Lagrangian Chaos and the Evolution of Advected Fields

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Problem Class - Advected Fields

Passive Scalar: \[ \frac{\partial}{\partial t} \phi(x,t) + \nabla \cdot (v \phi(x,t)) = \kappa \nabla^2 \phi(x,t) + S \]

Kinematic Dynamo: \[ \frac{\partial}{\partial t} B + v \cdot \nabla B = B \cdot \nabla v + \eta \nabla^2 B \]

Navier Stokes: \[ \frac{\partial}{\partial t} \omega + v \cdot \nabla \omega = \nu \nabla^2 \omega + S \quad \omega = \nabla \times v \]

Common Features: Field transported by flow
Dissipation at smallest scale length
\[ \kappa, \ \eta, \ \nu \quad \text{Small} \]

Important Restriction: Spatial scale of transported fields is smaller than that of flow
**Question:** What are the properties of the transported field in cases in which \( v(x,t) \) gives rise to chaotic fluid orbits?

**Transport versus Mixing**

**Transport**

RMS size of blob increases with time

**Enhanced Diffusion**

\[
\langle |x|^2 \Phi(x,t) \rangle \approx 2 D_{\text{enhanced}} t
\]

Blob becomes larger than spatial scale of flow

**Examples:**

- Transport in destroyed magnetic surfaces
Transport versus Mixing

Mixing

Blob acquires fine structure; high and low values of scalar are mixed together.

Reviews of Chaos and Mixing:


Decay of Passive Scalar

Passive Scalar: \[ \frac{\partial}{\partial t} \phi(x,t) + \mathbf{v} \cdot \nabla \phi(x,t) = \kappa \nabla^2 \phi(x,t) \]

Initial value of Scalar: \( \phi(x,0) = \phi_0(x) \) Periodic with period \( L_D \)

Fluid flow velocity: \( \mathbf{v}(x,t) \) Periodic with period \( L_f \)

Microscopic diffusion: \( \kappa \)

Fluid trajectories: \[ \frac{dx_i(t)}{dt} = \mathbf{v}(x_i,t) \] Chaotic for almost all initial conditions

**Question:** What is the long time behavior of \( \phi \) in the limit \( \kappa \to 0 \) ?
Time Decay of Scalar

- **Exponential decay of $\Phi$, “Strange Eigenfunction”**
  

- **Decay rate predicted based on local stretching rates.**
  

- **Validity of local stretching theory? Decay rate determined by longest scale.**
  

- **Experiment: decay determined by spatial diffusion**
  

- **Properties of strange eigenfunctions**
  
  Haynes and Vanneste, Phys Fluids 17, 097103 (2005)
What do you want to know?

<table>
<thead>
<tr>
<th>Decay rate</th>
<th>$\Phi(r,t) \sim e^{-\gamma t}$</th>
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**Power Spectrum**

- Fourier transform of two point correlation function
  \[ C(k,t) = \int d^2r \ e^{-ik \cdot r} \langle \Phi(x+r) \Phi(x) \rangle_x \]
- Averaged over angle in k-space
  \[ F(k,t) = \int \frac{d^2k'}{(2\pi)^2} \delta(k - |k'|) \ C(k',t) \]

**Structure Function**

\[ S_{2q}(r) = \left\langle \left| \Phi(x + r,t) - \Phi(x,t) \right|^{2q} \right\rangle \sim r^{\zeta_{2q}} \]

**Fractal Properties of Dissipation Field**

\[ \kappa |\nabla \Phi|^2 \]

All can be tied to the properties of the underlying flow
2D Model Flow

\[
v(x,t) = U \left[ \hat{x} f(t) \cos \left( \frac{2\pi y}{L_f} + \theta_1(t) \right) \\
+ \hat{y} \left( 1 - f(t) \right) \cos \left( \frac{2\pi x}{L_f} + \theta_2(t) \right) \right]
\]

Normalized displacement: \[ \frac{U T}{2L_f} \]

Random angles: \( \theta_1(t) \) and \( \theta_2(t) \)
Constant values randomly chosen for each interval \( n+1 > t/T \geq n \)
Sample Solutions - max resolution (6.4 \times 10^4)^2
Tsang et al. PRE 71, 066301 (2005).

Initial condition: \[ \phi_0 = 2 \sin \left[ \frac{2\pi(x + y)}{L_D} \right] \]

Scalar variance: \[ C(t) = \frac{1}{2L_d^2} \int dxdy\phi^2 \]

\[ UT/L_f = \pi, \]
\[ L_f = L_D \]

\[ \kappa T/L_D^2 \]
- 6.95 \times 10^{-8}
- 1.74 \times 10^{-8}
- 4.35 \times 10^{-9}
- 1.09 \times 10^{-9}
- 3.24 \times 10^{-10} \]

\[ t/T \]
- 0
- 25
- 50
- 75
- 100
- 125
Scale length of $\Phi$ is much smaller than scale length of $v$. WKB description is suggested

Action Density: $N(k,x,t)$,

$$\frac{\partial N}{\partial t} + \nabla \cdot \left( \frac{\partial \omega}{\partial k} N \right) - \frac{\partial}{\partial k} \left( \frac{\partial \omega}{\partial x} N \right) = -2\kappa k^2 N$$

Dispersion relation: $\omega = k \cdot v$

Characteristics:

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k} = v(x,t) \quad , \quad \frac{dk}{dt} = -\frac{\partial \omega}{\partial x} = - \nabla v \cdot k$$
Wave Packet Model

- Power spectrum constructed from an ensemble of trajectories labeled by the index $j$:

- $w_j(t)$ is the scalar variance associated with the $j^{th}$ trajectory.

Trajectory Equations:

\[
\frac{dx_j(t)}{dt} = v(x_j(t),t), \quad \frac{dk_j(t)}{dt} = -\nabla v(x_j(t),t) \cdot k_j(t)
\]
Local Stretching Theory

Chaotic Orbits

Fluid trajectories:
\[ \frac{dx(t)}{dt} = v(x(t), t) \]

Differential separation:
\[ \frac{d\delta x(t)}{dt} = \delta x(t) \cdot \nabla v(x, t) \]

Chaotic:
\[ \left| \delta x(t) \right| = \left| \delta x(0) \right| e^{ht}, \quad h > 0 \]

Compare with Eq. for \( k_j(t) \):
\[ \frac{d}{dt} k_j(t) = -\nabla v(x, t) \cdot k_j \]
\[ \frac{d}{dt} (\delta x \cdot k_j) = 0 \]

Lyapunov exponent:
\[ \bar{h} = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\delta x(t)}{\delta x(0)} \right| \]

Lyapunov exponent is the same for almost all initial conditions in a given chaotic region

\[ \delta x(t) \approx \delta x(0) e^{ht} \]

Neighboring trajectories diverge

\[ \delta x(0) \]

\[ \delta x(t) \]
Diverging and Converging Orbits

Differential separation: \[
\frac{d\delta x(t)}{dt} = \delta x(t) \cdot \nabla v(x,t)
\]

n Dimensions \((n = 2, 3)\)

n solutions for \(\delta x\)

\[
\delta x_1, \delta x_2, \ldots, \delta x_n, \quad \bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n
\]

Incompressible flows

\[
\sum_i \bar{h}_i = 0
\]

Converging solution

\[
\bar{h}_n < 0 \quad \text{[In 2D]} \quad |h_2| = h_1 = h
\]

Due to converging solution \(\nabla \Phi(x,t)\) grows exponentially

\[
|\nabla \Phi(x,t)| \approx |\nabla \Phi(x_0,0)| \exp[|h_2| t]
\]
Finite Time Lyapunov Exponents

Finite time Lyapunov exponent:

\[ h(x_0, t) = \frac{1}{t} \ln \frac{\delta x(t)}{\delta x(0)} \]

\[ \bar{h} = \lim_{t \to \infty} h(x_0, t) \]

Finite time Lyapunov exponents are characterized by a distribution, \( P(h,t) \).
Long Time Decay Based on Local Stretching

\[
F(k,t) = \sum_j w_j(t) \delta(k - |k_j(t)|)
\]

\[
w_j(t) = w_j(0) \exp[-2\kappa \int_0^t dt' k_j^2(t')]
\]

Typical trajectory: \( k_j(t) \) grows exponentially, leads to faster than exponential decay

\[
|k_j(t)| \approx \cos\theta |k_j(0)| \exp(ht)
\]

Dominant contribution from \( \theta = \pi/2 \)
\( k_j(0) \) is perpendicular to contracting direction.

Predicted decay rate:
\[
\gamma_0 = \min_{h \geq 0} [h + G(h)]
\]
Decay Rate Evaluation

\[ \exp[-\gamma t] \sim \int dhdt \, P(h,t) \exp[-2\kappa k_0^2 e^{2ht} \cos^2 \theta] \]

\[ P(h,t) \approx \exp[-tG(h)] \]

Evaluate by steepest descent

Predicted decay rate:

\[ \gamma_0 = \min_{h \geq 0}[h + G(h)] \]
Comparison with Numerics

\[ \frac{U_T}{L_f} = \pi, \]
\[ L_f = L_D \]

\[ \gamma_0 = \min_{h \geq 0} [h + G(h)] \]

Logarithmic correction,
Haynes and Vanneste,
Shekochihin, Haynes and Cowley
Upper Bound on Decay in $\kappa \to 0$ Limit
Tsang, TMA, and Ott, PRE 71 (2005)

For Strange Eigenfunctions: 

$$L_f^{-1} \ll k \ll (\hbar \kappa)^{-1/2}$$

$$S_0(k) \exp[-\gamma t] = \int_0^\infty dk' S_0(k') \left< \delta(k-k') | \partial x(0) / \partial x(t) \right>_{h,\theta}$$

Power spectrum of Scalar Eigenfunction
Actual Decay Rate
Local stretching of $k$

Assume Power Law: 

$$A k^{-\psi} > S_0(k) > B k^{-\psi}$$

Then can show: 

$$\gamma = \min_{h \geq 0} \left[ h + G(h) - |\psi| h \right] < \min_{h \geq 0} \left[ h + G(h) \right] = \gamma_0$$

$$\psi = 1 + \min_{h \geq 0} \left( \frac{G(h) - \gamma}{h} \right)$$
$$S_{\text{avg}}(k) = \langle S(k,t) / C(t) \rangle_t$$

$$90T < t < 100T$$

Flat Spectrum is signature of short wave length mechanism

$$S_0(k) \approx k^{-\psi}$$

$$\psi = 1 + \min_{h \geq 0} \left( \frac{G(h) - \gamma_0}{h} \right) = 0$$
Damping of Modes by Spatial Diffusion

Experiment: Decay determined by spatial diffusion

\[ \phi_0 = 2\sin\left[\frac{2\pi(x + y)}{L_D}\right] \]

\[ \nu(x,t) = U \left[ \hat{x}f(t)\cos\left(\frac{2\pi y}{L_f} + \theta_1(t)\right) + \hat{y}(1 - f(t))\cos\left(\frac{2\pi x}{L_f} + \theta_2(t)\right) \right] \]

Period of scalar greater than period of flow

\[ \frac{L_D}{L_f} = M > 1 \]
\[ \gamma_0 = \min_{h \geq 0} [h + G(h)] \]

Decay of coherent part of \( \Phi \)

\[ \gamma < -\frac{1}{T} \ln \left[ J_0 \left( \frac{\pi UT}{ML_f} \right) \right]^2 \]

For \( M \gg 1 \)

\[ \gamma = 2k^2 \kappa_{\text{eff}} \]

Where:

\[ k = \frac{2\pi}{ML_f} \]

\[ \kappa_{\text{eff}} = \frac{1}{8} U^2 T \]
Power Spectra for Slowly Decaying Modes

Power Law: \( S_0(k) \approx k^{-\psi} \)

\[
\psi = 1 + \min_{h \geq 0} \left( \frac{G(h) - \gamma}{h} \right)
\]
Intermittency of $\phi(x,t)$

1. Structure function exponents:

$$\left< \left| \phi(x + r) - \phi(x) \right|^q \right> \approx r^{\xi(q)}$$

$$\xi(q) = \min_{h \geq 0} \left( \frac{G(h) - q \gamma}{h} \right)$$

2. Multi-fractal dimension:

$$D_q = \lim_{\varepsilon \to 0} \left[ \left( 1 - q \right)^{-1} \ln \left( \sum_i \mu_i^q \right) \right] / \ln(L / \varepsilon)$$

$$\mu_i = \text{fraction of } \int dx dy \phi(x, y) \text{ \, in } \varepsilon \times \varepsilon \text{ box } i.$$
Forced and Damped Scalar

\[ \frac{\partial}{\partial t} \phi(x,t) + \nabla \left[ \cdot v \phi(x,t) \right] = \kappa \nabla^2 \phi(x,t) - T^{-1} \phi(x,t) + S \]

\[ F(k) \sim \frac{1}{k^{1+\xi}} \int_0^\infty d\tau \ M(\tau) \exp\left[ -2\kappa_\tau k^2 \right] \]

Correction to Batchelor’s Law \[ \xi = \min[ \frac{G(h) + T^{-1}}{h} ] \quad \text{Nam et al, PRL 83, 1999} \]


Diffusive Rollover - Pdf of “recent stretching” \[ M(\tau) \quad \tau = \int_0^t dt' k^2(t') / k^2(t) \]

Flow: 2D turbulence with drag

\[ F(k) \sim \frac{1}{k^{1+\xi}} \]

\[ \xi = \min \left[ \frac{G(h) + T^{-1}}{h} \right] \]

\[ \xi_{\text{th}} = 0.5 \]
High k Roll-Over

$$F(k) \sim \frac{1}{k^{1+\xi}} \int_{0}^{\infty} d\tau M(\tau) \exp\left[-2\kappa \tau k^2\right]$$

Pdf of “recent stretching” $M(\tau)$

$$\tau = \int_{0}^{t} dt' k^2(t') / k^2(t)$$
2D Turbulence with Drag

\[
\frac{\partial}{\partial t} \omega + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega - T^{-1} \omega + S \quad \omega = \nabla \cdot \mathbf{v} \times \mathbf{\hat{z}}
\]

Formally the same as the passive scalar problem, except \( \omega \) and \( \mathbf{v} \) are linked

If \( \mathbf{v} \) is smooth then:

\[
F(k) \sim \frac{1}{k^{1+\xi}} \int_0^\infty d\tau \, M(\tau) \exp\left[-2\kappa \tau k^2\right]
\]

If \( \xi > 0 \), then \( \mathbf{v} \) is smooth.

\[
h \sim \langle \|\nabla \mathbf{v}\|^2 \rangle^{1/2} \sim \sqrt{\int_{k_f}^\infty k^2 E(k) dk}, \quad E(k) = \frac{F(k)}{k^2}
\]
Numerical Parameters:

Simulation Domain:

$4096 \times 4096$

forcing at longest scale

drag $\mu=0.1$ for $|k|<6$

drag $\mu=0.1$ or 0.2, for $|k|>6$

$\mu = T^{-1}$  inverse life time
Stretching Distributions and Power Spectra

<table>
<thead>
<tr>
<th>$\mu(k &gt; 6)$</th>
<th>$\xi_{\text{th.}} (=\xi_{2,\text{th.}})$</th>
<th>$\xi_{\text{DNS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.63</td>
<td>0.61</td>
</tr>
<tr>
<td>0.2</td>
<td>1.10</td>
<td>1.12</td>
</tr>
</tbody>
</table>
Intermittency - Structure Function Exponent

\[ S_{2q}(r) = \langle |\omega(x + r) - \omega(x)|^{2q} \rangle \sim r^{\xi_{2q}} \]

For intermittent case exponent depends nonlinearly on \( q \)

\[ \min_h \left\{ \frac{G(h) + 2q\mu}{h} - \bar{\xi}_{2q} \right\} = 0. \]
Fractal Dimension of Dissipation Field

Vorticity Gradient Squared Field

$$|\nabla \omega(x, t = 61)|^2$$

$$\int_{cube-r} d^2x |\nabla \omega|^{2q} \sim r^{D_q}$$

$$D_q = 2 + \frac{\zeta_{2q} - q\zeta_q}{q - 1}$$
Other Problems

Kinematic Dynamo: \[ \frac{\partial}{\partial t} B + \mathbf{v} \cdot \nabla B = B \cdot \nabla \mathbf{v} + \eta \nabla^2 B \]

growth rate and fractal properties depend on stretching distribution

Linearized 3D Navier Stokes: \[ \frac{\partial}{\partial t} \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} = \nu \nabla^2 \mathbf{w} + \mathbf{S} \]

Reyl et al, PRL, 1997

Smooth flow, \( \mathbf{v}_0 \) satisfying NS with forcing is unstable to perturbed flow \( \mathbf{v}_1 \) that is intermittent with multiple spatial scales. Power spectrum, structure function, etc determined by finite time Lyapunov exponents
Conclusion

For a wide class of problems, the distribution of local stretching rates governs the fine scale structure of transported fields.

Passive Scalar
2D Turbulence with Drag
Kinematic Dynamo
Stability of smooth 3D NS flows

Decay Rates
Power Spectra
Structure Functions
Fractal Dimension of Dissipation Field