Shape Matters -- Building a Better Bottle?

Steve Cowley *Culham, Imperial*

Bill Dorland, K$^3$, Omar Hurricane and Pierre Gourdain,
Causes Without Results

“I am afraid I rather give myself away when I explain. Results without causes are much more impressive.”

Sherlock Holmes in The Stock-Broker’s Clerk, Arthur Conan Doyle.
Better Bottle?

We have a magnetic configuration that will take us to burning plasmas in ITER. This will probably be the configuration of the first generation of fusion reactors.

*Can we improve this? What would that mean?*

Better confinement? .................. $\tau_E$
Higher pressure -- lower B? .......... $\beta$
Less heat flux? ............ Divertor.

Has every configuration been tried? In 2D?
Requirements For Fusion.

\[ \text{Fusion Power} \propto n_D n_T T^2 \propto \beta^2 B^4 \]

\[ 10 \text{keV} < T < 20 \text{keV}. \]

Rough criterion for ignition. \[ nT\tau_E > 3 \times 10^{15} \text{ cm}^3 \text{ keV s} \]

Physics limits the achievable values of these quantities.

n : Density “Greenwald” limit.

\[ nT = \beta \frac{B^2}{8\pi} \quad \text{: Beta Limit.} \quad \beta = \beta_N \frac{I}{aB} \]

\( \tau_E \quad : \text{Turbulence.} \)
Raising Beta

\[ \mathbf{J} \times \mathbf{B} = \nabla p \]
Radial Force Balance \[ \nabla \psi \cdot (\mathbf{J} \times \mathbf{B}) = \nabla p \]

\[
\left[ R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2} \right] \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}
\]

Grad-Shafranov Equation.

where

\[
\mathbf{B} = \frac{\nabla \psi \times \mathbf{e}_T}{R} + \frac{F(\psi)}{R} \mathbf{e}_T
\]
The Small Parameter

\[ \frac{\epsilon}{q^2} = \frac{a}{q^2 R} \ll 1 \]

\[ B_p \ll B_T = F/R \]

Surfaces shift and Squash against edge.

\[ \nabla p + \frac{\nabla B_T^2}{2} + \frac{\nabla B_p^2}{2} + \frac{B_T^2 \nabla R}{R} + \frac{B_P \cdot \nabla B_P}{\mathcal{O}(\epsilon^2/q^2)} = 0 \]
\[
\left[ R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2} \right] \psi = -\mu_0 R^2 \frac{d\psi}{d\psi} - F \frac{dF}{d\psi}
\]

\[O(\epsilon)\]

\[\mu_0 R^2 \frac{d\psi}{d\psi} = -F \frac{dF}{d\psi}\]

Toroidal Field Confinement

\[R = R(\psi) \text{ or } \psi = \psi(R)\]

Straight vertical flux surfaces.
CORE

\[ F(\hat{R}) = \sqrt{2 \left( C - \mu_0 \int_{R_{\text{min}}}^{\hat{R}} \hat{R}'^2 \frac{dp}{d\hat{R}'} d\hat{R}' \right)} \]

C = constant & p increases and F decreases towards the axis

\[ R = R(\psi) \text{ or } \psi = \psi(R) \]

Straight vertical flux surfaces in core
Boundary Layer -- BL

Gradients are large perpendicular to wall $\xi = \text{distance to wall}$.

$$\frac{\partial^2 \psi}{\partial \xi^2} = -\mu_0 \left(R^2 - \hat{R}^2(\psi)\right) \frac{dp}{d\psi}$$

Width of Boundary Layer is small and Poloidal Field is strong

$$\left(\frac{\partial \psi}{\partial \xi}\right)^2 = -2\mu_0 \int_R^{\hat{R}} \left(R^2 - \hat{R}''^2\right) \frac{dp}{d\psi} \frac{\partial \psi}{\partial \xi} d\xi$$

Poloidal field pressure forces balance the residual force from Lack of cancellation of pressure and toroidal field forces.

$$\left|B_p\right| \sim \sqrt{\epsilon p} \ll \left|B_T\right|$$
Boundary Layer --- BL

\[ \xi(R, \hat{R}) = \int_{R_{\text{min}}}^{\hat{R}} \sqrt{-2\mu_0 \int_{R}^{\hat{R}'} d\hat{R}'' \frac{d}{d\hat{R}''} \frac{d\psi}{d\hat{R}'} \left( R^2 - \hat{R}''^2 \right)} \]

\[ \delta = a \frac{\epsilon}{\sqrt{q^2 \beta}} \]

Boundary layer width. Expansion works if \( \delta < a \)

Poloidal field increases outwards in Boundary Layer.
Comparison

Agreement gets better as we increase beta

FIG. 3: Comparison of a equilibrium solution computed in CUBE (top) and the same solution calculated using the analytic theory (bottom).
Good properties.

1. Good Average Curvature

Bad field line curvature in the boundary layer only. Core dominates average

\[ \langle \nabla p \cdot (b \cdot \nabla b) \rangle \sim -\frac{p}{aR} \]

Mercier stable and tearing mode stable.

Good properties.

2. Small trapped particle fraction

$|B|$ constant on flux surface in core $\Rightarrow$ no bounce points in core. Trapped particle fraction.....

$$f_T \sim (1 - \frac{B_{\text{min}}}{B_{\text{max}}})^{1/2} \frac{\Delta V}{V}$$

As beta increases both factors decrease. $|B|$ constant on flux surface in BL too (omnidigeneity). The volume fraction in BL is

$$\frac{\Delta V}{V} \sim \sqrt[2]{\frac{\epsilon}{q^2\beta}} \ll 1$$

Banana width is squeezed by strong $B_p$. Neoclassical transport reduced by more than $\sim \sqrt[2]{\frac{\epsilon}{q^2\beta}}$ SCC PPPL report.
Good properties.

3. Magnetic well

\[ B^2 = \frac{F^2}{R^2} + \frac{|\nabla \psi|^2}{R^2} \]

\[ \rho + \frac{B^2}{2} = \text{constant} \]

\[ |B| \text{ is small in the center of The plasma.} \]

\[ \mu = \frac{v_{\perp}^2}{B} = \text{constant} \]

Got to give particles energy To get them out. Helps Stability, Taylor 1963
Good properties.
4. Short Connection length

$B_p$ is large in BL so distance along field from bad to Good curvature is

$$L_c \sim qR \sqrt{\frac{\epsilon}{q^2 \beta}}$$

Stabilizing.
Good properties.

5. Strong negative local shear in BL

- **Shear length in boundary layer**
  \[ L_s \approx O(a/q\beta) \ll L_c \]

- **Instabilities are heavily sheared by Magnetic shear in BL** Stabilizes ballooning.
Internal Kink Stability

Analytic result just depends on shape -- not on $\beta$ or $q$.

Hurricane et. al.
Phys. Plas. 2000

What about Rotation?

External kinks are the problem.
Negative Triangularity - Reverse D.

Zero local Shear in Corner firmly in good Curvature region.

Particles drift reversed

Bill Dorland will show Interesting results on Similar configuration.
Negative Triangularity - Reverse D. TCV

Less transport In reverse D

Camenen et. al. Nucl Fus. 2007
Unity beta current hole equilibrium

This equilibrium is stable to all ideal MHD criteria including internal and external modes for \( n = 1, 2 \) and 3... *Note that the \( \beta_N \) is “small” despite the large value of beta.*

Pierre Gourdain’s work

<table>
<thead>
<tr>
<th>( R )</th>
<th>6 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>2 m</td>
</tr>
<tr>
<td>( B_T )</td>
<td>2.5 T</td>
</tr>
<tr>
<td>( \beta )</td>
<td>100%</td>
</tr>
<tr>
<td>( \langle \beta \rangle )</td>
<td>12%</td>
</tr>
<tr>
<td>( \beta_N )</td>
<td>4.6</td>
</tr>
<tr>
<td>( q_{\text{min}} )</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Unity beta current hole profiles

Diamagnetic | Paramagnetic

\[ p \text{ (Pa)} \]

\[ F \text{ (Tm)} \]

\[ q \]

\[ \psi \]

Diamagnetic

\[ J_{\text{tor}} \text{ (A/m\textsuperscript{2})} \]

\[ p \text{ (Pa)} \]

\[ F \text{ (Tm)} \]

\[ R \text{ (m)} \]
Internal and external kink stability

DCON finds stability for Mercier, high-n ballooning as well as fixed boundary kink modes (n=1).

The free boundary mode n=1 is also stable (stability criteria obtained for $\psi = 1$).

Stability for n=2 and n=3 was also demonstrated.
Better Bottle -- better shape?

• It certainly isn’t clear that we can find a better bottle. But we should use our best tools to look hard.

• Start with transport considerations -- gyro-kinetics?