Lattice Boltzmann approaches to magnetohydrodynamics and related models

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Overview

Lattice Boltzmann approach to hydrodynamics

- Derivation of hydrodynamics from the kinetic theory of gases
- From continuum to discrete kinetic theory
- Space/time discretisation
- Moment equations and matrix collision operators

Lattice Boltzmann magnetohydrodynamics

- Including the Lorentz force: Maxwell stress in the fluid equilibrium
- Simulating the induction equation: vector-valued distribution functions

Extensions:

- Braginskii magnetohydrodynamics
- Ohm’s law with current-dependent resistivity

Electromagnetism

- Moment equations imply Maxwell’s equations plus Ohm’s law

Nine velocity lattice Boltzman equation

One may simulate the nearly incompressible Navier–Stokes equations (with viscosity controlled by τ) using the lattice Boltzmann equation

$$\overline{f}_i(\mathbf{x}+\boldsymbol{\xi}_i\Delta t, t+\Delta t)-\overline{f}_i(\mathbf{x},t) = -\frac{\Delta t}{\tau+\Delta t/2}\left(\overline{f}_i(\mathbf{x},t)-f_i^{(0)}(\mathbf{x},t)\right),$$

on a nine velocity lattice ($i = 0, \ldots, 8$) in 2D with the equilibria

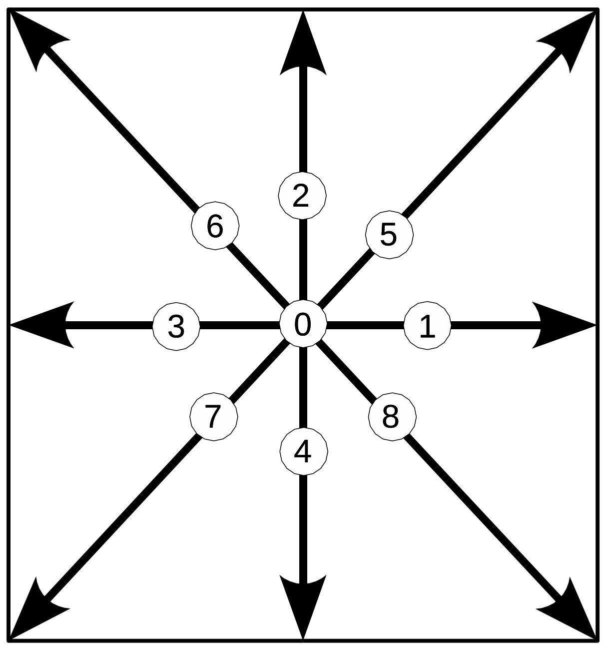
$$f_i^{(0)} = w_i\rho\left(1 + 3\boldsymbol{\xi}_i\cdot\mathbf{u} + \tfrac{9}{2}(\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \tfrac{1}{3}\mathbf{I}) : \mathbf{uu}\right),$$

where $\rho = \sum_i f_i$ and $\rho\mathbf{u} = \sum_i \boldsymbol{\xi}_i f_i$.

The weight factors w_i are

$$w_i = \begin{cases} 4/9, & \text{i=0,} \\ 1/9, & \text{i=1,2,3,4,} \\ 1/36, & \text{i=5,6,7,8,} \end{cases}$$

and the nine lattice vectors $\boldsymbol{\xi}_i$ are:



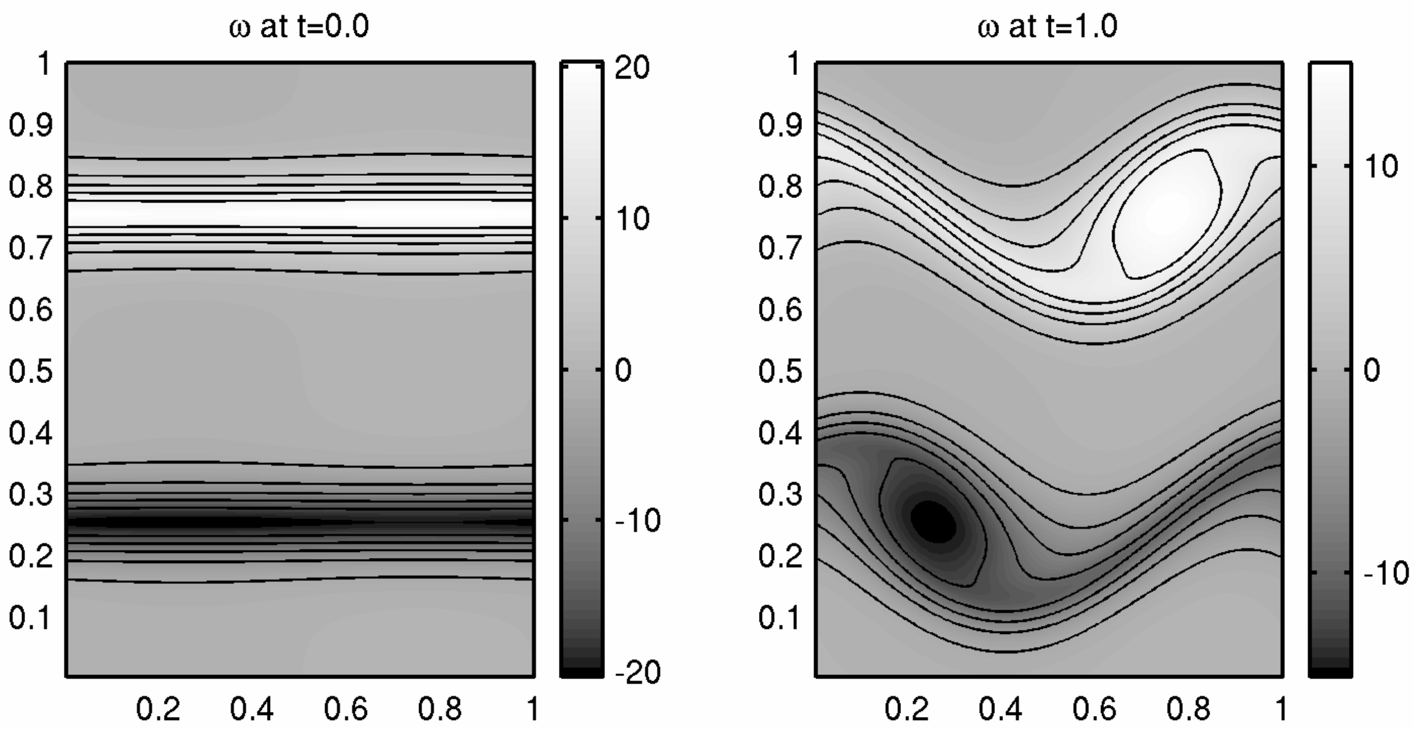
Minion & Brown (1997) benchmark

Roll-up of shear layers in Minion & Brown (1997) test problem,

$$u_x = \begin{cases} \tanh(\kappa(y - 1/4)), & y \le 1/2, \\ \tanh(\kappa(3/4 - y)), & y > 1/2, \end{cases}$$

$$u_y = \delta \sin(2\pi(x + 1/4)),$$

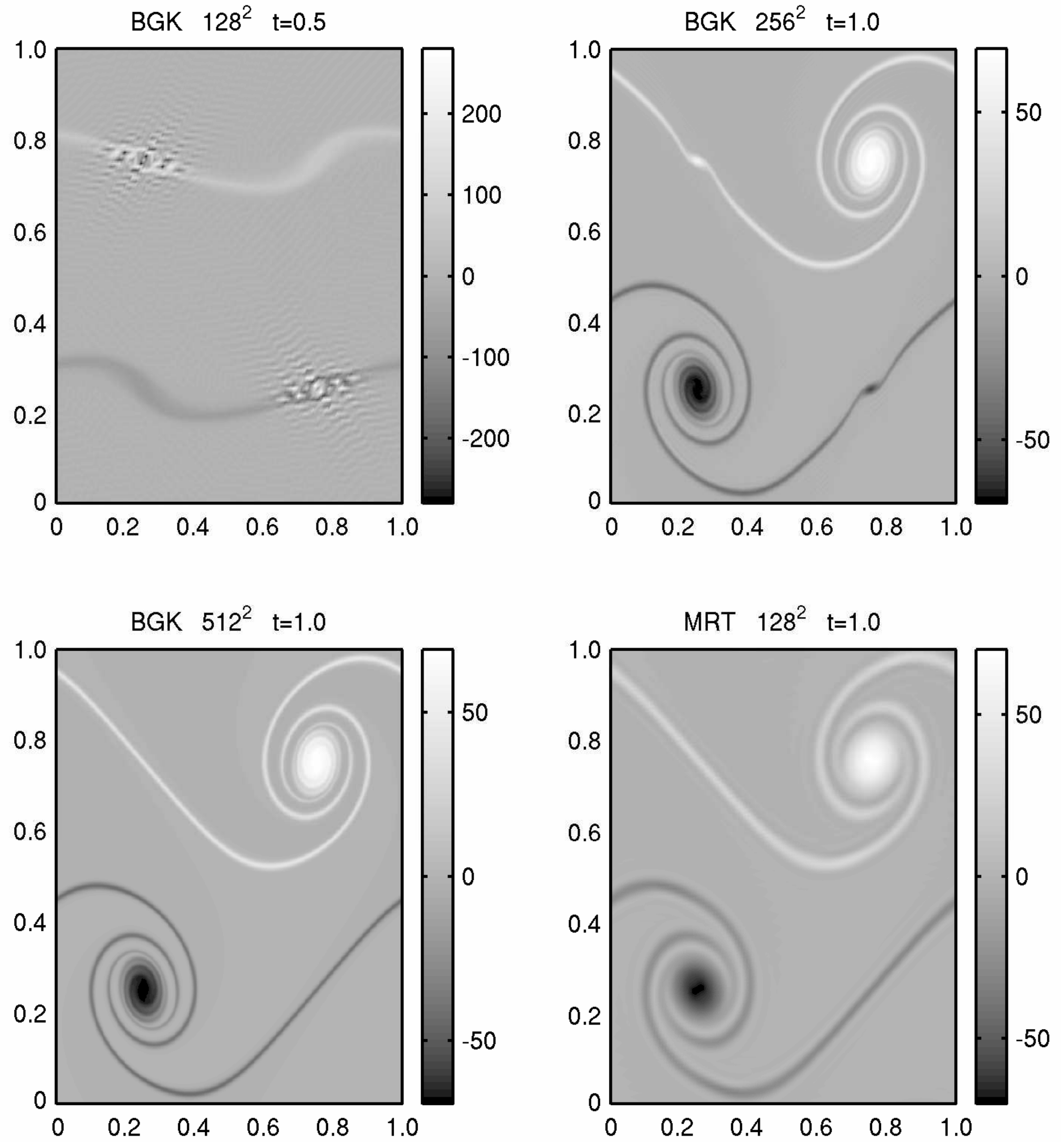
with $\kappa = 20$, $\delta = 0.05$, and $\mathrm{Re} = 1000$; so the solution remains well resolved



Modified Minion & Brown (1997) problem for roll-up of shear layers.

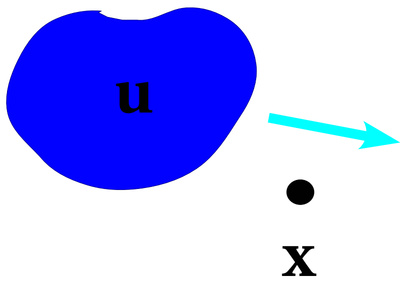
$Re = 10,000$ was marginal with $\kappa = 80$ and $\delta = 0.05$ on a 128×128 grid

$Re = 30,000$ on 128×128 and larger grids.



Newton's 2nd law, following a blob of fluid:

$$\rho\,\frac{d\mathbf{u}}{dt} = \mathbf{F}$$



Change attention to a fixed point $\mathbf{x}$ in space:

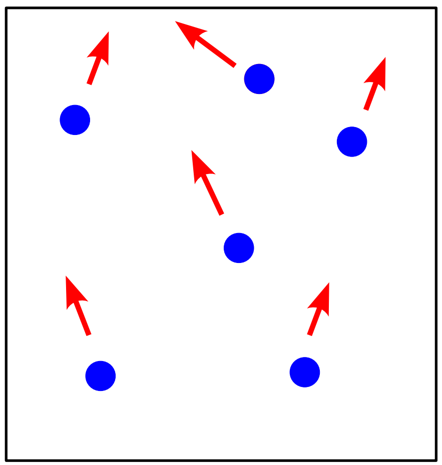
$$\rho\,\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \mathbf{F}$$

Intrinsic nonlinearity, even when $\mathbf{F}$ is linear (*eg* isothermal Newtonian fluids)

$$\mathbf{F} = \nabla\cdot\left[-c_{\mathsf{s}}^2\rho\mathbf{I} + \mu\{(\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\mathsf{T}}\}\right]$$

Boltzmann's equation from the kinetic theory of gases

$$\partial_t f + \boldsymbol{\xi}\cdot\nabla f = C[f,f]$$



Distribution function $f(\mathbf{x},\boldsymbol{\xi},t)$ instead of $\mathbf{u}(\mathbf{x},t)$.
Linear advection, but seven independent variables.

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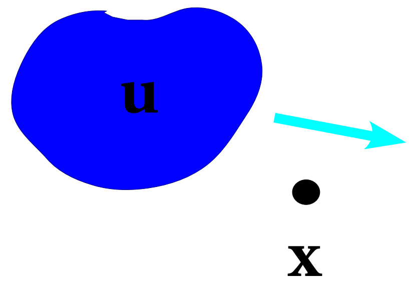

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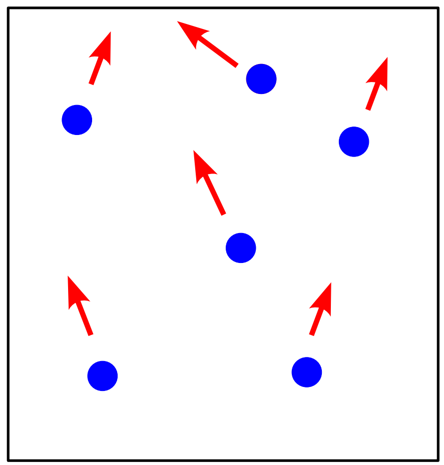
$$\mathbf{F} = \nabla\cdot\left[-c_{\mathsf{s}}^2\rho\mathbf{I} + \mu\{(\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\mathsf{T}}\}\right]$$

Lattice Boltzmann fits here
linear advection, few additional degrees of freedom



Boltzmann's equation from the kinetic theory of gases

$$\partial_t f + \boldsymbol{\xi}\cdot\nabla f = C[f,f]$$

Distribution function $f(\mathbf{x},\boldsymbol{\xi},t)$ instead of $\mathbf{u}(\mathbf{x},t)$.
Linear advection, but seven independent variables.

From (real) kinetic theory to fluid dynamics

Moments of $f(\mathbf{x}, \boldsymbol{\xi}, t)$ define functions of $(\mathbf{x}, t)$,

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi}, \quad \rho\mathbf{u} = \int \boldsymbol{\xi} f\, d\boldsymbol{\xi}, \quad \boldsymbol{\Pi} = \int \boldsymbol{\xi}\boldsymbol{\xi} f\, d\boldsymbol{\xi}.$$

Taking moments of Boltzmann's equation

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla f = C[f, f]$$

leads to exact conservation laws for mass and momentum

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \partial_t(\rho\mathbf{u}) + \nabla\cdot\boldsymbol{\Pi} = 0.$$

(RHS vanish because collisions conserve microscopic mass and momentum.)

Momentum flux is not conserved by collisions. It evolves according to

$$\partial_t \boldsymbol{\Pi} + \nabla\cdot \left(\int \boldsymbol{\xi}\boldsymbol{\xi}\boldsymbol{\xi} f\, d\boldsymbol{\xi}\right) = -\frac{1}{\tau}\left(\boldsymbol{\Pi} - \boldsymbol{\Pi}^{(0)}\right)$$

where $\boldsymbol{\Pi}^{(0)} = \rho\mathbf{u}\mathbf{u} + \rho\theta\mathbf{I}$, as given by a Maxwell–Boltzmann distribution.

An effective collision time τ may be calculated for approximations to $C[f, f]$.

Hydrodynamics follows by exploiting $\tau \ll T$ (a macroscopic timescale).

Derivation of hydrodynamics

Given the moment equations

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{u}) = 0, \quad \partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbf{\Pi} = 0,$$

$$\partial_t \mathbf{\Pi} + \nabla\cdot\mathsf{Q} = -\frac{1}{\tau}\left(\mathbf{\Pi} - \mathbf{\Pi}^{(0)}\right),$$

we derive hydrodynamics by seeking slowly varying solutions. We expand $\mathbf{\Pi}$, Q and all higher moments as series in τ,

$$\mathbf{\Pi} = \mathbf{\Pi}^{(0)} + \tau\mathbf{\Pi}^{(0)} + \cdots, \quad \mathsf{Q} = \mathsf{Q}^{(0)} + \tau\mathsf{Q}^{(1)} + \cdots,$$

and also expand the time derivative (multiple scales)

$$\partial_t = \partial_{t_0} + \tau\partial_{t_1} + \cdots.$$

E.g. the viscous stress is given by

$$\partial_{t_0}\mathbf{\Pi}^{(0)} + \nabla\cdot\mathsf{Q}^{(0)} = -\mathbf{\Pi}^{(1)}.$$

If we have the same moment system, and the same $\mathbf{\Pi}^{(0)}$ and $\mathsf{Q}^{(0)}$, it does not matter whether we started from the real Boltzmann equation.

What do we need from real kinetic theory?

We need $f \longrightarrow f^{(0)}$ under collisions, and we need some moments of the equilibrium distributions:

$$\int f^{(0)} d\boldsymbol{\xi} = \rho$$

$$\int \boldsymbol{\xi} f^{(0)} d\boldsymbol{\xi} = \rho \mathbf{u}$$

$$\int \boldsymbol{\xi}\boldsymbol{\xi} f^{(0)} d\boldsymbol{\xi} = \boldsymbol{\Pi}^{(0)} = \rho \mathbf{u}\mathbf{u} + \theta \rho \, \mathbf{I}$$

$$\int \boldsymbol{\xi}\boldsymbol{\xi}\boldsymbol{\xi} f^{(0)} d\boldsymbol{\xi} = \mathsf{Q}^{(0)}$$

where

$$Q^{(0)}_{\alpha\beta\gamma} = \theta\rho\left(u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\gamma\alpha} + u_\gamma \delta_{\alpha\beta}\right) + \partial_\alpha(\rho u_\alpha u_\beta u_\gamma)$$

All these things we can calculate from

$$f^{(0)} = \rho(2\pi\theta)^{-3/2} \exp\left(-|\boldsymbol{\xi} - \mathbf{u}|^2/(2\theta)\right) .$$

Simplifying the kinetic theory of gases

Replace $C[f,f]$ with the Bhatnagar–Gross–Krook (BGK) collision operator,

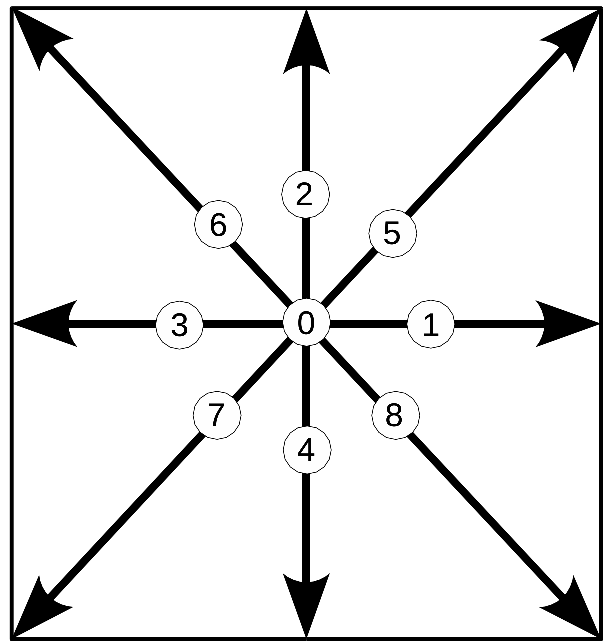
$$\partial_t f + \boldsymbol{\xi}\cdot\nabla f = -\tfrac{1}{\tau}\big(f - f^{(0)}\big).$$

f relaxes towards $f^{(0)}$ with a single relaxation time τ. Mass, momentum (and energy) are conserved, provided the ρ, $\mathbf{u}$, θ in $f^{(0)}$ are calculated from f.

We now have to supply $f^{(0)}$ explicitly,

$$f^{(0)} = \rho\,(2\pi\theta)^{-3/2}\exp[-|\boldsymbol{\xi}-\mathbf{u}|^2/(2\theta)]$$

Discretise the velocity space so that $\boldsymbol{\xi}$ is confined to a finite set $\boldsymbol{\xi}_0,\ldots,\boldsymbol{\xi}_N$, such as the 9 shown:



$f(\mathbf{x},\boldsymbol{\xi},t)$ is replaced by a set of $f_i(\mathbf{x},t)$.

Integral moments are replaced by sums,

$$\rho = \textstyle\sum_i f_i,\quad \rho\mathbf{u} = \sum_i \boldsymbol{\xi}_i f_i,\quad \mathbf{\Pi} = \sum_i \boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i.$$

All continuum calculations, rewritten using moments, go through unchanged.

Integration of the discrete Boltzmann equation in space and time

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = \Omega_i, \text{ where } \Omega_i = -\tfrac{1}{\tau}(f_i - f_i^{(0)})$$

Integrate along characteristics for time Δt,

$$f_i(\mathbf{x} + \boldsymbol{\xi}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = \int_0^{\Delta t} \Omega_i(\mathbf{x} + \boldsymbol{\xi}_i s, t + s) ds.$$

The left hand side is exact.

Approximating the integral by the trapezium rule (2nd order accuracy) gives

$$f_i(\mathbf{x}+\boldsymbol{\xi}_i \Delta t, t+\Delta t) - f_i(\mathbf{x}, t) = \tfrac{1}{2}\Delta t \Big(\Omega_i(\mathbf{x} + \boldsymbol{\xi}_i \Delta t, t+\Delta t) + \Omega_i(\mathbf{x}, t)\Big) + O(\Delta t^3).$$

[He, Chen, Doolen 1998]

Defining $\overline{f}_i(\mathbf{x}', t') = f_i(\mathbf{x}', t') - \tfrac{1}{2}\Delta t\, \Omega_i(\mathbf{x}', t')$ gives the explicit formula

$$\overline{f}_i(\mathbf{x}+\boldsymbol{\xi}_i \Delta t, t+\Delta t) - \overline{f}_i(\mathbf{x}, t) = -\frac{\Delta t}{\tau + \Delta t/2} \Big(\overline{f}_i(\mathbf{x}, t) - f_i^{(0)}(\mathbf{x}, t)\Big).$$

Reconstruct $f_i^{(0)}$ from $\rho = \sum_i f_i = \sum_i \overline{f}_i$ and $\rho \mathbf{u} = \sum_i \boldsymbol{\xi}_i f_i = \sum_i \boldsymbol{\xi}_i \overline{f}_i$.

Lattice Boltzmann versus discrete Boltzmann

Hydrodynamics follows from slowly varying solutions to the discrete Boltzmann equation

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = -\frac{1}{\tau}\left(f_i - f_i^{(0)}\right).$$

This is a partial differential equation (PDE) in space and time.

Only the particle velocities $\boldsymbol{\xi}_i$ are discrete in the “discrete Boltzmann equation”.

The lattice Boltzmann equation is an approximation to this PDE,

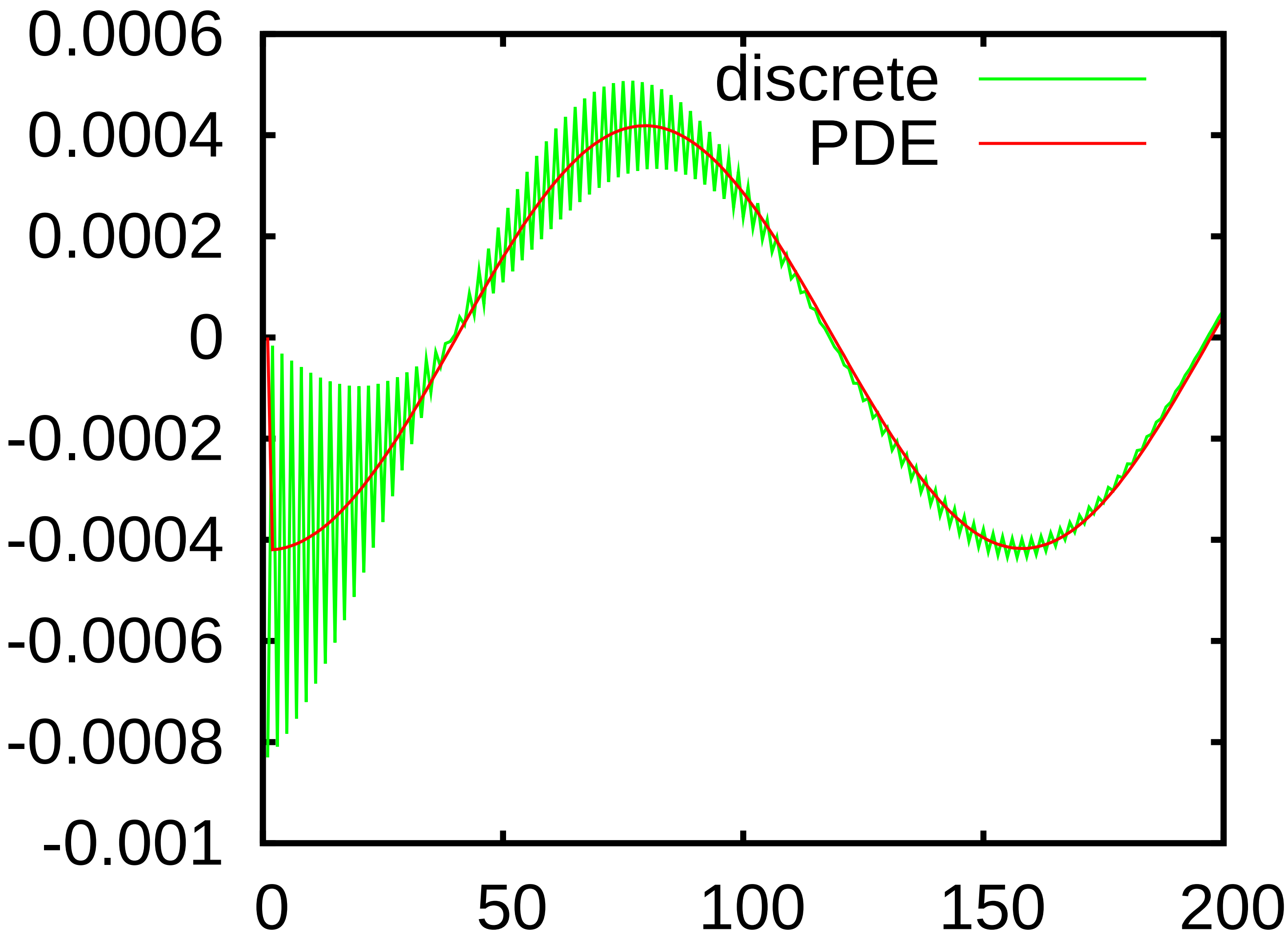
$$\overline{f}_i(\mathbf{x}+\boldsymbol{\xi}_i\Delta t, t+\Delta t) - \overline{f}_i(\mathbf{x}, t) = -\frac{\Delta t}{\tau+\Delta t/2}\left(\overline{f}_i(\mathbf{x}, t) - f_i^{(0)}(\mathbf{x}, t)\right).$$

For spatially uniform solutions, this last equation implies

$$\left(\overline{f}_i(t+\Delta t) - f_i^{(0)}(t)\right) = -\left(\frac{1-2\tau/\Delta t}{1+2\tau/\Delta t}\right)\left(\overline{f}_i(t) - f_i^{(0)}(t)\right).$$

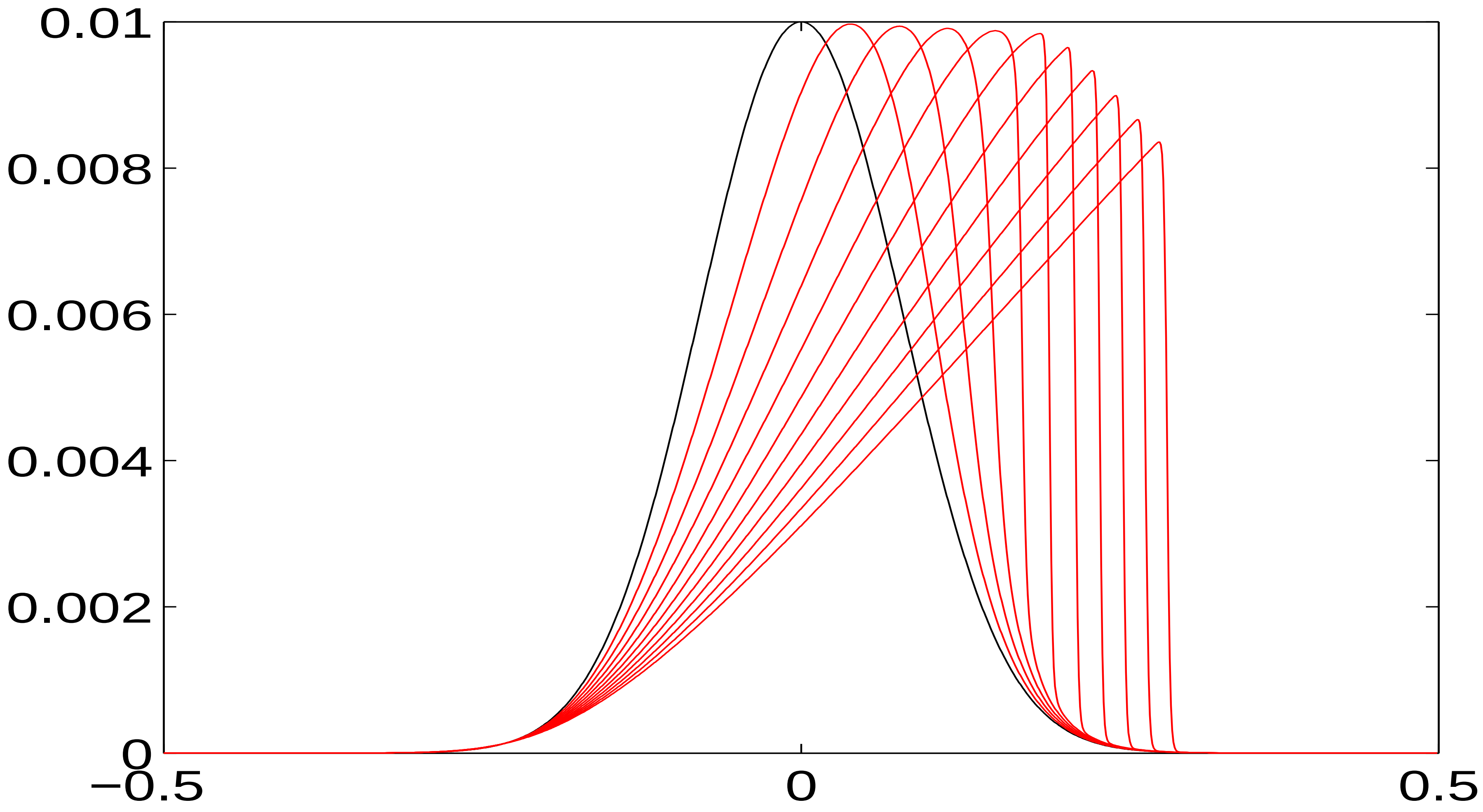
For $\tau \ll \Delta t$ the $\overline{f}_i$ oscillate around equilibrium from timestep to timestep.

In the discrete Boltzmann equation, $f_i \longrightarrow f_i^{(0)}$ monotonically.



Example with $\Delta t = 200\tau$. The LBE tracks slowly-varying solutions of the DBE, but with super-imposed oscillations.

Numerical example for Burgers equation



Computation performed using 1024 points with $\tau = 0.01$ in lattice units.

Ran on a “Type II” NMR quantum computer with 16 points [Chen *et al.* 2006].

MATLAB implementation for one timestep

$$\partial_t f_\pm + \xi_\pm \partial_x f_\pm = -\frac{1}{\tau}\left(f_\pm - f_\pm^{(0)}\right) \text{ with } f_\pm^{(0)} = \frac{1}{2}\left(\rho \pm \rho^2\right), \xi_\pm = \pm 1.$$

First compute $\rho = f_- + f_+$,

```
r = fm + fp;
```

Then compute the equilibria $f_\pm$ from ρ,

```
feqm = (1/2)*(r-r.^2);
feqp = (1/2)*(r+r.^2);
```

Next perform collisions, $f_\pm \mapsto f_\pm + (f_\pm^{(0)} - f_\pm)/(\tau + \frac{1}{2})$,

```
fm = fm + (feqm-fm)./(0.5+tau);
fp = fp + (feqp-fp)./(0.5+tau);
```

Finally perform advection by shifting the values onto the next gridpoint,

```
fm = circshift(fm',-1)';
fp = circshift(fp',1)';
```

Moment equations and matrix collision operators

From the discrete Boltzmann equation we derived

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u}) = 0,\;\; \partial_t(\rho\mathbf{u})+\nabla\cdot\mathbf{\Pi} = 0,\;\; \partial_t\mathbf{\Pi}+\nabla\cdot\mathsf{Q} = -\frac{1}{\tau}\left(\mathbf{\Pi}-\mathbf{\Pi}^{(0)}\right).$$

In 2D the moments ρ, $\mathbf{u}$, $\mathbf{\Pi}$ contain 6 degrees of freedom, but the lattice has 9.

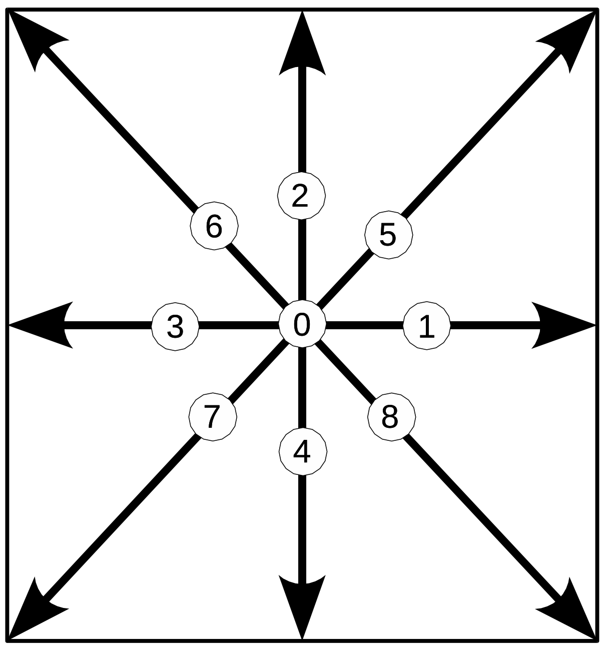
We define 2 more moments by

$$N = \sum_i g_i f_i,$$

$$\mathbf{J} = \sum_i g_i \boldsymbol{\xi}_i f_i,$$

where

$$g_i = (1, -2, -2, -2, -2, 4, 4, 4, 4).$$



Now we can reconstruct the distribution functions from these moments,

$$f_i = w_i\left(\rho+3(\rho\mathbf{u})\cdot\boldsymbol{\xi}_i+\frac{9}{2}\left[\mathbf{\Pi}-\frac{1}{3}\rho\mathbf{I}\right]:\left[\boldsymbol{\xi}_i\boldsymbol{\xi}_i-\frac{1}{3}\mathbf{I}\right]+g_i\left[\frac{1}{4}N+\frac{3}{8}\boldsymbol{\xi}_i\cdot\mathbf{J}\right]\right).$$

Equivalent moment system

More generally, we allow a matrix collision operator

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = -\Omega_{ij}\big(f_j - f_j^{(0)}\big),$$

designed to give a moment system of the form

$$\begin{aligned}
\partial_t \rho \;&+\; \nabla\cdot(\rho\mathbf{u}) = 0,\\
\partial_t(\rho\mathbf{u}) \;&+\; \nabla\cdot\boldsymbol{\Pi} = 0,\\
\partial_t \boldsymbol{\Pi} \;&+\; \nabla\cdot\left(\textstyle\sum_i \boldsymbol{\xi}_i\boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i\right) = -\frac{1}{\tau}\left(\boldsymbol{\Pi} - \boldsymbol{\Pi}^{(0)}\right),\\
\partial_t \mathbf{J} \;&+\; \nabla\cdot\left(\textstyle\sum_i g_i\boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i\right) = -\frac{1}{\tau_J}\left(\mathbf{J} - \mathbf{J}^{(0)}\right),\\
\partial_t N \;&+\; \nabla\cdot\mathbf{J} = -\frac{1}{\tau_N}\left(N - N^{(0)}\right).
\end{aligned}$$

The two sub-systems for ρ, $\mathbf{u}$, $\boldsymbol{\Pi}$, and for N, $\mathbf{J}$ are coupled through the higher moments in the equations for $\boldsymbol{\Pi}$ and $\mathbf{J}$. Project them onto the basis . . .

Discrete implementation

After discretising by integrating along characteristics, the post-collisional moments are

$$\overline{\mathbf{\Pi}}' = \overline{\mathbf{\Pi}} - \frac{1}{\tau + \frac{1}{2}\Delta t}\left(\overline{\mathbf{\Pi}} - \mathbf{\Pi}^{(0)}\right),$$

$$\overline{N}' = \overline{N} - \frac{1}{\tau_N + \frac{1}{2}\Delta t}\left(\overline{N} - N^{(0)}\right),$$

$$\overline{\mathbf{J}}' = \overline{\mathbf{J}} - \frac{1}{\tau_J + \frac{1}{2}\Delta t}\left(\overline{\mathbf{J}} - \mathbf{J}^{(0)}\right),$$

from which we can reconstruct the post-collision distribution functions,

$$f_i' = w_i\Big(\rho + 3(\rho\mathbf{u})\cdot\boldsymbol{\xi}_i + \tfrac{9}{2}\left[\overline{\mathbf{\Pi}}' - \tfrac{1}{3}\rho\mathbf{I}\right]:\left[\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \tfrac{1}{3}\mathbf{I}\right] + g_i\left[\tfrac{1}{4}\overline{N}' + \tfrac{3}{8}\boldsymbol{\xi}_i\cdot\overline{\mathbf{J}}'\right]\Big).$$

Choosing different values for τ, τ_N, τ_J, gives big gains in stability.

For example, taking $\tau_N = \tau_J = \frac{1}{2}\Delta t$ sets $\overline{N}' = N^{(0)}$ and $\overline{\mathbf{J}}' = \mathbf{J}^{(0)}$.

Magnetohydrodynamics

Magnetohydrodynamics (MHD)

MHD is a single fluid description of media containing at least two kinds of particles with opposite charges: liquid metals, electrolytes, ionised gases.

Applications to interiors of planets, stars, “space weather” etc. Nuclear fusion, industrial processing of liquid metals, producing aluminium, alloys . . .

Maxwell’s equations

$$-c^{-2}\partial_t\mathbf{E}+\nabla\times\mathbf{B}=\mathbf{J},\ \ \partial_t\mathbf{B}+\nabla\times\mathbf{E}=0,\ \ \nabla\cdot\mathbf{B}=0,\ \ \nabla\cdot\mathbf{E}=\rho_c/\epsilon_0$$

For non-relativistic ($\boldsymbol{v} \ll c$) and quasi-neutral ($\rho_c \ll 1$) phenomena we approximate by

$$\partial_t\mathbf{B}=-\nabla\times\mathbf{E},\quad \nabla\cdot\mathbf{B}=0,\quad \mathbf{J}=\nabla\times\mathbf{B}.$$

The electric field $\mathbf{E}$ is now just the flux of $\mathbf{B}$ in a conservation law

$$\partial_t\mathbf{B}+\nabla\cdot\boldsymbol{\Lambda}=0,\ \text{where}\ \Lambda_{\alpha\beta}=-\epsilon_{\alpha\beta\gamma}E_\gamma.$$

Ohm's law — the electron momentum equation

Constitutive relation for $\mathbf{E}$,

$$
\begin{aligned}
\mathbf{E} + \mathbf{u}\times\mathbf{B} = \quad & \eta\mathbf{J} && \text{resistivity} \\
+ \; & \alpha\mathbf{J}\times\mathbf{B} && \text{Hall effect} \\
- \; & \beta(\mathbf{J}\times\mathbf{B})\times\mathbf{B} && \text{ambipolar diffusion} \\
+ \; & \gamma\, d\mathbf{J}/dt && \text{electron inertia} \\
+ \; & \text{electron pressure} + \text{electron viscosity} + \cdots
\end{aligned}
$$

which emerges from multispecies kinetic theory for dilute plasmas.

Simplest common form is $\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta\mathbf{J}$,

$$
\begin{aligned}
\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) &= 0, \\
\partial_t(\rho\mathbf{u}) + \nabla\cdot(p\mathbf{I} + \rho\mathbf{u}\mathbf{u}) &= \mathbf{J}\times\mathbf{B} + \nabla\cdot(\mu\mathsf{S}), \\
\partial_t\mathbf{B} &= \nabla\times(\mathbf{u}\times\mathbf{B} - \eta\nabla\times\mathbf{B}).
\end{aligned}
$$

Compressible resistive MHD equations also include the Lorentz force $\mathbf{J}\times\mathbf{B}$.

Including the Lorentz force via the Maxwell stress

Lorentz force $\mathbf{J}\times\mathbf{B} = -\nabla\cdot\widetilde{\mathsf{M}}$ for $\widetilde{M}_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}|\mathbf{B}|^2 - B_\alpha B_\beta$.

Rewrite the inviscid momentum equation using the Maxwell stress $\widetilde{\mathsf{M}}$,

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\left(p\mathbf{I} + \rho\mathbf{u}\mathbf{u} + \tfrac{1}{2}B^2\mathbf{I} - \mathbf{B}\mathbf{B}\right) = 0.$$

Putting this desired second moment of the equilibrium distributions $f_i^{(0)}$,

$$\mathbf{\Pi}^{(0)} = (\theta\rho + \tfrac{1}{2}B^2)\mathbf{I} + \rho\mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B},$$

into the general formula

$$f_i^{(0)} = w_i\left(\rho\left[2 - \frac{3}{2}|\boldsymbol{\xi}_i|^2\right] + 3(\rho\mathbf{u})\cdot\boldsymbol{\xi}_i + \frac{9}{2}\mathbf{\Pi}^{(0)}:\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \frac{3}{2}\,\mathrm{Tr}\,\mathbf{\Pi}^{(0)}\right)$$

gives suitable two-dimensional equilibria (the same as before when $\mathbf{B} = 0$.)

We only use $\mathbf{B}$ at lattice points. In the induction equation we will only use $\mathbf{u}$ at lattice points. The two are coupled only through macroscopic variables.

The magnetic induction equation

The first moment of a Boltzmann equation

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = \mathcal{C}[f_i]$$

gives

$$\partial_t \sum_{i=0}^{N} \boldsymbol{\xi}_i f_i + \nabla \cdot \left(\sum_{i=0}^{N} \boldsymbol{\xi}_i \boldsymbol{\xi}_i f_i \right) = \sum_{i=0}^{N} \boldsymbol{\xi}_i \boldsymbol{\xi}_i \mathcal{C}[f_i] = 0.$$

Thus the momentum vector $\rho \mathbf{u}$ evolves as

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot \boldsymbol{\Pi} = 0,$$

where $\boldsymbol{\Pi} = \sum_i \boldsymbol{\xi}_i \boldsymbol{\xi}_i f_i$ is symmetric by construction.

By contrast, the evolution equation for $\mathbf{B}$ is

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \text{or} \quad \partial_t \mathbf{B} + \nabla \cdot \boldsymbol{\Lambda} = 0,$$

where $\boldsymbol{\Lambda}$ is antisymmetric and defined by $\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} E_\gamma$.

One cannot derive the induction equation from the usual Boltzmann equation.

Bi-directional streaming

First approach to lattice gas and lattice Boltzmann magnetohydrodynamics used bi-directional streaming in 2D with two sets of velocities, $\boldsymbol{v}_a^\sigma$ and $\mathbf{B}_a^\sigma$,

$$\rho \boldsymbol{v} = \sum_{a,\sigma} \boldsymbol{v}_a^\sigma f_a^\sigma, \quad \rho \mathbf{B} = \sum_{a,\sigma} \mathbf{B}_a^\sigma f_a^\sigma.$$

[Montgomery & Doolen 1987, Chen *et al.* 1991, Martínez *et al.* 1994]

The electric field tensor

$$\boldsymbol{\Lambda} = \sum_{a,\sigma} \boldsymbol{v}_a^\sigma \mathbf{B}_a^\sigma f_a^\sigma,$$

is no longer symmetric because $\boldsymbol{v}_a^\sigma$ and $\mathbf{B}_a^\sigma$ are different vectors.

Recent work of Mendoza & Munoz (2008) in 3D used three sets of velocities related by $\mathbf{B}_a^\sigma = \boldsymbol{v}_a^\sigma \times \mathbf{E}_a^\sigma$.

Also Succi *et al.* (1991) in 2D using a flux function and finite differences.

Vector Boltzmann equation for the magnetic field

Postulate some **vector-valued** distribution functions evolving by (PJD 2002)

$$\partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau_{\mathrm{b}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right).$$

Define the magnetic field by $\mathbf{B} = \sum_i \mathbf{g}_i$, and suppose that $\sum_i \mathbf{g}_i^{(0)} = \mathbf{B}$.

Summing the top equation gives

$$\partial_t \mathbf{B} + \nabla\cdot\boldsymbol{\Lambda} = 0,$$

with an electric field tensor defined by

$$\Lambda_{\alpha\beta} = \sum_i \xi_{i\alpha} g_{i\beta}.$$

$\boldsymbol{\Lambda}$ in turn evolves according to

$$\partial_t \boldsymbol{\Lambda} + \nabla\cdot\mathsf{M} = -\frac{1}{\tau_{\mathrm{b}}}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^{(0)}\right),\ \text{where } M_{\gamma\alpha\beta} = \sum_i \xi_{i\gamma}\xi_{i\alpha} g_{i\beta}.$$

Multiple-scales expansion

By analogy with hydrodynamics, we pose multiple-scales expansions of

$$\mathbf{g}_i = \mathbf{g}_i^{(0)} + \tau_{\mathrm{b}}\mathbf{g}_i^{(1)} + \cdots, \partial_t = \partial_{t_0} + \tau_{\mathrm{b}}\partial_{t_1} + \cdots$$

with the solvability conditions

$$\sum_{i=0}^{N} \mathbf{g}_i^{(n)} = 0 \text{ for } n = 1, 2, \ldots$$

This is equivalent to expanding

$$\mathbf{\Lambda} = \mathbf{\Lambda}^{(0)} + \tau_{\mathrm{b}}\mathbf{\Lambda}^{(1)} + \ldots, \quad \mathsf{M} = \mathsf{M}^{(0)} + \tau_{\mathrm{b}}\mathsf{M}^{(1)} + \cdots$$

while leaving $\mathbf{B}$ unexpanded.

Choosing $\Lambda_{\alpha\beta}^{(0)} = u_\alpha B_\beta - B_\alpha u_\beta$ gives ideal MHD at leading order,

$$\partial_t\mathbf{B} + \nabla\cdot\mathbf{\Lambda}^{(0)} = 0 \quad \Leftrightarrow \quad \partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}).$$

Expansion of the electric field

The equilibria $g_{i\beta}^{(0)} = w_i \left[B_\beta + \theta^{-1} \xi_{i\alpha} \Lambda_{\alpha\beta}^{(0)} \right]$ have the necessary moments,

$$\sum_{i=0}^{N} g_{i\beta}^{(0)} = B_\beta, \quad \sum_{i=0}^{N} \xi_{i\alpha} g_{i\beta}^{(0)} = \Lambda_{\alpha\beta}^{(0)}.$$

The first correction $\mathbf{\Lambda}^{(1)}$ is given by

$$\partial_{t_0} \mathbf{\Lambda}^{(0)} + \nabla \cdot \mathsf{M}^{(0)} = -\mathbf{\Lambda}^{(1)}.$$

The equilibria above give $M_{\gamma\alpha\beta}^{(0)} = \theta \, \delta_{\gamma\alpha} B_\beta$, so ($\theta$ is the lattice constant)

$$\Lambda_{\alpha\beta}^{(1)} = -\theta \partial_\alpha B_\beta + O(\mathrm{Ma}^3).$$

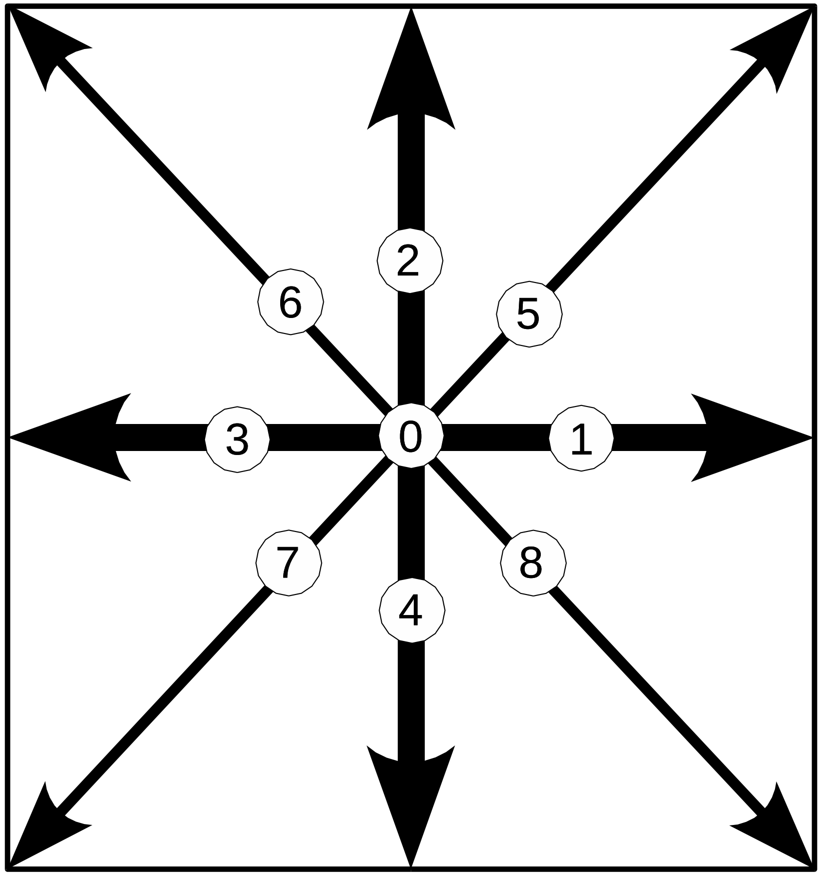
This scheme thus solves the resistive MHD induction equation in the form

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \text{ with } \eta = \theta \tau_{\mathrm{b}}.$$

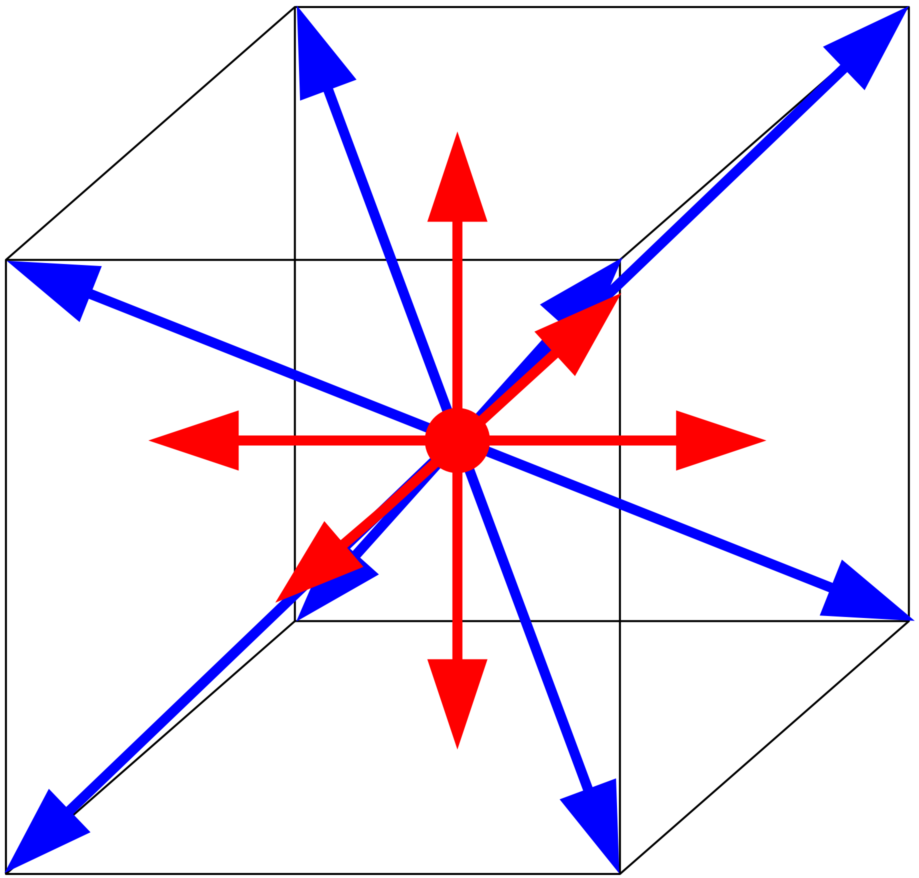
We also have $\mathbf{J} = \nabla \times \mathbf{B}$ available from $\epsilon_{\alpha\beta\gamma} \Lambda_{\alpha\beta}^{(1)} = \theta J_\gamma$.

Lattices for the magnetic distribution functions

Although the $\mathbf{g}_i$ are vectors while the f_i were scalars, we need fewer velocities for the magnetic distribution functions. $\left(\text{We do not need } \sum_i \boldsymbol{\xi}_i \boldsymbol{\xi}_i \boldsymbol{\xi}_i \mathbf{g}_i.\right)$



D2Q9 plus two D2Q5



D3Q15 plus three D3Q7

Picture from M. J. Pattison *et al.* (2008) Fusion Eng. & Design **83** 557–572

M.J. Pattison et al. / Fusion Engineering and Design 83 (2008) 557–572

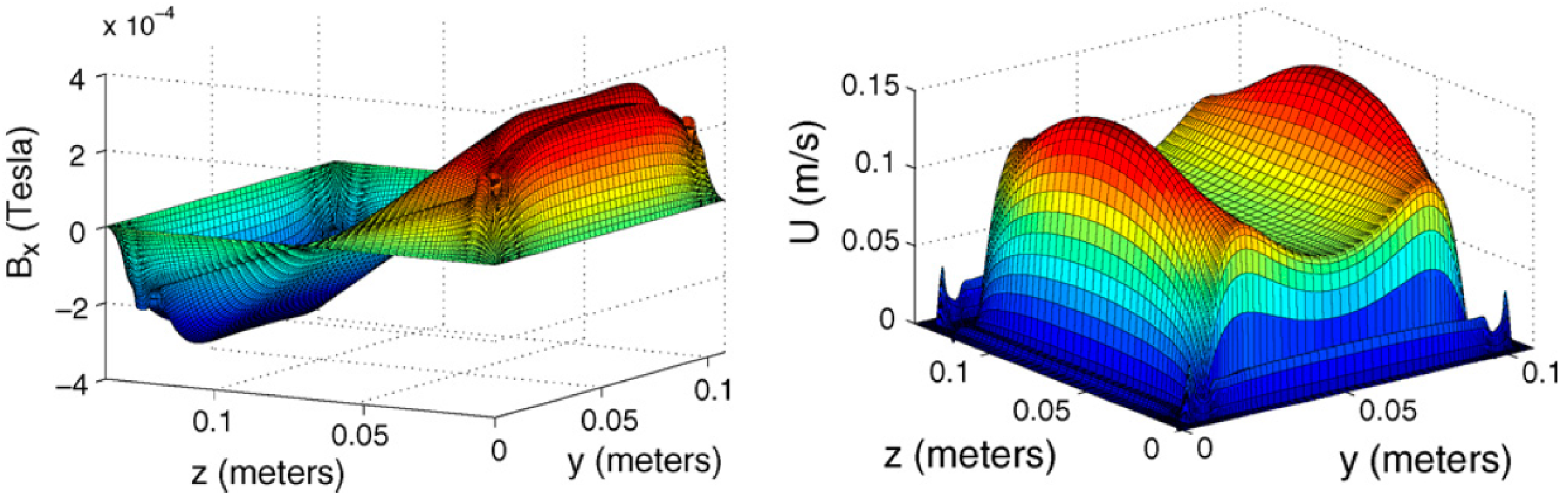
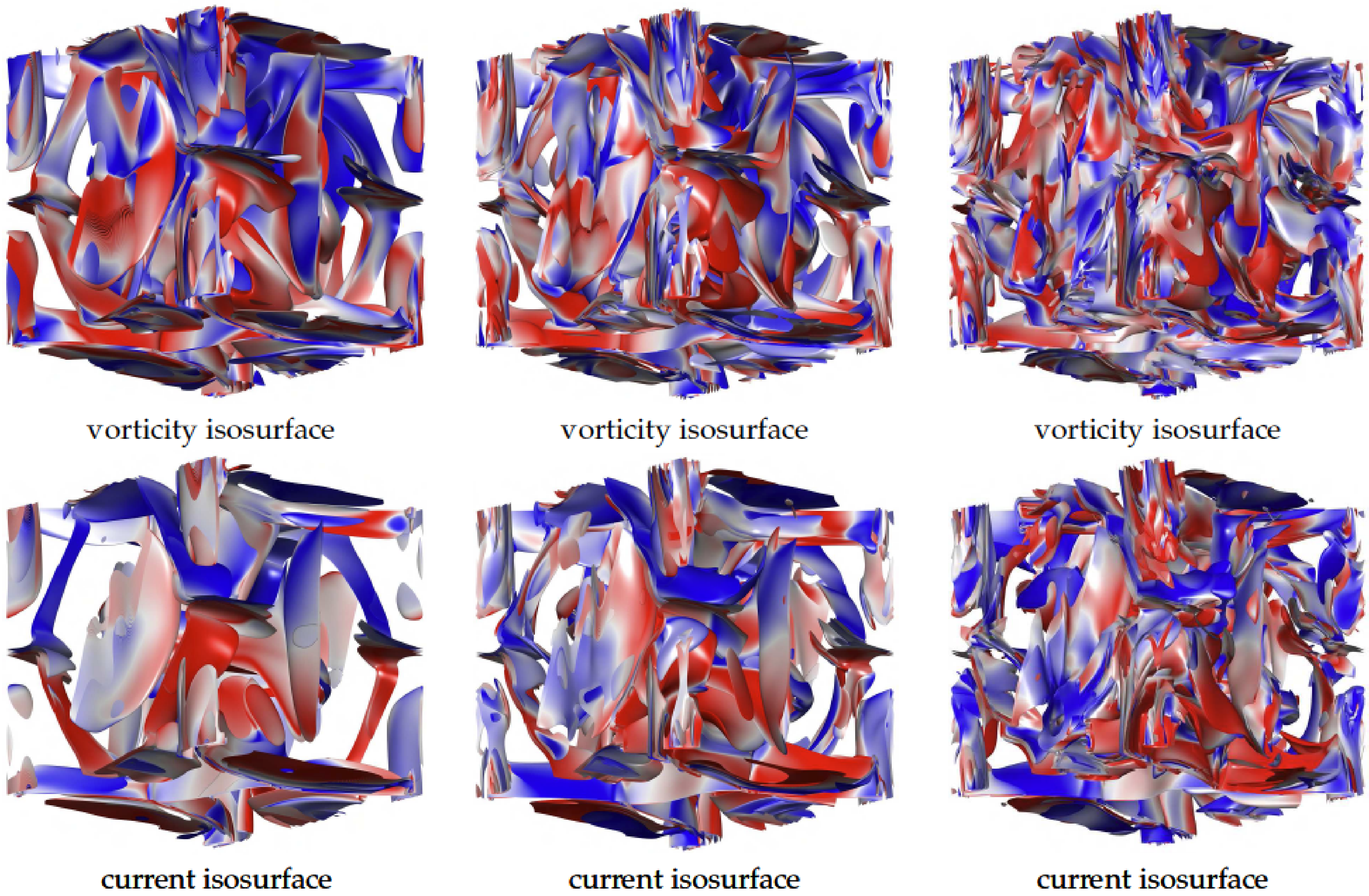


Fig. 19. Induced field (B_x) and velocity (U) plots on a cross-section of a thermal blanket module at $Ha = 100$.

Picture from G. Vahala *et al.* (2008) Commun. Comput. Phys. **4** 624–646



1800^3 simulation run on an SGI Altix with 9000 cores

Matrix (MRT) collision operators in magnetohydrodynamics

We may improve numerical stability by setting the "ghost" degrees of freedom other than ρ, $\rho\mathbf{u}$ and $\mathbf{\Pi}$ to equilibrium at every timestep. We relax the momentum flux towards its equilbrium value $\mathbf{\Pi}^{(0)}$,

$$\mathbf{\Pi}' = \mathbf{\Pi} - \frac{\Delta t}{\tau + \frac{1}{2}\Delta t}\left(\mathbf{\Pi} - \mathbf{\Pi}^{(0)}\right),$$

then reconstruct the post-collision distribution functions f_i' from ρ, $\mathbf{u}$ and $\mathbf{\Pi}'$

$$f_i' = w_i\left[\rho\left(2 - \frac{3}{2}|\boldsymbol{\xi}_i|^2\right) + 3\left(\rho\mathbf{u}\right)\cdot\boldsymbol{\xi}_i + \frac{9}{2}\mathbf{\Pi}':\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \frac{3}{2}\mathrm{Tr}\,\mathbf{\Pi}'\right].$$

Collisions conserve ρ and $\mathbf{u}$ so there are no tildes on these variables. Finally, we stream the post-collision distribution functions by setting

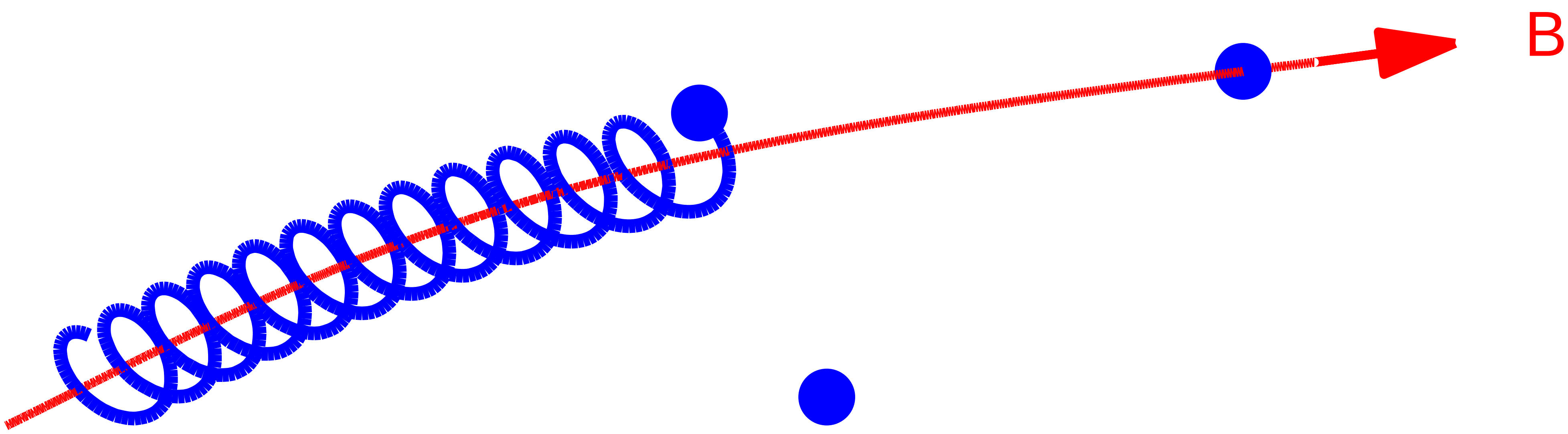
$$\overline{f}_i(\mathbf{x} + \boldsymbol{\xi}_i\Delta t, t + \Delta t) = f_i'(\mathbf{x}, t).$$

The magnetic field only enters through the definition of $\mathbf{\Pi}^{(0)}$, everything else is as it would be in pure hydrodynamics. [cf Pattison *et al.* 2008, Riley *et al.* 2008].

Braginskii magnetohydrodynamics

Braginskii magnetohydrodynamics

In a strongly magnetised plasmas the particles are tied to magnetic field lines.



The effective mean free path perpendicular to field lines is the gyroradius.

Mixing length theory gives a viscous stress aligned with the magnetic field,

$$\mathbf{\Pi}_{\text{visc}} \approx -2\mu_{\|}\,\hat{\mathbf{b}}\hat{\mathbf{b}}\,\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u},$$

where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$. Derived from kinetic theory by Braginskii (1965).

A simple model – parallel and perpendicular viscosities

Regularise Braginskii's (1965) leading order theory with a perpendicular viscosity $\mu_\perp \ll \mu_\parallel$. Write the stress as

$$\mathbf{\Pi}_{\mathsf{visc}} = -(\mu_\parallel - \mu_\perp)\hat{\mathbf{b}}\hat{\mathbf{b}}\,\hat{\mathbf{b}}\hat{\mathbf{b}} : \mathsf{S} - \mu_\perp \mathsf{S},$$

where $\mathsf{S} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}$.

In axes with the first axis aligned with the direction $\hat{\mathbf{b}}$,

$$\mathbf{\Pi}_{\mathsf{visc}} = -\begin{pmatrix} \mu_\parallel & & & \\ & \mu_\perp & & \\ & & \ddots & \\ & & & \mu_\perp \end{pmatrix} \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}\right).$$

Similar to liquid crystals, except $\mu_\perp \ll \mu_\parallel$ instead of $\mu_\perp \sim \mu_\parallel$.

Implementation

Implement by applying a larger relaxation time to $\hat{\mathbf{b}}\hat{\mathbf{b}} : \mathbf{\Pi}$ than to the rest of $\mathbf{\Pi}$.

Evaluate $\mathbf{\Pi}^{(0)}$ from the formula

$$\mathbf{\Pi}^{(0)} = (\theta\rho + \tfrac{1}{2}B^2)\mathbf{I} + \rho\mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B}$$

Construct the post-collision stress $\mathbf{\Pi}'$ using

$$\begin{aligned}\mathbf{\Pi}' = \mathbf{\Pi} \; - \; &\left(\mathbf{\Pi} - \mathbf{\Pi}^{(0)}\right)\frac{\Delta t}{\tau_\perp + \frac{1}{2}\Delta t} \\ &- \; \hat{\mathbf{b}}\hat{\mathbf{b}} \, (\mathbf{\Pi} : \hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{\Pi}^{(0)} : \hat{\mathbf{b}}\hat{\mathbf{b}}) \left(\frac{\Delta t}{\tau_\parallel + \frac{1}{2}\Delta t} - \frac{\Delta t}{\tau_\perp + \frac{1}{2}\Delta t}\right)\end{aligned}$$

where $\tau_\parallel = \theta^{-1}\mu_\parallel$ and $\tau_\perp = \theta^{-1}\mu_\perp$.

Reconstruct the post-collision distribution functions from the moments using

$$f_i' = w_i \left[\rho\left(2 - \tfrac{3}{2}|\boldsymbol{\xi}_i|^2\right) + 3\rho\mathbf{u}\cdot\boldsymbol{\xi}_i + \tfrac{9}{2}\,\mathbf{\Pi}' : \boldsymbol{\xi}_i\boldsymbol{\xi}_i - \tfrac{3}{2}\,\mathrm{Tr}\,\mathbf{\Pi}'\right].$$

Stream as usual.

Code fragment

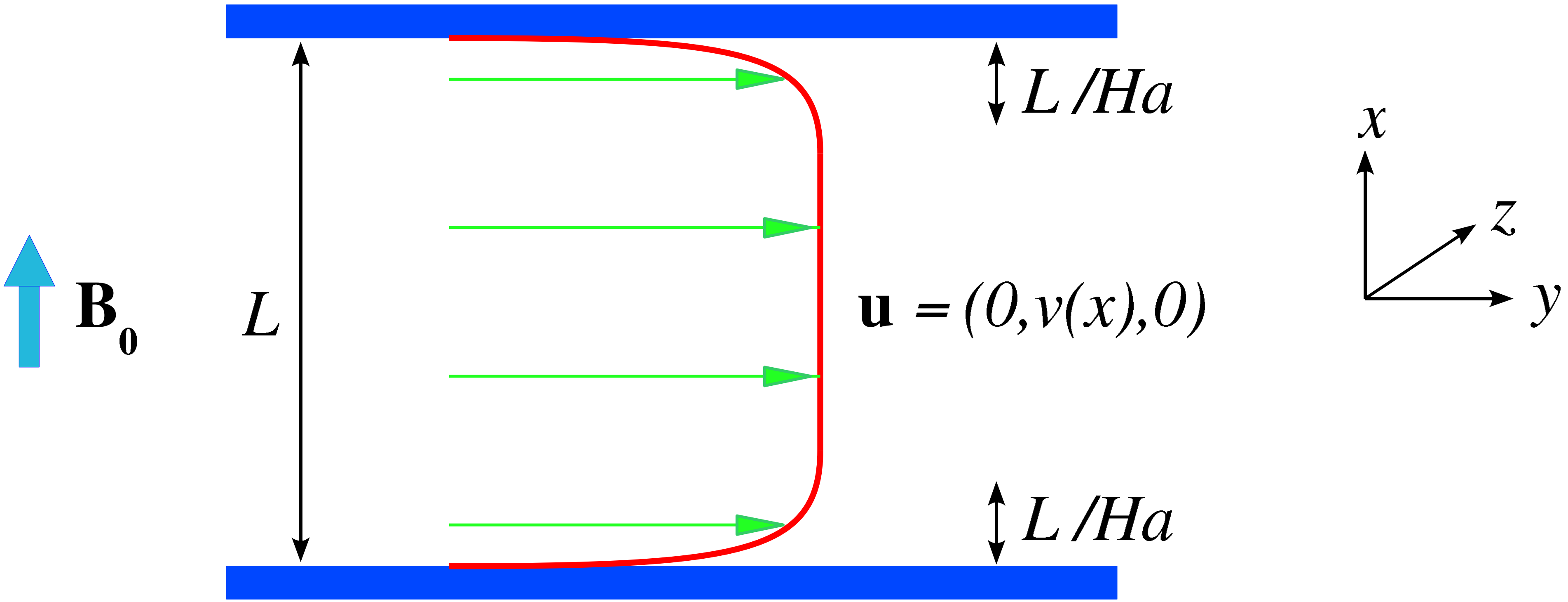
```
P0xx = (1/3)*rho+0.5*bsq + rho*ux*ux-bx*bx
P0xy = rho*ux*uy-bx*by
P0yy = (1/3)*rho+0.5*bsq + rho*uy*uy-by*by

Pbb = (bxhat**2*Pxx+2*bxhat*byhat*Pxy+byhat**2*Pyy)
P0bb = (bxhat**2*P0xx+2*bxhat*byhat*P0xy+byhat**2*P0yy)
Pbbt = - (Pbb-P0bb)*(1/(ptau+0.5) - 1/(tau+0.5))

Pxx = Pxx - (Pxx-P0xx)/(tau+0.5) + bxhat**2*Pbbt
Pxy = Pxy - (Pxy-P0xy)/(tau+0.5) + bxhat*byhat*Pbbt
Pyy = Pyy - (Pyy-P0yy)/(tau+0.5) + byhat**2*Pbbt

do k=0,8
  f(k,i,j)=w(k)*(2*rho-(3/2)*rho*(cx(k)**2+cy(k)**2)
    +3*rho*(ux*cx(k)+uy*cy(k)) -(3/2)*(Pxx+Pyy)
    +(9/2)*(Pxx*cx(k)**2+2*Pxy*cx(k)*cy(k)+Pyy*cy(k)**2))
enddo
```

Hartmann flow / planar channel flow

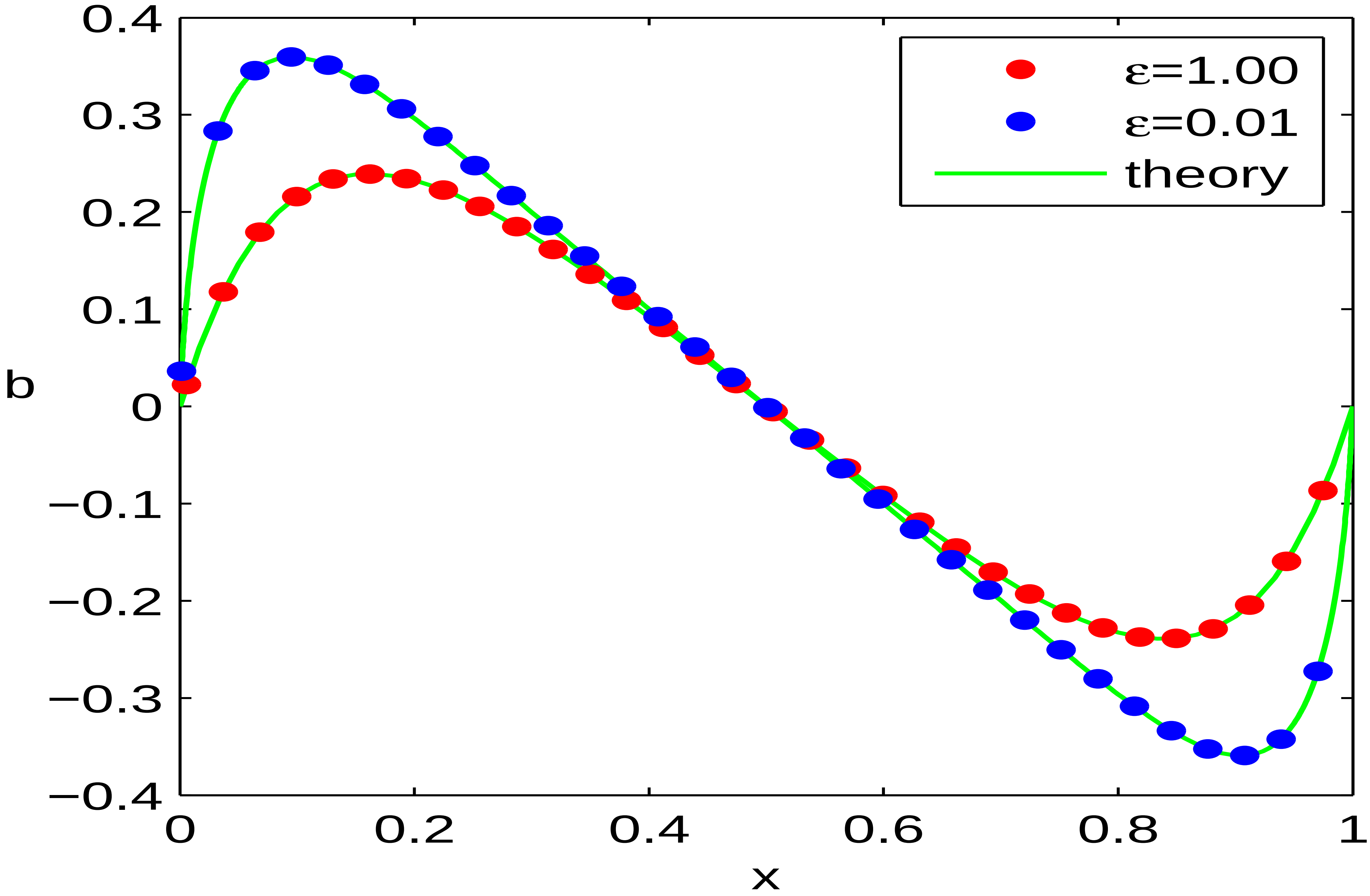


Planar fields $\mathbf{u} = U_0(0, v(x), 0)$ and $\mathbf{B} = B_0(1, b(x), 0)$.

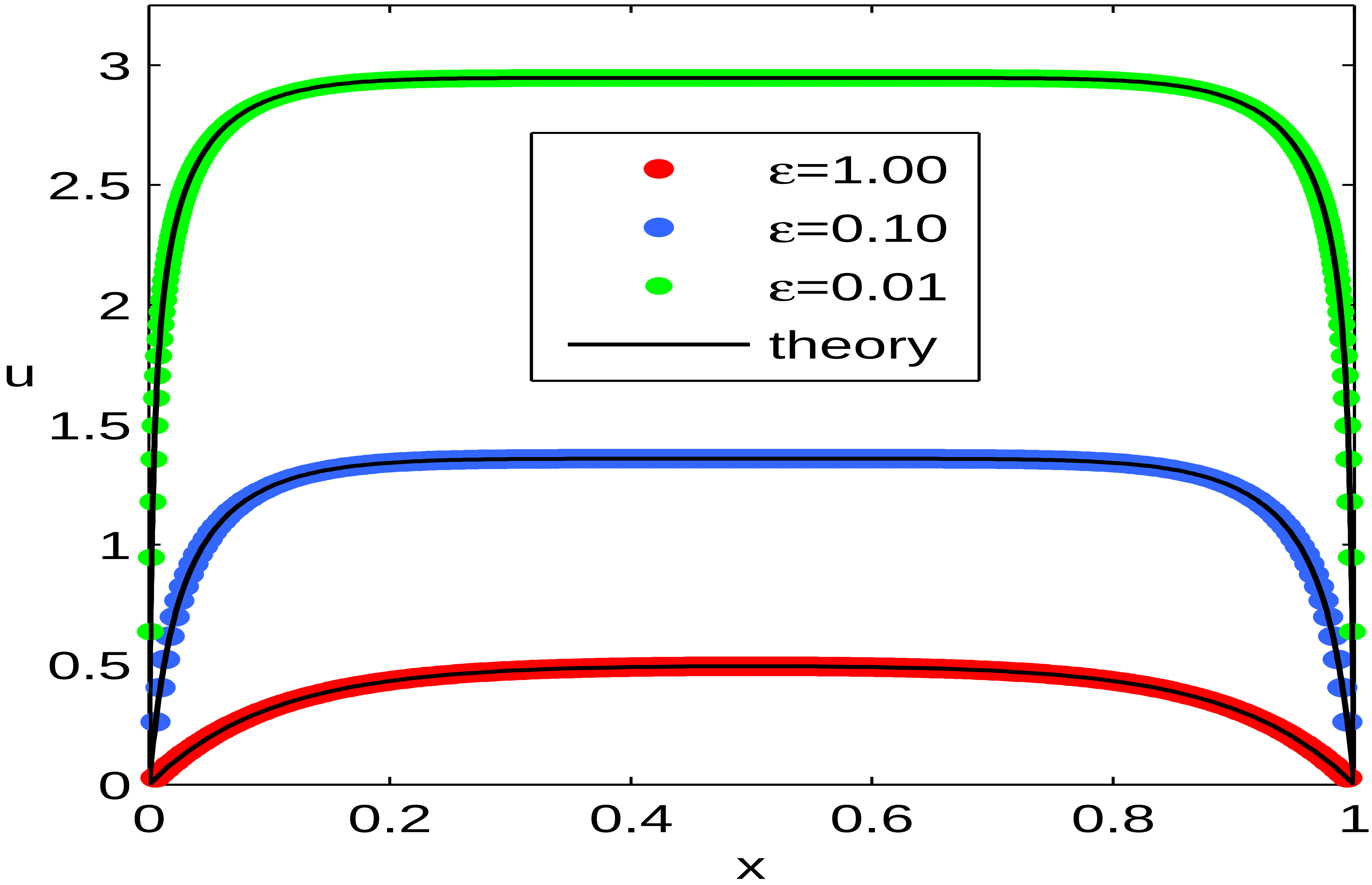
Magnetic field direction is $\hat{\mathbf{b}} = \dfrac{(1, b, 0)}{\sqrt{1+b^2}}$. Viscosity ratio $\epsilon = \left(\mu_\perp/\mu_\|\right)^{1/2}$.

Only nonzero components of S are $S_{xy} = S_{yx} = U_0 \dfrac{du}{dx}$.

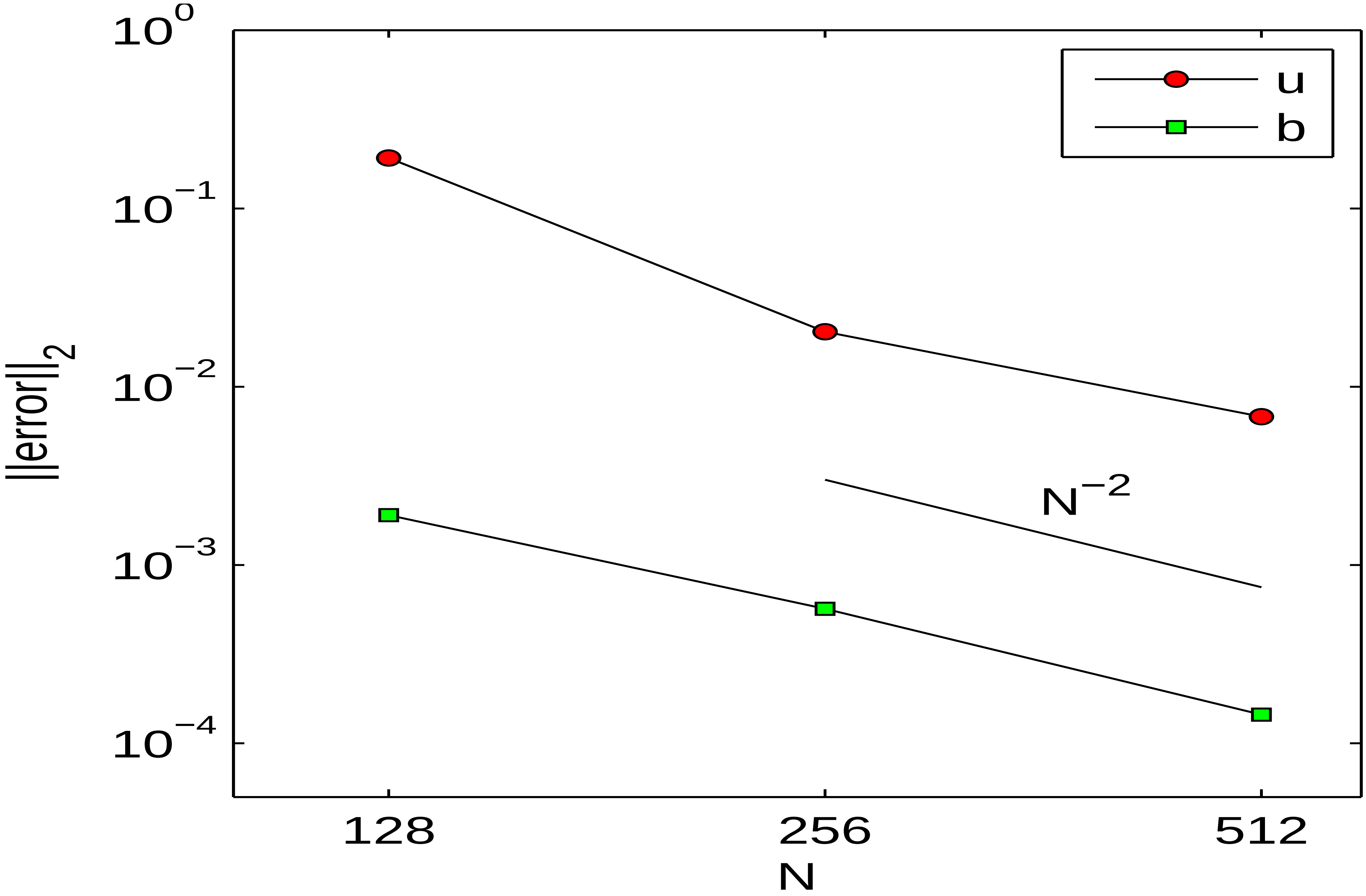
Hartmann flow with Braginskii's anisotropic viscosity – magnetic field



Hartmann flow with Braginskii's anisotropic viscosity – velocity



Convergence under grid refinement ($\epsilon = 0.1$)



Reference solutions from the TWPBVPC ODE solver [Cash & Mazzia 2005]

Current-dependent resistivity

Current-dependent resistivity

Slightly extended Ohm's law, $\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta(|\mathbf{J}|)\mathbf{J}$ [*eg* Otto 2001 JGR]

The resistivity $\eta(|\mathbf{J}|)$ is allowed to depend on the current $|\mathbf{J}|$.

[The viscosity in a generalised Newtonian fluid depends on the strain rate.]

Seems easy to implement – make τ depend on $|\mathbf{J}|$ obtained from $\mathbf{\Lambda}^{(1)}$.

[As in Aharonov & Rothman (1993), Hou *et al.* (1996), many others . . .]

This does not work. What we are really simulating is

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\cdot(\eta\nabla\mathbf{B}),$$

instead of

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}).$$

We get a spurious $\nabla\eta$ term, which vanished before when η = cst.

[Another discrepancy term always vanishes because $\nabla\cdot\mathbf{B} = 0$.]

Specifying the collision operator using moments

Earlier, we postulated

$$\partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau_{\mathrm{b}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right)$$

and took moments to obtain equations like

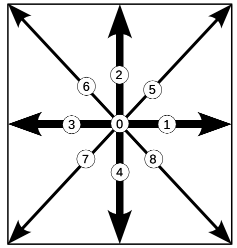
$$\partial_t \mathbf{B} + \nabla\cdot\boldsymbol{\Lambda} = 0, \quad \partial_t \boldsymbol{\Lambda} + \nabla\cdot\mathsf{M} = -\frac{1}{\tau_{\mathrm{b}}}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^{(0)}\right).$$

Perhaps we could do better with a more general (non-BGK) collision operator.

We can specify a collision operator by its action on a basis of moments.

First we need a basis of moments . . .

Moments of the D2Q5 scalar lattice



Lattice with $\boldsymbol{\xi}_0 = 0$, $\boldsymbol{\xi}_{1,3} = \pm\hat{\boldsymbol{x}}$, $\boldsymbol{\xi}_{2,4} = \pm\hat{\boldsymbol{y}}$.

The first five scalar moments are given by

$$\rho = \sum\nolimits_{i=0}^{4} f_i, \quad m_x = \sum\nolimits_{i=0}^{4} \xi_{ix} f_i, \quad m_y = \sum\nolimits_{i=0}^{4} \xi_{iy} f_i,$$

$$\Pi_{xx} = \sum\nolimits_{i=0}^{4} \xi_{ix}\xi_{ix} f_i, \quad \Pi_{yy} = \sum\nolimits_{i=0}^{4} \xi_{iy}\xi_{iy} f_i.$$

Π_{xy} is identically zero, because $\xi_{ix}\xi_{iy} = 0$ for every velocity in the lattice.

$$\begin{pmatrix} \rho \\ m_x \\ m_y \\ \Pi_{xx} \\ \Pi_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}.$$

This 5×5 matrix has full rank, so the above five moments form a basis.

Reconstructing the f_i from the moments

The f_i may be reconstructed from the moments $\rho, m_x, m_y, \mathbf{\Pi}_{xx}, \mathbf{\Pi}_{yy}$ by inverting the previous 5×5 matrix,

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \rho \\ m_x \\ m_y \\ \Pi_{xx} \\ \Pi_{yy} \end{pmatrix}.$$

Treating f_0 as a special case, this reconstruction may be written compactly as

$$f_i = \frac{1}{2}\left(\boldsymbol{\xi}_i \cdot \mathbf{m} + \boldsymbol{\xi}_i\boldsymbol{\xi}_i : \mathbf{\Pi}\right) \text{ for } i \neq 0, \quad f_0 = \rho - \left(\Pi_{xx} + \Pi_{yy}\right).$$

No choice of weights makes the moments $\rho, \mathbf{\Pi}_{xx}, \mathbf{\Pi}_{yy}$ mutually orthogonal.

Moment basis for D2Q5 MHD

Two components of the magnetic field,

$$B_x = \sum_{i=0}^{4} g_{ix}, \quad \text{and} \quad B_y = \sum_{i=0}^{4} g_{iy},$$

and four components of the electric field tensor,

$$\Lambda_{xx}, \Lambda_{xy}, \Lambda_{yx}, \Lambda_{yy}, \quad \text{where} \quad \Lambda_{\alpha\beta} = \sum_{i=0}^{4} \xi_{i\alpha} g_{i\beta}.$$

The evolution equation for $\mathbf{\Lambda}$,

$$\partial_t \mathbf{\Lambda} + \nabla \cdot \mathsf{M} = -\frac{1}{\tau}\left(\mathbf{\Lambda} - \mathbf{\Lambda}^{(0)}\right),$$

involves the third rank tensor

$$M_{\gamma\alpha\beta} = \sum_{i=0}^{4} \xi_{i\gamma} \xi_{i\alpha} g_{i\beta},$$

but $\xi_{i\alpha}\xi_{i\gamma} = 0$ when $\alpha \neq \gamma$.

This only leaves $M_{xxx}, M_{xxy}, M_{yyx}, M_{yyy}$ not identically zero.

Evolution of the higher moments

For any lattice,

$$\partial_t M_{\gamma\alpha\beta} + \partial_\mu N_{\mu\gamma\alpha\beta} = -\frac{1}{\tau}\left(M_{\gamma\alpha\beta} - M^{(0)}_{\gamma\alpha\beta}\right),$$

The fourth rank tensor N has components

$$N_{\mu\gamma\alpha\beta} = \sum\nolimits_{i=0}^{N} \xi_{i\mu}\xi_{i\gamma}\xi_{i\alpha}g_{i\beta}.$$

Specialising to the D2Q5 lattice, $\xi_{i\mu}\xi_{i\gamma}\xi_{i\alpha} = 0$ unless $\mu = \gamma = \alpha$.

$$N_{xxxx} = \sum\nolimits_{i=0}^{4} \xi_{ix}g_{ix} = \Lambda_{xx}, \quad N_{xxxy} = \sum\nolimits_{i=0}^{4} \xi_{ix}g_{iy} = \Lambda_{xy},$$
$$N_{yyyx} = \sum\nolimits_{i=0}^{4} \xi_{iy}g_{ix} = \Lambda_{yx}, \quad N_{yyyy} = \sum\nolimits_{i=0}^{4} \xi_{iy}g_{iy} = \Lambda_{yy},$$

with all other components vanishing.

We therefore have (no implied summation on α)

$$\partial_t M_{\alpha\alpha\beta} + \partial_\alpha \Lambda_{\alpha\beta} = -\frac{1}{\tau}\left(M_{\alpha\alpha\beta} - M^{(0)}_{\alpha\alpha\beta}\right),$$

which makes a closed system for $\mathbf{B}$, $\mathbf{\Lambda}$, and M. (Similarly in 3D.)

Specifying the collision operator using moments

We postulated

$$\partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau_{\mathrm{b}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right)$$

and took moments to obtain equations like

$$\partial_t \mathbf{B} + \nabla\cdot\boldsymbol{\Lambda} = 0, \quad \partial_t \boldsymbol{\Lambda} + \nabla\cdot\mathsf{M} = -\frac{1}{\tau_{\mathrm{b}}}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^{(0)}\right).$$

Instead, we can specify the collision operator by its action on the moments.

In 2D, a basis of moments was given by

$$B_x, B_y, \quad \Lambda_{xx}, \Lambda_{xy}, \Lambda_{yx}, \Lambda_{yy}, \quad M_{xxx}, M_{xxy}, M_{yyx}, M_{yyy}.$$

[four components vanish]

The $g_{i\beta}$ can be reconstructed using

$$g_{i\beta} = \tfrac{1}{2}\left(\xi_{i\alpha}\Lambda_{\alpha\beta} + \xi_{i\gamma}\xi_{i\alpha}M_{\gamma\alpha\beta}\right) \text{ for } i \neq 0,$$

$$g_{0\beta} = B_\beta - \left(M_{xx\beta} + M_{yy\beta}\right).$$

[Like a Gross–Jackson collision operator in continuum kinetic theory.]

Decomposition of the Λ tensor

We also need to decompose $\mathbf{\Lambda}$. We started with $\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} E_\gamma$.

This made $\partial_t \mathbf{B} = -\nabla\times\mathbf{E}$ become $\partial_t \mathbf{B} + \nabla\cdot\mathbf{\Lambda} = 0$.

In two dimensional MHD we expect $\begin{pmatrix} \Lambda_{xx} & \Lambda_{xy} \\ \Lambda_{yx} & \Lambda_{yy} \end{pmatrix} = \begin{pmatrix} 0 & -E_z \\ E_z & 0 \end{pmatrix}$.

$\Lambda^{(0)}_{\alpha\beta} = u_\alpha B_\beta - u_\beta B_\alpha$ is antisymmetric, consistent with the above.

However, $\Lambda^{(1)}_{\alpha\beta} = -\theta\tau\partial_\alpha B_\beta$ is not antisymmetric [cannot be made so].

We must treat $\mathbf{\Lambda}$ as a general rank-2 tensor, and decompose it into

$$\mathbf{\Lambda} = \text{antisymmetric} + \text{isotropic} + \text{symmetric traceless}$$

Isotropic part: $\mathrm{Tr}(\mathbf{\Lambda}^{(0)} + \mathbf{\Lambda}^{(1)}) = -\theta\tau\nabla\cdot\mathbf{B} \approx O(10^{-15})$ (round-off error)

Antisymmetric part: electric field

Symmetric traceless part?

Colliding the electric field

For resistivity $\tau + \lambda$ we collide the antisymmetric part of $\mathbf{\Lambda}$ according to

$$E'_z = -\frac{\Delta t}{\tau + 2\lambda + \frac{1}{2}\Delta t}\left(E_z - E_z^{(0)}\right)$$

(E'_z is a post-collision value) and the symmetric part $\widetilde{\mathbf{\Lambda}}$ according to

$$\widetilde{\mathbf{\Lambda}}' = -\frac{\Delta t}{\tau + \frac{1}{2}\Delta t}\left(\widetilde{\mathbf{\Lambda}} - \widetilde{\mathbf{\Lambda}}^{(0)}\right).$$

The post collision $\mathbf{\Lambda}$ tensor is thus

$$\Lambda'_{\alpha\beta} = \tilde{\Lambda}'_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}E'_\gamma$$

Finally, we reconstruct the $\mathbf{g}_i$ from $\mathbf{\Lambda}'$, M$'$, and $\mathbf{B}$ using

$$g_{i\beta} = \frac{1}{2}\left(\xi_{i\alpha}\Lambda'_{\alpha\beta} + \xi_{i\gamma}\xi_{i\alpha}M'_{\gamma\alpha\beta}\right) \text{ for } i \neq 0,$$

$$g_{0\beta} = B_\beta - \left(M'_{xx\beta} + M'_{yy\beta} + M'_{zz\beta}\right).$$

Current-dependent resistivity

We want to make the extra resistivity in $\tau + \lambda$ depend on the current $\mathbf{J}$.

We have $\mathbf{J} = \nabla\times\mathbf{B}$ from $J_\gamma = -\theta^{-1}\epsilon_{\gamma\alpha\beta}\Lambda^{(1)}_{\alpha\beta}$

In the $\mathbf{E}$ notation,

$$\mathbf{J} = \frac{1}{\theta}\mathbf{E}^{(1)} = \frac{1}{\theta}\,\frac{\Delta t}{\tau + 2\lambda + \frac{1}{2}\Delta t}\,\left(\mathbf{E} - \mathbf{E}^{(0)}\right).$$

We want λ to be a function of $\mathbf{J}$, so we solve (in 2D)

$$J_z\left(\tau + 2\lambda(J_z) + \frac{1}{2}\Delta t\right) = \frac{\Delta t}{\theta}\,\left(E_z - E_z^{(0)}\right)$$

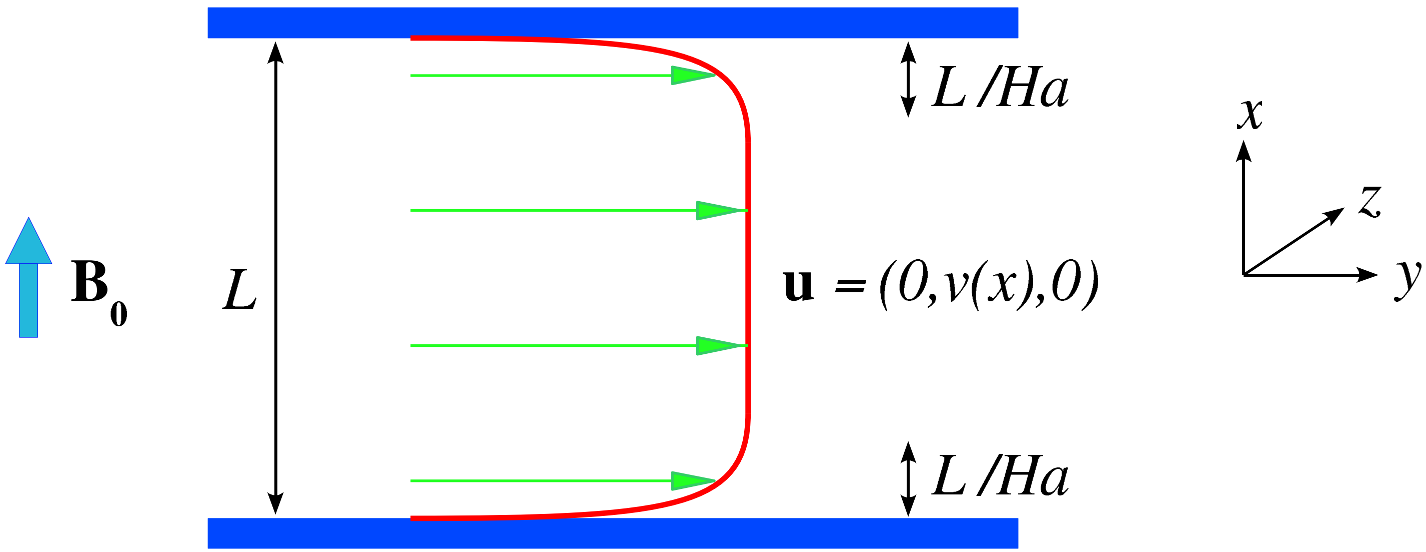
for J_z by Newton's method, then evaluate $\lambda(J_z)$. [in 3D need $|\mathbf{J}|$]

This simulates the induction equation in the correct form

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\cdot(\eta_0\nabla\mathbf{B}) - \nabla\times((\eta - \eta_0)\nabla\times\mathbf{B}).$$

Hartmann flow with current-dependent resistivity

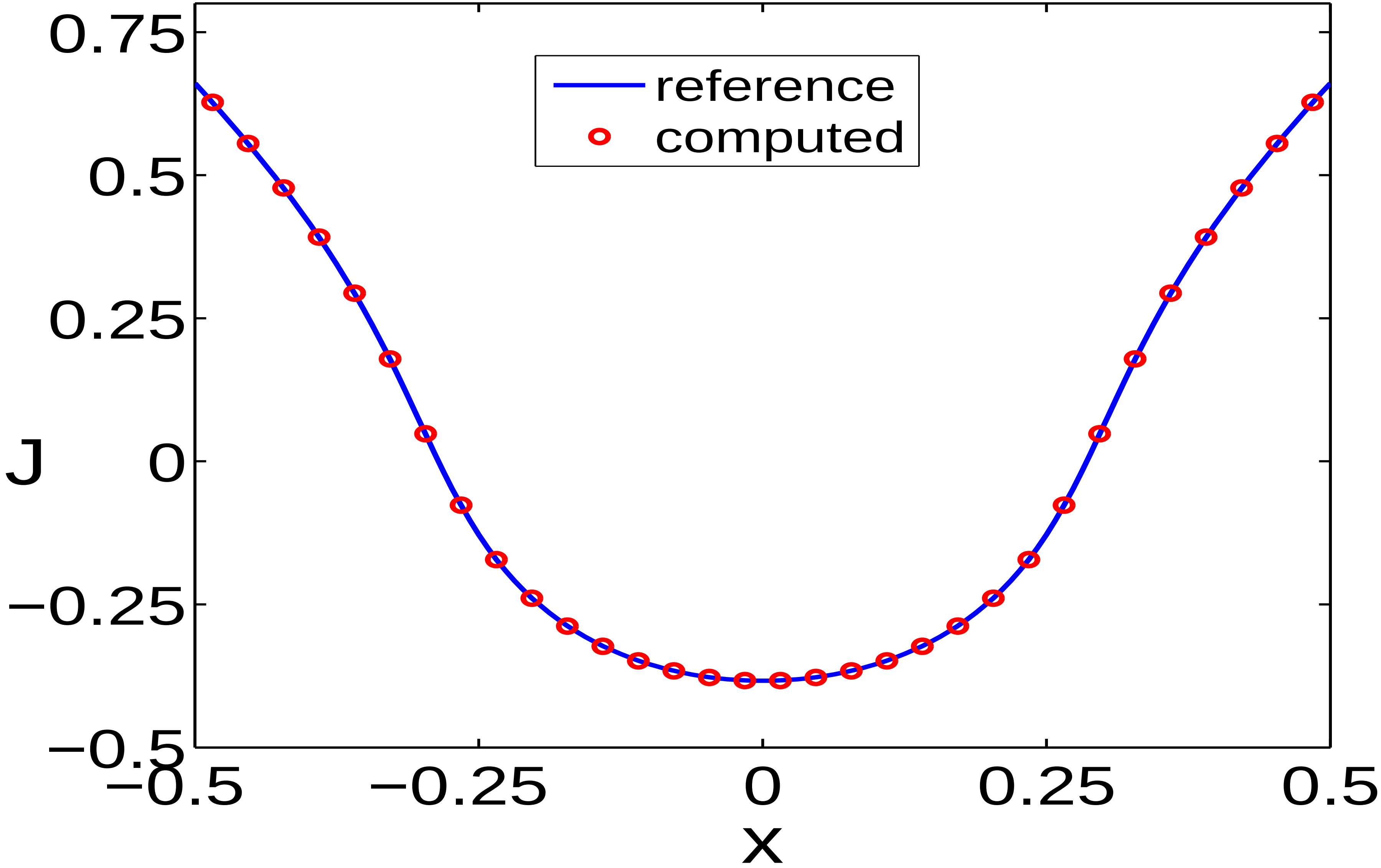
MHD analog of Poiseuille flow in a channel spanned by a magnetic field.



$\mathbf{u} = (0, v(x), 0)$ and $\mathbf{B} = (B_0, b(x), 0)$ $\qquad Ha = B_0 L/(\rho_0 \nu \eta_0)^{1/2}$

Momentum $\quad 0 = F + \rho_0 \nu \dfrac{d^2 v}{dx^2} + B_0 \dfrac{db}{dx}, \qquad |\mathbf{J}| = \left|\dfrac{db}{dx}\right|$

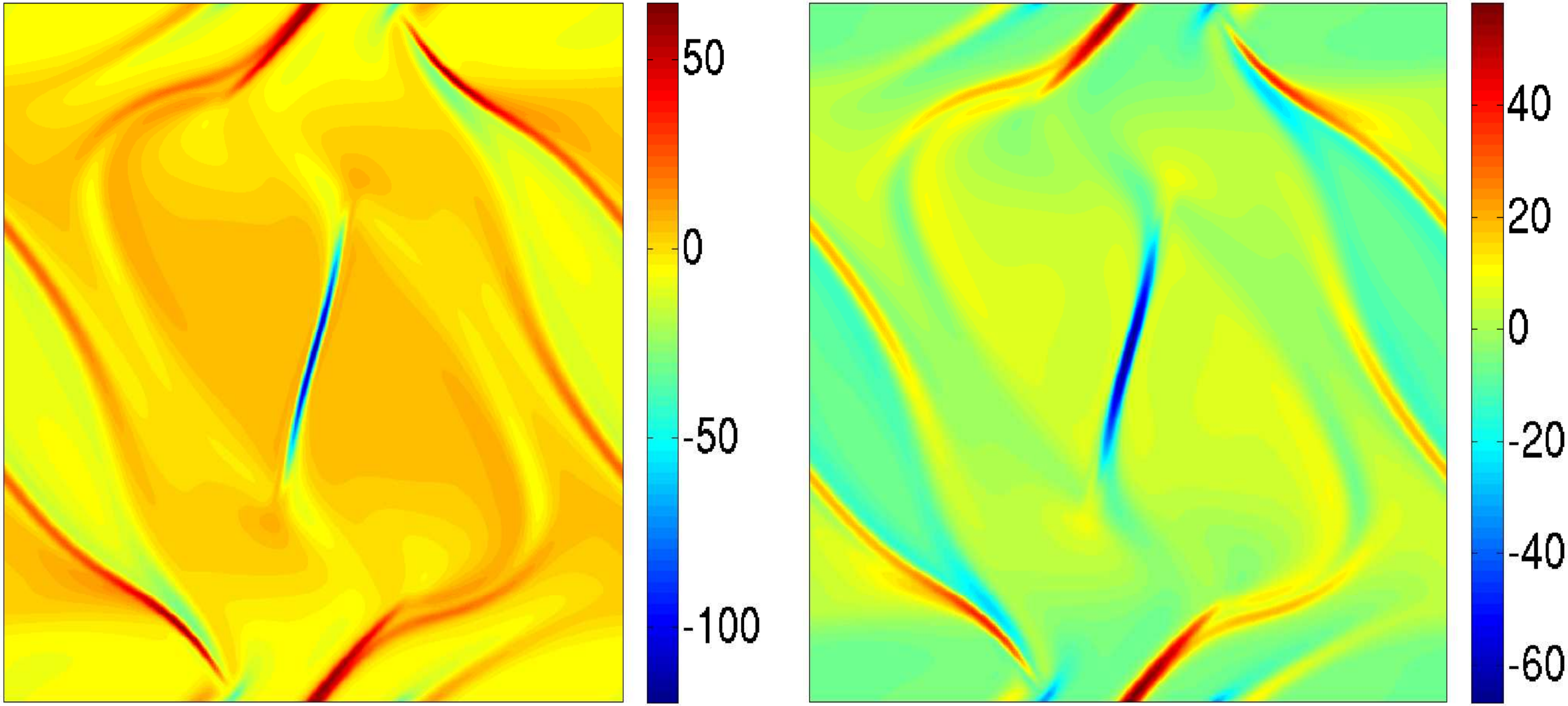
Induction $\quad 0 = B_0 \dfrac{dv}{dx} + \dfrac{d}{dx}\left[\eta\left(\dfrac{db}{dx}\right)\dfrac{db}{dx}\right].$



Resistivity $\eta(J) = \eta_0 \left(1 + |J/J_0|^n\right)$ with $n = 2$ and $J_0 = 0.6$.

$F = 1$, $\nu = 0.1$, $\rho = 1$, $\eta_0 = 0.5$, $B_0 = 1$, $m_x = 32$, $Ma = \sqrt{3}/100$.

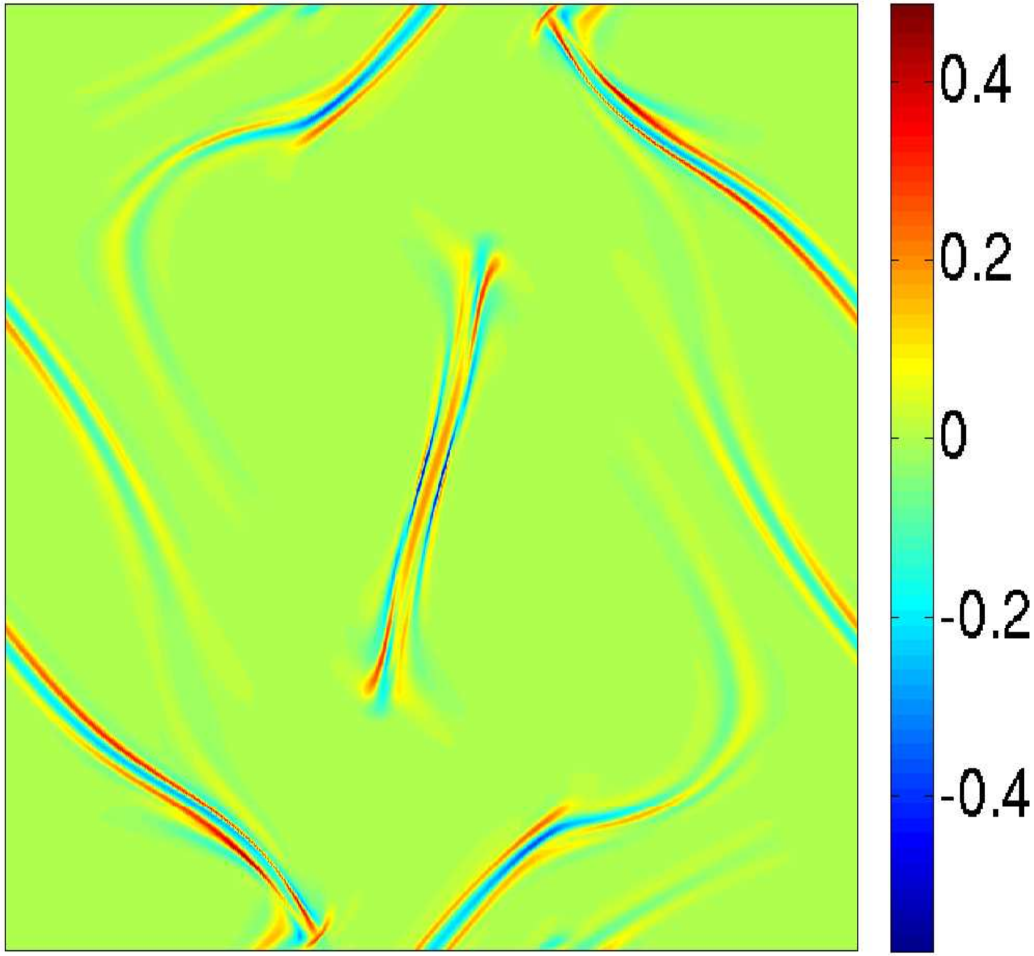
Orszag–Tang vortex without/with current-dependent resistivity



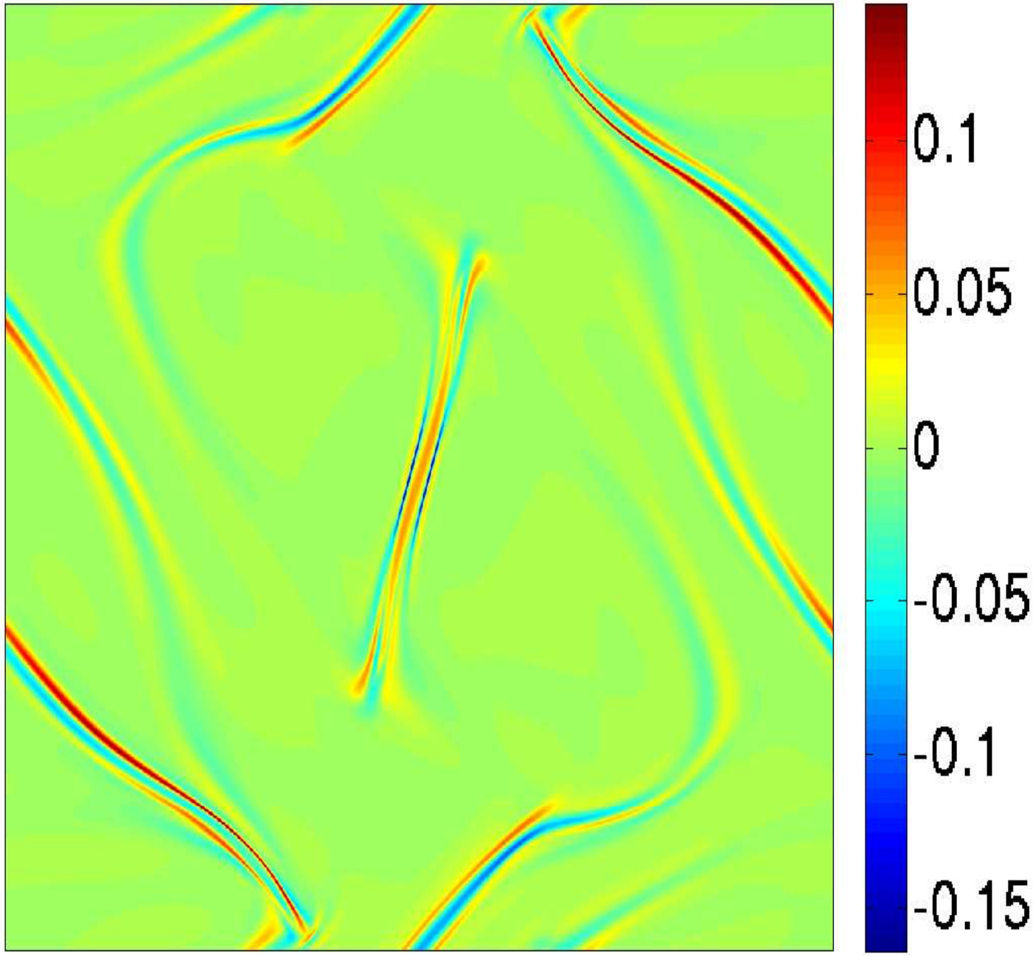
$\eta_0 = 0.0016$

$\eta(J) = \eta_0\,(1 + |J/J_0|^n)$
with $n = 2$ and $J_0 = 50$

Convergence to spectral solutions: errors in current



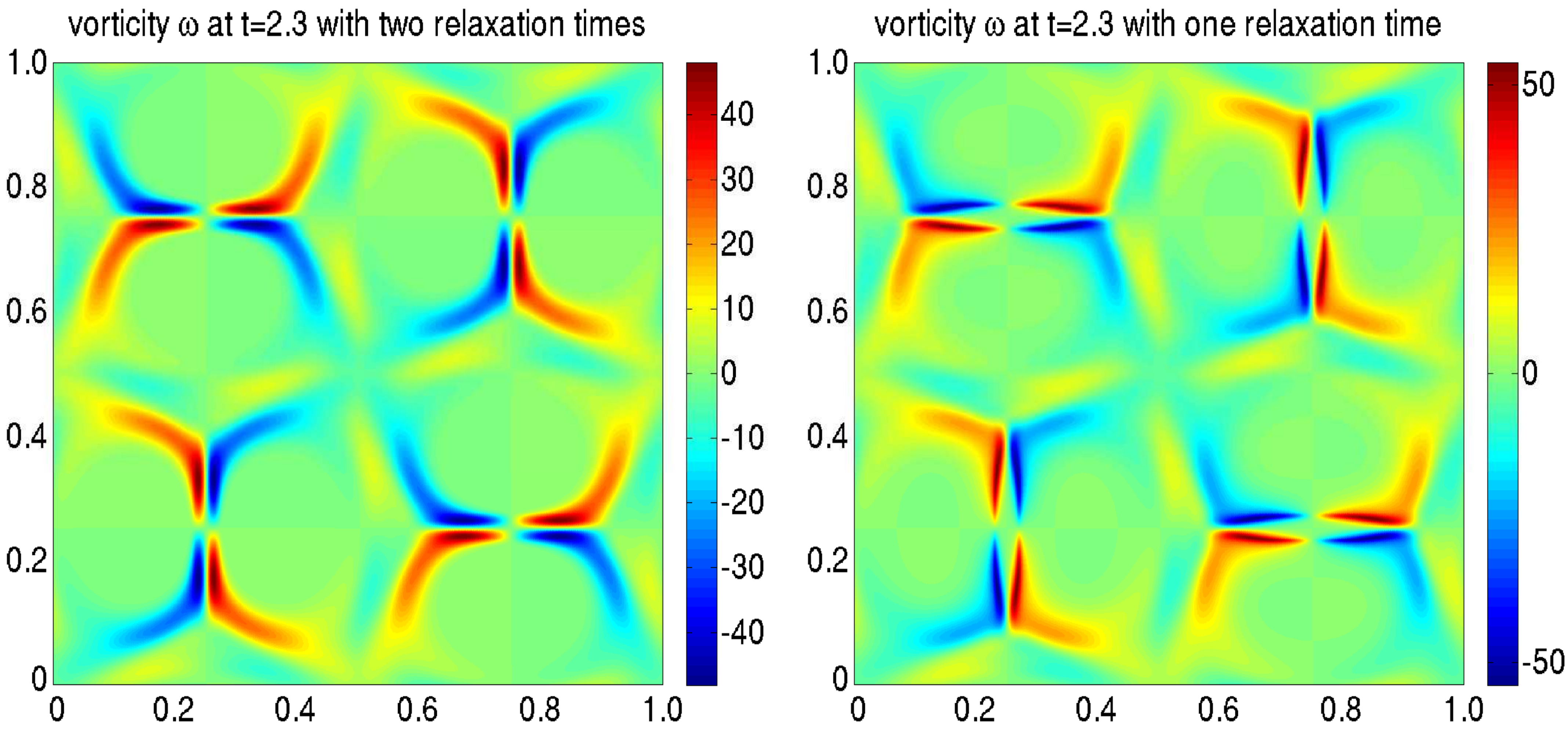
512×512



1024×1024

Roughly second order convergence

Reconnection of magnetic islands



correct treatment of $\Lambda - \Lambda^T$

single relaxation time for Λ

Electromagnetism

Recovering the full set of Maxwell equations

From the vector Boltzmann equation we found that $\mathbf{\Lambda}$ evolves according to

$$\partial_t \Lambda_{\alpha\beta} + \partial_\gamma M_{\gamma\alpha\beta} = -\frac{1}{\tau_E}\left(\Lambda_{\alpha\beta} - \Lambda^{(0)}_{\alpha\beta}\right).$$

The electric field given by $E_\gamma = -\frac{1}{2}\epsilon_{\gamma\alpha\beta}\Lambda_{\alpha\beta}$ thus evolves according to

$$\partial_t E_\gamma - \tfrac{1}{2}\epsilon_{\gamma\alpha\beta}\partial_\mu M_{\mu\alpha\beta} = -\frac{1}{\tau_E}\left(E_\gamma - E^{(0)}_\gamma\right).$$

We choose the collision operator so that

$$\partial_t M_{\alpha\alpha\beta} + \partial_\alpha \Lambda_{\alpha\beta} = -\frac{1}{\tau_M}\left(M_{\alpha\alpha\beta} - M^{(0)}_{\alpha\alpha\beta}\right),$$

(no implied sum on α) with

$$\tau_M \ll \tau_E.$$

We now seek solutions that vary on timescales $T \sim \tau_E \gg \tau_M$.

Recovering the full set of Maxwell equation – part 2

We thus leave $\mathbf{B}$, and now $\boldsymbol{\Lambda}$, unexpanded, while still expanding

$$\mathsf{M} = \mathsf{M}^{(0)} + \tau_M \mathsf{M}^{(1)} + \cdots .$$

We take

$$M_{\mu\alpha\beta} = M^{(0)}_{\mu\alpha\beta} = \theta\delta_{\mu\alpha}B_\beta$$

to sufficient accuracy in the evolution equation for $\mathbf{E}$,

$$\partial_t E_\gamma - \tfrac{1}{2}\epsilon_{\gamma\alpha\beta}\partial_\mu M^{(0)}_{\mu\alpha\beta} = -\frac{1}{\tau_E}\Big(E_\gamma - E^{(0)}_\gamma\Big).$$

This becomes

$$\partial_t \mathbf{E} - c^2\,\nabla\times\mathbf{B} = -\frac{1}{\tau_E}\Big(\mathbf{E} - \mathbf{E}^{(0)}\Big).$$

with speed of light $c = \left(\frac{1}{2}\theta\right)^{1/2}$. Substituting $\mathbf{E}^{(0)} = -\mathbf{u}\times\mathbf{B}$ gives

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \frac{1}{c^2\tau_E}\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right).$$

Maxwell's equation plus Ohm's law

In deriving

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \frac{1}{c^2\tau_E}(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

we have recovered the full Maxwell equation

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \mu_0 \mathbf{J}$$

with the relativistic displacement current, and Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

with conductivity

$$\sigma = \frac{1}{c^2\mu_0\tau_E}.$$

The kinetic equations for the moments $\mathbf{B}$ and $\mathbf{E}$ give Maxwell's equations plus Ohm's law for resistive MHD.

Making this more systematic

We previously obtained the resistive MHD equations by expanding

$$\mathbf{\Lambda} = \mathbf{\Lambda}^{(0)} + \tau_E \mathbf{\Lambda}^{(1)} + \cdots\,, \quad \mathsf{M} = \mathsf{M}^{(0)} + \tau_M \mathsf{M}^{(1)} + \cdots\,,$$

while leaving $\mathbf{B}$ unexpanded. The evolution equation for $\mathbf{\Lambda}$ gave

$$\mathbf{\Lambda}^{(1)} = -\tau_E \left(\partial_{t_0} \mathbf{\Lambda}^{(0)} + \nabla \cdot \mathsf{M}^{(0)} \right) = -\tau_E \theta \nabla \mathbf{B} + O(\mathrm{Ma}^3).$$

Using van Kampen's (1985) terminology, $\mathbf{B}$ is a slow variable (conserved by collisions) while $\mathbf{\Lambda}$ and M are fast variables.

Maxwell's equations follow from making $\tau_M \ll \tau_E$, then treating both $\mathbf{B}$ and $\mathbf{\Lambda}$ as unexpanded slow variables. M alone is fast.

We derived Maxwell's equation for $\mathbf{E}$ by substituting the leading order term of

$$\mathsf{M} = \mathsf{M}^{(0)} + \tau_M \mathsf{M}^{(1)} + \cdots\,,$$

into the exact evolution equation

$$\partial_t \mathbf{\Lambda} + \nabla \cdot \mathsf{M} = -\frac{1}{\tau_E}\left(\mathbf{\Lambda} - \mathbf{\Lambda}^{(0)}\right).$$

Radiation from an oscillating line dipole

Change collision operator to impose

$$\mathbf{E}^{(0)} = \mathbf{E} - \tau c^2 \mu_0 \mathbf{J}_0.$$

$\mathbf{E}$ now evolves according to

$$-c^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mathbf{J}_0$$

as in Maxwell's equations with a prescribed current source

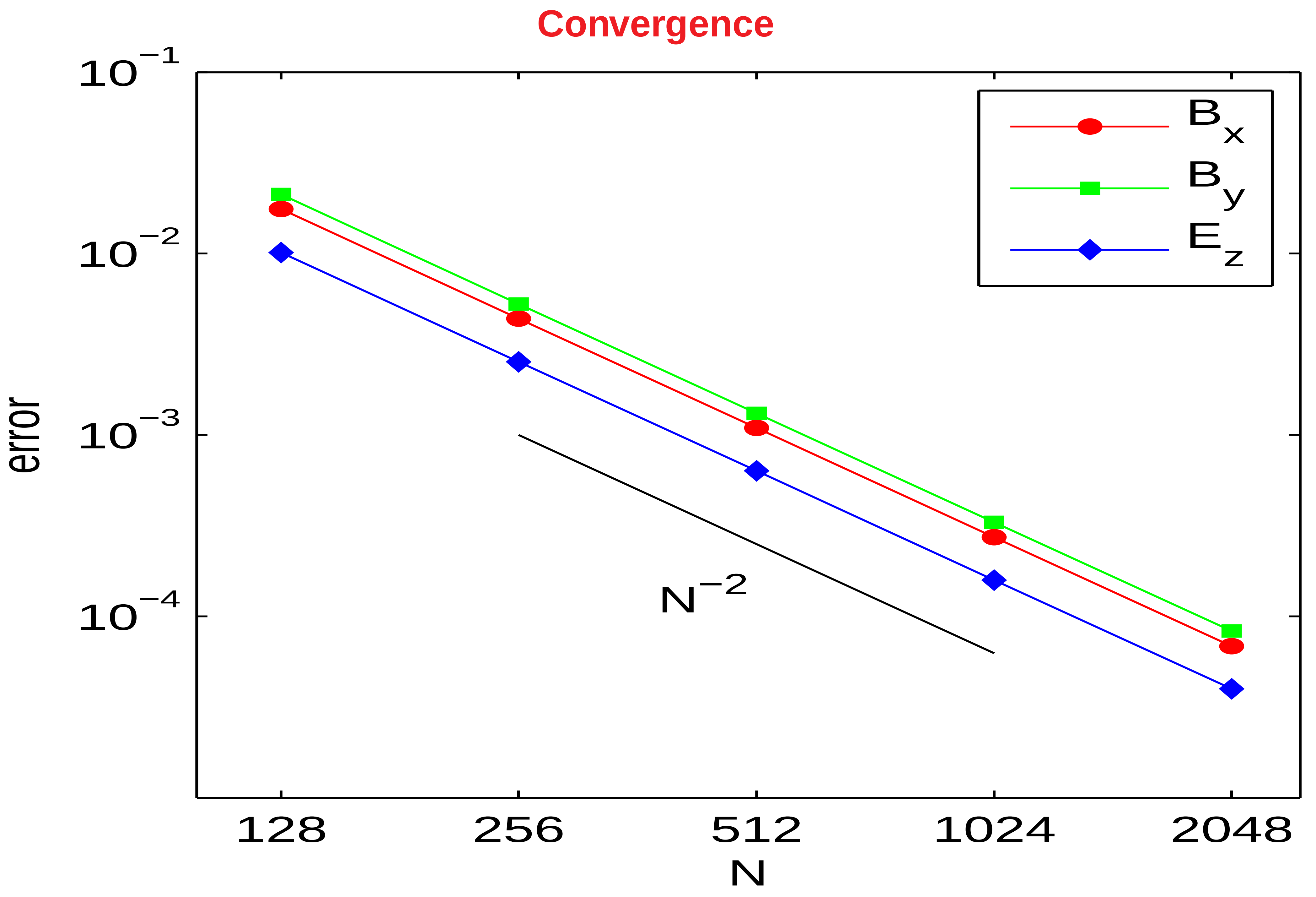
$$\mu_0 \mathbf{J}_0 = \frac{2}{\sqrt{\pi}\,\ell^2}\frac{x}{\ell}\exp\left(-\frac{x^2+y^2}{\ell^2}\right)\Theta(t)\sin(\omega t)\,\hat{\boldsymbol{z}}.$$

This particular source makes it easy to solve Maxwell's equations by expanding in 3D Fourier modes,

$$\mathbf{E} = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{B} = \sum_{\mathbf{k}} \mathbf{B}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

and using the exact solution of the ODE

$$c^{-2}\ddot{G} + k^2 G = \sin(\omega t), \quad \text{with } G(0) = \dot{G}(0) = 0.$$


Convergence
10^{-1}
10^{-2}
10^{-3}
10^{-4}
error
128
256
512
1024
2048
N
B_x
B_y
E_z
N^{-2}

Conclusions

Lattice Boltzmann approach offers linear, constant-coefficient advection. Nonlinearity is confined to the collision operator.

"Nonlinearity is local, nonlocality is linear" [Succi]

Keep just enough kinetic theory to recover a Navier–Stokes-level approximation.

Lattice Boltzmann magnetohydrodynamics uses vector distribution functions $\mathbf{g}_i$.

Parallel algorithm scales linearly across 32,768 processors on BlueGene/L.

The "electric field" tensor $\mathbf{\Lambda}$ has a symmetric part, as well as the antisymmetric part that carries the electric field via $E_\gamma = \epsilon_{\alpha\beta\gamma}\Lambda_{\alpha\beta}$.

More complicated collision operators implement improved plasma physics, but must pick out the antisymmetric part of $\mathbf{\Lambda}$.

Although only designed for non-relativistic magnetohydrodynamics, this lattice Boltzmann scheme contains the full Maxwell equations. We get more physics than we expected . . .