right\}$$

  - $\boldsymbol{v}$ Euler-Lagrange equation: $0 = \left(\boldsymbol{I} + \boldsymbol{\nabla}_v \boldsymbol{u}\right) \cdot \left[\dot{\boldsymbol{x}} - \left(\boldsymbol{u} + \boldsymbol{v}\right)\right]$
  - Our ordering $\Rightarrow \left\|\boldsymbol{\nabla}_v \boldsymbol{u}\right\| = 1$, so
    - $\boldsymbol{v} = \dot{\boldsymbol{x}} - \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{v}, t)$
    - but $\boldsymbol{x}$ is still the lab-frame position.

TOROIDAL EDGE ORDERING

- Basic ordering: $q\phi/T = O(\varepsilon^{-1})$, $V_{\mathrm{ExB}}/v_{\mathrm{th}} = O(\varepsilon^{0})$, $V'_{ExB}/\Omega = O(\varepsilon^{1})$
  - will use $\left(\partial/\partial t + \boldsymbol{V}_{\mathrm{ExB}}\cdot\boldsymbol{\nabla}\right)\ln(S)/\Omega \sim V'_{E\times B}/\Omega \sim \varepsilon \ll 1$
- low β - $\delta B/B_0 = O(\varepsilon^{1})\, \boldsymbol{A}(\boldsymbol{x},t) = \boldsymbol{A}_0(\boldsymbol{x}) + \delta A_{\parallel}(\boldsymbol{x},t)\hat{\boldsymbol{b}}_0$
- Edge ordering $qA_0/(mcv_{\mathrm{th}}) = O(\varepsilon^{-2})$; $\rho/L_p \sim L_p/R \sim \varepsilon$

LOWEST ORDER TRANSFORMATION

- Transform to $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{\rho}$, $\boldsymbol{v}$

$$L = \left[\boldsymbol{A}_0\left(\boldsymbol{R}+\boldsymbol{\rho}\right) + \delta A_{\|}\left(\boldsymbol{R}+\boldsymbol{\rho},t\right)\hat{\boldsymbol{b}}_0 + \boldsymbol{u}\left(\boldsymbol{R}+\boldsymbol{\rho},\boldsymbol{v},t\right) + \boldsymbol{v}\right]\cdot\left(\dot{\boldsymbol{R}}+\dot{\boldsymbol{\rho}}\right)$$
$$-\left[\frac{1}{2}\left(u^2+v^2\right) + \boldsymbol{u}\cdot\boldsymbol{v} + \phi\left(\boldsymbol{R}+\boldsymbol{\rho},t\right)\right].$$

- Separate $\phi\left(\boldsymbol{R}+\boldsymbol{\rho}\right)$ as $\phi\left(\boldsymbol{R}+\boldsymbol{\rho}\right) = \bar{\phi} + \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\bar{\phi} + \delta_1\tilde{\phi}$

  - $\delta_1\tilde{\phi} \equiv \tilde{\phi} - \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\bar{\phi} = O\left(\varepsilon = V'_{ExB}/\Omega\right)$; $\tilde{\phi} \equiv \phi\left(\boldsymbol{R}+\boldsymbol{\rho}\right) - \bar{\phi}\left(\boldsymbol{R},\mu\right)$,

- $\bar{\phi} \equiv \left\langle\phi\right\rangle = O\left(\varepsilon^{-1}\right)$ is gyrophase independent

- $\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\bar{\phi} = 0$ if $\boldsymbol{u} \equiv \hat{\boldsymbol{b}}\times\boldsymbol{\nabla}_R\bar{\phi}/\Omega_0$

- After this transformation and separation, all gyrophase dependences are at order ε^1 or higher

ALL GYROPHASE DEPENDENCES ARE AT FIRST OR HIGHTER ORDER

$$
\begin{aligned}
L = & \left\langle \boldsymbol{A}_0 \right\rangle \cdot \dot{\boldsymbol{R}} & \ldots\ldots O\left(\varepsilon^{-2}\right) \\
& - \left\langle \phi \right\rangle_0 & \ldots\ldots O\left(\varepsilon^{-1}\right) \\
& + \left[\boldsymbol{u} + \left(U_\parallel + \delta A_\parallel \right) \hat{\boldsymbol{b}}_0 \right] \cdot \dot{\boldsymbol{R}} - \mu \dot{\theta} - \left[\frac{1}{2} \left(U_\parallel^2 + \boldsymbol{u}^2 \right) + \mu \Omega \right] & \ldots\ldots O\left(\varepsilon^{0}\right) \\
& + \boldsymbol{u} \cdot \dot{\boldsymbol{\rho}} - \delta_1 \tilde{\phi} & \ldots\ldots O\left(\varepsilon^{1}\right) \\
& - \frac{1}{2} \left[\boldsymbol{\rho} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \left\langle \boldsymbol{A}_0 \right\rangle \right) \cdot \boldsymbol{\rho} \right] \cdot \dot{\boldsymbol{R}} & \ldots\ldots O\left(\varepsilon^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
L = & \left\langle \boldsymbol{A}_0 \right\rangle \cdot \dot{\boldsymbol{R}} & \ldots\ldots O\left(\varepsilon^{-2}\right) \\
& - \left\langle \phi \right\rangle_0 & \ldots\ldots O\left(\varepsilon^{-1}\right) \\
& + \left[\boldsymbol{u} + p_\parallel \hat{\boldsymbol{b}}_0 \right] \cdot \dot{\boldsymbol{R}} - \mu \dot{\theta} - \left[\frac{1}{2} \left(p_\parallel - \delta \bar{A}_\parallel \right)^2 + \frac{1}{2} \boldsymbol{u}^2 + \mu \Omega \right] & \ldots\ldots O\left(\varepsilon^{0}\right) \\
& + \boldsymbol{u} \cdot \dot{\boldsymbol{\rho}} - \left[\delta_1 \tilde{\phi} - \left(p_\parallel - \delta \bar{A}_\parallel \right) \delta \tilde{A}_\parallel \right] & \ldots\ldots O\left(\varepsilon^{1}\right) \\
& - \frac{1}{2} \left(\delta \tilde{A}_\parallel \right)^2 - \frac{1}{2} \left[\boldsymbol{\rho} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \left\langle \boldsymbol{A}_0 \right\rangle \right) \cdot \boldsymbol{\rho} \right] \cdot \dot{\boldsymbol{R}} & \ldots\ldots O\left(\varepsilon^{2}\right)
\end{aligned}
$$

LIE-TRANSFORM PERTURBATION THEORY

$$Z \to \bar{Z} = \left(\bar{\boldsymbol{R}}, \bar{U}_{\parallel}, \bar{\mu}, \bar{\theta}\right) = T\left(\varepsilon\right) Z$$

$$\bar{f}\left(Z\right) = T^{-1} \bar{f}\left(Z\right)$$

$$T = ...T_3 T_2 T_1$$

$$T_n\left(\varepsilon\right) = \exp\left(\varepsilon L_n\right)$$

$$\Gamma = T^{-1}\gamma + dS$$

$$\left(L_n \gamma\right)_\alpha = g_n^\beta \left(\frac{\partial \gamma_\alpha}{\partial Z^\beta} - \frac{\partial \gamma_\beta}{\partial Z^\alpha}\right)$$

- Due to spatial derivatives, $L_n \gamma = \left(L_n \gamma\right)_a + \varepsilon \left(L_n \gamma\right)_b + \varepsilon^2 \left(L_n \gamma\right)_c$

- Main results needed

$$\Gamma_{-2,-1,0} = \gamma_{-2,-1,0} + dS_{-2,-1,0},$$

$$\Gamma_1 = \gamma_1 - \left(L_1 \gamma_0\right)_a - \left(L_1 \gamma_{-1}\right)_b - \left(L_1 \gamma_{-2}\right)_c + dS_1,$$

$$\Gamma_2 = \left\langle \gamma_2 \right\rangle - \frac{1}{2}\left\langle \left(L_1 \gamma_1\right)_a \right\rangle$$

THE FIRST ORDER EQUATION YIELDS AND THE GAUGE FUNCTION AND THE GENERATORS

$$\boldsymbol{g}_1^{\perp} = -\frac{1}{\Omega_0}\hat{\boldsymbol{b}}_0 \times \boldsymbol{\nabla} S_1,$$

$$g_1^{\parallel} = -\frac{\partial S_1}{\partial p_{\parallel}},$$

$$g_1^{p_{\parallel}} = \nabla_{\parallel} S_1,$$

$$g_1^{\mu} = -\frac{\partial}{\partial \theta}\Big(S_1 + \boldsymbol{u}\cdot\boldsymbol{\rho}\Big),$$

$$g_1^{\theta} = \frac{\partial}{\partial \mu}\Big(S_1 + \boldsymbol{u}\cdot\boldsymbol{\rho}\Big)$$

$$0 = \frac{\partial S_1}{\partial t} + \boldsymbol{g}_1 \cdot \boldsymbol{\nabla}\overline{\phi} + g_1^{p_{\parallel}}\left(p_{\parallel} - \delta \overline{A}_{\parallel}\right) + \Omega g_1^{\mu} - \delta_1 \tilde{\psi},$$

$$\delta_1 \tilde{\psi} \equiv \delta_1 \tilde{\phi} - \left(p_{\parallel} - \delta \overline{A}_{\parallel}\right)\delta \tilde{A}_{\parallel}$$

THE FIRST ORDER EQUATION YIELDS AND THE GAUGE FUNCTION AND THE GENERATORS

$$\left(\frac{dS_1}{dt}\right)_{\text{slow}} - \Omega_0 \frac{\partial}{\partial\theta}\left(S_1 + \boldsymbol{u}\cdot\boldsymbol{\rho}\right) = \delta_1\tilde{\psi},$$

$$S_1 \approx -\left(\delta_1\tilde{\Psi}/\Omega_0\right) - \boldsymbol{\rho}\cdot\boldsymbol{u} = -\tilde{\Psi}/\Omega_0\,,$$

$$\tilde{\Psi} \equiv \Psi_i - \bar{\Psi}_i,$$

$$\Psi_i = \int_{\theta_0}^{\theta} d\theta\,\tilde{\psi},$$

$$\tilde{\psi} \equiv \tilde{\phi} - \left(p_\parallel - \delta\bar{A}_\parallel\right)\delta\tilde{A}_\parallel$$

$$\boldsymbol{g}_1^{\perp} = \frac{1}{\Omega_0}\hat{\boldsymbol{b}}_0 \times \boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega_0\right),$$

$$g_1^{\parallel} = -\Delta\tilde{A}_\parallel/\Omega_0\,,$$

$$g_1^{p_\parallel} = -\nabla_\parallel\left(\tilde{\Psi}/\Omega_0\right),$$

$$g_1^{\mu} = \left(\delta_1\tilde{\psi}/\Omega_0\right),$$

$$g_1^{\theta} = -\frac{1}{\Omega_0}\frac{\partial\delta_1\tilde{\Psi}}{\partial\mu}$$

ELECTROMAGNETIC TOROIDAL LAGRANGIAN COMPONENTS TO SECOND ORDER

$$\boldsymbol{\Gamma}_2 = \frac{1}{2\Omega_0}\left[\frac{\partial}{\partial\mu}\left\langle \boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\boldsymbol{\rho}\right)\delta_1\tilde{\psi}\right\rangle - \Omega_0\left\langle \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\left\langle \boldsymbol{A}_0\right\rangle\right)\cdot\boldsymbol{\rho}\right\rangle\right],$$

$$\Gamma_2^{\mu} = -\frac{1}{2\Omega_0}\left\{\left\langle \hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega_0\right)\cdot\boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\rho}}{\partial\mu}\right)\right\rangle - \left\langle \Delta\tilde{A}_{\|}\nabla_{\|}\left(\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\rho}}{\partial\mu}\right)\right\rangle\right.$$
$$\left. - \frac{\partial\boldsymbol{u}}{\partial\mu}\cdot\left\langle \boldsymbol{\rho}\,\frac{\partial\delta_1\tilde{\psi}}{\partial\mu}\right\rangle\right\},$$

$$\Gamma_2^{\theta} = \frac{1}{2\Omega_0}\left\{\left\langle \hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\left(\delta_1\tilde{\psi}/\Omega_0\right)\cdot\boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\boldsymbol{\rho}\right)\right\rangle - \left\langle \delta\tilde{A}_{\|}\nabla_{\|}\left(\boldsymbol{u}\cdot\boldsymbol{\rho}\right)\right\rangle\right.$$
$$\left. - \frac{\partial\boldsymbol{u}}{\partial\mu}\cdot\left\langle \frac{\partial\boldsymbol{\rho}}{\partial\theta}\,\delta_1\tilde{\psi}\right\rangle\right\},$$

$$H_2 = -\frac{1}{2\Omega_0}\left\{\left\langle \hat{\boldsymbol{b}}_0\times\boldsymbol{\nabla}\left(\tilde{\Psi}/\Omega_0\right)\cdot\boldsymbol{\nabla}\delta_1\tilde{\psi}\right\rangle - \left\langle \Delta\tilde{A}_{\|}\nabla_{\|}\left(\delta_1\tilde{\psi}\right)\right\rangle\right.$$
$$\left. + \frac{\partial}{\partial\mu}\left\langle \left(\delta_1\tilde{\psi}\right)^2\right\rangle + \frac{\partial\boldsymbol{u}}{\partial t}\cdot\frac{\partial}{\partial\mu}\left\langle \delta_1\tilde{\psi}\boldsymbol{\rho}\right\rangle\right\}.$$

CORE ORDERING – MORE "STANDARD" BUT HARDER

$$
\begin{aligned}
L = {} & \left\langle \boldsymbol{A}_0 \right\rangle \cdot \dot{\boldsymbol{R}} - \left\langle \phi \right\rangle_0 && \ldots\ldots O\left(\varepsilon^{-1}\right) \\
& + \left[\boldsymbol{u} + \left(U_{\parallel} + \delta A_{\parallel} \right) \hat{\boldsymbol{b}}_0 \right] \cdot \dot{\boldsymbol{R}} - \mu \dot{\theta} - \left[\frac{1}{2} \left(U_{\parallel}^2 + \boldsymbol{u}^2 \right) + \mu \Omega \right] && \ldots\ldots O\left(\varepsilon^{0}\right) \\
& + \boldsymbol{u} \cdot \dot{\boldsymbol{\rho}} - \delta_1 \tilde{\phi} - \frac{1}{2} \left[\boldsymbol{\rho} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \left\langle \boldsymbol{A}_0 \right\rangle \right) \cdot \boldsymbol{\rho} \right] \cdot \dot{\boldsymbol{R}} && \ldots\ldots O\left(\varepsilon^{1}\right)
\end{aligned}
$$

- Brings in
  - FLR corrections to equilibrium magnetic field
  - Higher-order equilibrium motion terms, e.g, Banos drift
  - Couplings of equilibrium magnetic field variation and fluctuations

SUMMARY OF (GYROAVERAGED) GYROKINETIC ORDERINGS AND DERIVATIONS

- $e\psi / T \ll 1, \quad \psi = \phi - \left(p_{\|} / mc\right) A_{\|}$ - most Hamiltonian GK derivations – core.
- $e\psi / T \sim 1, \quad V_{ExB} / v_{th} \ll 1$ - Dimits, Dubin, Lodestro `92, and extended here; many iterative derivations (e.g., Parra-Catto) – likely applicable to many edge situations without large flows.
- 2-scale, with $u_E / v_{th} \sim 1, \quad \delta V_{ExB} / v_{th} \ll 1$ - Brizard `94; Hahm `96; Qin *et. al*., `07; Hahm-Wang-Madsen `09. – addresses large flows.
- Our new ordering $V'_{ExB} / \Omega \ll 1$ for any perturbations – allows large perturbations and large flows. Captures new cross terms.
- Differences between new ordering/derivation and 2-scale
  - our $\bar{\boldsymbol{u}}$ not quite the same as 2-scale $\boldsymbol{u}_E$
  - $\nabla \cdot \left\langle \boldsymbol{\rho} \cdot \nabla \boldsymbol{u}_E \; \delta_1 \tilde{\phi} \right\rangle \sim \nabla \left\langle \left(\boldsymbol{\rho} \cdot \nabla u_E \right)^2 \right\rangle \sim \nabla \left\langle \delta_1 \tilde{\phi}^2 \right\rangle$

GYROKINETIC EQUATIONS HAVE BEEN DERIVED IN A NEW MORE GENERAL ORDERING

- Allows for
  - large flow velocities
  - large perturbation amplitudes
- Toroidal electromagneti gyrokinetic theory has been developed in a low-β ordering appropriate for MFE edge plasma conditions.
- Useful reduced and subsidiary orderings were found.
  - The “standard” toroidal Hamiltonian theory is valid for the more general DDL ordering, which allows for large relative perturbation amplitudes
  - Now have justification of the reduced “practical minimal model” as valid under the DDL ordering for ϕ and a subsidiary ordering for $\delta A_{||}$
- Numerical algorithms for the discretization of the second-order terms in the gyrokinetic equations of motion and Poisson’s equation were presented.