A tutorial on the method of multiple scales

H.G. Dudding

1 Introduction

This document aims to provide an introductory overview to the method of multiple scales (MMS) [1–3], including its motivation (2), a worked example for a system with an explicit small parameter (3) and worked examples of systems without an explicit small parameter (4).

2 Secular behaviour as a motivation for MMS

We wish to consider how to generate uniformly valid perturbative solutions to differential equations. We first analyse the damped harmonic oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \tag{1}$$

for which we would like to find a perturbative solution with respect to ε. We can write the solution $y = y\left(t; \varepsilon, p_j\right)$, where p_j represents the set of parameters in the system other than ε[1]. For clarity, we write the time derivative as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left.\frac{\partial y}{\partial t}\right|_{\varepsilon, p_j}. \tag{2}$$

For the damped oscillator the parameters p_j correspond to the two initial conditions $y\left(0\right)$ and $y'\left(0\right)$, where the prime notation has been used as a shorthand for 2.

We begin by considering an expansion of y in ε while holding t and p_j constant, such that

$$\begin{aligned} y\left(t; \varepsilon, p_j\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} \varepsilon^n \left.\frac{\partial^n y}{\partial \varepsilon^n}\right|_{t, p_j} \left(t; \varepsilon = 0, p_j\right) \\ &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots \end{aligned} \tag{3}$$

where we have defined the shorthand

$$y_n = \frac{1}{n!} \left.\frac{\partial^n y}{\partial \varepsilon^n}\right|_{t, p_j} \left(t; \varepsilon = 0, p_j\right). \tag{4}$$

Inserting the form of 3 into equation 1, we have

$$\left.\frac{\partial^2}{\partial t^2}\right|_{\varepsilon, p_j} \left(y_0 + \varepsilon y_1 + \ldots\right) + 2\varepsilon \left.\frac{\partial}{\partial t}\right|_{\varepsilon, p_j} \left(y_0 + \varepsilon y_1 + \ldots\right) + \left(y_0 + \varepsilon y_1 + \ldots\right) = 0. \tag{5}$$

The partial derivatives with respect to t commute with those with respect to ε in the definition of the y_n, and so gathering powers of ε,

$$\left[\left.\frac{\partial^2 y_0}{\partial t^2}\right|_{\varepsilon, p_j} + y_0\right] + \varepsilon \left[\left.\frac{\partial^2 y_1}{\partial t^2}\right|_{\varepsilon, p_j} + y_1 + 2 \left.\frac{\partial y_0}{\partial t}\right|_{\varepsilon, p_j}\right] + \varepsilon^2 \left[\left.\frac{\partial^2 y_2}{\partial t^2}\right|_{\varepsilon, p_j} + y_2 + 2 \left.\frac{\partial y_1}{\partial t}\right|_{\varepsilon, p_j}\right] + \cdots = 0. \tag{6}$$

For this to be true for non-zero ε each square bracket must itself be zero. The first bracket is solved by

$$y_0\left(t; p_j\right) = A_0 \cos t + B_0 \sin t \tag{7}$$

[1] The use of a semicolon represents a separation between coordinates and parameters, however this is largely aesthetic.

for integration constants A_0 and B_0. The approximate solution of y at this order is therefore $y\left(t;\varepsilon,p_j\right) = A_0\cos t + B_0\sin t + \mathcal{O}\left(\varepsilon\right)$, where the integration constants are related to the initial conditions via $y\left(0\right) = A_0 + \mathcal{O}\left(\varepsilon\right), y'\left(0\right) = B_0 + \mathcal{O}\left(\varepsilon\right)$ to give

$$y\left(t;\varepsilon,p_j\right) = y\left(0\right)\cos t + y'\left(0\right)\sin t + \mathcal{O}\left(\varepsilon\right). \tag{8}$$

Taking this to next order, the second bracket of 6 becomes

$$\left.\frac{\partial^2 y_1}{\partial t^2}\right|_{\varepsilon,p_j} + y_1 = 2A_0\sin t - 2B_0\cos t \tag{9}$$

which has the solution

$$y_1\left(t;p_j\right) = -A_0 t\cos t - B_0 t\sin t + A_1\cos t + B_1\sin t. \tag{10}$$

Combining this with the zeroth order and relating the integration constants to the initial conditions (appendix A.1.1) the solution to first order is

$$y\left(t;\varepsilon,p_j\right) = \left[y\left(0\right)\cos t + y'\left(0\right)\sin t\right] + \varepsilon\left[-y\left(0\right)t\cos t - y'\left(0\right)t\sin t + y\left(0\right)\sin t\right] + \mathcal{O}\left(\varepsilon^2\right). \tag{11}$$

We note however that the $t\cos t$ and $t\sin t$ terms grow linearly with t, and thus for times $t \sim 1/\varepsilon$ these terms will become comparable with the lowest order terms, and the validity of our expansion breaks down. This property is known as secularity, and in this case has arisen due to the terms on the RHS of equation 9 being proportional to the homogeneous solution for y_1 and thus driving an 'artificial' resonance in the solution. We can compare our result with the exact solution to 1,

$$y\left(t;\varepsilon,p_j\right) = \mathrm{e}^{-\varepsilon t}\left(y\left(0\right)\cos\left(\sqrt{1-\varepsilon^2}t\right) + \left[\frac{y'\left(0\right)+\varepsilon y\left(0\right)}{\sqrt{1-\varepsilon^2}}\right]\sin\left(\sqrt{1-\varepsilon^2}t\right)\right) \tag{12}$$

which we see is bounded for all t. Our expansion of the form in equation 3 has failed, and is shown graphically in figure 1. To remedy this we must try a different method, for which we turn to the method of multiple scales.

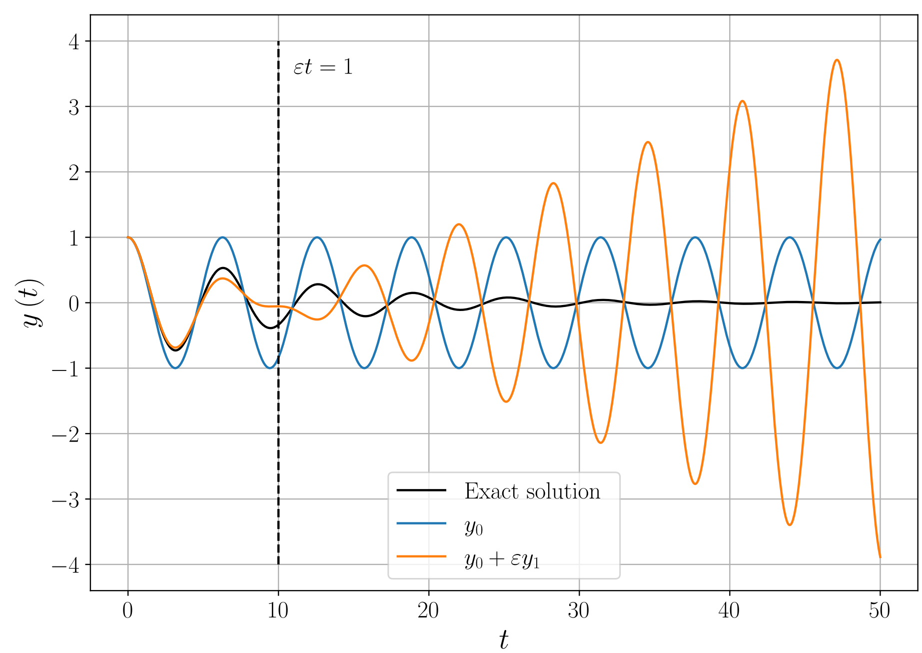

Figure 1: Attempts at finding a perturbative solution to the damped harmonic oscillator for $\varepsilon = 0.1$, $y\left(0\right) = 1$ and $y'\left(0\right) = 0$. Both approximations (equations 8 and 11) provide a reasonable result for $\varepsilon t \ll 1$, however fail at later times. Note the secular growth of the first order solution.

3 The method of multiple scales

3.1 Principles of the method

The method of multiple scales assumes that a function can be written in the form

$$y\left(t;\varepsilon,p_j\right)=y\left(t,\varepsilon t,\varepsilon^2 t,\ldots;\varepsilon,p_j\right). \tag{13}$$

Note, this is still the same function, this is just an assumption on how one can 'group' the variable dependence. As an example, consider expanding the functions of ε inside the trigonometric functions of equation 12. We see these can be grouped together,

$$y\left(t;\varepsilon,p_j\right)=\mathrm{e}^{-\varepsilon t}\left(y\left(0\right)\cos\left(t-\frac{1}{2}\varepsilon^2 t-\frac{1}{8}\varepsilon^4 t+\ldots\right)+\left[\frac{y'\left(0\right)+\varepsilon y\left(0\right)}{\sqrt{1-\varepsilon^2}}\right]\sin\left(t-\frac{1}{2}\varepsilon^2 t-\frac{1}{8}\varepsilon^4 t+\ldots\right)\right). \tag{14}$$

The method of multiple scales is based on generating and solving equations that treat these groupings as independent from one another, in such a way that eliminates secular terms, before then bringing them together to reconstruct a continuous, non-secular solution. To understand how to apply this method, we first for clarity relabel the groupings as $t=T_0, \varepsilon t=T_1, \varepsilon^2 t=T_2, \ldots$. The total differential of a function $y=y\left(t;\varepsilon,p_j\right)$ can be written

$$\mathrm{d}y=\left.\frac{\partial y}{\partial T_0}\right|_{T_1,T_2,\ldots,\varepsilon,p_j}\mathrm{d}T_0+\left.\frac{\partial y}{\partial T_1}\right|_{T_0,T_2,\ldots,\varepsilon,p_j}\mathrm{d}T_1+\cdots+\left.\frac{\partial y}{\partial \varepsilon}\right|_{T_0,T_1,T_2,\ldots,p_j}\mathrm{d}\varepsilon+\left.\frac{\partial y}{\partial p_j}\right|_{T_0,T_1,T_2,\ldots,\varepsilon}\mathrm{d}p_j. \tag{15}$$

We note that

$$\mathrm{d}y=\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j}\mathrm{d}t+\left.\frac{\partial y}{\partial \varepsilon}\right|_{t,p_j}\mathrm{d}\varepsilon+\left.\frac{\partial y}{\partial p_j}\right|_{t,\varepsilon}\mathrm{d}p_j \tag{16}$$

is still true, the difference just results from what is being held constant in our partial derivatives. Now evaluating the derivative with respect to t at constant ε and p_j in equation 15, we have via the chain rule

$$\begin{aligned}\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j}&=\left.\frac{\partial T_0}{\partial t}\right|_{\varepsilon,p_j}\left.\frac{\partial y}{\partial T_0}\right|_{T_1,T_2,\ldots,\varepsilon,p_j}+\left.\frac{\partial T_1}{\partial t}\right|_{\varepsilon,p_j}\left.\frac{\partial y}{\partial T_1}\right|_{T_0,T_2,\ldots,\varepsilon,p_j}+\left.\frac{\partial T_2}{\partial t}\right|_{\varepsilon,p_j}\left.\frac{\partial y}{\partial T_2}\right|_{T_0,T_1,\ldots,\varepsilon,p_j}+\ldots\\&=\left.\frac{\partial y}{\partial T_0}\right|_{T_1,T_2,\ldots,\varepsilon,p_j}+\varepsilon\left.\frac{\partial y}{\partial T_1}\right|_{T_0,T_2,\ldots,\varepsilon,p_j}+\varepsilon^2\left.\frac{\partial y}{\partial T_2}\right|_{T_0,T_1,\ldots,\varepsilon,p_j}+\ldots\\&=\sum_{n=0}^{\infty}\varepsilon^n\left.\frac{\partial y}{\partial T_n}\right|_{T_{m\neq n},\varepsilon,p_j}\end{aligned} \tag{17}$$

and so the time derivative has been turned into something that *looks like* a Taylor expansion. Our ability to do this relies on the fact that we are no longer dealing with the original coordinate t, as we have transformed the dependence of y into an infinite-dimensional function in the abstract domains T_n. We will see however that after solving in these abstract domains, the solution can be 'synthesised' back into the original coordinate t.

Now considering the expansion of y, our previous form (equation 3) no longer commutes with derivatives with respect to T_n. To enable an expansion of y that commutes with our derivatives, we Taylor expand y in ε while keeping the various T_n and p_j constant,

$$\begin{aligned}y\left(t;\varepsilon,p_j\right)&=y\left(T_0,T_1,\ldots;\varepsilon=0,p_j\right)+\varepsilon\left.\frac{\partial y}{\partial \varepsilon}\right|_{T_0,T_1,\ldots,p_j}\left(T_0,T_1,\ldots;0,p_j\right)+\frac{1}{2!}\varepsilon^2\left.\frac{\partial^2 y}{\partial \varepsilon^2}\right|_{T_0,T_1,\ldots,p_j}\left(T_0,T_1,\ldots;0,p_j\right)+\ldots\\&=\hat{y}_0\left(T_0,T_1,\ldots;p_j\right)+\varepsilon\hat{y}_1\left(T_0,T_1,\ldots;p_j\right)+\varepsilon^2\hat{y}_2\left(T_0,T_1,\ldots;p_j\right)+\ldots\\&=\sum_{n=0}^{\infty}\varepsilon^n\hat{y}_n\end{aligned} \tag{18}$$

where the definition of $\hat{y}_n$ is

$$\hat{y}_n = \frac{1}{n!}\left.\frac{\partial^n y}{\partial \varepsilon^n}\right|_{T_0,T_1,T_2,\dots,p_j}\left(T_0,T_1,T_2,\dots;\varepsilon=0,p_j\right). \tag{19}$$

Note the difference between this form and equation 3. Inserting 18 into 17, the partial derivatives with respect to T_n and ε now commute, and we get

$$\begin{aligned}\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j} &= \left[\left.\frac{\partial \hat{y}_0}{\partial T_0}\right|_{T_1,T_2,\dots,\varepsilon,p_j}\right] + \varepsilon\left[\left.\frac{\partial \hat{y}_1}{\partial T_0}\right|_{T_1,T_2,\dots,\varepsilon,p_j} + \left.\frac{\partial \hat{y}_0}{\partial T_1}\right|_{T_0,T_2,\dots,\varepsilon,p_j}\right] \\ &\quad + \varepsilon^2\left[\left.\frac{\partial \hat{y}_2}{\partial T_0}\right|_{T_1,T_2,\dots,\varepsilon,p_j} + \left.\frac{\partial \hat{y}_1}{\partial T_1}\right|_{T_0,T_2,\dots,\varepsilon,p_j} + \left.\frac{\partial \hat{y}_0}{\partial T_2}\right|_{T_0,T_1,\dots,\varepsilon,p_j}\right] + \mathcal{O}\left(\varepsilon^3\right) \\ &= \sum_{n=0}^{\infty}\varepsilon^n\sum_{j=0}^{n}\left.\frac{\partial \hat{y}_{n-j}}{\partial T_j}\right|_{T_{k\neq j},\varepsilon,p_j}\end{aligned} \tag{20}$$

with the second derivative written as

$$\begin{aligned}\left.\frac{\partial^2 y}{\partial t^2}\right|_{\varepsilon,p_j} &= \left[\left.\frac{\partial^2 \hat{y}_0}{\partial T_0^2}\right|_{T_1,T_2,\dots,\varepsilon,p_j}\right] + \varepsilon\left[\left.\frac{\partial^2 \hat{y}_1}{\partial T_0^2}\right|_{T_1,T_2,\dots,\varepsilon,p_j} + 2\left.\frac{\partial^2 \hat{y}_0}{\partial T_0\partial T_1}\right|_{T_2,\dots,\varepsilon,p_j}\right] \\ &\quad + \varepsilon^2\left[\left.\frac{\partial^2 \hat{y}_2}{\partial T_0^2}\right|_{T_1,T_2,\dots,\varepsilon,p_j} + 2\left.\frac{\partial^2 \hat{y}_1}{\partial T_0\partial T_1}\right|_{T_2,\dots,\varepsilon,p_j} + \left.\frac{\partial^2 \hat{y}_0}{\partial T_1^2}\right|_{T_0,T_2,\dots,\varepsilon,p_j} + 2\left.\frac{\partial^2 \hat{y}_0}{\partial T_0\partial T_2}\right|_{T_1,T_3,\dots,\varepsilon,p_j}\right] + \mathcal{O}\left(\varepsilon^3\right) \\ &= \sum_{n=0}^{\infty}\varepsilon^n\sum_{j=0}^{n}\sum_{k=0}^{n-j}\left.\frac{\partial^2 \hat{y}_{n-j-k}}{\partial T_j\partial T_k}\right|_{T_{m\neq\{j,k\}},\varepsilon,p_j}.\end{aligned} \tag{21}$$

3.2 Worked example of the damped harmonic oscillator

We now apply these considerations to equation 1. To reduce clutter we cease writing explicitly what is being held constant in our partial derivatives, however it should be unambiguous in the following. Applying our MMS expansion and gathering powers of ε, we obtain

$$\begin{aligned}&\left[\frac{\partial^2 \hat{y}_0}{\partial T_0^2} + \hat{y}_0\right] + \varepsilon\left[\frac{\partial^2 \hat{y}_1}{\partial T_0^2} + 2\frac{\partial^2 \hat{y}_0}{\partial T_0\partial T_1} + \hat{y}_1 + 2\frac{\partial \hat{y}_0}{\partial T_0}\right] \\ &\quad + \varepsilon^2\left[\frac{\partial^2 \hat{y}_2}{\partial T_0^2} + 2\frac{\partial^2 \hat{y}_1}{\partial T_0\partial T_1} + \frac{\partial^2 \hat{y}_0}{\partial T_1^2} + 2\frac{\partial^2 \hat{y}_0}{\partial T_0\partial T_2} + y_2 + 2\frac{\partial \hat{y}_1}{\partial T_0} + 2\frac{\partial \hat{y}_0}{\partial T_1}\right] + \cdots = 0.\end{aligned} \tag{22}$$

Setting each square bracket to zero as before, the first equation is solved by

$$\hat{y}_0 = A_0\left(T_1,T_2,\dots\right)\cos T_0 + B_0\left(T_1,T_2,\dots\right)\sin T_0 \tag{23}$$

which has a similar form to equation 7 however we now note that as this is the result of a partial differential equation with respect to T_0, the 'constants of integration' A_0 and B_0 are now functions of integration, having dependence on T_n, $n > 0$.

We consider truncating our expansion here and recovering a solution in the original coordinate t. Using 18, at this order we have

$$y\left(t;\varepsilon,p_j\right) = A_0\left(T_1,\dots\right)\cos T_0 + B_0\left(T_1,\dots\right)\sin T_0 + \mathcal{O}\left(\varepsilon\right). \tag{24}$$

The initial conditions refer to evaluating the function at $t = 0$ for constant ε and p_j, not when holding the T_n constant. Thus $t = 0$ corresponds to $T_0 = T_1 = T_2 = \cdots = 0$, and so $y\left(0\right) = A_0\left(0,\dots\right) + \mathcal{O}\left(\varepsilon\right)$ and, using 20 to evaluate the time derivative, $y'\left(0\right) = B_0\left(0,\dots\right) + \mathcal{O}\left(\varepsilon\right)$. We do not yet know the T_n, $n > 0$ dependence of A_0 and B_0, and so we can Taylor expand these functions in equation 24 around $T_1 = T_2 = \cdots = 0$, giving

$A_0(T_1,\dots) = A_0(0,\dots) + \mathcal{O}(T_1)$, $B_0(T_1,\dots) = B_0(0,\dots) + \mathcal{O}(T_1)$. The integration functions therefore become constants with an error of size T_1. Writing $T_0 = t$ and $T_1 = \varepsilon t$ by definition, our solution becomes $y(t;\varepsilon,p_j) = A_0(0,\dots)\cos t + B_0(0,\dots)\sin t + \mathcal{O}(\varepsilon,\varepsilon t)$, which after substituting in the initial conditions gives

$$\boxed{y(t;\varepsilon,p_j) = y(0)\cos t + y'(0)\sin t + \mathcal{O}(\varepsilon,\varepsilon t)} \tag{25}$$

which is simple harmonic motion, as expected. The solution at this order has the same form as equation 8, however we now have two errors associated with the solution. The first indicates that the solution has retained only the zeroth order part of the expansion of y, and the second indicates how far into the domain the solution is valid. We can see therefore that equation 25 is valid only for $\varepsilon \ll 1$ *and* $\varepsilon t \ll 1$.

3.2.1 First order

We now take this to next order. Using equation 23, the second bracket in equation 22 becomes

$$\begin{aligned}\frac{\partial^2 \hat{y}_1}{\partial T_0^2} + \hat{y}_1 &= -2\frac{\partial^2 \hat{y}_0}{\partial T_0 \partial T_1} - 2\frac{\partial \hat{y}_0}{\partial T_0}\\ &= 2\left(\frac{\partial A_0}{\partial T_1} + A_0\right)\sin T_0 - 2\left(\frac{\partial B_0}{\partial T_1} + B_0\right)\cos T_0\end{aligned} \tag{26}$$

which has the solution

$$\hat{y}_1 = -\left(\frac{\partial A_0}{\partial T_1} + A_0\right)T_0\cos T_0 - \left(\frac{\partial B_0}{\partial T_1} + B_0\right)T_0\sin T_0 + A_1\cos T_0 + B_1\sin T_0 \tag{27}$$

where $A_1 = A_1(T_1,\dots)$ and $B_1 = B_1(T_1,\dots)$. We note the appearance of secular terms, those that grow linearly with T_0. We find that these can be eliminated by enforcing

$$\frac{\partial A_0}{\partial T_1} + A_0 = 0 \implies A_0 = C_0(T_2,\dots)\,\mathrm{e}^{-T_1}, \qquad \frac{\partial B_0}{\partial T_1} + B_0 = 0 \implies B_0 = D_0(T_2,\dots)\,\mathrm{e}^{-T_1} \tag{28}$$

and so

$$\hat{y}_0 = \mathrm{e}^{-T_1}\left(C_0\cos T_0 + D_0\sin T_0\right), \qquad \hat{y}_1 = A_1\cos T_0 + B_1\sin T_0 \tag{29}$$

where we have the T_0 and T_1 dependence of $\hat{y}_0$ and the T_0 dependence of $\hat{y}_1$. Our solution for y is now

$$y(t;\varepsilon,p_j) = \mathrm{e}^{-T_1}\left(C_0\cos T_0 + D_0\sin T_0\right) + \varepsilon\left[A_1\cos T_0 + B_1\sin T_0\right] + \mathcal{O}\left(\varepsilon^2\right). \tag{30}$$

Taylor expanding the unknown T_n dependencies in the functions of integration and relating them to the initial conditions (appendix A.2.1), the solution becomes

$$\boxed{y(t;\varepsilon,p_j) = \mathrm{e}^{-\varepsilon t}\left(y(0)\cos t + y'(0)\sin t\right) + \varepsilon y(0)\sin t + \mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right).} \tag{31}$$

This is in agreement with the expansion of the exact solution, equation 12, now valid for $\varepsilon^2 \ll 1$ and $\varepsilon^2 t \ll 1$, as shown in figure 2.

3.2.2 Second order

We now continue this process to second order. Equating the third bracket in 22 to zero, we have

$$\begin{aligned}\frac{\partial^2 \hat{y}_2}{\partial T_0^2} + y_2 &= -2\frac{\partial^2 \hat{y}_1}{\partial T_0 \partial T_1} - \frac{\partial^2 \hat{y}_0}{\partial T_1^2} - 2\frac{\partial^2 \hat{y}_0}{\partial T_0 \partial T_2} - 2\frac{\partial \hat{y}_1}{\partial T_0} - 2\frac{\partial \hat{y}_0}{\partial T_1}\\ &= \left[C_0\mathrm{e}^{-T_1} - 2\frac{\partial D_0}{\partial T_2}\mathrm{e}^{-T_1} - 2B_1 - 2\frac{\partial B_1}{\partial T_1}\right]\cos T_0 + \left[D_0\mathrm{e}^{-T_1} + 2\frac{\partial C_0}{\partial T_2}\mathrm{e}^{-T_1} + 2A_1 + 2\frac{\partial A_1}{\partial T_1}\right]\sin T_0\end{aligned} \tag{32}$$

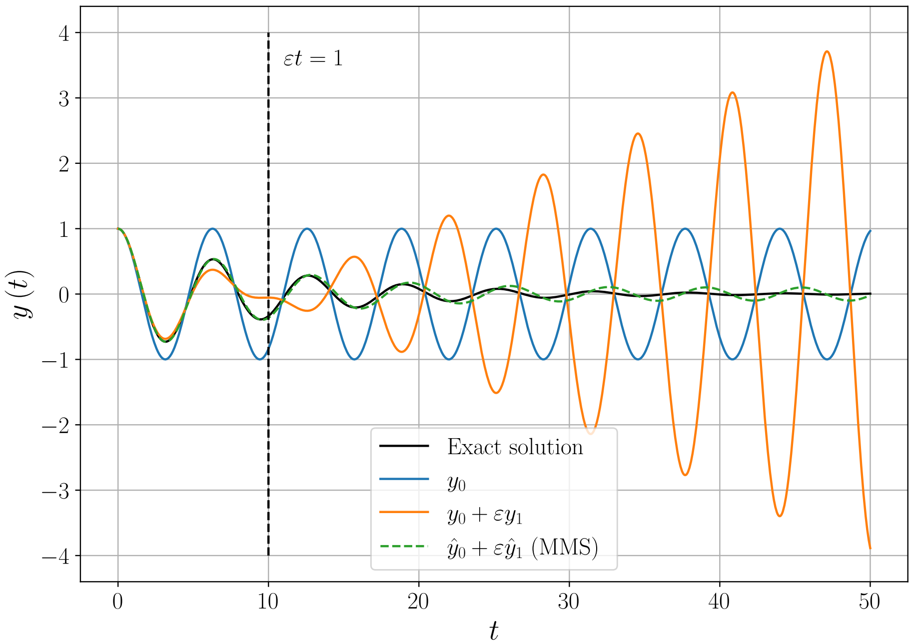

Figure 2: Verification of the first order solution (equation 31) to the damped harmonic oscillator using MMS (dashed green). Our approximation is no longer secular, and can be seen to agree with the exact solution for longer times than that of the lowest order (blue). A finite error persists at later times as the solution reaches the limits of its domain of validity.

which has the solution

$$\hat{y}_2 = \left[\frac{1}{2}C_0 \mathrm{e}^{-T_1} - \frac{\partial D_0}{\partial T_2}\mathrm{e}^{-T_1} - B_1 - \frac{\partial B_1}{\partial T_1}\right] T_0 \sin T_0 - \left[\frac{1}{2}D_0 \mathrm{e}^{-T_1} + \frac{\partial C_0}{\partial T_2}\mathrm{e}^{-T_1} + A_1 + \frac{\partial A_1}{\partial T_1}\right] T_0 \cos T_0 + A_2 \cos T_0 + B_2 \sin T_0. \tag{33}$$

where $A_2 = A_2(T_1, \dots)$ and $B_2 = B_2(T_1, \dots)$. The elimination of secular terms implies the two square brackets in 33 are zero, thus

$$\frac{\partial A_1}{\partial T_1} + A_1 = -\left(\frac{1}{2}D_0 + \frac{\partial C_0}{\partial T_2}\right)\mathrm{e}^{-T_1} \implies A_1 = -\left(\frac{1}{2}D_0 + \frac{\partial C_0}{\partial T_2}\right)T_1 \mathrm{e}^{-T_1} + C_1(T_2, \dots)\mathrm{e}^{-T_1} \tag{34}$$

$$\frac{\partial B_1}{\partial T_1} + B_1 = \left(\frac{1}{2}C_0 - \frac{\partial D_0}{\partial T_2}\right)\mathrm{e}^{-T_1} \implies B_1 = \left(\frac{1}{2}C_0 - \frac{\partial D_0}{\partial T_2}\right)T_1 \mathrm{e}^{-T_1} + D_1(T_2, \dots)\mathrm{e}^{-T_1}. \tag{35}$$

We now see we have secular terms in T_1, which are eliminated via

$$\frac{1}{2}D_0 + \frac{\partial C_0}{\partial T_2} = 0, \qquad \frac{1}{2}C_0 - \frac{\partial D_0}{\partial T_2} = 0. \tag{36}$$

These are solved by

$$C_0 = E_0 \cos\left(\frac{1}{2}T_2\right) - F_0 \sin\left(\frac{1}{2}T_2\right), \qquad D_0 = E_0 \sin\left(\frac{1}{2}T_2\right) + F_0 \cos\left(\frac{1}{2}T_2\right) \tag{37}$$

where $E_0 = E_0(T_3, \dots), F_0 = F_0(T_3, \dots)$. Bringing this together, we have

$$\hat{y}_0 = \mathrm{e}^{-T_1}\left(E_0 \cos\left(T_0 - \frac{1}{2}T_2\right) + F_0 \sin\left(T_0 - \frac{1}{2}T_2\right)\right) \tag{38}$$

$$\hat{y}_1 = \mathrm{e}^{-T_1}\left(C_1 \cos T_0 + D_1 \sin T_0\right), \qquad \hat{y}_2 = A_2 \cos T_0 + B_2 \sin T_0. \tag{39}$$

Our expression for y at this order is therefore

$$y(t;\varepsilon,p_j) = \mathrm{e}^{-T_1}\left(E_0\cos\left(T_0-\frac{1}{2}T_2\right)+F_0\sin\left(T_0-\frac{1}{2}T_2\right)\right)+\varepsilon\left[\mathrm{e}^{-T_1}\left(C_1\cos T_0+D_1\sin T_0\right)\right] + \varepsilon^2\left[A_2\cos T_0+B_2\sin T_0\right]+\mathcal{O}\left(\varepsilon^3\right) \tag{40}$$

which, expanding the unknown T_n dependencies and substituting the initial conditions (appendix A.2.2), becomes

$$\boxed{y(t;\varepsilon,p_j) = \mathrm{e}^{-\varepsilon t}\left(y(0)\cos\left(t-\frac{1}{2}\varepsilon^2 t\right)+y'(0)\sin\left(t-\frac{1}{2}\varepsilon^2 t\right)\right) + \varepsilon\mathrm{e}^{-\varepsilon t}y(0)\sin t+\varepsilon^2\frac{1}{2}y'(0)\sin t+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right)} \tag{41}$$

which is in agreement with our expansion of the exact solution, valid for $\varepsilon^3 \ll 1$ and $\varepsilon^3 t \ll 1$. This second order solution is plotted in figure 3.

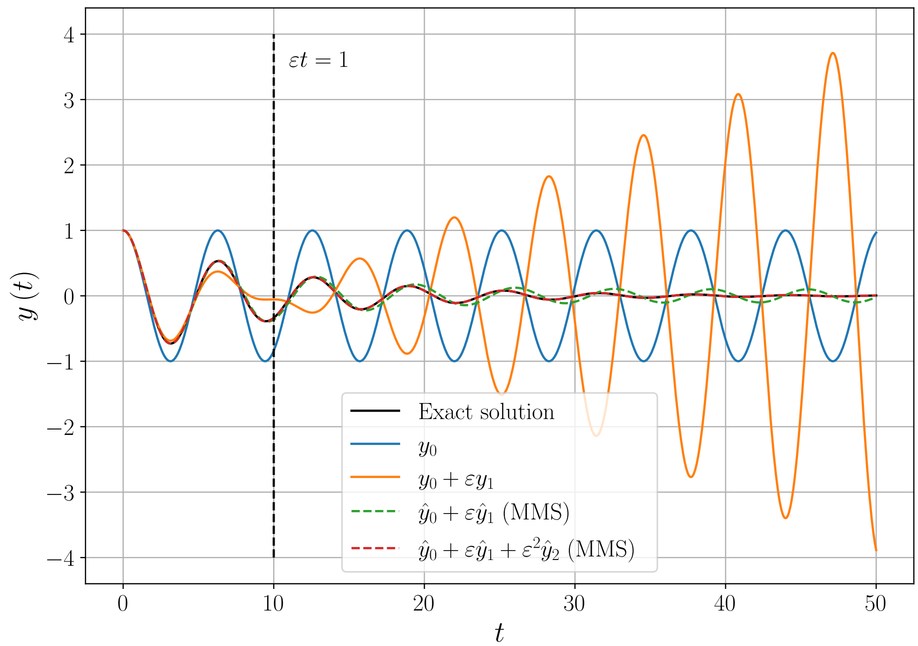

Figure 3: Verification of the second order solution (equation 41) to the damped harmonic oscillator using MMS (dashed red), which can be seen to essentially overlap with the exact solution throughout the domain considered.

4 The method of multiple scales for systems without an explicit small parameter

4.1 The simple pendulum

Having seen how MMS can be applied to cases in which an expansion parameter is explicit within the equation, we now consider its ability to deal with equations without one. As an example, we consider trying to solve the motion of a pendulum of length l under a constant force of gravity of field strength g,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\omega_0^2\sin(\theta)=0 \tag{42}$$

where $\omega_0^2 = g/l$ and $\theta = \theta(t;p_j)$, for which we know p_j must be ω_0 and the two initial conditions, $\theta(0)$ and $\theta'(0)$. To proceed, let us posit the existence of a parameter ε in which to expand. This appears as if we are creating a parameter from nothing, however as shall be seen, this amounts to 'splitting up' the parameters p_j into an expansion parameter ε and an as-yet undefined set of alternate parameters q_j, before then recombining to the original parameters p_j during the synthesis of the solution. Writing $\theta = \theta\left(T_0, T_1, \ldots; \varepsilon, q_j\right)$, we perform an MMS expansion on 42 using 17 and 18. The $\sin\theta$ term becomes

$$\begin{aligned}\sin\theta &= \sin\left(\hat{\theta}_0 + \varepsilon\hat{\theta}_1 + \varepsilon^2\hat{\theta}_2 + \ldots\right) \\ &= \left[\sin\hat{\theta}_0\right] + \varepsilon\left[\hat{\theta}_1\cos\hat{\theta}_0\right] + \varepsilon^2\left[\hat{\theta}_2\cos\hat{\theta}_0 - \frac{1}{2!}\hat{\theta}_1^2\sin\hat{\theta}_0\right] + \varepsilon^3\left[\hat{\theta}_3\cos\hat{\theta}_0 - \hat{\theta}_1\hat{\theta}_2\sin\hat{\theta}_0 - \frac{1}{3!}\hat{\theta}_1^3\cos\hat{\theta}_0\right] + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{43}$$

Gathering powers of ε and equating each bracket to zero in the expansion of 42, we generate the set of equations

$$\frac{\partial^2\hat{\theta}_0}{\partial T_0^2} + \omega_0^2\sin\hat{\theta}_0 = 0 \tag{44}$$

$$\frac{\partial^2\hat{\theta}_1}{\partial T_0^2} + 2\frac{\partial^2\hat{\theta}_0}{\partial T_0\partial T_1} + \omega_0^2\hat{\theta}_1\cos\hat{\theta}_0 = 0 \tag{45}$$

$$\frac{\partial^2\hat{\theta}_2}{\partial T_0^2} + 2\frac{\partial^2\hat{\theta}_1}{\partial T_0\partial T_1} + 2\frac{\partial^2\hat{\theta}_0}{\partial T_0\partial T_2} + \frac{\partial^2\hat{\theta}_0}{\partial T_1^2} + \omega_0^2\left(\hat{\theta}_2\cos\hat{\theta}_0 - \frac{1}{2}\hat{\theta}_1^2\sin\hat{\theta}_0\right) = 0 \tag{46}$$

$$\frac{\partial^2\hat{\theta}_3}{\partial T_0^2} + 2\frac{\partial^2\hat{\theta}_2}{\partial T_0\partial T_1} + 2\frac{\partial^2\hat{\theta}_1}{\partial T_0\partial T_2} + \frac{\partial^2\hat{\theta}_1}{\partial T_1^2} + 2\frac{\partial^2\hat{\theta}_0}{\partial T_0\partial T_3} + 2\frac{\partial^2\hat{\theta}_0}{\partial T_1\partial T_2} + \omega_0^2\left(\hat{\theta}_3\cos\hat{\theta}_0 - \hat{\theta}_1\hat{\theta}_2\sin\hat{\theta}_0 - \frac{1}{6}\hat{\theta}_1^3\cos\hat{\theta}_0\right) = 0 \tag{47}$$

and so on. Contrary to the case in which the expansion parameter appears explicitly in the equation, we see here that the lowest order equation, 44, is no simpler to generally solve than equation 42. We note however that the parameter ε we have posited does not yet have an explicit definition. Within this freedom, we may *choose* the function $\hat{\theta}_0$, provided that it satisfies 44, and this in turn implicitly defines the parameter ε. The definition becomes 'the parameter for which, when evaluated to be zero for the function $\theta\left(T_0, T_1, \ldots; \varepsilon, p_j\right)$, reduces it to $\hat{\theta}_0$'. This choosing of $\hat{\theta}_0$ essentially amounts to finding an 'equilibrium' of the original system. This choice is not unique, for example one may choose $\hat{\theta}_0 = 0$ or $\hat{\theta}_0 = \pi$, where each choice results in a different definition of the small parameter. Here we will choose $\hat{\theta}_0 = 0$. We also note that once this choice is made, there is no further freedom, and so no ability to choose anything else as we continue.

4.1.1 Zeroth order

As stated, we choose to proceed with $\hat{\theta}_0 = 0$. This trivially satisfies 44.

4.1.2 First order

Equation 45 becomes

$$\frac{\partial^2\hat{\theta}_1}{\partial T_0^2} + \omega_0^2\hat{\theta}_1 = 0 \tag{48}$$

which has the solution

$$\hat{\theta}_1 = A_1\left(T_1, \ldots\right)\cos\left(\omega_0 T_0 + \phi_1\left(T_1, \ldots\right)\right). \tag{49}$$

The solution for θ is therefore

$$\theta\left(t; \varepsilon, q_j\right) = \varepsilon A_1\cos\left(\omega_0 T_0 + \phi_1\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{50}$$

with derivative

$$\left.\frac{\partial\theta}{\partial t}\right|_{p_j} = -\omega_0\varepsilon A_1\sin\left(\omega_0 T_0 + \phi_1\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{51}$$

The initial conditions are

$$\theta\left(0\right) = \varepsilon A_1\left(0, \ldots\right)\cos\left(\phi_1\left(0, \ldots\right)\right) + \mathcal{O}\left(\varepsilon^2\right), \qquad \theta'\left(0\right) = -\omega_0\varepsilon A_1\left(0, \ldots\right)\sin\left(\phi_1\left(0, \ldots\right)\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{52}$$

Expanding the T_n dependence of A_1 and ϕ_1 in equation 50 and substituting the initial conditions using $\cos(x+y) = \cos x \cos y - \sin x \sin y$, we have

$$\begin{aligned}\theta(t;\varepsilon,q_j) &= \varepsilon A_1(0,\dots)\cos(\omega_0 T_0 + \phi_1(0,\dots)) + \mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right)\\ &= \varepsilon A_1(0,\dots)\cos(\omega_0 T_0)\cos(\phi_1(0,\dots)) - \varepsilon A_1(0,\dots)\sin(\omega_0 T_0)\sin(\phi_1(0,\dots)) + \mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right)\\ &= \theta(0)\cos(\omega_0 T_0) + \frac{\theta'(0)}{\omega_0}\sin(\omega_0 T_0) + \mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right)\end{aligned} \tag{53}$$

such that our solution at this order is

$$\boxed{\theta(t;p_j) = \theta(0)\cos(\omega_0 t) + \frac{\theta'(0)}{\omega_0}\sin(\omega_0 t) + \mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right).} \tag{54}$$

We see that ε does not appear in the solution except in the error terms, due to its combination with the expanded functions of integration being related to the initial conditions (52). The meaning of this error is currently somewhat hard to interpret, due to ε not having a numerical value. By taking this to higher order we will see more clearly what this error represents.

4.1.3 Second order

Turning to the second order equation, 46, we have

$$\begin{aligned}\frac{\partial^2\hat{\theta}_2}{\partial T_0^2} + \omega_0^2\hat{\theta}_2 &= -2\frac{\partial^2\hat{\theta}_1}{\partial T_0\partial T_1}\\ &= 2\omega_0\frac{\partial A_1}{\partial T_1}\sin(\omega_0 T_0 + \phi_1) + 2\omega_0 A_1\frac{\partial\phi_1}{\partial T_1}\cos(\omega_0 T_0 + \phi_1)\end{aligned} \tag{55}$$

which has the solution

$$\hat{\theta}_2 = -\frac{\partial A_1}{\partial T_1}T_0\cos(\omega_0 T_0 + \phi_1) + A_1\frac{\partial\phi_1}{\partial T_1}T_0\sin(\omega_0 T_0 + \phi_1) + A_2(T_1,\dots)\cos(\omega_0 T_0 + \phi_2(T_1,\dots)) \tag{56}$$

where to eliminate secular terms we require

$$\frac{\partial A_1}{\partial T_1} = \frac{\partial\phi_1}{\partial T_1} = 0. \tag{57}$$

The solution to this order is therefore

$$\theta(t;\varepsilon,q_j) = \varepsilon A_1\cos(\omega_0 T_0 + \phi_1) + \varepsilon^2 A_2\cos(\omega_0 T_0 + \phi_2) + \mathcal{O}\left(\varepsilon^3\right). \tag{58}$$

Expanding the unknown T_n dependencies and substituting the initial conditions (appendix A.3.1), it becomes

$$\boxed{\theta(t;p_j) = \theta(0)\cos(\omega_0 t) + \frac{\theta'(0)}{\omega_0}\sin(\omega_0 t) + \mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right)} \tag{59}$$

which is of the same form as equation 54, however we see that it is valid now for $\varepsilon^3 \ll 1$ and $\varepsilon^3 t \ll 1$.

4.1.4 Third order

Taking this procedure to third order, equation 47 reads

$$\begin{aligned}\frac{\partial^2\hat{\theta}_3}{\partial T_0^2} + \omega_0^2\hat{\theta}_3 &= \frac{1}{6}\omega_0^2\hat{\theta}_1^3 - 2\frac{\partial^2\hat{\theta}_1}{\partial T_0\partial T_2} - 2\frac{\partial^2\hat{\theta}_2}{\partial T_0\partial T_1}\\ &= \left(2\omega_0 A_1\frac{\partial\phi_1}{\partial T_2} + \frac{1}{8}\omega_0^2 A_1^3\right)\cos(\omega_0 T_0 + \phi_1) + 2\omega_0\frac{\partial A_1}{\partial T_2}\sin(\omega_0 T_0 + \phi_1)\\ &\quad + \frac{1}{24}\omega_0^2 A_1^3\cos(3(\omega_0 T_0 + \phi_1)) + 2\omega_0 A_2\frac{\partial\phi_2}{\partial T_1}\cos(\omega_0 T_0 + \phi_2) + 2\omega_0\frac{\partial A_2}{\partial T_1}\sin(\omega_0 T_0 + \phi_2)\end{aligned} \tag{60}$$

where we have used the identity $\cos^3 x = (3\cos x + \cos 3x)/4$. This has the solution

$$\begin{aligned}\hat{\theta}_3 = &-\frac{\partial A_1}{\partial T_2} T_0 \cos(\omega_0 T_0 + \phi_1) - \frac{1}{192} A_1^3 \cos(3(\omega_0 T_0 + \phi_1)) + \left(\frac{1}{16}\omega_0 A_1^2 + \frac{\partial \phi_1}{\partial T_2}\right) A_1 T_0 \sin(\omega_0 T_0 + \phi_1) \\ &+ A_2 \frac{\partial \phi_2}{\partial T_1} T_0 \sin(\omega_0 T_0 + \phi_2) - \frac{\partial A_2}{\partial T_1} T_0 \cos(\omega_0 T_0 + \phi_2) + A_3(T_1, \dots)\cos(\omega_0 T_0 + \phi_3(T_1, \dots))\end{aligned} \tag{61}$$

where the elimination of secular terms requires

$$\frac{\partial A_1}{\partial T_2} = \frac{\partial \phi_2}{\partial T_1} = \frac{\partial A_2}{\partial T_1} = 0 \tag{62}$$

$$\frac{1}{16}\omega_0 A_1^2 + \frac{\partial \phi_1}{\partial T_2} = 0 \implies \phi_1 = -\frac{1}{16}\omega_0 A_1^2 T_2 + C_1(T_3, \dots). \tag{63}$$

We therefore have

$$\begin{aligned}\theta(t; \varepsilon, q_j) = &\varepsilon A_1 \cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right) + \varepsilon^2 A_2 \cos(\omega_0 T_0 + \phi_2) \\ &+ \varepsilon^3 \left[A_3 \cos(\omega_0 T_0 + \phi_3) - \frac{1}{192} A_1^3 \cos\left(3\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right)\right)\right] + \mathcal{O}(\varepsilon^4).\end{aligned} \tag{64}$$

Expanding the unknown T_n dependencies and substituting the initial conditions (appendix A.3.2) we get our third order solution

$$\boxed{\begin{aligned}\theta(t; p_j) = &\theta(0) \cos\left(\left[1 - \frac{1}{16}\left(\theta(0)^2 + \left(\frac{\theta'(0)}{\omega_0}\right)^2\right)\right]\omega_0 t\right) + \frac{\theta'(0)}{\omega_0} \sin\left(\left[1 - \frac{1}{16}\left(\theta(0)^2 + \left(\frac{\theta'(0)}{\omega_0}\right)^2\right)\right]\omega_0 t\right) \\ &- \frac{1}{192}\left[\theta(0)^3 - 3\theta(0)\left(\frac{\theta'(0)}{\omega_0}\right)^2\right](\cos(3\omega_0 t) - \cos(\omega_0 t)) \\ &+ \frac{1}{192}\left[\left(\frac{\theta'(0)}{\omega_0}\right)^3 - 3\theta(0)^2\left(\frac{\theta'(0)}{\omega_0}\right)\right](\sin(3\omega_0 t) - 3\sin(\omega_0 t)) \\ &+ \frac{1}{16}\left(\frac{\theta'(0)}{\omega_0}\right)\left[\theta(0)^2 + \left(\frac{\theta'(0)}{\omega_0}\right)^2\right]\sin(\omega_0 t) + \mathcal{O}(\varepsilon^4, \varepsilon^4 t)\end{aligned}} \tag{65}$$

for which we see there is a correction to the lowest order frequency dependent on the initial condition. We can now also more easily see the meaning of our error terms. The $\mathcal{O}(\varepsilon^3)$ corrections can be seen to be terms cubic in the initial conditions. In this way, the ε error dependence is dealing with two degrees of freedom at once, $\theta(0)$ and $\theta'(0)$, and both are required to be small for our approximation to be valid. Our solutions to the pendulum are shown in figure 4.

4.2 The Lotka-Volterra system

A second example of a system without an explicit small parameter we consider is the Lotka-Volterra system, which is a pair of coupled nonlinear first order differential equations. Through normalisation (appendix A.3.3), the system can be written

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega_0^2 y + xy \tag{66}$$

to which we apply MMS. Again, there is no explicit small parameter, and so we expand in ε which as of yet lacks an explicit definition. We get

$$\left[\frac{\partial \hat{x}_0}{\partial T_0}\right] + \varepsilon\left[\frac{\partial \hat{x}_1}{\partial T_0} + \frac{\partial \hat{x}_0}{\partial T_1}\right] + \dots = [\hat{x}_0 - \hat{x}_0\hat{y}_0] + \varepsilon[\hat{x}_1 - \hat{x}_0\hat{y}_1 - \hat{x}_1\hat{y}_0] + \dots \tag{67}$$

$$\left[\frac{\partial \hat{y}_0}{\partial T_0}\right] + \varepsilon\left[\frac{\partial \hat{y}_1}{\partial T_0} + \frac{\partial \hat{y}_0}{\partial T_1}\right] + \dots = \left[-\omega_0^2\hat{y}_0 + \hat{x}_0\hat{y}_0\right] + \varepsilon\left[-\omega_0^2\hat{y}_1 + \hat{x}_0\hat{y}_1 + \hat{x}_1\hat{y}_0\right] + \dots. \tag{68}$$

These two sets of equations must be dealt with simultaneously.

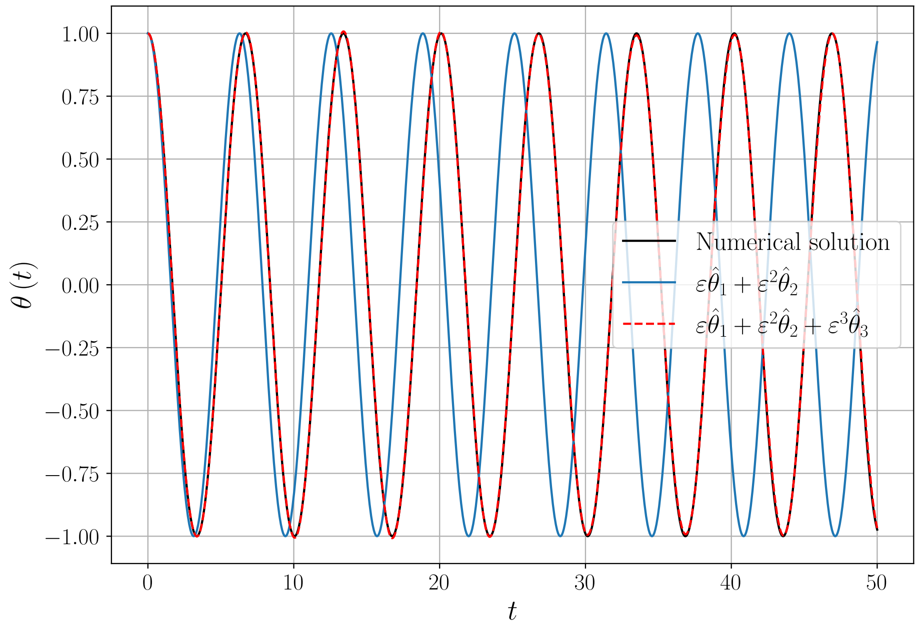

Figure 4: Verification of approximate solutions to the simple pendulum for $\omega_0 = 1$, $\theta(0) = 1$, $\theta'(0) = 0$ using MMS. At early times, the second order solution (equation 59, blue) of simple harmonic motion agrees with the numerical result (black), however can be seen to drift out of phase as time increases. The third order solution (equation 65, dashed red) which accounts for this change in phase can be seen to match the numerical result for far longer.

4.2.1 Zeroth order

At zeroth order, we have

$$\frac{\partial \hat{x}_0}{\partial T_0} = \hat{x}_0 - \hat{x}_0\hat{y}_0 \tag{69}$$

$$\frac{\partial \hat{y}_0}{\partial T_0} = -\omega_0^2\hat{y}_0 + \hat{x}_0\hat{y}_0 \tag{70}$$

where we are free to choose our lowest order solution within the lack of definition of ε. We choose to proceed with $\hat{x}_0 = \omega_0^2$, $\hat{y}_0 = 1$.

4.2.2 First order

At first order, using our zeroth order choices we have

$$\frac{\partial \hat{x}_1}{\partial T_0} + \frac{\partial \hat{x}_0}{\partial T_1} = \hat{x}_1 - \hat{x}_0\hat{y}_1 - \hat{x}_1\hat{y}_0 \implies \frac{\partial \hat{x}_1}{\partial T_0} = -\omega_0^2\hat{y}_1 \tag{71}$$

$$\frac{\partial \hat{y}_1}{\partial T_0} + \frac{\partial \hat{y}_0}{\partial T_1} = -\omega_0^2\hat{y}_1 + \hat{x}_0\hat{y}_1 + \hat{x}_1\hat{y}_0 \implies \frac{\partial \hat{y}_1}{\partial T_0} = \hat{x}_1 \tag{72}$$

and so

$$\frac{\partial^2 \hat{x}_1}{\partial T_0^2} + \omega_0^2\hat{x}_1 = 0 \implies \hat{x}_1 = A_1\cos(\omega_0 T_0 + \phi_1) \implies \hat{y}_1 = \frac{1}{\omega_0}A_1\sin(\omega_0 T_0 + \phi_1) \tag{73}$$

where $A_1 = A_1(T_1, \dots)$ and $\phi_1 = \phi_1(T_1, \dots)$. Synthesising the solution (appendix A.3.4), at this order we get

$$\boxed{x(t; p_j) = \omega_0^2 + \left(x(0) - \omega_0^2\right)\cos(\omega_0 t) - \omega_0\left(y(0) - 1\right)\sin(\omega_0 t) + \mathcal{O}\left(\varepsilon^2, \varepsilon^2 t\right)} \tag{74}$$

$$\boxed{y(t; p_j) = 1 + \frac{1}{\omega_0}\left(x(0) - \omega_0^2\right)\sin(\omega_0 t) + \left(y(0) - 1\right)\cos(\omega_0 t) + \mathcal{O}\left(\varepsilon^2, \varepsilon^2 t\right)} \tag{75}$$

where we note again that the combinations of ε with the expanded functions of integration characterise the proximity of the initial condition to the equilibrium

$$x(0) - \omega_0^2 = \varepsilon A_1(0,\dots)\cos(\phi_1(0,\dots)) + \mathcal{O}(\varepsilon^2) \tag{76}$$

$$\omega_0(y(0) - 1) = \varepsilon A_1(0,\dots)\sin(\phi_1(0,\dots)) + \mathcal{O}(\varepsilon^2) \tag{77}$$

which must both be small for our approximate solution to be valid.

A Additional derivation details

A.1 Section 2

A.1.1 Damped oscillator initial conditions at first order

Our solution for y is

$$y(t;\varepsilon,p_j) = [A_0\cos t + B_0\sin t] + \varepsilon[-A_0 t\cos t - B_0 t\sin t + A_1\cos t + B_1\sin t] + \mathcal{O}(\varepsilon^2) \tag{78}$$

which has the derivative

$$\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j} = [-A_0\sin t + B_0\cos t] + \varepsilon[-A_0\cos t + A_0 t\sin t - B_0\sin t - B_0 t\cos t - A_1\sin t + B_1\cos t] + \mathcal{O}(\varepsilon^2). \tag{79}$$

Relating these to the initial conditions, we have

$$y(0) = A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2), \qquad y'(0) = B_0 + \varepsilon[B_1 - A_0] + \mathcal{O}(\varepsilon^2). \tag{80}$$

We then rearrange such that $A_0 = y(0) - \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$ and $B_0 = y'(0) - \varepsilon[B_1 - y(0)] + \mathcal{O}(\varepsilon^2)$. Substituting these into 78, we get

$$y(t;\varepsilon,p_j) = \left[y(0)\cos t + y'(0)\sin t\right] + \varepsilon\left[-y(0)\,t\cos t - y'(0)\,t\sin t + y(0)\sin t\right] + \mathcal{O}(\varepsilon^2) \tag{81}$$

which is equation 11.

A.2 Section 3

A.2.1 First order MMS damped harmonic oscillator

Our solution at this order is

$$y(t;\varepsilon,p_j) = \mathrm{e}^{-T_1}(C_0\cos T_0 + D_0\sin T_0) + \varepsilon[A_1\cos T_0 + B_1\sin T_0] + \mathcal{O}(\varepsilon^2) \tag{82}$$

with time derivative (using equation 20)

$$\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j} = \mathrm{e}^{-T_1}(-C_0\sin T_0 + D_0\cos T_0) + \varepsilon\left[-\mathrm{e}^{-T_1}(C_0\cos T_0 + D_0\sin T_0) - A_1\sin T_0 + B_1\cos T_0\right] + \mathcal{O}(\varepsilon^2). \tag{83}$$

Imposing initial conditions, we obtain

$$y(0) = C_0(0,\dots) + \varepsilon A_1(0,\dots) + \mathcal{O}(\varepsilon^2) \tag{84}$$

$$y'(0) = D_0(0,\dots) + \varepsilon[B_1(0,\dots) - C_0(0,\dots)] + \mathcal{O}(\varepsilon^2). \tag{85}$$

We then expand C_0 and D_0 around $T_2 = T_3 = \cdots = 0$ as well as A_1 and B_1 around $T_1 = T_2 = \cdots = 0$ in equation 82,

$$y(t;\varepsilon,p_j) = \mathrm{e}^{-T_1}(C_0(0,\dots)\cos T_0 + D_0(0,\dots)\sin T_0) + \varepsilon[A_1(0,\dots)\cos T_0 + B_1(0,\dots)\sin T_0] + \mathcal{O}(\varepsilon^2, \varepsilon^2 t). \tag{86}$$

Multiplying equation 84 by $\mathrm{e}^{-T_1}\cos T_0$ we find

$$\begin{aligned}\mathrm{e}^{-T_1}y(0)\cos T_0 &= \mathrm{e}^{-T_1}C_0(0,\dots)\cos T_0+\varepsilon\mathrm{e}^{-T_1}A_1(0,\dots)\cos T_0+\mathcal{O}\left(\varepsilon^2\right)\\ &= \mathrm{e}^{-T_1}C_0(0,\dots)\cos T_0+\varepsilon A_1(0,\dots)\cos T_0+\mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right)\end{aligned} \tag{87}$$

where we have used $\mathrm{e}^{-T_1}=1+\mathcal{O}(\varepsilon t)$. Multiplying equation 85 by $\mathrm{e}^{-T_1}\sin T_0$,

$$\begin{aligned}\mathrm{e}^{-T_1}y'(0)\sin T_0 &= \mathrm{e}^{-T_1}D_0(0,\dots)\sin T_0+\varepsilon\mathrm{e}^{-T_1}\left[B_1(0,\dots)-C_0(0,\dots)\right]\sin T_0+\mathcal{O}\left(\varepsilon^2\right)\\ &= \mathrm{e}^{-T_1}D_0(0,\dots)\sin T_0+\varepsilon\left[B_1(0,\dots)-C_0(0,\dots)\right]\sin T_0+\mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right)\\ &= \mathrm{e}^{-T_1}D_0(0,\dots)\sin T_0+\varepsilon\left[B_1(0,\dots)-y(0)\right]\sin T_0+\mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right).\end{aligned} \tag{88}$$

Equation 86 therefore becomes

$$y(t;\varepsilon,p_j)=\mathrm{e}^{-T_1}\left(y(0)\cos T_0+y'(0)\sin T_0\right)+\varepsilon\left[y(0)\sin T_0\right]+\mathcal{O}\left(\varepsilon^2,\varepsilon^2 t\right) \tag{89}$$

which upon writing $T_0=t$, $T_1=\varepsilon t$ is equation 31.

A.2.2 Second order MMS damped harmonic oscillator

Our solution to this order is

$$\begin{aligned}y(t;\varepsilon,p_j)=&\,\mathrm{e}^{-T_1}\left(E_0\cos\left(T_0-\frac{1}{2}T_2\right)+F_0\sin\left(T_0-\frac{1}{2}T_2\right)\right)+\varepsilon\left[\mathrm{e}^{-T_1}\left(C_1\cos T_0+D_1\sin T_0\right)\right]\\ &+\varepsilon^2\left[A_2\cos T_0+B_2\sin T_0\right]+\mathcal{O}\left(\varepsilon^3\right)\end{aligned} \tag{90}$$

with time derivative

$$\begin{aligned}\left.\frac{\partial y}{\partial t}\right|_{\varepsilon,p_j}=&\,\mathrm{e}^{-T_1}\left(-E_0\sin\left(T_0-\frac{1}{2}T_2\right)+F_0\cos\left(T_0-\frac{1}{2}T_2\right)\right)\\ &+\varepsilon\left[-\mathrm{e}^{-T_1}\left(E_0\cos\left(T_0-\frac{1}{2}T_2\right)+F_0\sin\left(T_0-\frac{1}{2}T_2\right)\right)+\mathrm{e}^{-T_1}\left(-C_1\sin T_0+D_1\cos T_0\right)\right]\\ &+\varepsilon^2\left[\mathrm{e}^{-T_1}\frac{1}{2}\left(E_0\sin\left(T_0-\frac{1}{2}T_2\right)-F_0\cos\left(T_0-\frac{1}{2}T_2\right)\right)\right.\\ &\qquad\left.-\mathrm{e}^{-T_1}\left(C_1\cos T_0+D_1\sin T_0\right)-A_2\sin T_0+B_2\cos T_0\right]+\mathcal{O}\left(\varepsilon^3\right).\end{aligned} \tag{91}$$

The initial conditions are

$$y(0)=E_0(0,\dots)+\varepsilon C_1(0,\dots)+\varepsilon^2 A_2(0,\dots)+\mathcal{O}\left(\varepsilon^3\right) \tag{92}$$

$$y'(0)=F_0(0,\dots)+\varepsilon\left[-E_0(0,\dots)+D_1(0,\dots)\right]+\varepsilon^2\left[-\frac{1}{2}F_0(0,\dots)-C_1(0,\dots)+B_2(0,\dots)\right]+\mathcal{O}\left(\varepsilon^3\right). \tag{93}$$

We expand E_0, F_0, C_1, D_1, A_2 and B_2 in all of their T_n arguments in equation 90,

$$\begin{aligned}y(t;\varepsilon,p_j)=&\,\mathrm{e}^{-T_1}\left(E_0(0,\dots)\cos\left(T_0-\frac{1}{2}T_2\right)+F_0(0,\dots)\sin\left(T_0-\frac{1}{2}T_2\right)\right)\\ &+\varepsilon\left[\mathrm{e}^{-T_1}\left(C_1(0,\dots)\cos T_0+D_1(0,\dots)\sin T_0\right)\right]\\ &+\varepsilon^2\left[A_2(0,\dots)\cos T_0+B_2(0,\dots)\sin T_0\right]\\ &+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right).\end{aligned} \tag{94}$$

Multiplying 92 by $\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right)$,

$$\begin{aligned} y(0)\,\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right) &= E_0(0,\dots)\,\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right)+\varepsilon C_1(0,\dots)\,\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right)\\ &\quad+\varepsilon^2 A_2(0,\dots)\,\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right)+\mathcal{O}\left(\varepsilon^3\right)\\ &= E_0(0,\dots)\,\mathrm{e}^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right)+\varepsilon C_1(0,\dots)\,\mathrm{e}^{-T_1}\cos T_0\\ &\quad+\varepsilon^2 A_2(0,\dots)\cos T_0+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right) \end{aligned} \tag{95}$$

and 93 by $\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)$,

$$\begin{aligned} y'(0)\,\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right) &= F_0(0,\dots)\,\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)+\varepsilon\left[-E_0(0,\dots)+D_1(0,\dots)\right]\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)\\ &\quad+\varepsilon^2\left[-\frac{1}{2}F_0(0,\dots)-C_1(0,\dots)+B_2(0,\dots)\right]\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)+\mathcal{O}\left(\varepsilon^3\right)\\ &= F_0(0,\dots)\,\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)+\varepsilon\left[-E_0(0,\dots)+D_1(0,\dots)\right]\mathrm{e}^{-T_1}\sin T_0\\ &\quad+\varepsilon^2\left[-\frac{1}{2}F_0(0,\dots)-C_1(0,\dots)+B_2(0,\dots)\right]\sin T_0+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right) \end{aligned} \tag{96}$$

where we use $y'(0)=F_0(0,\dots)+\mathcal{O}(\varepsilon)$ and

$$\begin{aligned} -\varepsilon\mathrm{e}^{-T_1}y(0)\sin T_0 &= -\varepsilon\mathrm{e}^{-T_1}E_0(0,\dots)\sin T_0-\varepsilon^2\mathrm{e}^{-T_1}C_1(0,\dots)\sin T_0+\mathcal{O}\left(\varepsilon^3\right)\\ &= -\varepsilon\mathrm{e}^{-T_1}E_0(0,\dots)\sin T_0-\varepsilon^2 C_1(0,\dots)\sin T_0+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right) \end{aligned} \tag{97}$$

such that we re-arrange equation 96 to get

$$\begin{aligned} y'(0)\,\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)+\varepsilon\mathrm{e}^{-T_1}y(0)\sin T_0+\varepsilon^2\frac{1}{2}y'(0)\sin T_0 &= F_0(0,\dots)\,\mathrm{e}^{-T_1}\sin\left(T_0-\frac{1}{2}T_2\right)\\ &\quad+\varepsilon\left[D_1(0,\dots)\right]\mathrm{e}^{-T_1}\sin T_0+\varepsilon^2\left[B_2(0,\dots)\right]\sin T_0+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right). \end{aligned} \tag{98}$$

Substituting 95 and 98 into equation 94 gives

$$\begin{aligned} y(t;\varepsilon,p_j) &= \mathrm{e}^{-T_1}\left(y(0)\cos\left(T_0-\frac{1}{2}T_2\right)+y'(0)\sin\left(T_0-\frac{1}{2}T_2\right)\right)\\ &\quad+\varepsilon\mathrm{e}^{-T_1}y(0)\sin T_0+\varepsilon^2\frac{1}{2}y'(0)\sin T_0+\mathcal{O}\left(\varepsilon^3,\varepsilon^3 t\right) \end{aligned} \tag{99}$$

which, writing $T_n=\varepsilon^n t$, is equation 41.

A.3 Section 4

A.3.1 Second order pendulum solution

The solution to this order is

$$\theta(t;\varepsilon,q_j)=\varepsilon A_1\cos(\omega_0 T_0+\phi_1)+\varepsilon^2 A_2\cos(\omega_0 T_0+\phi_2)+\mathcal{O}\left(\varepsilon^3\right). \tag{100}$$

with derivative

$$\frac{\partial\theta}{\partial t}=-\omega_0\varepsilon A_1\sin(\omega_0 T_0+\phi_1)-\omega_0\varepsilon^2 A_2\sin(\omega_0 T_0+\phi_2)+\mathcal{O}\left(\varepsilon^3\right) \tag{101}$$

such that

$$\theta(0) = \varepsilon A_1(0,\dots)\cos(\phi_1(0,\dots)) + \varepsilon^2 A_2(0,\dots)\cos(\phi_2(0,\dots)) + \mathcal{O}\left(\varepsilon^3\right) \tag{102}$$

$$\theta'(0) = -\omega_0\varepsilon A_1(0,\dots)\sin(\phi_1(0,\dots)) - \omega_0\varepsilon^2 A_2(0,\dots)\sin(\phi_2(0,\dots)) + \mathcal{O}\left(\varepsilon^3\right). \tag{103}$$

Expanding 100 and substituting the initial conditions, we get

$$\begin{aligned}\theta(t;\varepsilon,q_j) &= \varepsilon A_1(0,\dots)\cos(\omega_0 T_0 + \phi_1(0,\dots)) + \varepsilon^2 A_2(0,\dots)\cos(\omega_0 T_0 + \phi_2(0,\dots)) + \mathcal{O}\left(\varepsilon^3, \varepsilon^3 t\right)\\ &= \varepsilon A_1(0,\dots)\cos(\omega_0 T_0)\cos(\phi_1(0,\dots)) - \varepsilon A_1(0,\dots)\sin(\omega_0 T_0)\sin(\phi_1(0,\dots))\\ &\quad + \varepsilon^2 A_2(0,\dots)\cos(\omega_0 T_0)\cos(\phi_2(0,\dots)) - \varepsilon^2 A_2(0,\dots)\sin(\omega_0 T_0)\sin(\phi_2(0,\dots)) + \mathcal{O}\left(\varepsilon^3, \varepsilon^3 t\right)\\ &= \theta(0)\cos(\omega_0 T_0) + \frac{\theta'(0)}{\omega_0}\sin(\omega_0 T_0) + \mathcal{O}\left(\varepsilon^3, \varepsilon^3 t\right)\end{aligned} \tag{104}$$

which upon writing $T_0 = t$ is equation 59.

A.3.2 Third order pendulum solution

Our solution at this order is

$$\begin{aligned}\theta(t;\varepsilon,q_j) = &\varepsilon A_1 \cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right) + \varepsilon^2 A_2\cos(\omega_0 T_0 + \phi_2)\\ &+ \varepsilon^3\left[A_3\cos(\omega_0 T_0 + \phi_3) - \frac{1}{192}A_1^3\cos\left(3\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right)\right)\right] + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{105}$$

with derivative

$$\begin{aligned}\left.\frac{\partial\theta}{\partial t}\right|_{p_j} = &-\omega_0\varepsilon A_1\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right) - \omega_0\varepsilon^2 A_2\sin(\omega_0 T_0 + \phi_2)\\ &+ \varepsilon^3\Bigg[-\omega_0 A_3\sin(\omega_0 T_0 + \phi_3) + \frac{3\omega_0}{192}A_1^3\sin\left(3\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right)\right)\\ &+ \frac{1}{16}\omega_0 A_1^3\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2 T_2 + C_1\right)\Bigg] + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{106}$$

and initial conditions

$$\begin{aligned}\theta(0) = &\varepsilon A_1(0,\dots)\cos(C_1(0,\dots)) + \varepsilon^2 A_2(0,\dots)\cos(\phi_2(0,\dots))\\ &+ \varepsilon^3\left[A_3(0,\dots)\cos(\phi_3(0,\dots)) - \frac{1}{192}A_1^3(0,\dots)\cos(3C_1(0,\dots))\right] + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{107}$$

$$\begin{aligned}\frac{\theta'(0)}{\omega_0} = &-\varepsilon A_1(0,\dots)\sin(C_1(0,\dots)) - \varepsilon^2 A_2(0,\dots)\sin(\phi_2(0,\dots))\\ &+ \varepsilon^3\left[-A_3(0,\dots)\sin(\phi_3(0,\dots)) + \frac{3}{192}A_1^3(0,\dots)\sin(3C_1(0,\dots)) + \frac{1}{16}A_1^3(0,\dots)\sin(C_1(0,\dots))\right]\\ &+ \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{108}$$

Expanding the unknown T_n dependence in equation 105 as well as the T_2 dependence of $\hat{\theta}_3$ and using $\cos(x+y) = \cos x \cos y - \sin x \sin y$,

$$\begin{aligned}
\theta(t;\varepsilon,q_j) =& \varepsilon A_1(0,\dots)\cos(C_1(0,\dots))\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) \\
&- \varepsilon A_1(0,\dots)\sin(C_1(0,\dots))\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) \\
&+ \varepsilon^2 A_2(0,\dots)\cos(\omega_0 T_0)\cos(\phi_2(0,\dots)) - \varepsilon^2 A_2(0,\dots)\sin(\omega_0 T_0)\sin(\phi_2(0,\dots)) \\
&+ \varepsilon^3 A_3(0,\dots)\cos(\omega_0 T_0)\cos(\phi_3(0,\dots)) - \varepsilon^3 A_3(0,\dots)\sin(\omega_0 T_0)\sin(\phi_3(0,\dots)) \\
&- \frac{1}{192}\varepsilon^3 A_1^3(0,\dots)\cos(3\omega_0 T_0)\cos(3C_1(0,\dots)) + \frac{1}{192}\varepsilon^3 A_1^3(0,\dots)\sin(3\omega_0 T_0)\sin(3C_1(0,\dots)) \\
&+ \mathcal{O}\left(\varepsilon^4,\varepsilon^4 t\right).
\end{aligned} \tag{109}$$

Multiplying the first initial condition by $\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right)$ and expanding to $\mathcal{O}\left(\varepsilon^4 t\right)$,

$$\begin{aligned}
\theta(0)\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) =& \varepsilon A_1(0,\dots)\cos(C_1(0,\dots))\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) \\
&+ \varepsilon^2 A_2(0,\dots)\cos(\phi_2(0,\dots))\cos(\omega_0 T_0) \\
&+ \varepsilon^3\left[A_3(0,\dots)\cos(\phi_3(0,\dots))\right]\cos(\omega_0 T_0) \\
&- \varepsilon^3\left[\frac{1}{192}A_1^3(0,\dots)\cos(3C_1(0,\dots))\right]\cos(\omega_0 T_0) \\
&+ \mathcal{O}\left(\varepsilon^4,\varepsilon^4 t\right)
\end{aligned} \tag{110}$$

and similarly for the second initial condition, multiplying by $\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right)$,

$$\begin{aligned}
\frac{\theta'(0)}{\omega_0}\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) =& -\varepsilon A_1(0,\dots)\sin(C_1(0,\dots))\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) \\
&- \varepsilon^2 A_2(0,\dots)\sin(\phi_2(0,\dots))\sin(\omega_0 T_0) \\
&- \varepsilon^3\left[A_3(0,\dots)\sin(\phi_3(0,\dots))\right]\sin(\omega_0 T_0) \\
&+ \varepsilon^3\left[\frac{3}{192}A_1^3(0,\dots)\sin(3C_1(0,\dots))\right]\sin(\omega_0 T_0) \\
&+ \varepsilon^3\left[\frac{1}{16}A_1^3(0,\dots)\sin(C_1(0,\dots))\right]\sin(\omega_0 T_0) \\
&+ \mathcal{O}\left(\varepsilon^4,\varepsilon^4 t\right).
\end{aligned} \tag{111}$$

Substituting these into 109, we get

$$\begin{aligned}
\theta(t;\varepsilon,q_j) =& \theta(0)\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) + \frac{\theta'(0)}{\omega_0}\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2(0,\dots)T_2\right) \\
&- \frac{1}{192}\varepsilon^3 A_1^3(0,\dots)\cos(3C_1(0,\dots))\left(\cos(3\omega_0 T_0) - \cos(\omega_0 T_0)\right) \\
&+ \frac{1}{192}\varepsilon^3 A_1^3(0,\dots)\sin(3C_1(0,\dots))\left(\sin(3\omega_0 T_0) - 3\sin(\omega_0 T_0)\right) \\
&- \frac{1}{16}\varepsilon^3 A_1^3(0,\dots)\sin(C_1(0,\dots))\sin(\omega_0 T_0) \\
&+ \mathcal{O}\left(\varepsilon^4,\varepsilon^4 t\right).
\end{aligned} \tag{112}$$

The terms involving $A_1(0,\dots)$ and $C_1(0,\dots)$ are dealt with via the trigonometric identities

$$\cos(3C_1(0,\dots)) = -3\cos(C_1(0,\dots)) + 4\cos^3(C_1(0,\dots)) \tag{113}$$

$$\sin\left(3C_1\left(0,\dots\right)\right) = 3\sin\left(C_1\left(0,\dots\right)\right) - 4\sin^3\left(C_1\left(0,\dots\right)\right) \tag{114}$$

and

$$\theta\left(0\right) = \varepsilon A_1\left(0,\dots\right)\cos\left(C_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{115}$$

$$\frac{\theta'\left(0\right)}{\omega_0} = -\,\varepsilon A_1\left(0,\dots\right)\sin\left(C_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{116}$$

such that

$$\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2 = \varepsilon^2 A_1^2\left(0,\dots\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{117}$$

$$\theta\left(0\right)^3 = \varepsilon^3 A_1^3\left(0,\dots\right)\cos^3\left(C_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^4\right) \tag{118}$$

$$\left(\frac{\theta'\left(0\right)}{\omega_0}\right)^3 = -\,\varepsilon^3 A_1^3\left(0,\dots\right)\sin^3\left(C_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^4\right). \tag{119}$$

The terms in 112 therefore become

$$\begin{aligned}\varepsilon^3 A_1^3\left(0,\dots\right)\cos\left(3C_1\left(0,\dots\right)\right) &= -3\varepsilon^3 A_1^3\left(0,\dots\right)\cos\left(C_1\left(0,\dots\right)\right) + 4\varepsilon^3 A_1^3\left(0,\dots\right)\cos^3\left(C_1\left(0,\dots\right)\right)\\ &= -3\theta\left(0\right)\left[\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right] + 4\theta\left(0\right)^3 + \mathcal{O}\left(\varepsilon^4\right)\\ &= \theta\left(0\right)^3 - 3\theta\left(0\right)\left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2 + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{120}$$

$$\begin{aligned}\varepsilon^3 A_1^3\left(0,\dots\right)\sin\left(3C_1\left(0,\dots\right)\right) &= 3\varepsilon^3 A_1^3\left(0,\dots\right)\sin\left(C_1\left(0,\dots\right)\right) - 4\varepsilon^3 A_1^3\left(0,\dots\right)\sin^3\left(C_1\left(0,\dots\right)\right)\\ &= -3\left(\frac{\theta'\left(0\right)}{\omega_0}\right)\left[\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right] + 4\left(\frac{\theta'\left(0\right)}{\omega_0}\right)^3 + \mathcal{O}\left(\varepsilon^4\right)\\ &= \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^3 - 3\theta\left(0\right)^2\left(\frac{\theta'\left(0\right)}{\omega_0}\right) + \mathcal{O}\left(\varepsilon^4\right)\end{aligned} \tag{121}$$

$$\varepsilon^3 A_1^3\left(0,\dots\right)\sin\left(C_1\left(0,\dots\right)\right) = -\left(\frac{\theta'\left(0\right)}{\omega_0}\right)\left[\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right] + \mathcal{O}\left(\varepsilon^4\right). \tag{122}$$

such that we have

$$\begin{aligned}\theta\left(t;\varepsilon,q_j\right) =& \,\theta\left(0\right)\cos\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2\left(0,\dots\right)T_2\right) + \frac{\theta'\left(0\right)}{\omega_0}\sin\left(\omega_0 T_0 - \frac{1}{16}\omega_0 A_1^2\left(0,\dots\right)T_2\right)\\ &- \frac{1}{192}\left[\theta\left(0\right)^3 - 3\theta\left(0\right)\left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right]\left(\cos\left(3\omega_0 T_0\right) - \cos\left(\omega_0 T_0\right)\right)\\ &+ \frac{1}{192}\left[\left(\frac{\theta'\left(0\right)}{\omega_0}\right)^3 - 3\theta\left(0\right)^2\left(\frac{\theta'\left(0\right)}{\omega_0}\right)\right]\left(\sin\left(3\omega_0 T_0\right) - 3\sin\left(\omega_0 T_0\right)\right)\\ &+ \frac{1}{16}\left(\frac{\theta'\left(0\right)}{\omega_0}\right)\left[\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right]\sin\left(\omega_0 T_0\right)\\ &+ \mathcal{O}\left(\varepsilon^4, \varepsilon^4 t\right).\end{aligned} \tag{123}$$

The final term to deal with is the $A_1^2\left(0,\dots\right)T_2$ dependency in the trigonometric functions. This is done by

$$\begin{aligned}A_1^2\left(0,\dots\right)T_2 &= \varepsilon^2 A_1^2\left(0,\dots\right)t\\ &= \left[\theta\left(0\right)^2 + \left(\frac{\theta'\left(0\right)}{\omega_0}\right)^2\right]t + \mathcal{O}\left(\varepsilon^3 t\right).\end{aligned} \tag{124}$$

When this is substituted, the error term combines with the ε dependence of the coefficients of the trigonometric functions to always be $\mathcal{O}\left(\varepsilon^4 t\right)$. Making this substitution and writing $T_0 = t$, we get equation 65.

A.3.3 Lotka-Volterra normalisation

The Lotka-Volterra equations are typically written in terms of four parameters,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy \tag{125}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y + \delta xy. \tag{126}$$

Defining $\bar{x} = \delta x/\alpha$, $\bar{y} = \beta y/\alpha$, $\bar{t} = \alpha t$ and $\omega_0^2 = \gamma/\alpha$, these become

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \bar{x} - \bar{x}\bar{y} \tag{127}$$

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{t}} = -\omega_0^2\bar{y} + \bar{x}\bar{y}. \tag{128}$$

Dropping the bar notation gives 66.

A.3.4 Lotka-Volterra initial conditions at first order

Using 73, the solutions for x and y are

$$x\left(t;\varepsilon,q_j\right) = \omega_0^2 + \varepsilon A_1 \cos\left(\omega_0 T_0 + \phi_1\right) + \mathcal{O}\left(\varepsilon^2\right), \qquad y\left(t;\varepsilon,q_j\right) = 1 + \frac{1}{\omega_0}\varepsilon A_1 \sin\left(\omega_0 T_0 + \phi_1\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{129}$$

with initial conditions

$$x\left(0\right) - \omega_0^2 = \varepsilon A_1\left(0,\dots\right)\cos\left(\phi_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{130}$$

$$\omega_0\left(y\left(0\right) - 1\right) = \varepsilon A_1\left(0,\dots\right)\sin\left(\phi_1\left(0,\dots\right)\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{131}$$

Expanding the unknown T_n dependencies in 129, using $\cos(x+y) = \cos x \cos y - \sin x \sin y$, $\sin\left(x+y\right) = \sin x \cos y + \sin y \cos x$ and substituting the initial conditions, we get

$$x\left(t;p_j\right) = \omega_0^2 + \left(x\left(0\right) - \omega_0^2\right)\cos\left(\omega_0 t\right) - \omega_0\left(y\left(0\right) - 1\right)\sin\left(\omega_0 t\right) + \mathcal{O}\left(\varepsilon^2, \varepsilon^2 t\right) \tag{132}$$

$$y\left(t;p_j\right) = 1 + \frac{1}{\omega_0}\left(x\left(0\right) - \omega_0^2\right)\sin\left(\omega_0 t\right) + \left(y\left(0\right) - 1\right)\cos\left(\omega_0 t\right) + \mathcal{O}\left(\varepsilon^2, \varepsilon^2 t\right) \tag{133}$$

which are equations 74 and 75.

References

[1] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory.* Springer, 1999.

[2] Ali H. Nayfeh. *Perturbation Methods.* John Wiley & Sons, 2008.

[3] A Salih. *The Method of Multiple Scales.* 2014.