Lagrangian Chaos and the Evolution of Advected Fields

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Problem Class - Advected Fields

Passive Scalar: $$\frac{\partial}{\partial t}\phi(\mathbf{x},t)+\nabla\left[\cdot\mathbf{v}\phi(\mathbf{x},t)\right]=\kappa\nabla^2\phi(\mathbf{x},t)+S$$

Kinematic Dynamo: $$\frac{\partial}{\partial t}\mathbf{B}+\mathbf{v}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{v}+\eta\nabla^2\mathbf{B}$$

Navier Stokes: $$\frac{\partial}{\partial t}\omega+\mathbf{v}\cdot\nabla\omega=\nu\nabla^2\omega+\mathbf{S} \qquad \omega=\nabla\times\mathbf{v}$$

Common Features: Field transported by flow
Dissipation at smallest scale length

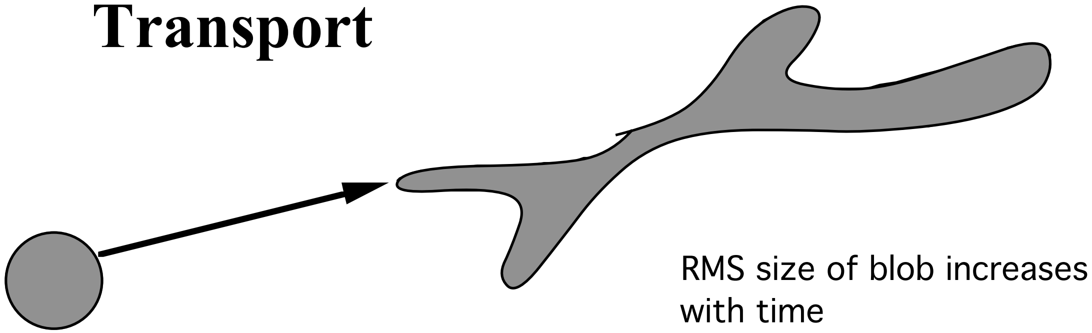
$\kappa,\quad\eta,\quad\nu$ Small

Important Restriction: Spatial scale of transported fields is smaller than that of flow

Question: What are the properties of the transported field in cases in which v($\mathbf{x}$,t) gives rise to chaotic fluid orbits?

Transport versus Mixing

Transport



Examples: Transport in destroyed magnetic surfaces

M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, G. M. Zaslavsky, Nucl. Fusion **6**, 297 (1966).
A. B. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).

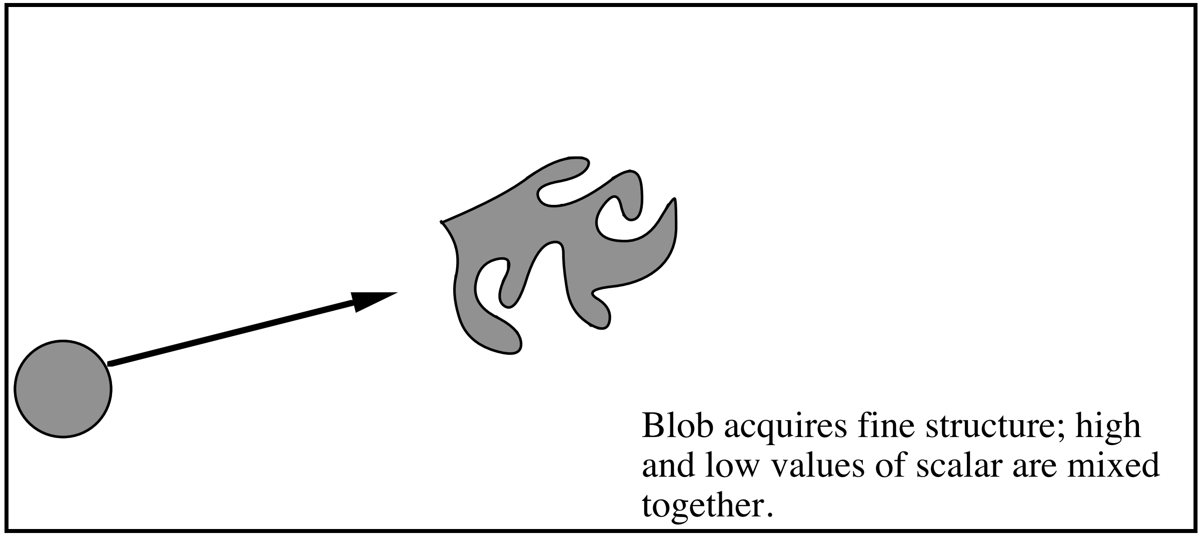
Enhanced Diffusion

$$\langle \, |\mathbf{x}|^2 \, \Phi(\mathbf{x},t) \, \rangle \approx 2 \, D_{enhanced} t$$

Blob becomes larger than spatial scale of flow

Transport versus Mixing

Mixing



Reviews of Chaos and Mixing:

H. Aref, J. Fluid Mech. **143**, 1 (1984)

J. M. Ottino, *The Kinematics of Mixing: Stretching, Chaos, and Transport* (Cambridge U. P., 1989)

IUTAM Symposium of Fluid Mechanics of Stirring and Mixing, Phys Fluids **A3**, 1009-1469 (1991).

Decay of Passive Scalar

Passive Scalar: $$\frac{\partial}{\partial t}\phi(\boldsymbol{x},t)+\boldsymbol{v}\cdot\nabla\phi(\boldsymbol{x},t)=\kappa\nabla^2\phi(\boldsymbol{x},t)$$

Initial value of Scalar: $\phi(\boldsymbol{x},0)=\phi_0(\boldsymbol{x})$ Periodic with period L_D

Fluid flow velocity: $\boldsymbol{v}(\boldsymbol{x},t)$ Periodic with period L_f

Microscopic diffusion: κ

Fluid trajectories: $$\frac{d\boldsymbol{x}_i(t)}{dt}=\boldsymbol{v}(\boldsymbol{x}_i,t)$$ Chaotic for almost all initial conditions

Question: What is the long time behavior of ϕ in the limit $\kappa \to 0$?

Time Decay of Scalar

- **Exponential decay of *Φ*, "Strange Eigenfunction"**

  R. T. Pierrehumbert, Chaos Solitons and Fractals 4, 1091 (1994).

- **Decay rate predicted based on local stretching rates.**

  TMA, Fan and Ott, PRL **75**, 1751 (1995).

- **Validity of local stretching theory? Decay rate determined by longest scale.**

  J. Sukhatme and R. T. Pierrehumbert (2002)
  J. -L. Thiffeault and S. Childress (2003)
  D. R. Fereday, P. H. Haynes, A. Wonhas, J. C. Vasilicos (2002)
  D. R. Fereday and P. H. Haynes, Phys. Fluids 16, (2004)

- **Experiment: decay determined by spatial diffusion**

  Voth et al. Phys Fluids (2003).

- **Properties of strange eigenfunctions**

  Chertkov and Lebedev, PRL 90, (2003)
  Balkovsky and Flouxon,Phys Rev. E 60 (1999).
  A. Pikovsky and O. Popovych, Europhys. Lett. 61, 625 (2003).
  A. A. Schekocihin, P. H. Haynes and S. C. Cowley, PRE 046304 (2004).
  Haynes and Vanneste, Phys Fluids 17, 097103 (2005)

What do you want to know?

Decay rate $\Phi(\mathbf{r},t) \sim e^{-\gamma t}$

Power Spectrum

Fourier transform of two point correlation function $C(\mathbf{k},t) = \int d^2r\ e^{-i\mathbf{k}\cdot\mathbf{r}} \langle \Phi(\mathbf{x}+\mathbf{r})\ \Phi(\mathbf{x}) \rangle_x$

Averaged over angle in k-space $F(k,t) = \int \frac{d^2k'}{(2\pi)^2}\ \delta(k - |\mathbf{k}'|)\ C(\mathbf{k}',t)$

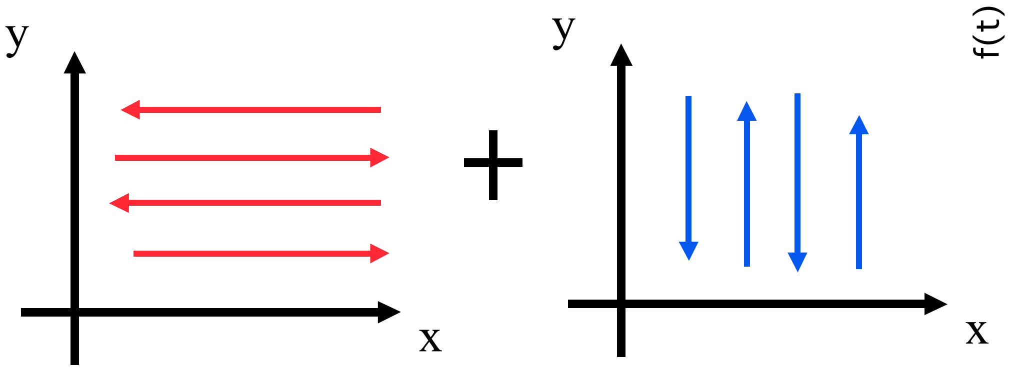
Structure Function $S_{2q}(r) = \left\langle \left| \Phi(\mathbf{x}+\mathbf{r},t) - \Phi(\mathbf{x},t) \right|^{2q} \right\rangle \sim r^{\varsigma_{2q}}$

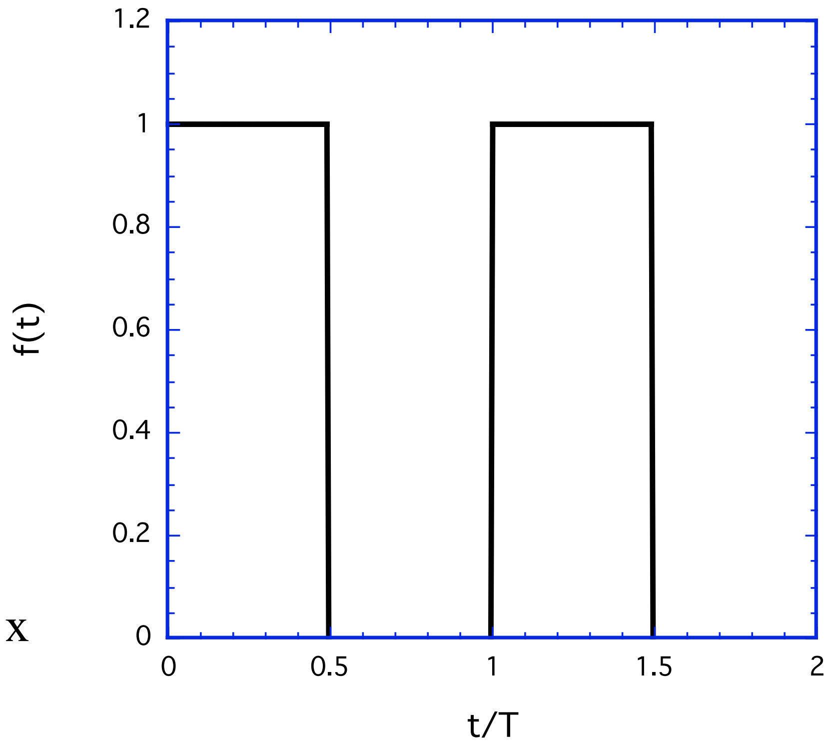
Fractal Properties of Dissipation Field $\kappa |\nabla\Phi|^2$

All can be tied to the properties of the underlying flow

2D Model Flow

$$\boldsymbol{v}(x,t) = U\begin{bmatrix} \hat{\boldsymbol{x}}\, f(t)\cos\left(2\pi y / L_f + \theta_1(t)\right) \\ +\hat{\boldsymbol{y}}\left(1 - f(t)\right)\cos\left(2\pi x / L_f + \theta_2(t)\right) \end{bmatrix}$$





Normalized displacement: $\dfrac{UT}{2L_f}$

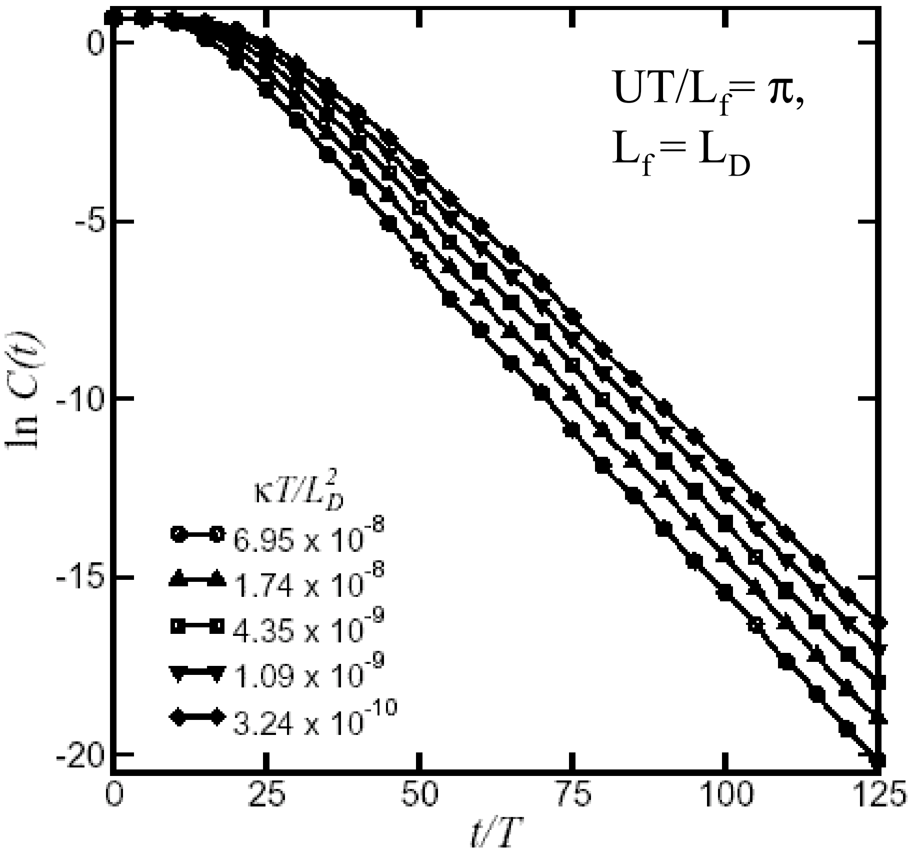
Random angles: θ_1(t) and θ_2(t)
Constant values randomly chosen for each interval n+1 > t/T ≥ n

Sample Solutions - max resolution $(6.4 \times 10^4)^2$

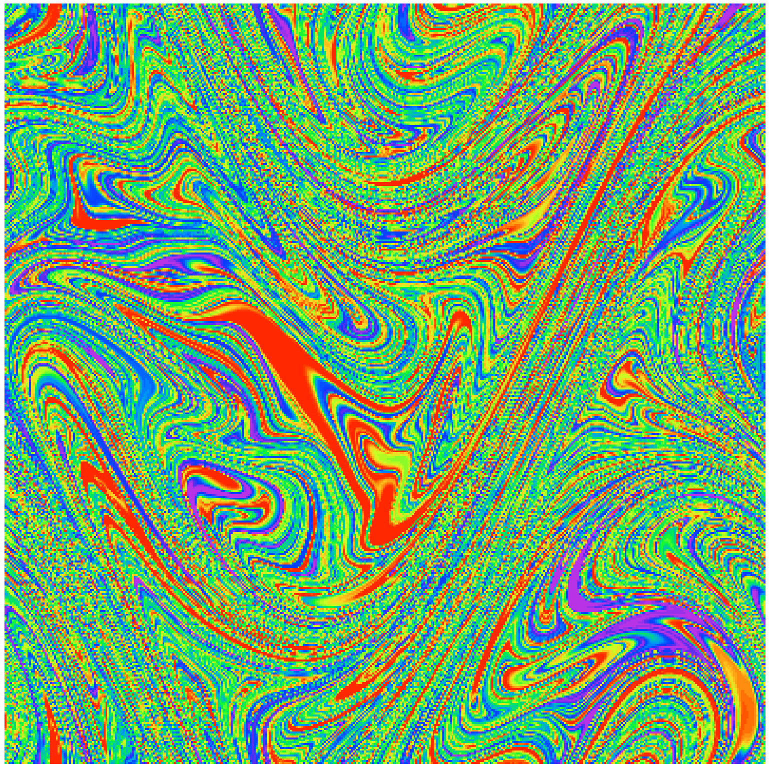
Tsang et al. PRE **71**, 066301 (2005).

Initial condition $\phi_0 = 2 sin\left[2\pi(x+y)/L_D\right]$

Scalar variance: $C(t) = \frac{1}{2L_d^2}\int dxdy\phi^2$



t = 20T



Wave Packet Model -Reduced k Spectrum [RKS]

Scale length of Φ is much smaller than scale length of v.
WKB description is suggested
Action Density: N(**k**,**x**,t),

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\frac{\partial \omega}{\partial \mathbf{k}} N\right) - \frac{\partial}{\partial \mathbf{k}}\cdot\left(\frac{\partial \omega}{\partial \mathbf{x}} N\right) = -2\kappa k^2 N$$

Dispersion relation: $\omega = \mathbf{k}\cdot\mathbf{v}$

Characteristics:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \omega}{\partial \mathbf{k}} = \mathbf{v}(\mathbf{x},t) \quad , \quad \frac{dk}{dt} = -\frac{\partial \omega}{\partial \mathbf{x}} = -\nabla \mathbf{v}\cdot\mathbf{k}$$

Wave Packet Model

TMA, Z. Fan, E. Ott and E Garcia-Lopez, Phys Fluids 8, 3094 (1996).

- Power spectrum constructed from an ensemble of trajectories labeled by the index j:

$$F(k,t) = \int \frac{d^2k'}{(2\pi)^2}\, \delta(k - |\mathbf{k}'|)\; C(\mathbf{k}',t)$$

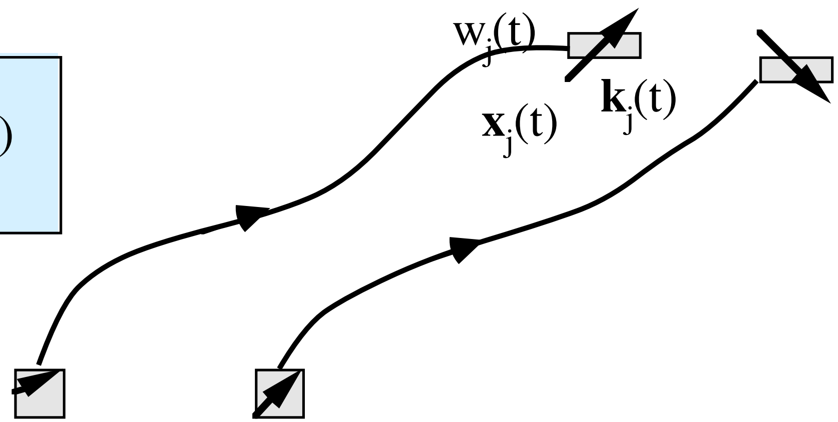
$$F(k,t) = \sum_j w_j(t)\; \delta(k - |k_j(t)|)$$

$$w_j(t) = w_j(0) \exp[\, -2\kappa \int_0^t dt'\; k_j^2(t')]$$

- $w_j(t)$ is the scalar variance associated with the j^{th} trajectory.

Trajectory Equations:

$$\frac{d\,\mathbf{x}_j(t)}{dt} = \mathbf{v}(\mathbf{x}_j(t),t)\;, \qquad \frac{d\mathbf{k}_j(t)}{dt} = -\nabla\mathbf{v}(\mathbf{x}_j(t),t)\cdot\mathbf{k}_j(t)$$



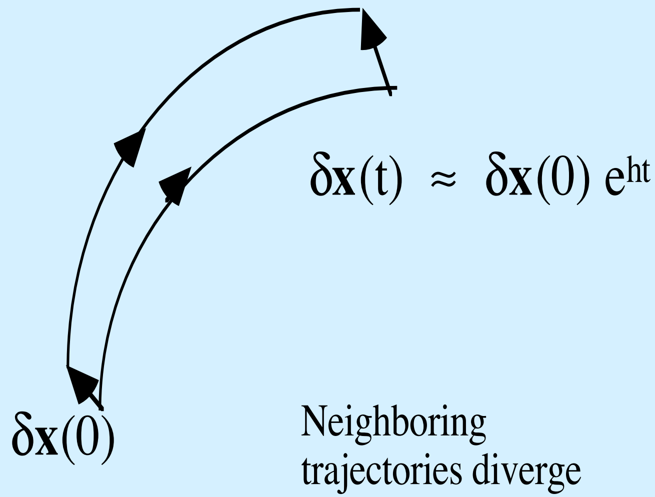
Local Stretching Theory

Chaotic Orbits

Fluid trajectories: $\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t),t)$

Differential separation: $\frac{d\delta\mathbf{x}(t)}{dt} = \delta\mathbf{x}(t)\cdot\nabla\,\mathbf{v}(\mathbf{x},t)$

Chaotic: $|\delta\mathbf{x}(t)| \approx |\delta\mathbf{x}(0)|\, e^{ht}$, $h>0$



Compare with Eq. for $k_j(t)$: $\frac{d}{dt}\mathbf{k}_j(t) = -\nabla\mathbf{v}(\mathbf{x},t)\cdot\mathbf{k}_j$

$$\frac{d}{dt}\left(\delta\mathbf{x}\cdot\mathbf{k}_j\right) = 0$$

Lyapunov exponent:

$$\bar{h} = \lim_{t\to\infty} \frac{1}{t}\, ln\left|\frac{\delta\mathbf{x}(t)}{\delta\mathbf{x}(0)}\right|$$

Lyapunov exponent is the same for almost all initial conditions in a given chaotic region

Diverging and Converging Orbits

Differential separation: $$\frac{d\delta\mathbf{x}(t)}{dt} = \delta\mathbf{x}(t)\cdot\nabla\,\mathbf{v}(\mathbf{x},t)$$

n Dimensions (n = 2,3)

n solutions for $\delta\mathbf{x}$ $\quad \delta\mathbf{x}_1, \delta\mathbf{x}_2, \ldots \delta\mathbf{x}_n, \qquad \bar{h}_1, \bar{h}_2, \ldots \bar{h}_n$

Incompressible flows $\quad \sum_i \bar{h}_i = 0$

Converging solution $\quad \bar{h}_n < 0 \qquad [\text{In 2D } |h_2| = h_1 = h\,]$

Due to converging solution $\nabla\Phi(\mathbf{x},t)$ grows exponetially

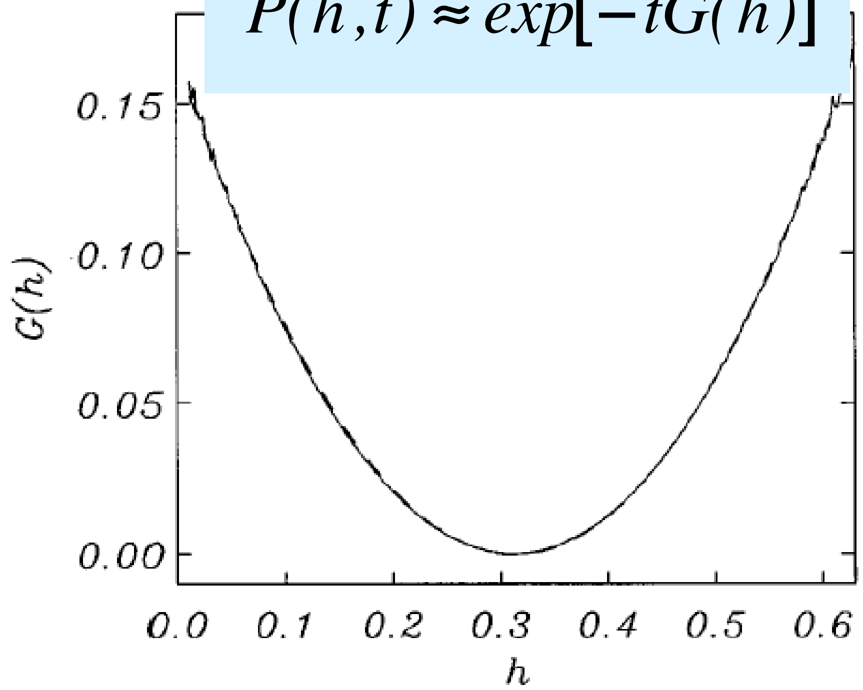
$$|\nabla\Phi(\mathbf{x},t)| \approx |\nabla\Phi(\mathbf{x}_0,0)|\,\exp[\,|h_2|\,t]$$

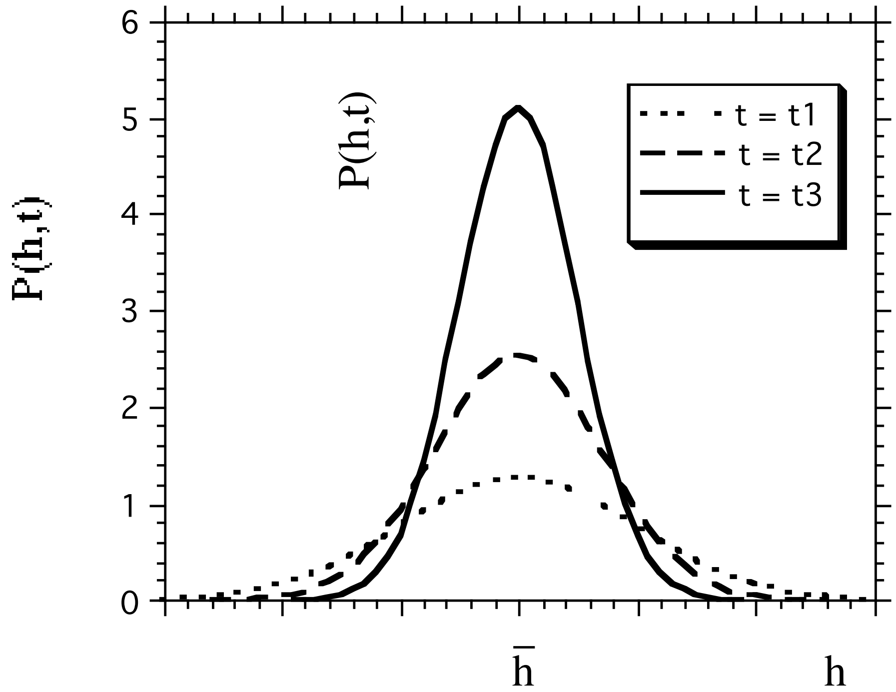
Finite Time Lyapunov Exponents

Finite time Lyapunov exponent: $h(x_0,t) = \frac{1}{t}\, ln \left| \frac{\delta \mathbf{x}(t)}{\delta \mathbf{x}(0)} \right|$ $\bar{h} = \lim_{t\to\infty} h(\mathbf{x}_0,t)$

Finite time Lyapunov exponents are characterized by a distribution, P(h,t).

$$P(h,t) \approx exp\left[-tG(h)\right]$$



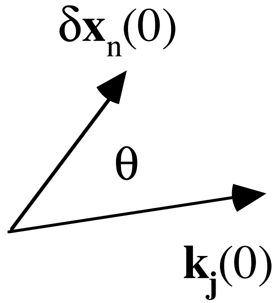


Long Time Decay Based on Local Stretching

$$F(k,t) = \sum_j w_j(t)\ \delta(k - |k_j(t)|)$$

$$w_j(t) = w_j(0)\exp[-2\kappa \int_0^t dt'\ k_j^2(t')]$$

Typical trajectory: $k_j(t)$ grows exponentially, leads to faster than exponential decay



$$|\mathbf{k}_\mathbf{j}(t)| \approx \cos\theta |\mathbf{k}_\mathbf{j}(0)| \exp(ht)$$

$$P(h,t) \approx exp[-tG(h)]$$

Dominant contribution from $\theta = \pi/2$
$k_j(0)$ is perpendicular to contracting direction.

Predicted decay rate: $\gamma_0 = min_{h \geq 0}[h + G(h)]$

Decay Rate Evaluation

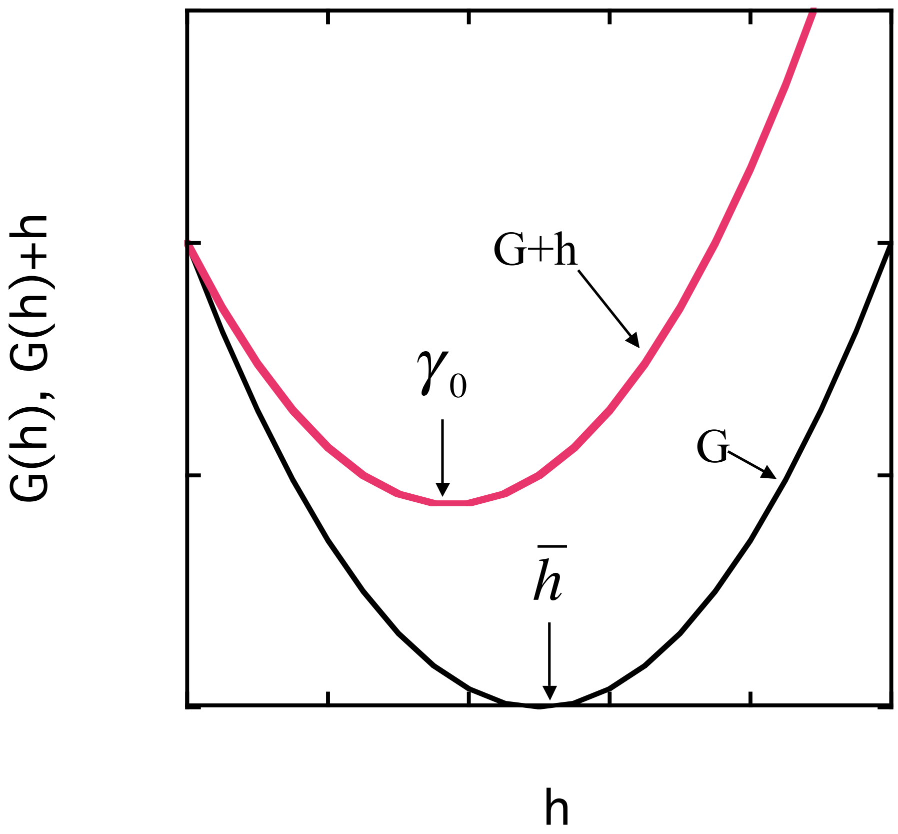
$$\exp[-\gamma t] \sim \int dh d\theta \, P(h,t) \exp[-2\kappa k_0^2 e^{2ht} \cos^2\theta]$$

$$P(h,t) \approx exp[-tG(h)]$$

Evaluate by steepest descent

Predicted decay rate:

$$\gamma_0 = min_{h \geq 0}[h + G(h)]$$

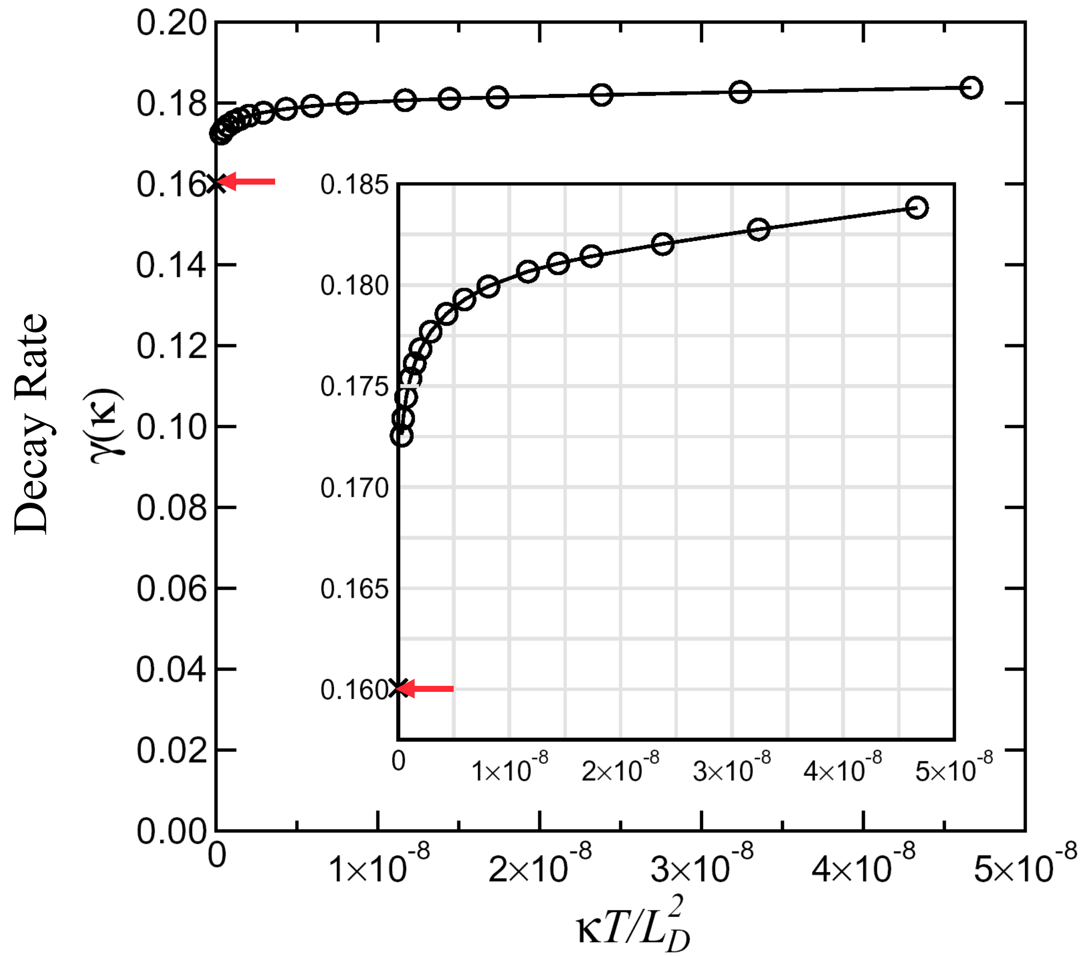


Comparison with Numerics

$UT/L_f = \pi$,
$L_f = L_D$

$$\gamma_0 = min_{h \geq 0}[h + G(h)]$$

Logarithmic correction,
Haynes and Vanneste,
Shekochihin, Haynes and Cowley



Upper Bound on Decay in κ→0 Limit

Tsang, TMA, and Ott, PRE 71 (2005)

For Strange Eigenfunctions: $L_f^{-1} \ll k \ll \left(\bar{h}\kappa\right)^{-1/2}$

$$S_0(k)\,exp\left[-\gamma t\right] = \int_0^\infty dk' S_0(k') \left\langle \delta(k - k' \left|\partial \mathbf{x}(0) / \partial \mathbf{x}(t)\right| \right\rangle_{h,\theta}$$

Power spectrum of Scalar Eigenfunction

Actual Decay Rate

Local stretching of *k*

Assume Power Law: $Ak^{-\psi} > S_0(k) > Bk^{-\psi}$

Then can show: $\gamma = min_{h\geq 0}\left[h + G(h) - |\psi| h\right] < min_{h\geq 0}\left[h + G(h)\right] = \gamma_0$

$$\psi = 1 + min_{h\geq 0}\left(\frac{G(h) - \gamma}{h}\right)$$

Power Spectra

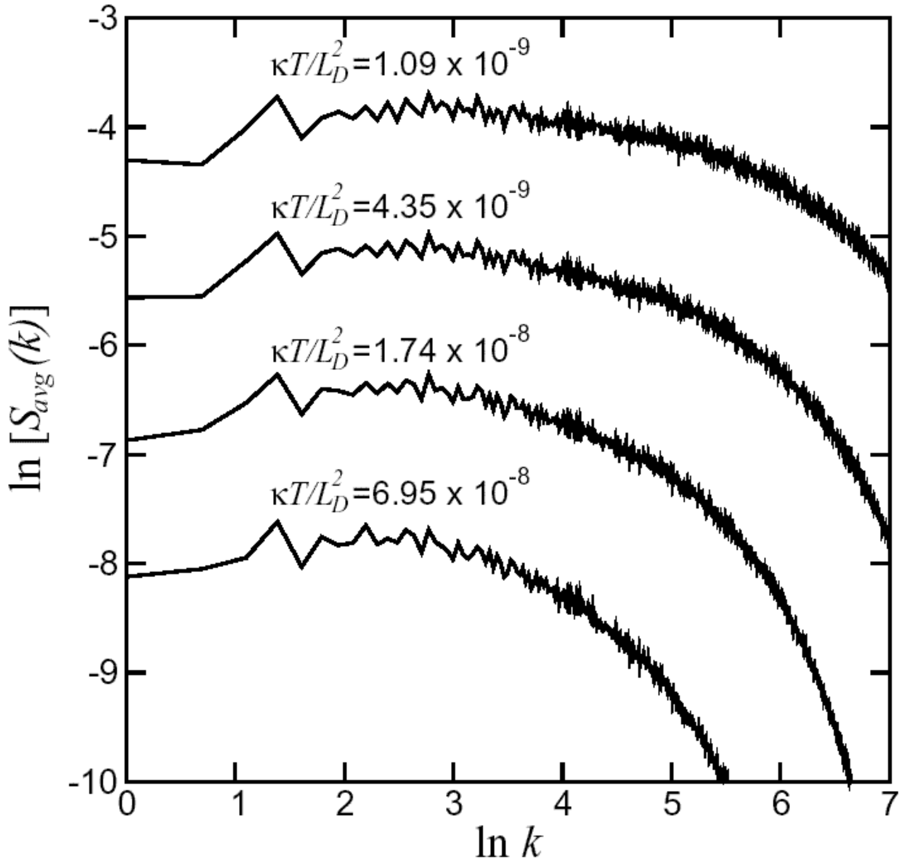
$$S_{avg}(k) = \langle S(k,t) / C(t) \rangle_t$$

$$90T < t < 100T$$

Flat Spectrum is signature of short wave length mechanism

$$S_0(k) \approx k^{-\psi}$$

$$\psi = 1 + min_{h \geq 0}\left(\frac{G(h) - \gamma_0}{h}\right) = 0$$

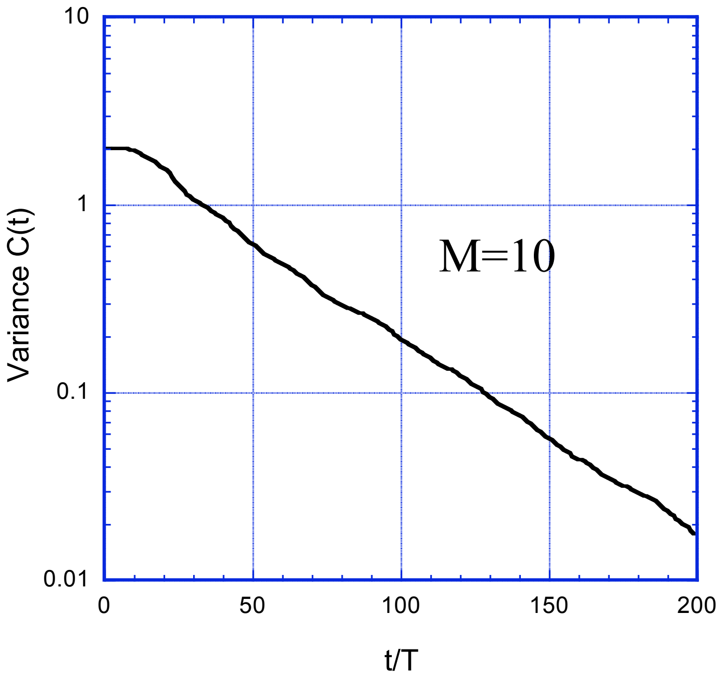


Damping of Modes by Spatial Diffusion

Experiment: Decay determined by spatial diffusion
Voth et al. Phys Fluids (2003).

$$\phi_0 = 2\sin\left[2\pi(x+y)/L_D\right]$$

$$\boldsymbol{v}(x,t) = U\begin{bmatrix}\hat{\boldsymbol{x}}\, f(t)\cos\left(2\pi y/L_f + \theta_1(t)\right) \\ +\hat{\boldsymbol{y}}\left(1-f(t)\right)\cos\left(2\pi x/L_f + \theta_2(t)\right)\end{bmatrix}$$

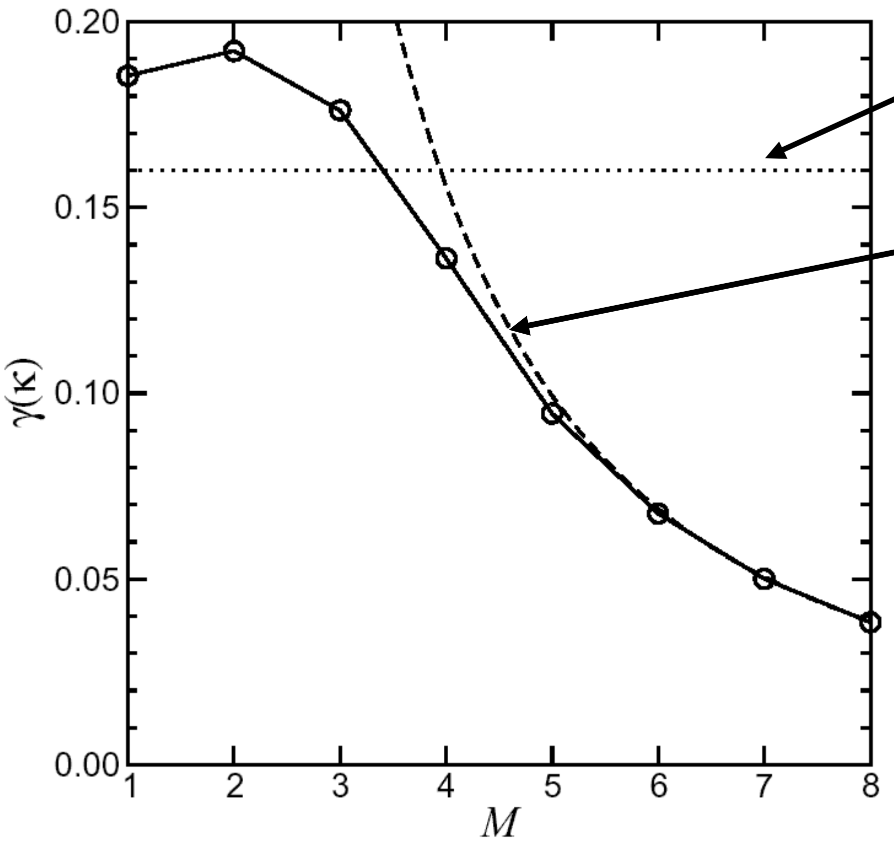


t=100T

Period of scalar greater than period of flow

$$L_D / L_f = M > 1$$

Decay Rate vs $M=L_D/L_f$



$M=L_D/L_f$

$$\gamma_0 = min_{h\geq 0}[h + G(h)]$$

Decay of coherent part of ***Φ***

$$\gamma < -\frac{1}{T} ln\left[J_0\left(\frac{\pi UT}{ML_f}\right)\right]^2$$

For M>>1 $\gamma = 2k^2\kappa_{eff}$

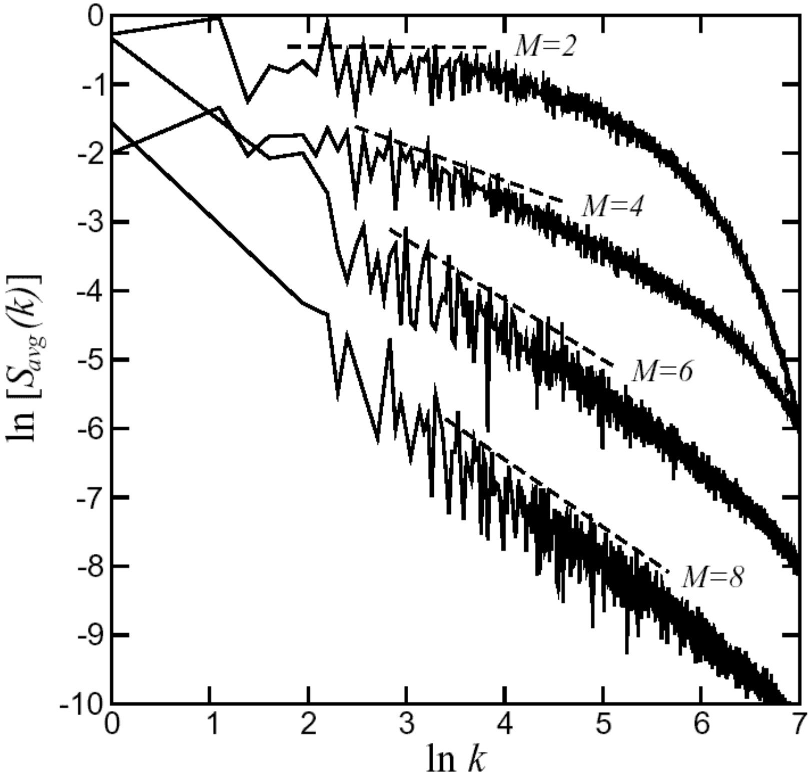
Where:

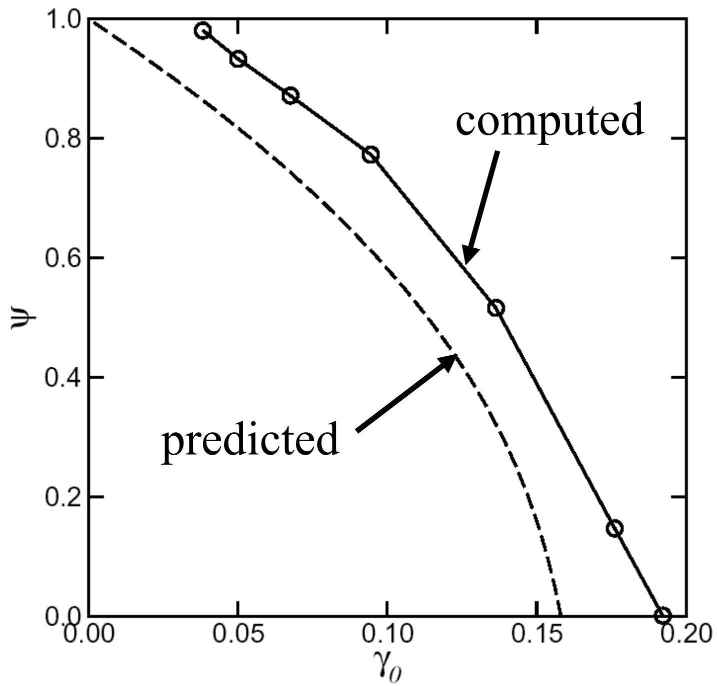
$$k = 2\pi / ML_f \quad \kappa_{eff} = \frac{1}{8}U^2T$$

Power Spectra for Slowly Decaying Modes

Power Law: $S_0(k) \approx k^{-\psi}$

$$\psi = 1 + min_{h \geq 0}\left(\frac{G(h) - \gamma}{h}\right)$$





Intermittency of φ(x,t)

1. Structure function exponents: $\left\langle \left|\phi(\boldsymbol{x}+\boldsymbol{r})-\phi(\boldsymbol{x})\right|^{q}\right\rangle \approx |r|^{\xi(q)}$

$$\xi(q)= min_{h\geq 0}\left(\frac{G(h)-q\gamma}{h}\right)$$

2. Multi-fractal dimension:

$$D_q = lim_{\varepsilon\to 0}\left[(1-q)^{-1} ln\left(\sum_i \mu^q\right)\right] / \, ln(L/\varepsilon)$$

μ_i = fraction of $\int dxdy\left|\phi(x,y)\right|$ in ε × ε box i.

$$D_q = 2-\frac{\xi(q)-q\xi(1)}{1-q}$$

Forced and Damped Scalar

finite lifetime

$$\frac{\partial}{\partial t}\phi(\mathbf{x},t)+\nabla\left[\cdot\mathbf{v}\phi(\mathbf{x},t)\right]=\kappa\nabla^2\phi(\mathbf{x},t)-T^{-1}\phi(\mathbf{x},t)+S$$

Source

diffusive rollover

Power Spectrum: $$F(k)\sim\frac{1}{k^{1+\xi}}\overline{\int_0^\infty d\tau\ M(\tau)\exp\left[-2\kappa\tau k^2\right]}$$

Correction to Batchelor's Law $$\xi=\min\left[\frac{G(h)+T^{-1}}{h}\right]$$ Nam et al, PRL 83, (1999)

Leads to intermittency: Abraham, Nature (1998), Chertkov Phys Fluids (1998)

Diffusive Rollover - Pdf of "recent stretching" M(τ) $$\tau=\int_0^t dt' k^2(t')/k^2(t)$$

Yuan et al. Chaos, (2000)

Power Law Spectrum

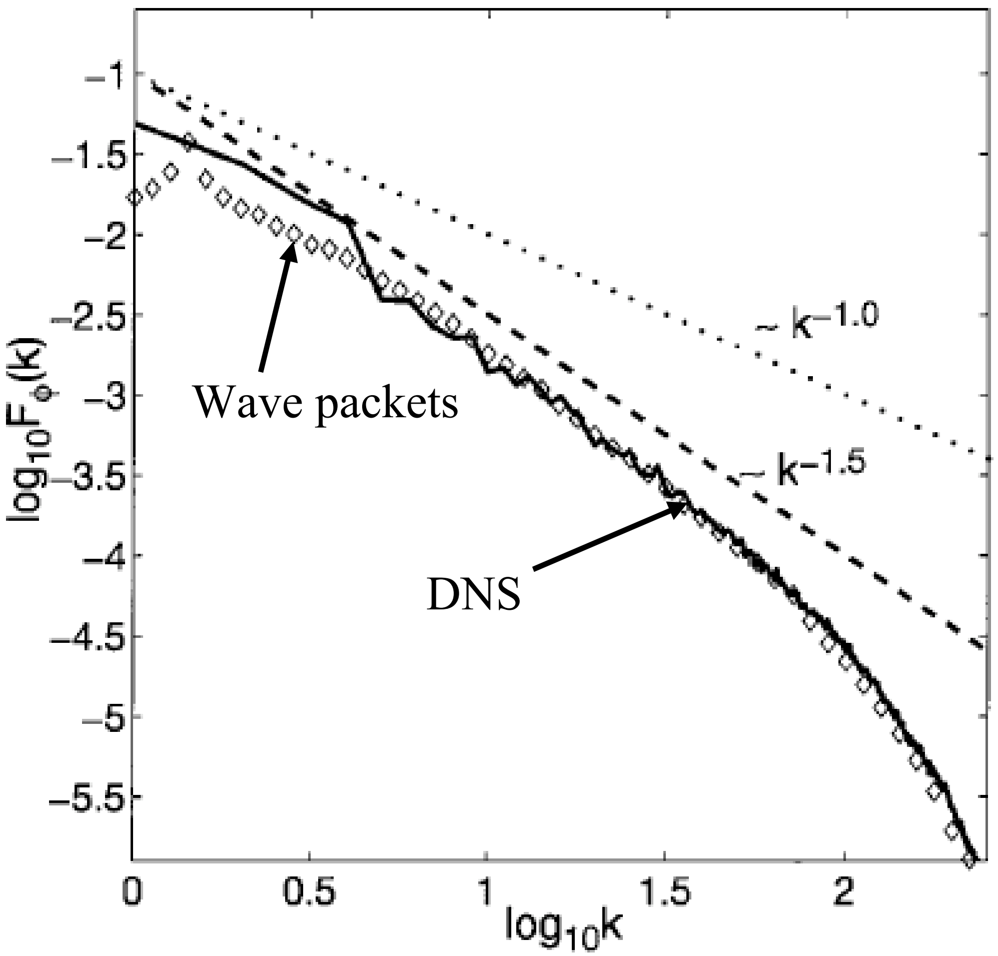
Nam et al, PRL 83, (1999)

Flow: 2D turbulence with drag

$$F(k) \sim \frac{1}{k^{1+\xi}}$$

$$\xi = \min[\frac{G(h)+T^{-1}}{h}]$$

ξ_{th} =0.5



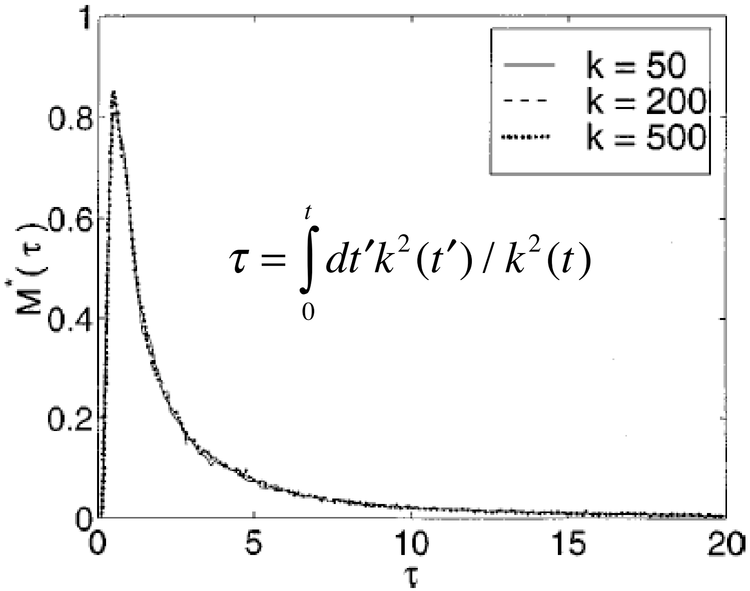
High k Roll-Over

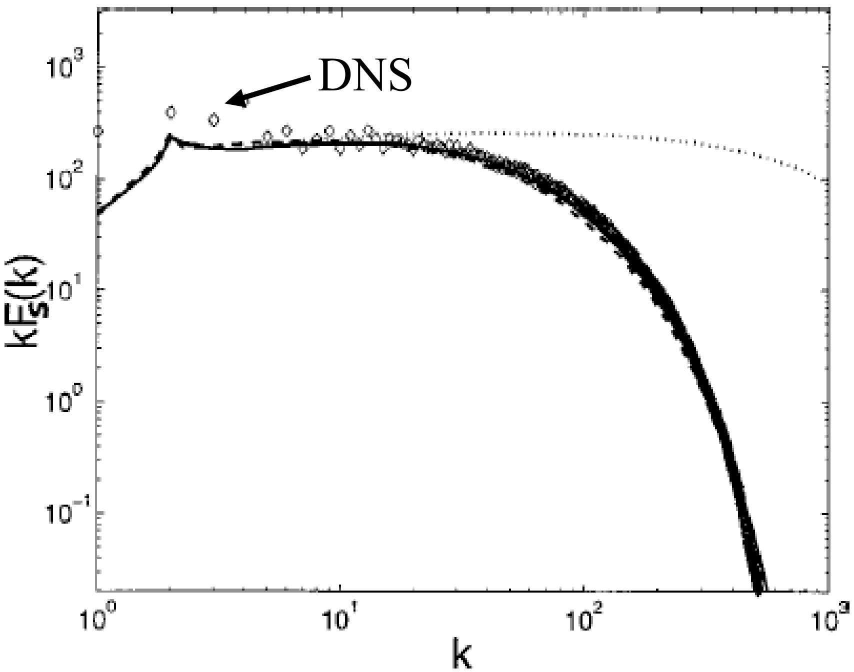
Yuan et al. Chaos, (2000)

diffusive rollover

$$F(k) \sim \frac{1}{k^{1+\xi}} \int_0^\infty d\tau \; M(\tau) \exp\left[-2\kappa\tau k^2\right]$$

Pdf of “recent stretching” M(τ)





2D Turbulence with Drag

Nam et al. PRL (2000), Bernard EuroPhys Ltr. (2000)

Vorticity Evolution $$\frac{\partial}{\partial t}\omega + \mathbf{v}\cdot\nabla\omega = \nu\nabla^2\omega - T^{-1}\omega + S \qquad \omega = \nabla\cdot\mathbf{v}\times\hat{\mathbf{z}}$$

Formally the same as the passive scalar problem, except ω and **v** are linked

If **v** is smooth then: $$F(k) \sim \frac{1}{k^{1+\xi}}\int_0^\infty d\tau\, M(\tau)\exp\left[-2\kappa\tau k^2\right]$$

If $\xi > 0$, then **v** is smooth.

$$h \sim \langle \|\vec{\nabla}\vec{v}\|^2\rangle^{1/2} \sim \sqrt{\int_{k_f}^{\infty} k^2 E(k)dk}, \qquad E(k) = F(k)/k^2$$

Numerical solution of 2D NS Equation
Tsang et al. PRE 71, 066313 (2005)

Numerical Parameters:

Simulation Domain:

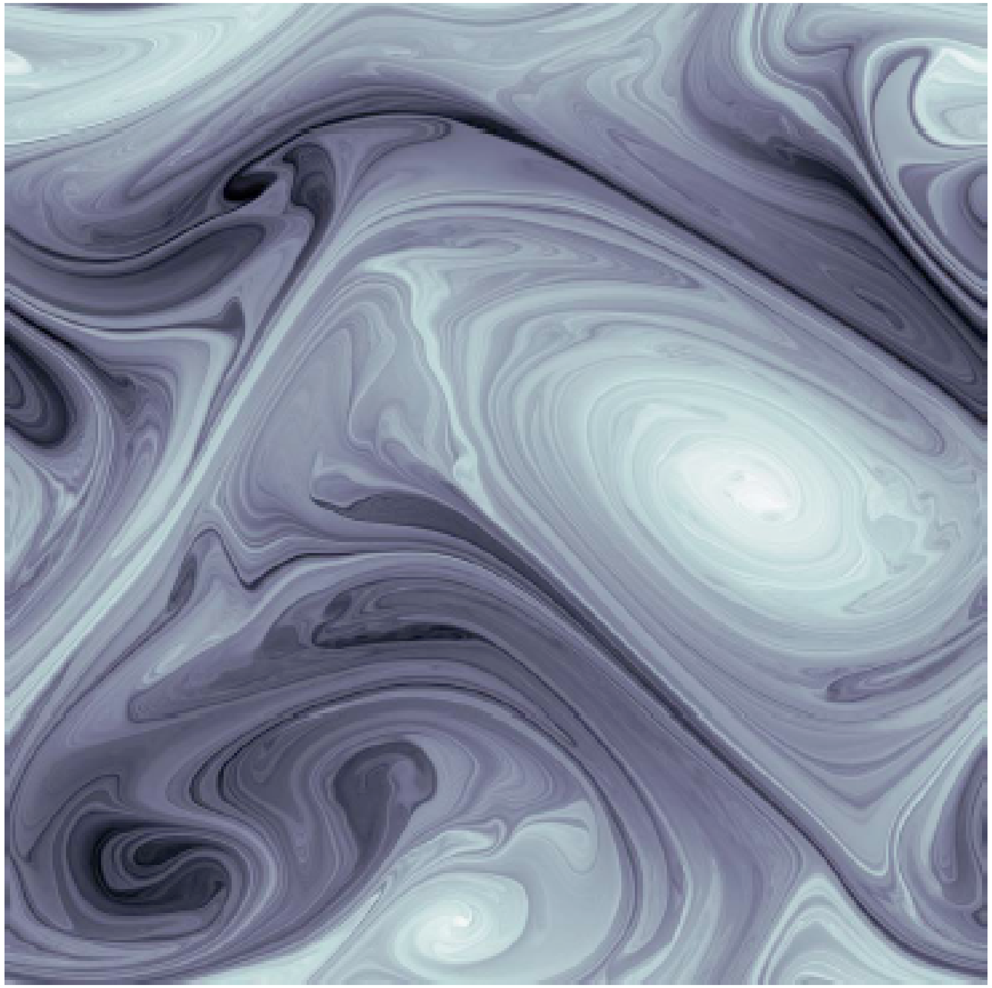
4096×4096

forcing at longest scale

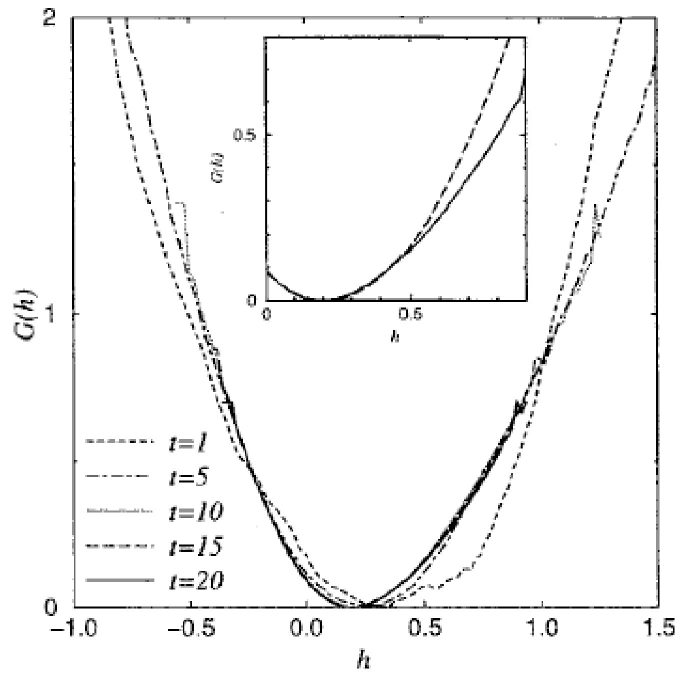
drag μ=0.1 for |k|<6

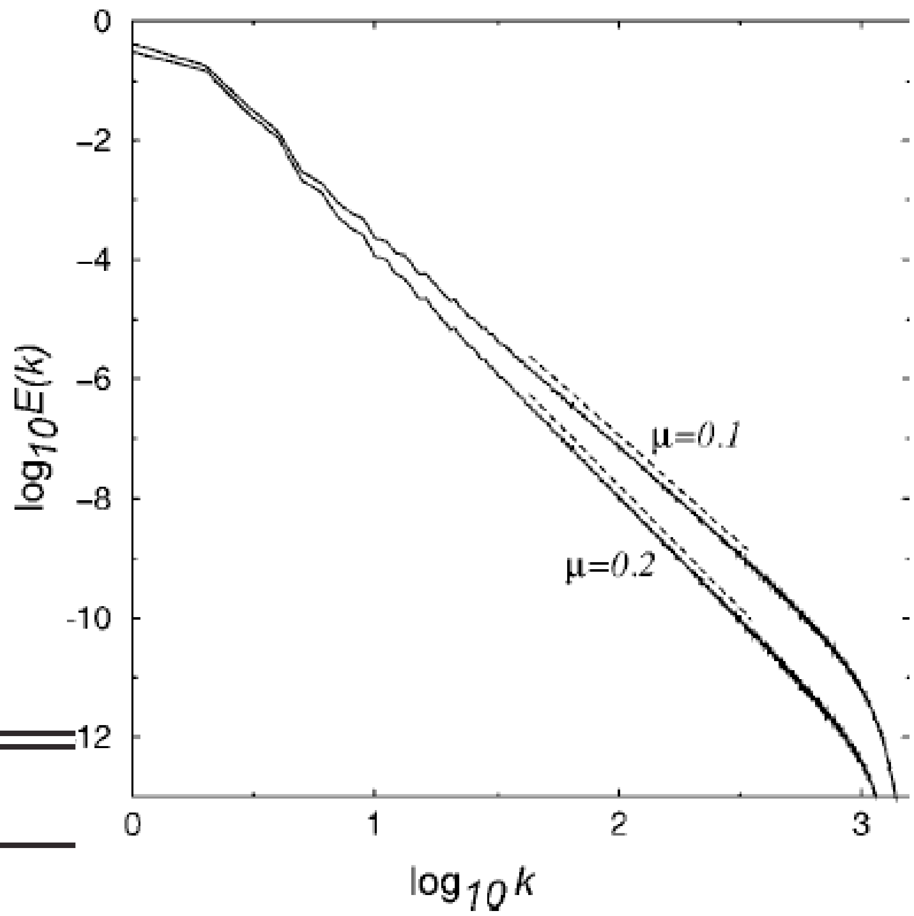
drag μ=0.1 or 0.2, for |k|>6

$\mu = T^{-1}$ inverse life time



Stretching Distributions and Power Spectra





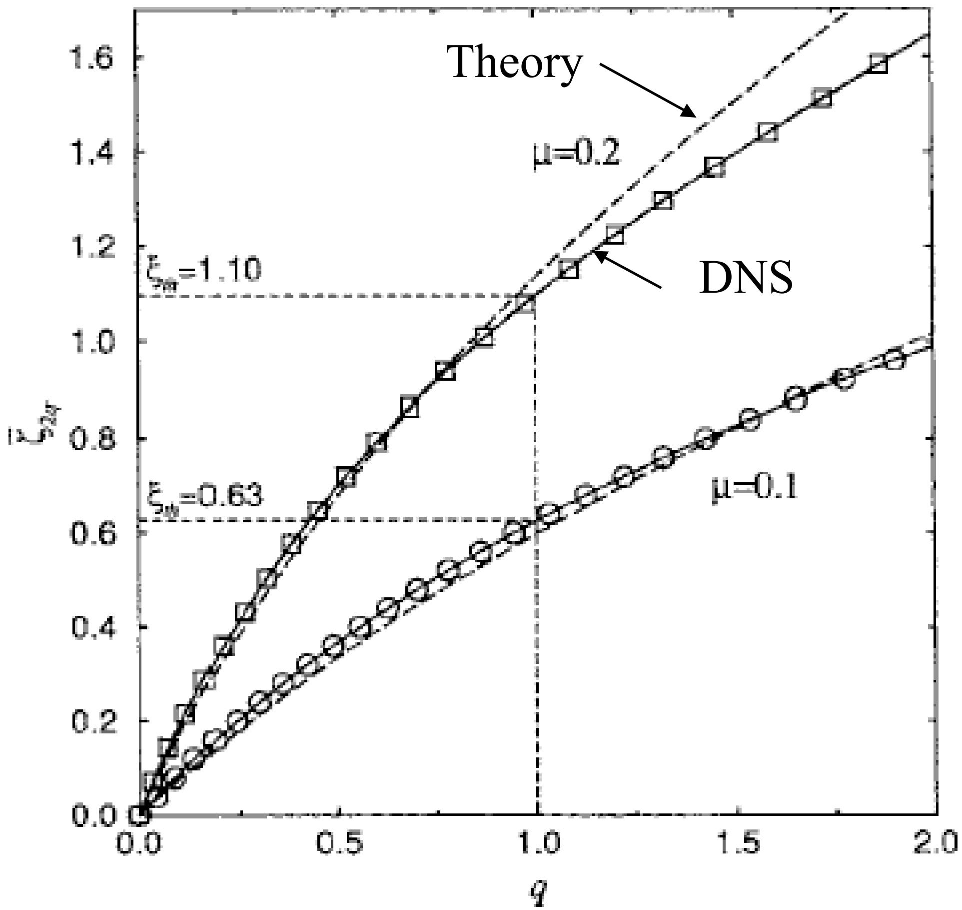
$\mu(k>6)$	$\xi_{\text{th.}}(=\zeta_{2,\text{th.}})$	ξ_{DNS}
0.1	0.63	0.61
0.2	1.10	1.12

Intermittency - Structure Function Exponent

$$S_{2q}(r) = \left\langle \left|\omega(\mathbf{x}+\mathbf{r}) - \omega(\mathbf{x})\right|^{2q} \right\rangle \sim r^{\zeta_{2q}}$$

For intermittent case exponent depends nonlinearly on q

$$\min_h \left\{ \frac{G(h) + 2q\mu}{h} - \bar{\zeta}_{2q} \right\} = 0.$$



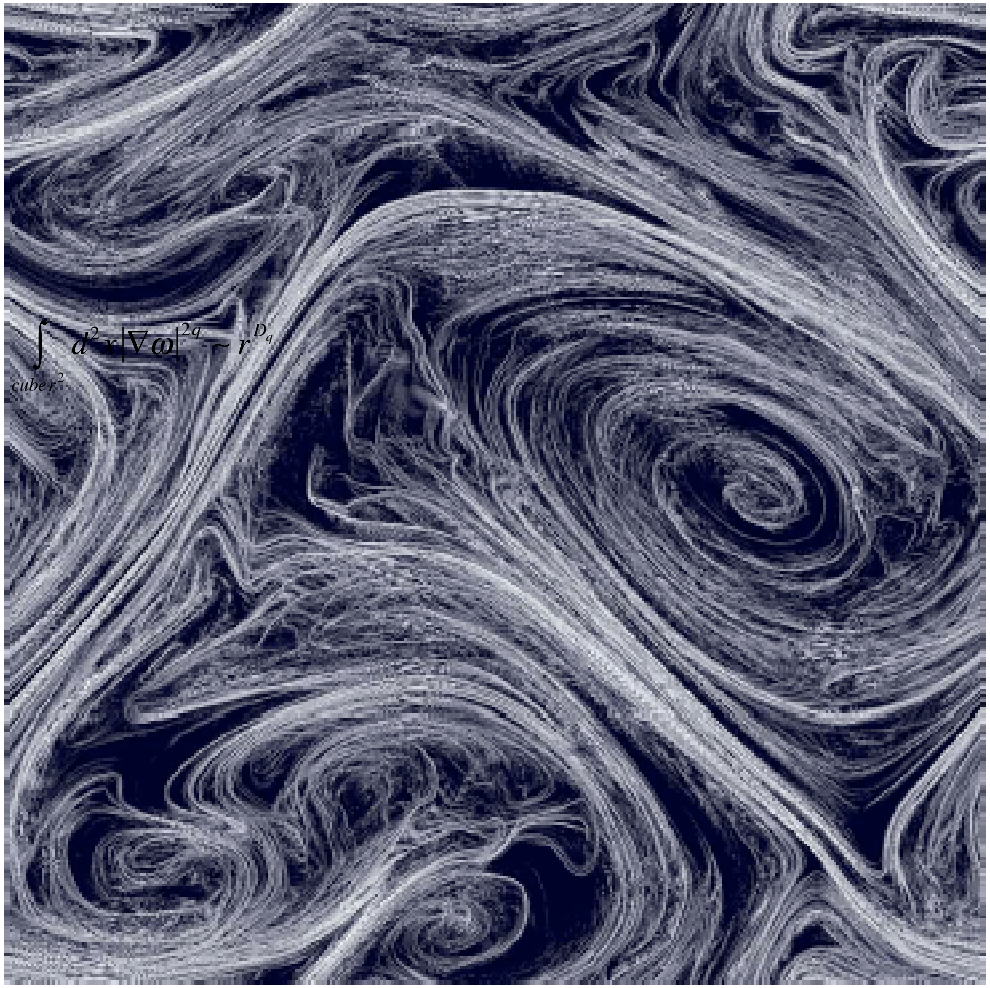
Fractal Dimension of Dissipation Field

Vorticity Gradient
Squared Field

$$\left|\nabla\omega(x,t=61)\right|^2$$

$$\int_{cube-r} d^2x\left|\nabla\omega\right|^{2q} \sim r^{D_q}$$

$$D_q = 2 + \frac{\varsigma_{2q} - q\varsigma_q}{q-1}$$



Other Problems

Kinematic Dynamo: $$\frac{\partial}{\partial t}\mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v} + \eta\nabla^2\mathbf{B}$$

growth rate and fractal properties depend on stretching distribution

Linearized 3D Navier Stokes:
Reyl et al, PRL, 1997

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{S} \qquad \mathbf{w} = \nabla\times\mathbf{v}$$

Smooth flow, $\mathbf{v}_0$ satisfying NS with forcing is unstable to perturbed flow $\mathbf{v}_1$ that is intermittent with multiple spatial scales. Power spectrum, structure function, etc determined by finite time Lyapunov exponents

Conclusion

For a wide class of problems, the distribution of local stretching rates governs the fine scale structure of transported fields.

Passive Scalar
2D Turbulence with Drag
Kinematic Dynamo
Stability of smooth 3D NS flows

Decay Rates
Power Spectra
Structure Functions
Fractal Dimension of Dissipation Field