Lagrangian formulation of gyrokinetic theory with a single expansion parameter

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Outline

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Setting of the talk: Time independent magnetic field. We only deal with the dynamics of ions.

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Gyrokinetic ordering

Notation: $L \sim |\nabla(\ln|\mathbf{A}|)|^{-1}$, v_t the thermal speed, ρ_i a typical ion gyroradius, Ω a typical gyrofrequency, $\varphi(\mathbf{r}, t)$ the electrostatic potential.

- $\epsilon = \rho_i/L \ll 1$.
- $Ze\varphi/T_i \sim \epsilon$.
- $\mathcal{F}[\varphi(\cdot, t)](\mathbf{k})$ is localized in the region
  - $|\mathbf{k}_\perp|\rho_i \sim 1$, i.e. $|\mathbf{k}_\perp| \sim \epsilon^{-1}/L$,
  - $|\mathbf{k}_{||}| \sim L^{-1}$.
- $\mathcal{F}[\varphi(\mathbf{r}, \cdot)](\omega)$ localized in the region $\omega/\Omega \sim \epsilon$.

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However, modern derivations of the gyrokinetic equations are carried out by performing two independent expansions:

(I) Guiding-center expansion in $\epsilon = \rho_i/L$.

(II) Gyrokinetic expansion in powers of a parameter, ϵ_φ, giving the size of the electrostatic fluctuations.

Motivation

We will implement the gyrokinetic ordering, defined only by ϵ, in the phase-space Lagrangian formalism and give a fully explicit expression for the ϵ^2 term of the gyrokinetic Hamiltonian, $\overline{H}^{(2)}$, in general geometry.

- Calculations are much more complicated.
- Guiding-center and gyrokinetic dynamics are inextricably tied together and new terms arise that are absent in the customary derivations.
    - Similar remark made by A. M. Dimits in *Phys. Plasmas* (2010).

Outline

Phase-space Lagrangian formalism

Take a dynamical system defined by a Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ and the canonical Poisson bracket $\{q^i, p_j\} = \delta^i_j$. Then, Hamilton equations can be obtained by making stationary the functional

$$S[\mathbf{q}, \mathbf{p}] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \mathbf{p}(t), \dot{\mathbf{q}}(t), \dot{\mathbf{p}}(t), t)\mathrm{d}t,$$

where

$$\mathcal{L}(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}, t) = p_i \dot{q}^i - H(\mathbf{q}, \mathbf{p}, t).$$

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This formulation combines

- scalar invariance and variational principle from Lagrangian mechanics,
- flexibility of coordinate transformations in phase-space from Hamiltonian mechanics.

Lagrangian of a charged particle in an electromagnetic field

The Lagrangian in Gaussian units is given by

$$\mathcal{L}(\mathbf{r},\mathbf{v},\dot{\mathbf{r}},\dot{\mathbf{v}},t) = \Big((Ze/c)\mathbf{A}(\mathbf{r}) + M\mathbf{v}\Big)\cdot\dot{\mathbf{r}} - H(\mathbf{r},\mathbf{v},t),$$

with the Hamiltonian

$$H(\mathbf{r},\mathbf{v},t) = Mv^2/2 + Ze\varphi(\mathbf{r},t).$$

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In dimensionless variables

$$\check{\mathbf{r}} = \frac{\mathbf{r}}{L}, \check{\mathbf{v}} = \frac{\mathbf{v}}{v_t}, \check{t} = \frac{v_t t}{L}, \check{\varphi} = \frac{Ze\varphi}{\epsilon M v_t^2}, \check{\mathbf{A}} = \frac{\mathbf{A}}{B_0 L}, \check{H} = \frac{H}{M v_t^2}, \check{\mathcal{L}} = \frac{\mathcal{L}}{M v_t^2},$$

$$\check{\mathcal{L}} = \left(\epsilon^{-1}\check{\mathbf{A}}(\check{\mathbf{r}}) + \check{\mathbf{v}}\right)\cdot\dot{\check{\mathbf{r}}} - \left(\check{v}^2/2 + \epsilon\check{\varphi}(\check{\mathbf{r}},\check{t})\right).$$

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$$\check{\mathcal{L}} = \left(\epsilon^{-1}\check{\mathbf{A}}(\check{\mathbf{r}}) + \check{\mathbf{v}}\right)\cdot\dot{\check{\mathbf{r}}} - \left(\check{v}^2/2 + \epsilon\check{\varphi}(\check{\mathbf{r}},\check{t})\right).$$

Our ordering is completed by assuming that

$$\nabla_{\mathbf{r}_\perp}\check{\varphi} \sim O(\epsilon^{-1}).$$

Formally, $\check{\varphi}(\check{\mathbf{r}}_\perp/\epsilon, \check{\mathbf{r}}_{||}, \check{t})$.

Outline

General strategy of the change of coordinates

$$\mathcal{L}(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}, t) = p_i \dot{q}^i - H(\mathbf{q}, \mathbf{p}, t).$$

We obtain equivalent Lagrangians via:

- Change of coordinates: $\mathbf{q}(\mathbf{Q}, \mathbf{P}, t)$, $\mathbf{p}(\mathbf{Q}, \mathbf{P}, t)$.
- Addition of total time derivatives: $\mathcal{L} \mapsto \mathcal{L} + \mathrm{d}S/\mathrm{d}t$.

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Applying both transformations:

$$\mathcal{L}'(\mathbf{Q},\mathbf{P},\dot{\mathbf{Q}},\dot{\mathbf{P}},t) = \left(p_i(\mathbf{Q},\mathbf{P},t)\frac{\partial q^i}{\partial Q^j} + \frac{\partial S}{\partial Q^j}\right)\dot{Q}^j + \left(p_i(\mathbf{Q},\mathbf{P},t)\frac{\partial q^i}{\partial P^j} + \frac{\partial S}{\partial P^j}\right)\dot{P}^j$$
$$- \left(H(\mathbf{q}(\mathbf{Q},\mathbf{P},t),\mathbf{p}(\mathbf{Q},\mathbf{P},t),t) - p_i(\mathbf{Q},\mathbf{P},t)\frac{\partial q^i}{\partial t} - \frac{\partial S}{\partial t}\right).$$

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$$- \left(H(\mathbf{q}(\mathbf{Q},\mathbf{P},t),\mathbf{p}(\mathbf{Q},\mathbf{P},t),t) - p_i(\mathbf{Q},\mathbf{P},t)\frac{\partial q^i}{\partial t} - \frac{\partial S}{\partial t}\right).$$

Roughly speaking, in gyrokinetics $(\mathbf{Q},\mathbf{P}) \equiv \{\mathbf{R},u,\mu,\theta\}$ and one wants to answer the following question: *Do new coordinates $\{\mathbf{R},u,\mu,\theta\}$ and a function S exist such that $\mathcal{L}'$ is independent of θ and μ is a constant of motion?*

General strategy of the change of coordinates

Just for technical convenience, we will write the gyrokinetic transformation T_ϵ as the composition of two transformations, $T_\epsilon = T_{1\epsilon} T_{2\epsilon}$.

- $T_{1\epsilon}$ is a prescribed, preparatory change of variables adapted to the magnetic field structure.
- $T_{2\epsilon}$ is determined order by order by imposing the gyrophase-invariance of the Lagrangian and the conservation of μ.

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Remark: In order to determine $\overline{H}^{(2)}$, computation of the term proportional to $\dot{\theta}$ in the Lagrangian is needed at order ϵ^3. We will use the notation $\mathcal{L} = \cdots + O(\epsilon^3, \epsilon^4)$.

Transformed Lagrangian under $T_{1\epsilon}$

$\hat{\mathbf{e}}_1(\mathbf{r})$, $\hat{\mathbf{e}}_2(\mathbf{r})$, orthonormal vector fields such that $\hat{\mathbf{e}}_1(\mathbf{r}) \times \hat{\mathbf{e}}_2(\mathbf{r}) = \hat{\mathbf{b}}(\mathbf{r})$. Then, $(\mathbf{r}, \mathbf{v}) = T_{1\epsilon}(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g)$ is defined by

$$(\mathbf{r}, \mathbf{v}) = \Big(\mathbf{R}_g + \epsilon \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g), \quad v_{||g}\hat{\mathbf{b}}(\mathbf{R}_g) + \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g) \times \mathbf{B}(\mathbf{R}_g)\Big).$$

$$\boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g) := -\sqrt{2\mu_g / B(\mathbf{R}_g)}\left[\sin\theta_g \hat{\mathbf{e}}_1(\mathbf{R}_g) - \cos\theta_g \hat{\mathbf{e}}_2(\mathbf{R}_g)\right].$$

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Now, we compute the transformed Lagrangian at the required order and add the total time derivative

$$-\frac{d}{dt}\Bigg\{\mathbf{A}_g \cdot \boldsymbol{\rho}_g + \frac{\epsilon}{2}\boldsymbol{\rho}_g\boldsymbol{\rho}_g : \nabla_{\mathbf{R}_g}\mathbf{A}_g + \frac{\epsilon^2}{6}\boldsymbol{\rho}_g\boldsymbol{\rho}_g : \nabla_{\mathbf{R}_g}\nabla_{\mathbf{R}_g}\mathbf{A}_g \cdot \boldsymbol{\rho}_g$$
$$-\epsilon^3 \int_0^{\theta_g}\left[\tilde{\mathbf{A}}_3 \cdot (\boldsymbol{\rho}_g \times \hat{\mathbf{b}}) - \langle\tilde{\mathbf{A}}_3 \cdot (\boldsymbol{\rho}_g \times \hat{\mathbf{b}})\rangle\right] d\theta'_g\Bigg\}$$

Transformed Lagrangian under $T_{1\epsilon}$

$$\mathcal{L}\circ T_{1\epsilon}(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g) = \left(\frac{1}{\epsilon}\mathbf{A}_g + v_{||g}\hat{\mathbf{b}}_g + \epsilon\mathbf{\Gamma}^{(1)}_{\mathbf{R}_g} + \epsilon^2\mathbf{\Gamma}^{(2)}_{\mathbf{R}_g}\right)\cdot\dot{\mathbf{R}}_g$$

$$+\epsilon\left(-\mu_g + \epsilon\Gamma^{(1)}_{\theta_g} + \epsilon^2\Gamma^{(2)}_{\theta_g}\right)\dot{\theta}_g - H^{(0)} - \epsilon H^{(1)} + O(\epsilon^3, \epsilon^4),$$

where

$$\mathbf{\Gamma}^{(1)}_{\mathbf{R}_g} = \mu_g\nabla_{\mathbf{R}_g}\hat{\mathbf{e}}_{2g}\cdot\hat{\mathbf{e}}_{1g} - v_{||g}\nabla_{\mathbf{R}_g}\hat{\mathbf{b}}_g\cdot\boldsymbol{\rho}_g - \frac{1}{2}(\boldsymbol{\rho}_g\cdot\nabla_{\mathbf{R}_g}B_g)\boldsymbol{\rho}_g\times\hat{\mathbf{b}}_g + \frac{B_g}{2}\boldsymbol{\rho}_g\cdot\nabla_{\mathbf{R}_g}\hat{\mathbf{b}}_g\times\boldsymbol{\rho}_g,$$

$$\mathbf{\Gamma}^{(2)}_{\mathbf{R}_g} = \frac{1}{6}\boldsymbol{\rho}_g\boldsymbol{\rho}_g : \nabla_{\mathbf{R}_g}\nabla_{\mathbf{R}_g}\mathbf{B}_g\times\boldsymbol{\rho}_g - \frac{2\mu_g}{3B_g}(\boldsymbol{\rho}_g\cdot\nabla_{\mathbf{R}_g}B_g)\nabla_{\mathbf{R}_g}\hat{\mathbf{e}}_{2g}\cdot\hat{\mathbf{e}}_{1g}$$

$$-\frac{B_g}{3}[\boldsymbol{\rho}_g\cdot\nabla_{\mathbf{R}_g}\hat{\mathbf{b}}_g\cdot(\boldsymbol{\rho}_g\times\hat{\mathbf{b}}_g)]\nabla_{\mathbf{R}_g}\hat{\mathbf{b}}_g\cdot\boldsymbol{\rho}_g,$$

$$\Gamma^{(1)}_{\theta_g} = \frac{2\mu_g}{3B_g}\boldsymbol{\rho}_g\cdot\nabla_{\mathbf{R}_g}B_g, \quad \Gamma^{(2)}_{\theta_g} = \frac{\mu_g^2}{4B_g^2}(\mathbf{I} - \hat{\mathbf{b}}_g\hat{\mathbf{b}}_g) : \nabla_{\mathbf{R}_g}\nabla_{\mathbf{R}_g}\mathbf{B}_g\cdot\hat{\mathbf{b}}_g,$$

$$H^{(0)} = \frac{1}{2}v_{||g}^2 + \mu_g B_g, \quad H^{(1)} = \langle\phi\rangle(\mathbf{R}_{g\perp}/\epsilon, R_{g||}, \mu_g, t) + \widetilde{\phi}(\mathbf{R}_{g\perp}/\epsilon, R_{g||}, \mu_g, \theta_g, t).$$

$$\phi(\mathbf{R}_{g\perp}/\epsilon, R_{g||}, \mu_g, \theta_g, t) := \varphi(\mathbf{R}_{g\perp}/\epsilon + \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g), R_{g||}, t).$$

Transformed Lagrangian under $T_{1\epsilon}T_{2\epsilon}$

The operator $(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g) = T_{2\epsilon}(\mathbf{R}, u, \mu, \theta)$ can be written as a Lie-transform

$$(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g) = \exp \sum_{n=1}^{\infty} \epsilon^n \Big(\mathbf{R}_n \cdot \nabla_{\mathbf{R}} + u_n \partial_u + \mu_n \partial_\mu + \theta_n \partial_\theta \Big) (\mathbf{R}, u, \mu, \theta),$$

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but the following variables are more useful for us ($\mathbf{R}_1 \equiv 0$):

$$\tilde{\mathbf{R}}_3 = \mathbf{R}_3 + \frac{1}{2}\mathbf{R}_2 \cdot \nabla_{(\mathbf{R}_\perp/\epsilon)}\mathbf{R}_2 + \frac{u_1}{2}\frac{\partial \mathbf{R}_2}{\partial u} + \frac{\mu_1}{2}\frac{\partial \mathbf{R}_2}{\partial \mu} + \frac{\theta_1}{2}\frac{\partial \mathbf{R}_2}{\partial \theta}.$$

$$\tilde{u}_2 = u_2 + \frac{u_1}{2}\frac{\partial u_1}{\partial u} + \frac{\mu_1}{2}\frac{\partial u_1}{\partial \mu} + \frac{\theta_1}{2}\frac{\partial u_1}{\partial \theta},$$

$$\tilde{\mu}_2 = \mu_2 + \frac{1}{2}\mathbf{R}_2 \cdot \nabla_{(\mathbf{R}_\perp/\epsilon)}\mu_1 + \frac{u_1}{2}\frac{\partial \mu_1}{\partial u} + \frac{\mu_1}{2}\frac{\partial \mu_1}{\partial \mu} + \frac{\theta_1}{2}\frac{\partial \mu_1}{\partial \theta},$$

$$\tilde{\theta}_2 = \theta_2 + \frac{1}{2}\mathbf{R}_2 \cdot \nabla_{(\mathbf{R}_\perp/\epsilon)}\theta_1 + \frac{u_1}{2}\frac{\partial \theta_1}{\partial u} + \frac{\mu_1}{2}\frac{\partial \theta_1}{\partial \mu} + \frac{\theta_1}{2}\frac{\partial \theta_1}{\partial \theta}$$

$$(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g) =$$

$$(\mathbf{R} + \epsilon^2 \mathbf{R}_2 + \epsilon^3 \tilde{\mathbf{R}}_3, u + \epsilon u_1 + \epsilon^2 \tilde{u}_2, \mu + \epsilon \mu_1 + \epsilon^2 \tilde{\mu}_2, \theta + \epsilon \theta_1 + \epsilon^2 \tilde{\theta}_2) + \ldots$$

$$
\begin{aligned}
&\mathcal{L}\circ T_{1\epsilon}\circ T_{2\epsilon} = \\
&\Big[\epsilon^{-1}\mathbf{A}_{\mathbf{R}} + u\hat{\mathbf{b}}_{\mathbf{R}} + \epsilon\Big(\mathbf{\Gamma}^{(1)}_{\mathbf{R}} + \mathbf{R}_2\cdot\nabla_{\mathbf{R}}\mathbf{A}_{\mathbf{R}} + u_1\hat{\mathbf{b}}_{\mathbf{R}}\Big) + \epsilon^2\Big(\mathbf{\Gamma}^{(2)}_{\mathbf{R}} + \tilde{\mathbf{R}}_3\cdot\nabla_{\mathbf{R}}\mathbf{A}_{\mathbf{R}} \\
&+ \tilde{u}_2\hat{\mathbf{b}}_{\mathbf{R}} + u\mathbf{R}_2\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}} + u_1\partial_u\mathbf{\Gamma}^{(1)}_{\mathbf{R}} + \mu_1\partial_\mu\mathbf{\Gamma}^{(1)}_{\mathbf{R}} + \theta_1\partial_\theta\mathbf{\Gamma}^{(1)}_{\mathbf{R}}\Big)\Big]\cdot\dot{\mathbf{R}} + \epsilon\mathbf{A}_{\mathbf{R}}\cdot\dot{\mathbf{R}}_2 \\
&+ \epsilon^2\Big[u\hat{\mathbf{b}}_{\mathbf{R}}\cdot\dot{\mathbf{R}}_2 + \mathbf{A}_{\mathbf{R}}\cdot\dot{\tilde{\mathbf{R}}}_3\Big] + \epsilon^3\Big[\mathbf{A}_{\mathbf{R}}\cdot\dot{\tilde{\mathbf{R}}}_4 + u\hat{\mathbf{b}}_{\mathbf{R}}\cdot\dot{\tilde{\mathbf{R}}}_3 \\
&+ \Big(\mathbf{\Gamma}^{(1)}_{\mathbf{R}} + \mathbf{R}_2\cdot\nabla_{\mathbf{R}}\mathbf{A}_{\mathbf{R}} + u_1\hat{\mathbf{b}}_{\mathbf{R}}\Big)\cdot\dot{\mathbf{R}}_2\Big] + \epsilon\Big[-\mu + \epsilon(\Gamma^{(1)}_{\theta} - \mu_1) \\
&+ \epsilon^2\Big(\Gamma^{(2)}_{\theta} - \tilde{\mu}_2 + \mu_1\partial_\mu\Gamma^{(1)}_{\theta} + \theta_1\partial_\theta\Gamma^{(1)}_{\theta}\Big)\Big]\dot{\theta} - \epsilon^2\mu\dot{\theta}_1 \\
&+ \epsilon^3\Big[(\Gamma^{(1)}_{\theta} - \mu_1)\dot{\theta}_1 - \mu\dot{\tilde{\theta}}_2\Big] - H^{(0)}_{\mathbf{R}} - \epsilon\Big(H^{(1)}_{\mathbf{R}} + uu_1 + \mu_1 B_{\mathbf{R}}\Big) \\
&- \epsilon^2\Big(u\tilde{u}_2 + \frac{u_1^2}{2} + \tilde{\mu}_2 B_{\mathbf{R}} + \mu\mathbf{R}_2\cdot\nabla_{\mathbf{R}}B_{\mathbf{R}} + \mathbf{R}_2\cdot\nabla_{(\mathbf{R}_\perp/\epsilon)}H^{(1)}_{\mathbf{R}} \\
&+ \mu_1\partial_\mu H^{(1)}_{\mathbf{R}} + \theta_1\partial_\theta H^{(1)}_{\mathbf{R}}\Big) + O(\epsilon^3, \epsilon^4).
\end{aligned}
$$

Transformed Lagrangian under $T_{1\epsilon}T_{2\epsilon}$

Allow the addition of a total derivative to the Lagrangian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon S^{(1)} + \epsilon^2 S^{(2)} + \epsilon^3 S^{(3)}\right](\mathbf{R}_\perp/\epsilon, \mathbf{R}, u, \mu, \theta, t),$$

and impose that the Lagrangian be of the form

$$\mathcal{L} = \left[\frac{1}{\epsilon}\mathbf{A}_{\mathbf{R}} + u\hat{\mathbf{b}}_{\mathbf{R}} + \epsilon\overline{\mathbf{\Gamma}}^{(1)}_{\mathbf{R}}\right] \cdot \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} - \epsilon\mu\frac{\mathrm{d}\theta}{\mathrm{d}t} - \overline{H}^{(0)} - \epsilon\overline{H}^{(1)} - \epsilon^2\overline{H}^{(2)} + O(\epsilon^3, \epsilon^4),$$

where $\overline{H}^{(0)}, \overline{H}^{(1)}, \overline{H}^{(2)}$ are gyrophase independent and we choose

$$\overline{\mathbf{\Gamma}}^{(1)}_{\mathbf{R}} = \mu\nabla_{\mathbf{R}}\hat{\mathbf{e}}_{2\mathbf{R}} \cdot \hat{\mathbf{e}}_{1\mathbf{R}} - \frac{\mu}{2}\hat{\mathbf{b}}_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}_{\mathbf{R}}.$$

These conditions allow to determine $\mathbf{R}_2, u_1, \mu_1, \theta_1$, $\tilde{\mathbf{R}}_{3\perp}, \tilde{u}_2, \tilde{\mu}_2$, $\overline{H}^{(0)}, \overline{H}^{(1)}, \overline{H}^{(2)}$, $S^{(1)}, S^{(2)}$ and $S^{(3)}$.

Explicit expressions for the change of coordinates

$$\mathbf{R}_{2\perp} = -\frac{u}{B_{\mathbf{R}}}\hat{\mathbf{b}}_{\mathbf{R}} \times \nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}} \cdot \boldsymbol{\rho} - \frac{1}{2B_{\mathbf{R}}}\boldsymbol{\rho}\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B_{\mathbf{R}} - \frac{1}{B_{\mathbf{R}}^2}\hat{\mathbf{b}}_{\mathbf{R}} \times \nabla_{(\mathbf{R}_\perp/\epsilon)}\widetilde{\Phi},$$

$$R_{2||} = -\frac{2u}{B_{\mathbf{R}}}\hat{\mathbf{b}}_{\mathbf{R}} \cdot \nabla\hat{\mathbf{b}}_{\mathbf{R}} \cdot (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}}) - \frac{1}{8}\left[\boldsymbol{\rho}\boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})\right] : \nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}$$

$$u_1 = u\hat{\mathbf{b}}_{\mathbf{R}} \cdot \nabla\hat{\mathbf{b}}_{\mathbf{R}} \cdot \boldsymbol{\rho} - \frac{B_{\mathbf{R}}}{4}\left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})\boldsymbol{\rho}\right] : \nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}$$

$$\mu_1 = -\frac{\widetilde{\phi}}{B_{\mathbf{R}}} - \frac{u^2}{B_{\mathbf{R}}}\hat{\mathbf{b}}_{\mathbf{R}} \cdot \nabla\hat{\mathbf{b}}_{\mathbf{R}} \cdot \boldsymbol{\rho} + \frac{u}{4}\left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})\boldsymbol{\rho}\right] : \nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}.$$

$$\theta_1 = \frac{1}{B_{\mathbf{R}}}\frac{\partial\widetilde{\Phi}}{\partial\mu} + \frac{u^2}{2\mu B_{\mathbf{R}}}\hat{\mathbf{b}}_{\mathbf{R}} \cdot \nabla\hat{\mathbf{b}}_{\mathbf{R}} \cdot (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})$$
$$+ \frac{u}{8\mu}\left[\boldsymbol{\rho}\boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}})\right] : \nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}} + \frac{1}{B_{\mathbf{R}}}(\boldsymbol{\rho} \times \hat{\mathbf{b}}_{\mathbf{R}}) \cdot \nabla_{\mathbf{R}} B_{\mathbf{R}}.$$

$$\widetilde{\Phi} := \int^{\theta} \widetilde{\phi}\, d\theta'.$$

Explicit expressions for the gyrokinetic Hamiltonian

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$$\overline{H}^{(2)} = \frac{1}{2B_{\mathbf{R}}^2}\left\langle \nabla_{(\mathbf{R}_\perp/\epsilon)}\widetilde{\Phi}\cdot\left(\hat{\mathbf{b}}_{\mathbf{R}}\times\nabla_{(\mathbf{R}_\perp/\epsilon)}\widetilde{\phi}\right)\right\rangle - \frac{1}{2B_{\mathbf{R}}}\frac{\partial\langle\widetilde{\phi}^2\rangle}{\partial\mu}$$

$$- \frac{u}{B_{\mathbf{R}}}\left\langle\left(\nabla_{(\mathbf{R}_\perp/\epsilon)}\widetilde{\phi}\times\hat{\mathbf{b}}_{\mathbf{R}}\right)\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}\cdot\boldsymbol{\rho}\right\rangle - \frac{\mu}{2B_{\mathbf{R}}^2}\nabla_{\mathbf{R}}B_{\mathbf{R}}\cdot\nabla_{(\mathbf{R}_\perp/\epsilon)}\langle\phi\rangle$$

$$- \frac{1}{4B_{\mathbf{R}}}\left\langle\nabla_{(\mathbf{R}_\perp/\epsilon)}\widetilde{\phi}\cdot\left[\boldsymbol{\rho}\boldsymbol{\rho}-(\boldsymbol{\rho}\times\hat{\mathbf{b}}_{\mathbf{R}})(\boldsymbol{\rho}\times\hat{\mathbf{b}}_{\mathbf{R}})\right]\cdot\nabla_{\mathbf{R}}B_{\mathbf{R}}\right\rangle - \frac{1}{B_{\mathbf{R}}}\nabla_{\mathbf{R}}B_{\mathbf{R}}\cdot\langle\widetilde{\phi}\,\boldsymbol{\rho}\rangle$$

$$- \frac{u^2}{B_{\mathbf{R}}}\hat{\mathbf{b}}_{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}\cdot\frac{\partial}{\partial\mu}\langle\widetilde{\phi}\,\boldsymbol{\rho}\rangle + \frac{u}{4}\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}:\frac{\partial}{\partial\mu}\left\langle\widetilde{\phi}\left[\boldsymbol{\rho}(\boldsymbol{\rho}\times\hat{\mathbf{b}}_{\mathbf{R}})+(\boldsymbol{\rho}\times\hat{\mathbf{b}}_{\mathbf{R}})\boldsymbol{\rho}\right]\right\rangle$$

$$- \frac{3u^2\mu}{2B_{\mathbf{R}}^2}(\hat{\mathbf{b}}_{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}})\cdot\nabla_{\mathbf{R}}B_{\mathbf{R}} + \frac{\mu^2}{4B_{\mathbf{R}}}(\mathbf{I}-\hat{\mathbf{b}}_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}):\nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\mathbf{B}_{\mathbf{R}}\cdot\hat{\mathbf{b}}_{\mathbf{R}}$$

$$- \frac{3\mu^2}{4B_{\mathbf{R}}^2}|\nabla_{\mathbf{R}\perp}B_{\mathbf{R}}|^2 + \frac{u^2\mu}{2B_{\mathbf{R}}}\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}:\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}} + \left(\frac{\mu^2}{8}-\frac{\mu u^2}{4B_{\mathbf{R}}}\right)\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}:(\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}})^{\mathrm{T}}$$

$$- \left(\frac{3\mu u^2}{8B_{\mathbf{R}}}+\frac{\mu^2}{16}\right)(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}_{\mathbf{R}})^2 + \left(\frac{3\mu u^2}{2B_{\mathbf{R}}}-\frac{u^4}{2B_{\mathbf{R}}^2}\right)|\hat{\mathbf{b}}_{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}_{\mathbf{R}}|^2$$

$$+ \left(\frac{\mu u^2}{8B_{\mathbf{R}}}-\frac{\mu^2}{16}\right)(\hat{\mathbf{b}}_{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}_{\mathbf{R}})^2.$$

Outline

Gyrokinetic Poisson equation

In dimensionless variables and denoting by λ_D the Debye length:

$$\nabla^2\phi(\mathbf{r},t) = -ZL^2/\lambda_D^2\left(\int f(\mathbf{r},\mathbf{v},t)\mathrm{d}^3\mathbf{v} - n_e(\mathbf{r},t)\right).$$

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If $\lambda_D/\rho_i \ll 1$ and $\nabla_{\mathbf{R}}F \sim O(1)$, this equation gives

$$\int \mathcal{B}_{||}^*(\mathbf{R},u)\left[F(\mathbf{R},u,\mu,t) - \epsilon u_{1\mathbf{R}}\partial_2 F(\mathbf{R},u,\mu,t) - \epsilon\mu_{1\mathbf{R}}\partial_3 F(\mathbf{R},u,\mu,t)\right]$$

$$\times\delta(\mathbf{R}+\epsilon\boldsymbol{\rho}(\mathbf{R},\mu,\theta)-\mathbf{r})\mathrm{d}^3\mathbf{R}\mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta - n_e(\mathbf{r},t) + O(\epsilon^2) = 0,$$

where $\mathcal{B}_{||}^*(\mathbf{R},u) = \Big(\mathbf{B}(\mathbf{R}) + \epsilon u\nabla\times\mathbf{b}(\mathbf{R})\Big)\cdot\mathbf{b}(\mathbf{R})$.

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A careful elimination of the delta function yields

$$0 = -n_e(\mathbf{r},t) + \int \mathcal{B}_{||}^*(\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta),u)\Big[F(\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta),u,\mu,t)$$

$$-\epsilon u_{1\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta)}\partial_2 F(\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta),u,\mu,t)$$

$$-\epsilon\mu_{1\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta)}\partial_3 F(\mathbf{r}-\epsilon\boldsymbol{\rho}(\mathbf{r},\mu,\theta),u,\mu,t)\Big]\Big(1-\epsilon\nabla_{\mathbf{r}}\cdot\boldsymbol{\rho}(\mathbf{r},\mu,\theta)\Big)\mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta + O(\epsilon^2).$$

Gyrokinetic Poisson equation

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Gyrokinetic Poisson equation

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Then,

$$\int B_{||}(\mathbf{r}) F_0(\mathbf{r}, u, \mu, t) \mathrm{d}u \mathrm{d}\mu \mathrm{d}\theta + \epsilon \int B_{||}(\mathbf{r}) F_1(\mathbf{r} - \epsilon \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), u, \mu, t) \mathrm{d}u \mathrm{d}\mu \mathrm{d}\theta$$

$$+ \frac{\epsilon}{B_{\mathbf{r}}} \int B_{||}(\mathbf{r}) \tilde{\phi}(\mathbf{r} - \epsilon \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), t) \partial_3 F_0(\mathbf{r}, u, \mu, t) \mathrm{d}u \mathrm{d}\mu \mathrm{d}\theta$$

$$+ \epsilon \mathbf{b}_{\mathbf{r}} \cdot (\nabla \times \mathbf{b}_{\mathbf{r}}) \int u F_0(\mathbf{r}, u, \mu, t) \mathrm{d}u \mathrm{d}\mu \mathrm{d}\theta - n_e(\mathbf{r}, t) + O(\epsilon^2) = 0.$$

- The Jacobian coming from the delta function does not contribute.
- From the final expression we **deduce** that $\nabla_{\mathbf{R}_\perp} F_1 \sim \epsilon^{-1}$.

Outline

Conclusions and further work

- We have formulated a gyrokinetic theory with a single expansion parameter, ϵ, in the Lagrangian formalism.
  - No splitting between guiding-center and gyrokinetic dynamics is possible.
- The ϵ^2 term of the gyrokinetic Hamiltonian in general geometry has been explicitly computed.
  - By setting $\varphi \equiv 0$ we get for free the guiding-center Lagrangian with zero electrostatic potential to an order higher than available in the literature, as far as we know.

Next steps:

- Work out the electromagnetic case.
- Include external flows.
- Implications?