Covariant gyrokinetic theory of magnetoplasmas

Claudio Cremaschini[a], Massimo Tessarotto[b] and John C. Miller[a,c]

[a] *International School for Advanced Studies (SISSA) and INFN, Trieste, Italy*
[b] *University of Trieste and Consortium for Magnetofluid Dynamics, Trieste, Italy*
[c] *Department of Physics (Astrophysics), University of Oxford, Oxford, U.K.*

Abstract. A basic prerequisite for investigating relativistic astrophysical magnetoplasmas is the achievement of an accurate description of single-particle covariant dynamics, based on gyrokinetic theory. In astrophysical contexts, these magnetized plasmas occur in accretion discs, plasma inflows and outflows and relativistic jets, close to neutron stars and black holes (both of stellar mass and in galactic nuclei). If radiation-reaction effects are negligible, the covariant theory is developed on the assumption that both the space-time metric and the EM fields are suitably prescribed, while allowing for the possible presence of gravitational/EM perturbations driven by collective plasma interactions which may arise naturally in such systems. An interesting issue concerns the situation when the background electric field (produced either by the plasma itself or by other sources) is suitably small (or vanishing) with respect to the magnetic field, while at the same time short-wavelength EM perturbations can be present. In the present work, we extend the relativistic gyrokinetic theory developed by Beklemishev at al. [1999-2005] to include also the treatment of such a case. In particular, we show that this requires the development of a perturbative expansion involving simultaneously both the particle 4-position vector and the corresponding 4-velocity vector. For treating this, we derive the asymptotic dynamical equations using a synchronous form of the relativistic Hamilton variational principle, which allows one to satisfy exactly the physical realizability condition for the 4-velocity and to display the inner relationships between the gyrokinetic variables.

INTRODUCTION

- Gyrokinetic transformation: phase-space transformation to hybrid variables.
- Simpler equations of motion.
- Gyrophase angle ϕ is ignorable.
- Asymptotic theory based on perturbative expansions.
- Covariant formulation of GKT theory for magnetoplasmas.

MOTIVATIONS

- **Astrophysical problems: relativistic plasma flows in curved space-time.**
- **Accretion discs and relativistic jets close to compact objects.**
- **Collisionless relativistic plasmas in strong magnetic fields:**

$$B \sim 10^{12} G$$

- **Relativistic Vlasov-Maxwell equations in GKT variables.**

ASSUMPTIONS

- **Covariant GKT theory.**
- **Background space-time metric and EM fields prescribed.**
- **Short-wavelength EM and gravitational perturbations included:**

$$\varepsilon \sim \frac{r_L}{L_{EM}} \sim \frac{r_L}{L_g} \ll 1$$

$$\xi \equiv \frac{\lambda_{EM}}{L_{EM}} \ll 1 \qquad \delta \equiv \frac{\lambda_g}{L_g} \ll 1$$

HAMILTON VARIATIONAL PRINCIPLE

- **Synchronous hybrid variational principle.**
- **Constrained dynamics – Lagrange multipliers.**

$$S(\mathbf{y}) \equiv \int_1^2 \gamma(\mathbf{y}) + dF,$$

$$\gamma(\mathbf{y}) = g_{\mu\nu}\left[qA^{\nu} + u^{\nu}\right] dr^{\mu} + \lambda \left[g_{\mu\nu} u^{\mu} u^{\nu} - 1\right].$$

$$\{\mathbf{y}\} \equiv \Big\{\mathbf{x}(s) \equiv \left(r^{\mu}(s), u^{\mu}(s), \lambda(x)\right) : \mathbf{x}(s) \in C^{(2)}(I),$$
$$\mathbf{x}(s) \in \mathbb{R}^{9}, I \equiv [s_1, s_2] \subseteq \mathbb{R}, \mathbf{x}(s_i) = \mathbf{x}_1 (i = 1, 2)\Big\},$$

GKT TRANSFORMATION

- **Extended phase-space transformation:**

$$\left(r^{\alpha}, u^{\alpha}\right) \leftrightarrow \mathbf{y}^{i} \equiv \left(r'^{\alpha}, u'^{\alpha}\right)$$

In particular:

$$\begin{aligned} r^{\mu} &= r'^{\mu} + \sum_{s=1}^{\infty} \varepsilon^{s} r_{s}^{\mu} \equiv r'^{\mu} + \varepsilon \widehat{r}_{1}^{\mu}(\mathbf{y}', \varepsilon), \\ u^{\mu} &= u'^{\mu} \oplus \sum_{s=1}^{\infty} \xi^{s} v_{s}^{\mu}(\mathbf{y}) \equiv u'^{\mu} \oplus \xi \widehat{v}_{1}^{\mu}(\mathbf{y}', \varepsilon) \end{aligned}$$

EM FUNDAMENTAL TETRAD

- **Basis tetrad formalism:** $(\tau^{\mu}, l^{\mu}, l'^{\mu}, l''^{\mu})$

- **Eigenvectors of the Faraday tensor.**

- **Leading-order 4-velocity expressed as:**

$$u'^{\mu} = a^{\mu}\cos\phi' + b^{\mu}\sin\phi' + \overline{u}^{\mu}$$

$$a^{\mu}a_{\mu} = b^{\mu}b_{\mu} = 1 - g_{\mu\nu}\overline{u}^{\mu}\overline{u}^{\nu} = -w^{2}.$$

$$\overline{u}^{\mu} = u^{0}e_{0}^{\mu} + u^{\|}e_{1}^{\mu}$$

GKT FUNCTIONAL

- **Example: perturbative theory correct to *O(ε)*.**
- **GKT functional independent of the gyrophase angle:**

$$\gamma_g\left(u_0, u_\|, \overset{\wedge}{\mu}, r'^{\mu}, \lambda\right) = \left\{\left(\frac{q}{\varepsilon}A'_\mu + u_\| l_\mu + u_0 \tau_\mu\right) dr^\mu + \right.$$
$$\left. + \overset{\wedge}{\mu} d\phi + \lambda\left[u_0^2 - u_\|^2 - 2qB\overset{\wedge}{\mu} - 1\right] ds\right\}$$

- **Constraint equation:**

$$\overset{\sim}{u'_\mu} - q r_1^\nu F'_{\mu\nu} = 0,$$

AVERAGED QUANTITIES

- **Larmor-radius 4-vector – using the fundamental EM tetrad:**

$$r_1^\nu = -\frac{w}{qB}\left(l''^\nu \sin\phi' - l'^\nu \cos\phi'\right)$$

- **Relativistic magnetic moment:**

$$\hat{\mu} \equiv \frac{1}{2}\left\langle \tilde{u}_\mu \frac{\partial r_1^\mu}{\partial \phi'} \right\rangle = \frac{w^2}{2qB}$$

DYNAMICAL EQUATIONS

- **Particular case: slowly-varying EM fields and nearly flat space-time:**

$$d\hat{\mu} = 0 \Rightarrow \hat{\mu} = const.$$

$$\tau_\mu dr'^\mu - u_0 ds = 0 \Rightarrow u_0 = \frac{dr'^\mu}{ds}\tau_\mu,$$

$$l_\mu dr'^\mu + u_\parallel ds = 0 \Rightarrow u_\parallel = -\frac{dr'^\mu}{ds} l_\mu,$$

$$d\phi' + qB ds = 0 \Rightarrow \frac{d\phi'}{ds} = -\Omega,$$

$$\frac{q}{\varepsilon} F'_{\mu\nu} dr'^\nu - l_\mu du_\parallel - \tau_\mu du_0 - 2\chi q \hat{\mu} B_{,\mu} = 0,$$

$$u_0^2 - u_\parallel^2 - 2qB\hat{\mu} - 1 = 0,$$