Isaac Newton Institute for Mathematical Sciences
Gyrokinetics in Laboratory and Astrophysical Plasmas

August 2-13, 2010

GYROFLUID MOMENTUM CONSERVATION LAW

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The self-regulation of anomalous transport processes by plasma flows in turbulent axisymmetric magnetized plasmas has been intensively investigated in the past decade.

Because a strong coupling has been observed between toroidal-momentum transport and energy transport in such plasmas, it is natural to investigate the link between these two global conservation laws through an application of the Noether method on a suitable Lagrangian density.

The general variational formulation of nonlinear dissipationless reduced fluid models describing nonlinear turbulent dynamics of strongly magnetized plasmas is expressed in terms of the multi-component field

$$\psi^{\alpha} \equiv (\mathbf{\Phi}, \mathbf{A}, \mathbf{E}, \mathbf{B}; n, \mathbf{u}, p_{\|}, p_{\perp})$$

REDUCED FLUID LAGRANGIAN

$$\mathcal{L} \;\equiv\; \mathcal{L}_{\mathsf{M}}(\mathbf{E},\mathbf{B}) \;+\; \mathcal{L}_{\Psi}(\Phi,\mathbf{A};n,\mathbf{u}) \;+\; \mathcal{L}_{\mathsf{F}}(n,\mathbf{u},p_{\|},p_{\perp};\mathbf{E},\mathbf{B})$$

- **Electromagnetic Lagrangian density**

$$\mathcal{L}_{\mathsf{M}}(\mathbf{E},\mathbf{B}) \;\equiv\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right)$$

- **Interaction (gauge-dependent) Lagrangian density**

$$\mathcal{L}_{\Psi}(\Phi,\mathbf{A};n,\mathbf{u}) \;\equiv\; -\sum q\,n\left(\Phi \;-\; \mathbf{A}\cdot\frac{\mathbf{u}}{c}\right) \;\equiv\; -\sum q\,n\,\Psi$$

- **Reduced-fluid Lagrangian density**

$$\mathcal{L}_{\mathsf{F}}(n,\mathbf{u},p_{\|},p_{\perp};\mathbf{E},\mathbf{B})$$

- **Lagrangian Derivatives**

◦ **Reduced charge-current densities**

$$\left(\frac{\partial \mathcal{L}}{\partial \Phi},\ \frac{\partial \mathcal{L}}{\partial \mathbf{A}}\right) \ \equiv\ \left(-\,\varrho, \frac{\mathbf{J}}{c}\right) \ \equiv\ \left(\sum qn,\ \sum qn\,\frac{\mathbf{u}}{c}\right)$$

Note: Gauge transformation $(\Phi, \mathbf{A}) \rightarrow (\Phi', \mathbf{A}') \ \Rightarrow$

$$\begin{aligned}\mathcal{L}' \ &\equiv\ \mathcal{L} \ +\ \varrho\,\frac{\partial}{\partial t}\left(\frac{\chi}{c}\right) \ +\ \mathbf{J}\cdot\nabla\left(\frac{\chi}{c}\right) \\ &\equiv\ \mathcal{L} \ +\ \frac{\partial}{\partial t}\left(\varrho\,\frac{\chi}{c}\right) \ +\ \nabla\cdot\left(\mathbf{J}\,\frac{\chi}{c}\right)\end{aligned}$$

so that action $\mathcal{A} \equiv \int\ \mathcal{L}(\psi^{\alpha})\ d^4x$ is gauge-invariant.

- **Reduced polarization and magnetization**

$$\left(\frac{\partial\mathcal{L}}{\partial\mathbf{E}},\ \frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right) \equiv \left(\frac{\mathbf{D}}{4\pi},\ -\frac{\mathbf{H}}{4\pi}\right) \equiv \left(\frac{\mathbf{E}}{4\pi} + \mathbf{P},\ -\frac{\mathbf{B}}{4\pi} + \mathbf{M}\right)$$

where

$$\boxed{(\mathbf{P},\ \mathbf{M}) \equiv \left(\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\mathbf{E}},\ \frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\mathbf{B}}\right)}$$

Note: Higher-order multipole contributions require the reduced-fluid Lagrangian density to depend on gradients of the electromagnetic fields (not considered here).

We note that gyrofluid models and gyrokinetic models implicitly contain all multipole polarization and magnetization contributions whenever finite-Larmor-radius effects are retained to all orders.

- **Reduced-fluid kinetic momentum and energy**

$$\left(\mathbf{p},\, K\right) \;\equiv\; \left(n^{-1}\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\mathbf{u}},\;\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial n}\right)$$

- **Reduced-fluid (symmetric) pressure tensor**

$$\mathsf{P}_{*} \;\equiv\; -\,2\,p_{\|}\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial p_{\|}}\,\widehat{\mathsf{b}}\widehat{\mathsf{b}} \;-\; p_{\perp}\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial p_{\perp}}\;\left(\mathbf{I}-\widehat{\mathsf{b}}\widehat{\mathsf{b}}\right)$$

Note: Regular fluid $\mathcal{L}_{\mathsf{F}} \equiv mn\,|\mathbf{u}|^2/2 - \mathrm{Tr}(\mathsf{P})$

$$(\mathbf{p}, K, \mathsf{P}_{*}) \;\equiv\; \left(m\mathbf{u},\, m|\mathbf{u}|^2/2,\, \mathsf{P}\right)$$

REDUCED VARIATIONAL PRINCIPLE

- **Constraint Equations**

◦ **Electromagnetic constraints**

$$\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0$$

$$\boxed{\delta\mathbf{E} \equiv -\nabla\delta\Phi - c^{-1}\partial\delta\mathbf{A}/\partial t \quad \text{and} \quad \delta\mathbf{B} \equiv \nabla \times \delta\mathbf{A}}$$

- **Fluid constraints for $\eta^a \equiv (n, p_{\|}, p_\perp)$**

$$
\begin{aligned}
\frac{\partial n}{\partial t} &= -\nabla\cdot\left(n\,\mathbf{u}\right) \\
\frac{\partial p_{\|}}{\partial t} &= -\nabla\cdot\left(p_{\|}\,\mathbf{u}\right) - 2\,p_{\|}\,\widehat{\mathsf{b}}\widehat{\mathsf{b}} : \nabla\mathbf{u} \\
\frac{\partial p_\perp}{\partial t} &= -\nabla\cdot\left(p_\perp\,\mathbf{u}\right) - p_\perp\left(\mathbf{I} - \widehat{\mathsf{b}}\widehat{\mathsf{b}}\right) : \nabla\mathbf{u}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\delta n &= -\nabla\cdot(n\,\boldsymbol{\xi}) \\
\delta\mathbf{u} &= (\partial/\partial t + \mathbf{u}\cdot\nabla)\,\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{u} \\
\delta p_\perp &= -\nabla\cdot(p_\perp\,\boldsymbol{\xi}) - p_\perp\,(\mathbf{I} - \widehat{\mathsf{b}}\widehat{\mathsf{b}}) : \nabla\boldsymbol{\xi} \\
\delta p_{\|} &= -\nabla\cdot\left(p_{\|}\,\boldsymbol{\xi}\right) - 2\,p_{\|}\,\widehat{\mathsf{b}}\widehat{\mathsf{b}} : \nabla\boldsymbol{\xi}
\end{aligned}
}
$$

where $\boldsymbol{\xi}$ generates a virtual spatial displacement for a fluid element of each particle species

- **Eulerian Variation of Reduced Lagrangian Density**

$$
\begin{aligned}
\delta\mathcal{L} \;&\equiv\; \delta n\,\frac{\partial\mathcal{L}}{\partial n} \;+\; \cdots \;+\; \delta\mathbf{B}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{B}} \\
&=\; \delta\Phi\left(\frac{\partial\mathcal{L}}{\partial\Phi} \;+\; \nabla\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right) \;+\; \delta\mathbf{A}\cdot\left(\frac{\partial\mathcal{L}}{\partial\mathbf{A}} \;+\; \frac{1}{c}\,\frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\mathbf{E}} \;+\; \nabla\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right) \\
&\quad -\; \sum\,\boldsymbol{\xi}\cdot\left[\,\frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\mathbf{u}} \;+\; \nabla\cdot\left(\mathbf{u}\,\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right) \;+\; \nabla\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right. \\
&\quad \left. -\; \left(\eta^{a}\,\nabla\frac{\partial\mathcal{L}}{\partial\eta^{a}}\right) \;+\; \nabla\cdot\mathsf{P}_{*}\,\right] \;+\; \underbrace{\frac{\partial\Lambda}{\partial t} \;+\; \nabla\cdot\boldsymbol{\Gamma}}_{\text{Noether}},
\end{aligned}
$$

- **Reduced variational principle**

$$
\boxed{\int\,\delta\mathcal{L}\;d^{4}x \;\equiv\; 0}
$$

EULER-POINCARE EQUATIONS

- **Reduced Maxwell Equations**

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \Phi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \\
0 &= \frac{\partial \mathcal{L}}{\partial \mathbf{A}} + \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{E}} \right) + \nabla \times \frac{\partial \mathcal{L}}{\partial \mathbf{B}}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\nabla \cdot \mathbf{D} &= 4\pi\, \varrho \\
\nabla \times \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{J}
\end{aligned}
}
$$

- ∘ **Microscopic form**

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &\equiv 4\pi \left(\varrho - \nabla \cdot \mathbf{P} \right) \\
\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &\equiv \frac{4\pi}{c} \left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + c\, \nabla \times \mathbf{M} \right)
\end{aligned}
$$

- **Reduced Force Equation**

$$0 = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right) + \nabla\cdot\left(\mathbf{u}\,\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right) + \nabla\mathbf{u}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{u}}$$
$$- \left(\eta^a\,\nabla\frac{\partial \mathcal{L}}{\partial \eta^a}\right) + \nabla\cdot\mathsf{P}_*$$

$$\boxed{\begin{aligned}\frac{d\mathbf{p}}{dt} &= q\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right)\\ &\quad + (\nabla K - \nabla\mathbf{u}\cdot\mathbf{p})\\ &\quad + n^{-1}\left(p_\perp\,\nabla\frac{\partial \mathcal{L}_\mathsf{F}}{\partial p_\perp} + p_\|\,\nabla\frac{\partial \mathcal{L}_\mathsf{F}}{\partial p_\|} - \nabla\cdot\mathsf{P}_*\right)\end{aligned}}$$

- **Regular particle-fluid model**

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right) - n^{-1}\,\nabla\cdot\mathsf{P}$$

ENERGY-MOMENTUM CONSERVATION LAWS BY NOETHER METHOD

- **Noether Equation**

$$\boxed{\delta\mathcal{L} = \frac{\partial\Lambda}{\partial t} + \nabla\cdot\boldsymbol{\Gamma}}$$

- **Noether components**

$$\Lambda = \sum \boldsymbol{\xi}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - \frac{1}{c}\,\delta\mathbf{A}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}}$$

$$\begin{aligned}\boldsymbol{\Gamma} = & \sum\left[\mathbf{u}\left(\boldsymbol{\xi}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right) - \boldsymbol{\xi}\left(\eta^a\frac{\partial\mathcal{L}}{\partial\eta^a}\right) + \mathsf{P}_*\cdot\boldsymbol{\xi}\right] \\ & - \delta\Phi\,\frac{\partial\mathcal{L}}{\partial\mathbf{E}} + \delta\mathbf{A}\times\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\end{aligned}$$

- **Space-time translations**

$$\boldsymbol{\xi} \;=\; \delta\mathbf{x} \;-\; \mathbf{u}\,\delta t$$

$$\delta\mathcal{L} \;=\; -\,\delta t\,(\partial\mathcal{L}/\partial t - \partial'\mathcal{L}/\partial t) \;-\; \delta\mathbf{x}\cdot(\nabla\mathcal{L} - \nabla'\mathcal{L})$$

$$\delta\Phi \;=\; \delta\mathbf{x}\cdot\mathbf{E} \;-\; c^{-1}\partial\delta\chi/\partial t$$

$$\delta\mathbf{A} \;=\; \delta\mathbf{x}\times\mathbf{B} \;+\; c\,\delta t\,\mathbf{E} \;+\; \nabla\delta\chi$$

- **Gauge-dependent term**

$$\delta\chi \;\equiv\; c\,\delta t\,\Phi \;-\; \delta\mathbf{x}\cdot\mathbf{A}$$

- **Background space-time dependence**

$$\left(\nabla'\mathcal{L}, \frac{\partial'\mathcal{L}}{\partial t}\right) \;\equiv\; \left(\nabla\mathcal{L} \;-\; \nabla\psi^{\alpha}\,\frac{\partial\mathcal{L}}{\partial\psi^{\alpha}},\; \frac{\partial\mathcal{L}}{\partial t} \;-\; \frac{\partial\psi^{\alpha}}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\psi^{\alpha}}\right)$$

- **Noether gauge-invariance transformation**

$$\boxed{\begin{aligned} \overline{\Lambda} &\equiv \Lambda + \nabla\cdot\left(\frac{\delta\chi}{c}\,\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right) \\ \overline{\mathbf{\Gamma}} &\equiv \mathbf{\Gamma} - \frac{\partial}{\partial t}\left(\frac{\delta\chi}{c}\,\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right) - c\,\nabla\times\left(\frac{\delta\chi}{c}\,\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right) \end{aligned}}$$

- **Gauge-invariant Noether components ($\overline{\mathcal{L}} \equiv \mathcal{L} - \mathcal{L}_{\Psi}$)**

$$\overline{\Lambda} \equiv -\frac{1}{c}\left(\delta\mathbf{x}\times\mathbf{B} + c\,\delta t\,\mathbf{E}\right)\cdot\frac{\mathbf{D}}{4\pi} + \sum\left[\left(\delta\mathbf{x} - \mathbf{u}\,\delta t\right)\cdot n\,\mathbf{p}\right]$$

$$\begin{aligned} \overline{\mathbf{\Gamma}} \equiv &\sum\left\{\left[\mathsf{P}_{*} + n\,\mathbf{u}\,\mathbf{p} - \left(\eta^{a}\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\eta^{a}}\right)\mathbf{I}\right]\cdot\left(\delta\mathbf{x} - \mathbf{u}\,\delta t\right)\right\} \\ &- \left[\mathbf{D}\,\mathbf{E} + \mathbf{B}\,\mathbf{H} - (\mathbf{B}\cdot\mathbf{H})\,\mathbf{I}\right]\cdot\frac{\delta\mathbf{x}}{4\pi} - \delta t\,\left(\frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}\right) \end{aligned}$$

- **Reduced Energy Conservation Law**

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\mathbf{S} = -\,\frac{\partial'\overline{\mathcal{L}}}{\partial t}}$$

◦ **Reduced energy density**

$$\mathcal{E} \equiv \sum n\,\mathbf{p}\cdot\mathbf{u} + \mathbf{E}\cdot\mathbf{P} + \frac{1}{8\pi}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) - \mathcal{L}_{\mathsf{F}}$$

◦ **Reduced energy-density flux**

$$\mathbf{S} \equiv \sum\left[\mathbf{u}\left(n\,\mathbf{p}\cdot\mathbf{u} - \eta^a\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\eta^a}\right) + \mathsf{P}_*\cdot\mathbf{u}\right] + \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}$$

- **Reduced Momentum Conservation Law**

$$\boxed{\frac{\partial \boldsymbol{\Pi}}{\partial t} + \nabla\cdot\mathsf{T} = \nabla'\overline{\mathcal{L}}}$$

- **Reduced momentum density**

$$\boldsymbol{\Pi} \equiv \sum n\,\mathbf{p} + \frac{\mathbf{D}\times\mathbf{B}}{4\pi\,c}$$

- **Reduced canonical momentum-stress tensor**

$$\mathsf{T} \equiv \left[\left(\mathcal{L}_{\mathsf{F}} - \sum \eta^{a}\,\frac{\partial\mathcal{L}_{\mathsf{F}}}{\partial\eta^{a}}\right) + \frac{1}{8\pi}\left(|\mathbf{E}|^{2} + |\mathbf{B}|^{2}\right) - \mathbf{B}\cdot\mathbf{M}\right]\mathbf{I} + \sum \mathsf{P}_{*} + \left[\sum n\,\mathbf{u}\,\mathbf{p} - \frac{1}{4\pi}\left(\mathbf{D}\,\mathbf{E} + \mathbf{B}\,\mathbf{H}\right)\right]$$

- **Regular particle-fluid momentum density and symmetric canonical momentum-stress tensor**

$$\mathbf{\Pi}_{\mathsf{p}} \;=\; \sum m n\, \mathbf{u} \;+\; \frac{\mathbf{E}\times\mathbf{B}}{4\pi\, c}$$

$$\mathsf{T}_{\mathsf{p}} \;=\; \frac{1}{4\pi}\left[\left(|\mathbf{E}|^2 \;+\; |\mathbf{B}|^2\right)\frac{\mathbf{I}}{2} \;-\; (\mathbf{E}\,\mathbf{E}+\mathbf{B}\,\mathbf{B})\right] \;+\; \sum\left(\mathsf{P} \;+\; m n\,\mathbf{u}\,\mathbf{u}\right)$$

Asymmetry of the reduced canonical momentum-stress tensor is due to the dynamical reduction that led to the reduced-fluid Lagrangian density $\mathcal{L}_{\mathsf{F}}$.

- **Reduced Toroidal Angular Momentum Equation**

  - **Axisymmetric tokamak geometry**

$$\frac{\partial' \overline{\mathcal{L}}}{\partial \varphi} \equiv \frac{\partial \mathbf{x}}{\partial \varphi} \cdot \nabla' \overline{\mathcal{L}} = 0$$

  - **Reduced toroidal momentum density** $\Pi_\varphi \equiv \mathbf{\Pi} \cdot \partial \mathbf{x} / \partial \varphi$

$$\frac{\partial \Pi_\varphi}{\partial t} = -\frac{\partial \mathbf{x}}{\partial \varphi} \cdot \left(\nabla \cdot \mathsf{T} \right) = -\nabla \cdot \left(\mathsf{T} \cdot \frac{\partial \mathbf{x}}{\partial \varphi} \right) + \mathsf{T}^\top : \nabla \frac{\partial \mathbf{x}}{\partial \varphi}$$

$$\boxed{\frac{\partial \Pi_\varphi}{\partial t} + \nabla \cdot \left(\mathsf{T} \cdot \frac{\partial \mathbf{x}}{\partial \varphi} \right) = -\mathsf{T}_\mathsf{A} : \nabla \frac{\partial \mathbf{x}}{\partial \varphi} \equiv \hat{\mathsf{z}} \cdot \boldsymbol{\tau}}$$

  - **Reduced-asymmetry torque**

$$\boldsymbol{\tau} \equiv \sum n \, \mathbf{u} \times \mathbf{p} + \left(\mathbf{E} \times \mathbf{P} + \mathbf{B} \times \mathbf{M} \right)$$

- **Surface-averaged reduced toroidal-momentum transport equation**

$$\boxed{\frac{\partial\langle\Pi_\varphi\rangle}{\partial t} + \frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\left\langle\nabla\psi\cdot\mathsf{T}\cdot\frac{\partial\mathbf{x}}{\partial\varphi}\right\rangle\right) = \left\langle\widehat{\mathsf{z}}\cdot\boldsymbol{\tau}\right\rangle}$$

$$\langle\cdots\rangle \equiv \mathcal{V}^{-1}\oint(\cdots)\,\mathcal{J}\,d\vartheta\,d\varphi$$

$$\mathcal{J} \equiv (\nabla\psi\times\nabla\vartheta\cdot\nabla\varphi)^{-1}$$

$$\mathcal{V} \equiv \oint\mathcal{J}\,d\vartheta\,d\varphi$$

$$\boxed{\nabla\psi\cdot\mathsf{T}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} \equiv \nabla\psi\cdot\left[\sum n\,\mathbf{u}\,\mathbf{p} - \frac{1}{4\pi}(\mathbf{D}\,\mathbf{E} + \mathbf{B}\,\mathbf{H})\right]\cdot\frac{\partial\mathbf{x}}{\partial\varphi}}$$

REDUCED GYROFLUID MODEL

(Brizard, 2005 & 2008)

- **Gyrofluid Lagrangian density**

$$\mathcal{L} = \frac{1}{8\pi}\left(|\mathbf{E}_\perp|^2 - |\mathbf{B}|^2\right) + \sum\left[\frac{q}{c} n\mathbf{u}\cdot\left(\mathbf{A}_0 + A_\| \,\widehat{\mathsf{b}}_0\right) + \frac{1}{2} mn\left|u_\|\left(\widehat{\mathsf{b}}_0 + \mathbf{B}_\perp/B_0\right) + \mathbf{u}_E\right|^2 - \left(q\,n\,\Phi + \mathcal{P}\right)\right]$$

$$\equiv \frac{1}{8\pi}\left(|\mathbf{E}_\perp|^2 - |\mathbf{B}|^2\right) + \sum\left[\frac{q}{c} n\mathbf{u}\cdot\left(\mathbf{A}_0 + A_{\|\rho}\,\widehat{\mathsf{b}}_0\right) + \frac{1}{2} mn\,u_\|^2 - \left(q\,n\,\Phi_\rho + n\,\mathcal{K}_\rho + \mathcal{P}\right)\right]$$

- **Perturbed magnetic fields**

$$\mathbf{B}_\perp = \nabla\times\left(A_\|\,\widehat{\mathsf{b}}_0\right) \text{ and } \mathbf{B}_{\perp\rho} \equiv \nabla\times\left(A_{\|\rho}\,\widehat{\mathsf{b}}_0\right)$$

- **Perturbed electric fields**

$$\mathbf{E} = -\nabla\Phi - \frac{\widehat{\mathsf{b}}_0}{c}\frac{\partial A_\|}{\partial t} \text{ and } \mathbf{E}_\rho = -\nabla\Phi_\rho - \frac{\widehat{\mathsf{b}}_0}{c}\frac{\partial A_{\|\rho}}{\partial t}$$

- **Nonlinear FLR-corrected potentials**

$$\begin{pmatrix} \Phi_{\rho} \\ \\ A_{\|\rho} \end{pmatrix} \equiv \begin{pmatrix} \Phi - \boldsymbol{\rho}_{\perp} \cdot \mathbf{E}_{\perp} \\ \\ A_{\|} - \widehat{\mathsf{b}}_0 \cdot \boldsymbol{\rho}_{\perp} \times \mathbf{B}_{\perp} \end{pmatrix}$$

- **Low-frequency ponderomotive potential**

$$\mathcal{K}_{\rho} \equiv \frac{1}{2} m \, \Omega_0^2 \, |\boldsymbol{\rho}_{\perp}|^2 \equiv \frac{m}{2} |\mathbf{U}_{\perp}|^2$$

- **Gyrofluid displacement**

$$\boldsymbol{\rho}_{\perp} = \frac{c}{B_0 \Omega_0} \left(\mathbf{E}_{\perp} + \frac{u_{\|}}{c} \widehat{\mathsf{b}}_0 \times \mathbf{B}_{\perp} \right) \equiv \frac{\widehat{\mathsf{b}}_0}{\Omega_0} \times \mathbf{U}_{\perp}$$

- **Reduced kinetic energy**

$$K \equiv \frac{m}{2} u_\parallel^2 + \mathcal{K}_\rho$$

- **Reduced kinetic momentum**

$$\mathbf{p} = m \left(u_\parallel + \mathbf{U}_\perp \cdot \frac{\mathbf{B}_\perp}{B_0} \right) \widehat{\mathsf{b}}_0 \equiv m \, u_\parallel^* \, \widehat{\mathsf{b}}_0$$

- **Reduced polarization and magnetization**

$$\mathbf{P} \equiv \sum qn \, \boldsymbol{\rho}_\perp = \sum mn \, \frac{c\widehat{\mathsf{b}}_0}{B_0} \times \mathbf{U}_\perp$$

$$\mathbf{M} \equiv \sum qn \, \boldsymbol{\rho}_\perp \times \frac{u_\parallel}{c} \widehat{\mathsf{b}}_0 = \sum mn \, \frac{u_\parallel}{B_0} \, \mathbf{U}_\perp$$

- **Reduced Euler-Poincaré Fluid Equation**

$$\boxed{mn\,\widehat{\mathsf{b}}_0\,\frac{du_\|}{dt} \;=\; qn\,\left(\mathbf{E}_\rho + \frac{\mathbf{u}}{c}\times\mathbf{B}_\rho^*\right) \;-\; \nabla\cdot\mathsf{P}_\rho}$$

$$\mathbf{B}_\rho^* \;\equiv\; \mathbf{B}_0 \;+\; u_\|\,(B_0/\Omega_0)\,\nabla\times\widehat{\mathsf{b}}_0 \;+\; \mathbf{B}_{\perp\rho}$$

- **Gyrofluid parallel-force equation**

$$mn\,\frac{du_\|}{dt} \;=\; \frac{\mathbf{B}_\rho^*}{B_0}\cdot\left(qn\,\mathbf{E}_\rho \;-\; \nabla\cdot\mathsf{P}_\rho\right) \;\equiv\; \mathsf{b}_\rho^*\cdot\left(qn\,\mathbf{E}_\rho \;-\; \nabla\cdot\mathsf{P}_\rho\right)$$

- **Gyrofluid fluid velocity**

$$\mathbf{u} \;=\; u_\|\,\mathsf{b}_\rho^* \;+\; \left(qn\,\mathbf{E}_\rho \;-\; \nabla\cdot\mathsf{P}_\rho\right)\times\frac{\widehat{\mathsf{b}}_0}{mn\,\Omega_0}$$

$$\equiv\;\; \text{parallel} + \text{curvature} + \mathrm{E}\times\mathrm{B} + \cdots$$

- **Gyrofluid Momentum Conservation Law**

  - **Gyrofluid momentum density**

$$\boldsymbol{\Pi} \;=\; \sum m\,n\,\left(u_{\|}\,\widehat{\mathsf{b}}_0 \;+\; \mathbf{U}_{\perp}\right) \;+\; \frac{\mathbf{E}\times\mathbf{B}}{4\pi\;c}$$

$$\;=\; m\,n\,\left(u_{\|} \;+\; \frac{\mathbf{B}_{\perp}}{B_0}\cdot\mathbf{U}_{\perp}\right)\widehat{\mathsf{b}}_0 \;+\; \frac{\mathbf{D}\times\mathbf{B}}{4\pi\;c}$$

  - **Gyrofluid canonical momentum-stress tensor**

$$\mathsf{T} \;=\; \mathbf{I}\left[\frac{1}{8\pi}\left(|\mathbf{E}_{\perp}|^2 + |\mathbf{B}|^2\right) \;-\; \mathbf{B}\cdot\mathbf{M}\right] \;+\; \sum\;\left(mn\;u_{\|}^{*}\,\mathbf{u}\,\widehat{\mathsf{b}}_0 \;+\; \mathsf{P}\right)$$

$$-\;\frac{1}{4\pi}\left(\mathbf{D}\,\mathbf{E} \;+\; \mathbf{B}\,\mathbf{H}\right)$$

  - **Reduced toroidal angular-momentum flux**

$$\nabla\psi\cdot\mathsf{T}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} \;\equiv\; \nabla\psi\cdot\left[\sum\;mn\;u_{\|}^{*}\,\mathbf{u}\,\widehat{\mathsf{b}}_0 \;-\; \frac{1}{4\pi}\left(\mathbf{D}\,\mathbf{E} \;+\; \mathbf{B}_{\perp}\,\mathbf{H}\right)\right]\cdot\frac{\partial\mathbf{x}}{\partial\varphi}$$

- **Gyrofluid Momentum-stress Asymmetry**

$$\boxed{\boldsymbol{\tau} \equiv \sum mn\, u_\parallel^* \mathbf{u} \times \widehat{\mathsf{b}}_0 + \mathbf{B}_0 \times \mathbf{M} + (\mathbf{E}_\perp \times \mathbf{P} + \mathbf{B}_\perp \times \mathbf{M})}$$

  - Nonlinear electromagnetic asymmetry vanishes

$$\mathbf{E}_\perp \times \mathbf{P} \;=\; =\; mn\, u_\parallel \left(\frac{c\mathbf{E}_\perp}{B_0} \cdot \frac{\mathbf{B}_\perp}{B_0} \right) \widehat{\mathsf{b}}_0 \;=\; -\, \mathbf{B}_\perp \times \mathbf{M}$$

  - Linear magnetization asymmetry

$$\mathbf{B}_0 \times \mathbf{M} \;=\; \sum \frac{qn\, u_\parallel}{\Omega_0} \left(\mathbf{E}_\perp \;+\; \frac{u_\parallel}{c} \widehat{\mathsf{b}}_0 \times \mathbf{B}_\perp \right)$$

  - Gyrofluid asymmetry

$$\sum mn\, u_\parallel^* \, \mathbf{u} \times \widehat{\mathsf{b}}_0 \;=\; -\, \sum \frac{u_\parallel^*}{\Omega_0} \Big[qn \left(\mathbf{E}_{\perp\rho} \;+\; \frac{u_\parallel}{c} \widehat{\mathsf{b}}_0 \times \mathbf{B}^*_{\perp\rho} \right) \\ -\; (\nabla \cdot \mathsf{P}_\rho)_\perp \Big]$$

- **Net gyrofluid momentum-stress asymmetry**

$$\boldsymbol{\tau} \equiv \boldsymbol{\tau}_E + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P$$

$$\boxed{\begin{aligned}
\boldsymbol{\tau}_E &= \sum \frac{qn}{\Omega_0} \left(u_\parallel \mathbf{E}_\perp - u_\parallel^* \mathbf{E}_{\perp\rho} \right) \\
\boldsymbol{\tau}_B &= \sum \frac{qn}{\Omega_0} \frac{u_\parallel \widehat{\mathsf{b}}_0}{c} \times \left(u_\parallel \mathbf{B}_\perp - u_\parallel^* \mathbf{B}_{\perp\rho}^* \right) \\
\boldsymbol{\tau}_P &= \sum \frac{u_\parallel^*}{\Omega_0} \left[(\nabla \cdot \mathsf{P})_\perp + mn \, \nabla_\perp \mathbf{U}_\perp \cdot \mathbf{U}_\perp \right]
\end{aligned}}$$

$$\mathbf{E}_{\perp\rho} = \mathbf{E}_\perp + \nabla_\perp \left(\boldsymbol{\rho}_\perp \cdot \mathbf{E}_\perp \right)$$

$$\mathbf{B}_{\perp\rho}^* = \mathbf{B}_\perp + \frac{\mathbf{B}_0}{\Omega_0} \times \left[u_\parallel^* \left(\widehat{\mathsf{b}}_0 \cdot \nabla \widehat{\mathsf{b}}_0 \right) + \nabla \left(u_\parallel^* - u_\parallel \right) \right]$$

CONCLUSIONS AND FUTURE WORK

- Gyrofluid dynamical reduction introduces net momentum-stress asymmetries that may drive spontaneous toroidal rotation in axisymmetric toroidal magnetized plasmas.

Fictitious or Real Gyrofluid Torques ?

- Need to calculate $\langle \widehat{z} \cdot \boldsymbol{\tau} \rangle$ in gyrofluid model.

- On-going work includes gyrokinetic momentum conservation laws.