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TRANSPORT PROCESSES IN A PLASMA

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§ 1. Transport Equations

The state of an ionized gas (plasma) can be specified by the distribution functions $f_a(\mathbf{t}, \mathbf{r}; \mathbf{v})$ that characterize each particle component. These functions describe the density of particles of species a at time t at the point $\mathbf{r}, \mathbf{v}$ in phase space; the quantity $f_a(t, \mathbf{r}, \mathbf{v})\, d\mathbf{r} d\mathbf{v}$ then represents the number of particles in the six-dimensional volume element $d\mathbf{r}\, d\mathbf{v}$. In the simplest case the plasma consists of electrons (a = e) and a single ion species (a = i); in more complicated cases the plasma may contain several ion species in addition to neutral particles (a = n) such as atoms, molecules, excited atoms, and so on. The behavior of the ionized gas is described by a system of kinetic equations (Boltzmann equations) which carry the distribution functions forward in time (cf. for example [1, 2, 3, 39]):

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial x_\beta}(v_\beta f_a) + \frac{\partial}{\partial v_\beta}\left(\frac{F_{a\beta}}{m_a} f_a\right) = C_a. \tag{1.1}$$

Here, $\mathbf{F}_a$ is the force exerted at point $\mathbf{r}$ on a particle of species a and velocity $\mathbf{v}$; m_a is the particle mass. For particles that carry a charge e_a and are located in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$

$$\mathrm{F}_a = e_a \mathrm{E} + \frac{e_a}{c}[\mathbf{v}\mathbf{B}]. \tag{1.2}$$

The kinetic equation does not take account of thermal fluctuations. The function $f_a(t, \mathbf{r}, \mathbf{v})$ that appears in Eq. (1.1) represents a smoothed density averaged over a volume containing a large number of particles.

The force $\mathbf{F}_a$ on the left side of the kinetic equation is also a "smoothed" macroscopic force and represents an average over a volume containing many particles and over times long compared with the appropriate time of flight; the same is true of the fields $\mathbf{E}$ and $\mathbf{B}$. The force $\mathbf{F}_a$ does not take account of rapidly fluctuating microfields and microforces that arise when particles come very close to each other. These effects (which we will simply call collisions) are taken into account by the collision term C_a on the right side of the equation. The problem of separating the self-consistent field from the microfields is an extremely complicated one and has been treated by many authors, for example, Kadomtsev [37].

Particles of species a can collide with each other and with other particle species. Thus, one must actually write

$$C_a = \sum_b C_{ab}(f_a, f_b), \quad (1.3)$$

where C_{ab} gives the change per unit time in the distribution function for particles of species a due to collisions with particles of species b. The C_{ab} terms can describe either elastic or inelastic collisions.* The so-called Boltzmann collision term, which describes elastic collisions, is given in the Appendix. In the case of elastic collisions between charged particles we shall use the collision term in the relatively simple form first given by Landau [11]. The collision term for inelastic collisions is extremely complicated and cannot always be written in explicit form. Inelastic collisions will be neglected in this review.

Certain properties of the collision term can be deduced even when its explicit form is not known. If processes that convert particles of one species into another, (ionization, dissociation, etc.) are neglected the collision terms satisfy the conditions

$$\int C_{ab}\, d\mathbf{v} = 0; \quad (1.4)$$

$$\int m_a \mathbf{v} C_{aa}\, d\mathbf{v} = 0; \quad (1.5)$$

$$\int \frac{m_a v^2}{2} C_{aa}\, d\mathbf{v} = 0. \quad (1.6)$$

*For example, excited atoms are treated as a different "species" from the unexcited atoms and are assigned a different subscript. We also note that Eq. (1.1) does not make explicit reference to the rotational degrees of freedom, which can be important, for example, in dealing with molecules. In order to take these rotational effects into account it would be necessary to introduce a distribution function that would depend on the total rotational moment of the particle **M** (in addition to **r** and **v**). Formally it can be assumed that **M** and the internal degrees of freedom are taken into account by the subscript a; actually, however, taking account of rotation is extremely complicated and will not done here. We shall simply assume that appropriate averages have been taken over the rotational variable.

When multiplied by $d\mathbf{r}$ the integral in (1.4) represents the change in the total number of particles of species a in a volume element $d\mathbf{r}$ due to collisions with particles of species b; in elastic collisions, however, no such change occurs and the integral vanishes. The integrals in (1.5) and (1.6) denote the change in momentum and energy, respectively, for particles of species a resulting from collisions between like particles; since momentum and energy are conserved in such collisions these integrals must also vanish. Similarly, the following relations hold for elastic collisions between different particle species, a and b, in which the total momentum and energy are conserved:

$$\int m_a \mathbf{v} C_{ab}\, d\mathbf{v} + \int m_b \mathbf{v} C_{ba}\, d\mathbf{v} = 0; \quad (1.5')$$

$$\int (m_a v^2/2)\, C_{ab}\, d\mathbf{v} + \int (m_b v^2/2)\, C_{ba}\, d\mathbf{v} = 0. \quad (1.6')$$

It is a general result of statistical mechanics that the particles of any gas in thermal equilibrium are characterized by a Maxwellian velocity distribution f^0:

$$f^0 = \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{m}{2T}(\mathbf{v}-\mathbf{V})^2}. \quad (1.7)$$

The subscript a has been omitted; n is the density, i.e., the number of particles of a given species per unit volume; T is the temperature of the gas; **V** is the velocity of the gas as a whole. The temperature will always be expressed in energy units so that the Boltzmann constant will not appear in the formulas. When the Maxwellian distribution is used the left side of the kinetic equation vanishes. Thus, regardless of the actual form of the collision term, when a Maxwellian distribution is used the collision term must vanish. Furthermore, if the distribution function changes only by virtue of collisions, it can be shown that no matter what the initial conditions are the distribution function must approach a Maxwellian in the course of time; this is a statement of the well-known H-theorem of Boltzmann, a proof of which can be found in [1, 2, 3]. The approach of the distribution function to a Maxwellian by means of collisions is called relaxation. Relaxation generally occurs in a time of the order of the mean time between collisions.

The description of a plasma by means of distribution functions is a rather detailed one and may not always be necessary. It is frequently sufficient to describe a plasma more simply in terms of certain average quantities, for example, the number of particles of a given species per unit volume

$$n_a(t, \mathbf{r}) = \int f_a(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \tag{1.8}$$

the mean velocity of these particles

$$\mathbf{V}_a(t, \mathbf{r}) = \frac{1}{n_a} \int \mathbf{v} f(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v} = \langle \mathbf{v} \rangle_a, \tag{1.9}$$

and the mean energy or temperature. In thermal equilibrium, i.e., when the distribution function is a Maxwellian, the mean kinetic energy per particle $m\langle v^2\rangle/2$ can be related simply to the temperature; furthermore, in the coordinate system in which $V = 0$ the simple relation $m\langle v^2\rangle/2 = (3/2)T$ holds. If the gas is not in thermal equilibrium, it is possible to define a temperature by introducing the quantity $m\langle v^2\rangle/3$ in the coordinate system in which $V = 0$. The temperature defined in this way is a function of t and **r** and other local macroscopic characteristics of the gas

$$T_a(t, \mathbf{r}) = \frac{1}{n_a} \int \frac{m_a}{3} (\mathbf{v} - \mathbf{V}_a)^2 f_a(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v} = \frac{m_a}{3} \langle (\mathbf{v} - \mathbf{V}_a)^2 \rangle. \tag{1.10}$$

In general, the macroscopic parameters n_a, $\mathbf{V}_a$, and T_a in a nonequilibrium state are different for different particle species. In some cases, these parameters, which have simple physical meanings, are supplemented by other more complicated parameters.

The equations that describe the behavior of the macroscopic parameters, which are called the transport equations, can be obtained from the kinetic equation. Equation (1.1) is multiplied by 1, $m_a\mathbf{V}$, and $m_a v^2/2$, respectively, and integrated over velocity. Carrying out this procedure and using Eq. (1.4) we find

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{V}) = 0, \tag{1.11}$$

$$\frac{\partial}{\partial t}(mnV_\alpha) + \frac{\partial}{\partial x_\beta}(mn\langle v_\alpha v_\beta\rangle) - en\left(E_\alpha + \frac{1}{c}[\mathbf{VB}]_\alpha\right) = \int m v_\alpha C\, d\mathbf{v}, \tag{1.12}$$

$$\frac{\partial}{\partial t}\left(\frac{mn}{2}\langle v^2\rangle\right) + \operatorname{div}\left(\frac{mn}{2}\langle v^2 \mathbf{v}\rangle\right) - en\mathbf{E}\cdot\mathbf{V} = \int \frac{mv^2}{2} C\, d\mathbf{v}. \tag{1.13}$$

The subscript a will be omitted for reasons of simplicity hereinafter. The angle brackets denote averages over the velocity distribution function.

The order of integration over velocity and differentiation over time and coordinates are interchanged in the first two terms in Eq. (1.1); the third term is integrated by parts and it is assumed that the distribution function vanishes rapidly as $v \to \infty$. Equation (1.11) expresses the conservation of particles and is called the particle transport equation or the equation of continuity. If particles are produced or annihilated Eq. (1.4) no longer applies and the zero on the right side of Eq. (1.11) must be replaced by an appropriate intensity for the particle source.

It will be found convenient to transform Eqs. (1.12) and (1.13) as follows. The velocity is divided into two components – a mean velocity **V** and a random velocity $\mathbf{v}' = \mathbf{v} - \mathbf{V}$; it is evident that $\langle \mathbf{v}' \rangle = 0$. The second term in Eq. (1.12) is written

$$\langle v_\alpha v_\beta \rangle = \langle (V_\alpha + v'_\alpha)(V_\beta + v'_\beta) \rangle = V_\alpha V_\beta + \langle v'_\alpha v'_\beta \rangle, \text{ because } \langle v'_\alpha \rangle = \langle v'_\beta \rangle = 0.$$

Expressing $\partial n/\partial t$ by means of the equation of continuity it is now possible to write Eq. (1.12) in the form

$$mn\frac{dV_\alpha}{dt} = -\frac{\partial p}{\partial x_\alpha} - \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} + en\left(E_\alpha + \frac{1}{c}[\mathbf{VB}]_\alpha\right) + R_\alpha, \tag{1.14}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_\beta \frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla) \tag{1.15}$$

is the so-called substantive derivative, and

$$p = nm\langle v'^2 \rangle/3 = nT, \tag{1.16}$$

$$\pi_{\alpha\beta} = nm\langle v'_\alpha v'_\beta - (v'^2/3)\,\delta_{\alpha\beta} \rangle, \tag{1.17}$$

$$\mathbf{R} = \int m\mathbf{v}' C\, d\mathbf{v}. \tag{1.18}$$

The quantity p is the scalar pressure for particles of a given species. The complete pressure tensor for a given species is

$$P_{\alpha\beta} = \int m v'_\alpha v'_\beta f(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v} = nm \langle v'_\alpha v'_\beta \rangle = p\delta_{\alpha\beta} + \pi_{\alpha\beta}. \tag{1.19}$$

If the velocity distribution function (for the random velocity) is isotropic, then $\langle v_x'^2\rangle = \langle v_y'^2\rangle = \langle v_z'^2\rangle = (1/3)\langle v'^2\rangle$, $\langle v'_x v'_y\rangle = \langle v'_x v'_z\rangle = \langle v'_y v'_z\rangle = 0$, so that $P_{\alpha\beta} = p\delta_{\alpha\beta}$. The tensor $\pi_{\alpha\beta}$ represents the part of $P_{\alpha\beta}$ that arises as a result of the deviation of the distribution from spherical symmetry. The quantity $\pi_{\alpha\beta}$ will be called the stress tensor. Like $P_{\alpha\beta}$ this tensor is symmetric $\pi_{\alpha\beta} = \pi_{\beta\alpha}$.

The quantity **R** represents the mean change in the momentum of the particles of a given species due to collisions with all other particles.

Equation (1.14) is called the momentum transport equation or simply the equation of motion. It represents a generalization of the corresponding equation in gas dynamics.

Carrying out similar transformations

$$\left\langle \frac{v^2}{2} v_\beta \right\rangle = \frac{1}{2} V^2 V_\beta + V_\alpha \langle v'_\alpha v'_\beta \rangle + \frac{1}{2} \langle v'^2 \rangle V_\beta +$$

$$+ \left\langle \frac{1}{2} v'^2 v'_\beta \right\rangle = \left(\frac{1}{2} V^2 + \frac{5}{2} \frac{P}{mn} \right) V_\beta + \frac{1}{mn} V_\alpha \pi_{\alpha\beta} + \left\langle \frac{1}{2} v'^2 v'_\beta \right\rangle,$$

we can reduce Eq. (1.13) to the form

$$\frac{\partial}{\partial t}\left(\frac{nm}{2} V^2 + \frac{3}{2} nT \right) + \frac{\partial}{\partial x_\beta} \left\{ \left(\frac{nm}{2} V^2 + \frac{5}{2} nT \right) V_\beta + \right.$$

$$\left. + (\pi_{\alpha\beta} \cdot V_\alpha) + q_\beta \right\} = en\, \mathbf{E}\mathbf{V} + \mathbf{R}\mathbf{V} + Q. \tag{1.20}$$

Here we have introduced the notation

$$\mathbf{q} = \int \frac{m}{2} v'^2 \mathbf{v}' f(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v} = nm \left\langle \frac{v'^2}{2} \mathbf{v}' \right\rangle, \tag{1.21}$$

$$Q = \int \frac{mv'^2}{2} C\, d\mathbf{v}. \tag{1.22}$$

The vector **q** is the flux density of heat carried by particles of a given species and represents the transport of the energy associated with the random motion in the coordinate system in which the particle gas as a whole is at rest at a given point in space.

The quantity Q is the heat generated in a gas of particles of a given species as a consequence of collisions with particles of other species.

Equation (1.20) is called the energy transport equation. The first term in Eq. (1.20) represents the change in the total energy of particles of a given species: this consists of the kinetic energy $nmV^2/2$ and the internal energy $(3/2)nT$ (per unit volume). The term in the curly brackets represents the total energy flux and consists of the macroscopic transport of the total energy with velocity **V**, the microscopic energy flux, i.e., the heat flux **q**, and the work done by the total pressure forces

$$\frac{\partial V_\alpha P_{\alpha\beta}}{\partial x_\beta} = \frac{\partial (pV_\beta + \pi_{\alpha\beta} V_\alpha)}{\partial x_\beta}.$$

The term on the right side takes account of the work done by any other forces and the heat generation.

In some cases it is convenient to eliminate the kinetic energy from Eq. (1.20) by means of the equation of continuity and the equation of motion. We then obtain an equation for the transport of internal energy, or the heat-balance equation:

$$\frac{3}{2} \frac{\partial nT}{\partial t} + \operatorname{div}\left(\frac{3}{2} nT\mathbf{V} \right) + nT \operatorname{div} \mathbf{V} +$$

$$+ \pi_{\alpha\beta} \frac{\partial V_\alpha}{\partial x_\beta} + \operatorname{div} \mathbf{q} = Q. \tag{1.23}$$

The equation of continuity (1.11) can now be used to obtain

$$\frac{3}{2} \frac{\partial nT}{\partial t} + \operatorname{div}\left(\frac{3}{2} nT\mathbf{V} \right) = \frac{3}{2} n \frac{dT}{dt}.$$

Introducing the quantity

$$s = \ln (T^{3/2}/n) = \ln (p^{3/2}/n^{5/2}),$$

and again using (1.11) we can also write Eq. (1.23) in the form

$$Tn \frac{ds}{dt} = T \left\{ \frac{\partial ns}{\partial t} + \operatorname{div}(ns\mathbf{V}) \right\} = -\operatorname{div} \mathbf{q} - \pi_{\alpha\beta} \frac{\partial V_\alpha}{\partial x_\beta} + Q. \tag{1.23'}$$

To within an unimportant constant the quantity s represents the entropy per particle.

Let $\mathbf{R}_{ab}$ be the change of momentum and Q_{ab} the heat generated in a gas of particles of species a as a consequence of collisions with particles of species b. Then $\mathbf{R}_a = \sum_b \mathbf{R}_{ab}$, and $Q_a = \sum_b Q_{ab}$. Using the fact that particles, momentum, and energy are conserved in collision (1.4)-(1.6) we find

$$R_{ba} = -R_{ab},$$

$$Q_{ab} + Q_{ba} = -\mathbf{R}_{ab}\mathbf{V}_a - \mathbf{R}_{ba}\mathbf{V}_b = -\mathbf{R}_{ab}(\mathbf{V}_a - \mathbf{V}_b). \quad (1.24)$$

If Eqs. (1.11), (1.14), and (1.23) are to be actually used to find the parameters n, **V**, and T it is first necessary to establish the relation between $\pi_{\alpha\beta}$, **q**, **R**, and Q and the parameters n, **V**, and T. This relation can be stated phenomenologically or by kinetic methods. If the second approach is used an approximate solution for the kinetic equations must be obtained in order to express the distribution function at a given point in terms of n, **V**, and T; this relation is then used in Eqs. (1.17), (1.18), (1.21), and (1.22) to obtain an expression for $\pi_{\alpha\beta}$, **q**, **R**, and Q at the same point. In principle this approximate local solution of the kinetic equation is valid in the case of practical importance in which certain requirements pertinent to the macroscopic analysis of a plasma are satisfied. Essentially these requirements state that all quantities must vary slowly in space (small gradients) and time. The possibility of using a local solution derives from the existence of the relaxation process, which causes any arbitrary distribution to become a Maxwellian as a consequence of collisions. The Maxwellian distribution represents the solution of the kinetic equation for the case in which the gradients and time derivatives vanish identically. If these quantities are nonvanishing, but small, the distribution function will still be close to a Maxwellian and the difference (proportional to the small gradient) will also be small. Thus, if one is interested in changes occurring in time intervals much greater than the collision time and if all quantities vary slowly over distances traversed by the particles between collisions, the solution of the kinetic equation will approximate a Maxwellian; specifically, the solution will be of the form

$$f_a(t, \mathbf{r}, \mathbf{v}) = f_a^0 + f_a^1 = \frac{n_a(t, \mathbf{r})}{(2\pi T_a/m_a)^{3/2}} e^{-\frac{m_a}{2T_a(t,\mathbf{r})}[\mathbf{v}-\mathbf{V}_a(t,\mathbf{r})]^2} + f_a^1, \quad (1.25)$$

where $|f_a^1| \ll f_a^0$. The first-order term f_a^1 can be treated as a small correction or perturbation on the zero-order distribution function f_a^0. This correction will be proportional to effects that disturb the Maxwellian distribution, i. e., gradients, electric fields, etc. The Maxwellian function and its derivatives are determined uniquely by the parameters n, **V**, and T and by the derivatives of these parameters; hence, these same quantities can be used to express the correction f^1 and, in the final analysis, $\pi_{\alpha\beta}$, **q**, **R**, and Q. These latter quantities are then proportional to the effects that produce the deviations from equilibrium. The corresponding coefficients of proportionality (for example, the coefficient of friction between particles of different species, the thermal conductivity, viscosity etc.) are called the transport coefficients, and determination of these coefficients is the basic goal of kinetic theory.

The program we have mapped out can only be carried to a successful conclusion in a fully ionized gas with one ion species.

Such a system will be called a simple plasma and will be the primary subject of discussion in the present review.

The transport coefficients for a simple plasma are given in §§2 and 4. These coefficients are given qualitative physical interpretations and evaluated in order-of-magnitude terms in §3, and computed numerically from the kinetic equation in §4. The use of the transport equations to describe a plasma in a strong magnetic field frequently leads to paradoxes which have been the source of various errors and ambiguities in the literature. Some of these paradoxes are considered in §5. The application of the transport equations for particles of different species in analyses which assume a plasma model based on a single complex gas is described in §6 (fully ionized plasma) and 7 (partially ionized plasma). This magnetohydrodynamic model of a plasma is frequently used in practice and can, in some cases, be justified by means of the kinetic equations and the transport equations; in some cases the model is used purely in the interests of simplicity. The individual sections of this review are more or less independent of each other so that §§4 and 5 can be omitted without loss of understanding of the remaining text.

§ 2. Transport Equations for a Simple Plasma (Summary of Results)

For purposes of reference, in this section we list the transport equations for a fully ionized plasma consisting of electrons and a single ion species with charge Ze. The transport coefficients of a fully ionized plasma have been computed by many authors. A method for obtaining the transport equations from the kinetic equations is given in detail in the monograph by Chapman and Cowling [1]. This same work contains expressions for the heat flux and the stress tensor for a single-component ionized gas in a magnetic field; the electrical conductivity in a magnetic field is also derived. The transport coefficients for a fully ionized gas have also been computed in [12-22] and in other places. Although these coefficients have been derived by various methods and in various forms, in all cases they apply only when the local distribution is very close to a Maxwellian. The results listed here are derived in §4 following the method used in [17].

With the exception of the electrical conductivity, at the present time no transport coefficient for a simple plasma has yet been measured experimentally.

The transport equations for a simple plasma comprise the equations of continuity, motion, and heat balance for the electrons and for the ions:

$$\frac{\partial n_e}{\partial t} + \operatorname{div}(n_e \mathbf{V}_e) = 0, \tag{2.1e}$$

$$\frac{\partial n_i}{\partial t} + \operatorname{div}(n_i \mathbf{V}_i) = 0, \tag{2.1i}$$

$$m_e n_e \frac{d_e V_{e\alpha}}{dt} = -\frac{\partial p_e}{\partial x_\alpha} - \frac{\partial \pi_{e\alpha\beta}}{\partial x_\beta} - e n_e \left(E_\alpha + \frac{1}{c}[\mathbf{V}_e \mathbf{B}]_\alpha\right) + R_\alpha, \tag{2.2e}$$

$$m_i n_i \frac{d_i V_{i\alpha}}{dt} = -\frac{\partial p_i}{\partial x_\alpha} - \frac{\partial \pi_{i\alpha\beta}}{\partial x_\beta} + Zen_i \left(E_\alpha + \frac{1}{c}[\mathbf{V}_i \mathbf{B}]_\alpha\right) - R_\alpha, \tag{2.2i}$$

$$\frac{3}{2} n_e \frac{d_e T_e}{dt} + p_e \operatorname{div} \mathbf{V}_e = -\operatorname{div} \mathbf{q}_e - \pi_{e\alpha\beta} \frac{\partial V_{e\alpha}}{\partial x_\beta} + Q_e, \tag{2.3e}$$

$$\frac{3}{2} n_i \frac{d_i T_i}{dt} + p_i \operatorname{div} \mathbf{V}_i = -\operatorname{div} \mathbf{q}_i - \pi_{i\alpha\beta} \frac{\partial V_{i\alpha}}{\partial x_\beta} + Q_i, \tag{2.3i}$$

where

$$p_e = n_e T_e, \quad p_i = n_i T_i,$$

$$\frac{d_e}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_e \nabla), \quad \frac{d_i}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_i \nabla). \tag{2.4}$$

In the expressions for the transport coefficients used below we shall make use of the fact that the plasma is neutral, writing $n = n_e = Zn_i$. We also exploit the fact that the ratio m_e/m_i is small.

The electron and ion collision times (in seconds) can be written in the form:*

$$\tau_e = \frac{3\sqrt{m_e}\, T_e^{3/2}}{4\sqrt{2\pi}\, \lambda e^4 Z^2 n_i} = \frac{3.5 \cdot 10^4}{(\lambda/10)} \cdot \frac{T_e^{3/2}}{Zn}, \tag{2.5e}$$

$$\tau_i = \frac{3\sqrt{m_i}\, T_i^{3/2}}{4\sqrt{\pi}\, \lambda e^4 Z^4 n_i} = \frac{3.0 \cdot 10^6}{(\lambda/10)} \left(\frac{m_i}{2m_p}\right)^{1/2} \frac{T_i^{3/2}}{Z^3 n}, \tag{2.5i}$$

where m_p is the mass of the proton and λ is the Coulomb logarithm [6] (for $T_e < 50\text{eV}$, $\lambda = 23.4 - 1.15 \log n + 3.45 \log T_e$; for $T_e > 50\text{eV}$, $\lambda = 25.3 - 1.15 \log n + 2.3 \log T_e$).

The cyclotron frequencies (sec^{-1}) for the electrons and ions are

$$\omega_e = \frac{eB}{m_e c} = 1.76 \cdot 10^7 B, \tag{2.6e}$$

$$\omega_i = \frac{ZeB}{m_i c} = 0.96 \cdot 10^4 \frac{ZB}{m_i/m_p}. \tag{2.6i}$$

In a magnetic field the transport coefficients depend on the quantity $\omega\tau$. In this section we shall only give the limiting expressions for large values of $\omega_e\tau_e$ and $\omega_i\tau_i$. These can also be used to obtain expressions for the case $B = 0$ by assuming that the transport coefficients in the direction of the magnetic field are equal to the transport coefficients in the absence of the field. Expressions for arbitrary values of $\omega\tau$ are given in §4.

The symbols $\parallel$ and $\perp$ on the vectors mean that we are to take the component parallel or perpendicular to the magnetic field respectively; for example $\mathbf{u}_\parallel = \mathbf{h}(\mathbf{uh})$, $\mathbf{u}_\perp = [\mathbf{h}[\mathbf{uh}]]$, where $\mathbf{h} = \mathbf{B}/B$ is a unit vector in the direction of the magnetic field.

The transfer of momentum from ions to electrons by collisions $\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T$ is made up of two parts: the force of friction $\mathbf{R}_u$ due to the existence of a relative velocity $\mathbf{u} = \mathbf{V}_e - \mathbf{V}_i$, and a thermal force $\mathbf{R}_T$,

*In all the practical formulas here and below the temperature is expressed in electron volts, the magnetic field in gauss, and all other quantities in cgs units.

TABLE 1

Z	Formula number					
	(2. 8)	(2. 9) and (2. 10)		(2. 11)	(2. 12)	(2. 13)
1	0.51	0.71	3/2	5/2	3.16	4.66
2	0.44	0.9	3/2	5/2	4.9	4.0
3	0.40	1.0	3/2	5/2	6.1	3.7
4	0.38	1.1	3/2	5/2	6.9	3.6
∞	0.29	1.5	3/2	5/2	12.5	3.2

which arises by virtue of a gradient in the electron temperature. The electron heat flux is made up of two analogous parts; $\mathbf{q}_e = \mathbf{q}_u^e + \mathbf{q}_T^e$. The relative velocity of the electrons and ions is related simply to the current density; specifically, $\mathbf{j} = -en\mathbf{u}$.

At large values of $\omega_e\tau_e$ the relations derived in §4 give the following expressions for the momentum transfer via collisions and for the electron heat flux (Z = 1):

The friction force:

$$\mathbf{R}_u = -\frac{m_e n_e}{\tau_e}(0.51\mathbf{u}_\parallel + \mathbf{u}_\perp) = en\left(\frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp}\right), \quad (2.6)$$

where the electrical conductivities are

$$\sigma_\perp = \frac{e^2 n_e \tau_e}{m_e} = \sigma_1 T_e^{3/2}, \quad (2.7)$$

$$\sigma_\parallel = 1.96\sigma_\perp = 1.96\sigma_1 T_e^{3/2}, \quad (2.8)$$

where

$$\sigma_1 = \frac{0.9\cdot 10^{13}}{(\lambda/10)Z}\ \mathrm{sec}^{-1}\cdot \mathrm{eV}^{-3/2}.$$

The thermal force:

$$\mathbf{R}_T = -0.71 n_e \nabla_\parallel T_e - \frac{3}{2}\frac{n_e}{\omega_e\tau_e}[\mathbf{h}\nabla T_e]. \quad (2.9)$$

The electron heat flux:

$$\mathbf{q}_u^e = 0.71 n_e T_e \mathbf{u}_\parallel + \frac{3}{2}\frac{n_e T_e}{\omega_e\tau_e}[\mathbf{h}\mathbf{u}], \quad (2.10)$$

$$\mathbf{q}_T^e = -\varkappa_\parallel^e \nabla_\parallel T_e - \varkappa_\perp^e \nabla_\perp T_e - \frac{5}{2}\frac{c n_e T_e}{eB}[\mathbf{h}\nabla T_e], \quad (2.11)$$

where the thermal conductivities are:

$$\varkappa_\parallel^e = 3.16\frac{n_e T_e \tau_e}{m_e}, \quad (2.12)$$

$$\varkappa_\perp^e = 4.66\frac{n_e T_e}{m_e \omega_e^2 \tau_e}. \quad (2.13)$$

If Z ≠ 1 the coefficients in these expressions are modified in accordance with Table 1.

When $\omega_i\tau_i \gg 1$ the ion heat flux is

$$\mathbf{q}_i = -\varkappa_\parallel^i \nabla_\parallel T_i - \varkappa_\perp^i \nabla_\perp T_i + \frac{5}{2}\frac{c n_i T_i}{ZeB}[\mathbf{h}\nabla T_i], \quad (2.14)$$

$$\varkappa_\parallel^i = 3.9\frac{n_i T_i \tau_i}{m_i}, \quad (2.15)$$

$$\varkappa_\perp^i = 2\frac{n_i T_i}{m_i \omega_i^2 \tau_i}. \quad (2.16)$$

The heat acquired by the ions in collisions with electrons is

$$Q_i = Q_\Delta = \frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_e - T_i). \quad (2.17)$$

The heat generated in the electrons as a consequence of collisions with ions is

$$Q_e = -\mathbf{R}\mathbf{u} - Q_\Delta = \frac{j_\parallel^2}{\sigma_\parallel} + \frac{j_\perp^2}{\sigma_\perp} + \frac{1}{cn_e}\mathbf{j}\mathbf{R}_T - \frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_e - T_i). \quad (2.18)$$

The stress tensor in the absence of a magnetic field is

$$\pi_{\alpha\beta} = -\eta_0 W_{\alpha\beta}, \quad (2.19)$$

where the rate-of-strain tensor

$$W_{\alpha\beta} = \frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\,\mathrm{div}\,\mathbf{V}. \tag{2.20}$$

In a strong magnetic field ($\omega\tau \gg 1$) the components of the tensor $\pi_{\alpha\beta}$ have the following form in the coordinate system with z axis parallel to the magnetic field:

$$\left.\begin{aligned}
\pi_{zz} &= -\eta_0 W_{zz},\\
\pi_{xx} &= -\eta_0 \frac{1}{2}(W_{xx} + W_{yy}) - \eta_1 \frac{1}{2}(W_{xx} - W_{yy}) - \eta_3 W_{xy},\\
\pi_{yy} &= -\eta_0 \frac{1}{2}(W_{xx} + W_{yy}) - \eta_1 \frac{1}{2}(W_{yy} - W_{xx}) + \eta_3 W_{xy},\\
\pi_{xy} &= \pi_{yx} = -\eta_1 W_{xy} + \eta_3 \frac{1}{2}(W_{xx} - W_{yy}),\\
\pi_{xz} &= \pi_{zx} = -\eta_2 W_{xz} - \eta_4 W_{yz},\\
\pi_{yz} &= \pi_{zy} = -\eta_2 W_{yz} + \eta_4 W_{xz}.
\end{aligned}\right\} \tag{2.21}$$

The expressions in (2.21) apply for both ions and electrons but the tensors $W_{\alpha\beta}$ and the viscosity coefficients are obviously different for the two species.

The ion viscosity coefficients are

$$\eta_0^i = 0.96 n_i T_i \tau_i, \tag{2.22}$$

$$\eta_1^i = \frac{3}{10}\frac{n_i T_i}{\omega_i^2 \tau_i}, \quad \eta_2^i = 4\eta_1^i, \tag{2.23}$$

$$\eta_3^i = \frac{1}{2}\frac{n_i T_i}{\omega_i}, \quad \eta_4^i = 2\eta_3^i. \tag{2.24}$$

The electron viscosity coefficients are (Z = 1)

$$\eta_0^e = 0.73 n_e T_e \tau_e, \tag{2.25}$$

$$\eta_1^e = 0.51 \frac{n_e T_e}{\omega_e^2 \tau_e}, \quad \eta_2^e = 4\eta_1^e, \tag{2.26}$$

$$\eta_3^e = -\frac{1}{2}\frac{n_e T_e}{\omega_e}; \quad \eta_4^e = 2\eta_3^e. \tag{2.27}$$

The heat generated as a result of viscosity is

$$Q_{\mathrm{vis}} = -\pi_{\alpha\beta}\frac{\partial V_\alpha}{\partial x_\beta} = -\frac{1}{2}\pi_{\alpha\beta}W_{\alpha\beta}$$

or, neglecting terms of order $(\omega\tau)^{-2}$,

$$Q_{\mathrm{vis}} = -\pi_{\alpha\beta}\frac{\partial V_\alpha}{\partial x_\beta} = \frac{3}{4}\eta_0 W_{zz}^2 = \frac{\eta_0}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial V_z}{\partial z}\right)^2. \tag{2.28}$$

An expression for $\pi_{\alpha\beta}$ in an arbitrary coordinate system is given in §4.

§ 3. Kinetics of a Simple Plasma (Qualitative Description)

Particle Motion and Collisions. The transport coefficients in §2 can be derived through the use of some simple ideas based on the motion of individual particles and the properties of Coulomb collisions.

In the absence of a magnetic field a free particle moves in a straight line with constant velocity. Collisions distort the particle trajectory and change the magnitude of the velocity. The resulting motion can be represented roughly as taking place along a broken line consisting of randomly directed segments with lengths of order $l = v\tau$ where $v \sim (2T/m)^{1/2}$ is the characteristic thermal velocity and τ is the characteristic time interval between collisions which change the direction of motion.

In a magnetic field the charged particle moves without collisions along a helix with radius of order $r = mvc/eB$ that winds around the magnetic line of force. Collisions disturb this regular motion; one way of looking at the situation is to say that after a time interval $\sim\tau$ the particle starts to describe a new helix that is not an extension of the earlier one. Two limiting cases must be distinguished. In a weak magnetic field $r \gg l$ or, $\omega\tau \ll 1$ ($\omega = eB/mc$ is the cyclotron frequency). In a strong field $r \ll l$, $\omega\tau \gg 1$. In a weak field the portions of the helix traversed by the particle between collisions are not very different from segments of straight lines. In a strong field the particle can describe many turns between collisions. When $\omega\tau \gg 1$ the magnetic field has a strong effect on transport properties in the transverse direction; on the other hand, particles can still move freely along the field, traveling a distance $\sim l$ between collisions as if $B = 0$. Thus, the magnetic field does not affect the longitudinal flow;

the transport coefficients are the same for longitudinal flows and for flows in an arbitrary direction with B = 0.

It should be noted that Coulomb collisions are not really true collisions (in the sense of instantaneous collisions); because of the long range of the Coulomb force the stochastic interaction between charged particles goes on continuously and causes a continuous randomization of particle velocities. However, this feature only affects the actual form of the collision term and is not important as far as our qualitative description is concerned. A qualitative description requires only that we know the characteristic time intervals between collisions; these times may be conveniently taken to be the time required for the total integrated change in the direction of the velocity to add up to an angle of order unity.

A quantitative analysis of Coulomb collisions requires the use of an appropriate expression for the collision term (cf. §4). Speaking roughly, we may say that the effective scattering cross section for Coulomb collisions is about one order of magnitude greater than $\pi(e_1e_2/\varepsilon)^2$ where e_1 and e_2 are the charges of the colliding particles, ε is the distance of closest approach. Thus, the mean free path for Coulomb collisions is proportional to the square of the energy of the particles or the square of the temperature.

As in §2, we shall make use of two characteristic times: τ_e, the electron-ion scattering time and τ_i, the ion-ion scattering time. The first of these depends only on the electron temperature because the electrons have much higher velocities and the relative velocity is determined by the electrons in electron-ion collisions. The second characteristic time depends on the ion temperature. All of the other characteristic times can be expressed conveniently in terms of τ_e and τ_i.

A characteristic feature of a fully ionized plasma is the very small ratio of the masses of the plasma components, the electrons and ions. Because this ratio is so small, the electron gas and the ion gas reach equilibria separately in a time much shorter than that required for the two gases to come to equilibrium with each other. Say that the electron equilibration time is $\sim\tau_{ee}$ and that the ion equilibration time is $\sim\tau_{ii}$ while the electron-ion equilibration time is $\sim\tau_{ei}^{\varepsilon}$. If the electron and ion temperatures are of the same order we find

$$\tau_{ee} : \tau_{ii} : \tau_{ei}^{e} = 1 : (m_i/m_e)^{1/2} : (m_i/m_e).$$

On the other hand, the mean free path is determined by the particle energies and is thus of the same order for the electrons and ions even

though the ion velocity is $(m_e/m_i)^{1/2}$ times the electron velocity so that $\tau_{ii} \sim (m_i/m_e)^{1/2}\tau_{ee}$. The relative velocity in electron-ion collisions is of the same order as for electron-electron collisions so that both processes have approximately the same probability. [Large fractional energy exchanges occur between like particles in a single collision]; hence $\tau_{ee} \sim \tau_e$, $\tau_{ii} \sim \tau_i$. On the other hand, only a small fraction of the energy is transferred (the order of the mass ratio) in collisions of a light particle with a heavy particle so that

$$\tau_{ei}^{e} \sim (m_i/m_e)\,\tau_e \sim (m_i/m_e)\,\tau_{ee}.$$

If the ion temperature is smaller than the electron temperature, as is frequently the case, the ion path is smaller and τ_{ii} is reduced. However, both of the other characteristic times remain unchanged so that $\tau_{ee} \ll \tau_{ei}^{\varepsilon}$ and $\tau_{ii} \ll \tau_{ei}^{\varepsilon}$ as before. If the ion temperature is greater than the electron temperature the quantity τ_{ii} increases, but the condition $\tau_{ii} \ll \tau_{ei}^{\varepsilon}$ is still satisfied so long as $T_i/T_e \ll (m_i/m_e)^{1/3}$.

Thus, a local equilibrium (Maxwellian distribution) is established within each of the components in a simple plasma before it is established between the components. It is precisely this circumstance that makes it possible to obtain transport equations when the electron and ion temperatures are different. The transfer of momentum from the ions to the electrons occurs in about the same time $\sim\tau_{ei}^{\varepsilon}$ as the transfer of energy; hence ion-electron momentum transfer is small compared with ion-ion momentum transfer. For this reason collisions of ions with electrons generally have very little effect on the form of the ion distribution function. On the other hand, the transfer of momentum from the electrons to the ions occurs in a time of the same order as the electron-electron momentum transfer time $\tau_e \sim \tau_{ee}$, so that collisions of electrons with ions have an important effect on the form of the electron distribution function.

The Friction Force $\mathbf{R}_u$. In collisions of electrons with ions which have zero mean velocity ($V_i = 0$) the electron velocities remain essentially unchanged in magnitude but do undergo random changes in direction. Thus, the electrons lose their ordered velocity with respect to the ions $\mathbf{u} = \mathbf{V}_e - \mathbf{V}_i$ in a time $\sim\tau_e$ and consequently lose momentum $m_e\mathbf{u}$ per electron (which is given to the ions). This means that a frictional force $(m_e n_e/\tau_e)\mathbf{u}$ is exerted on the electrons; this force is equal and opposite to the force exerted on the ions. We note that the quantity τ_e defined by Eq. (2.5e) is chosen in such a way that the frictional force $\mathbf{R}^0$ that appears in the interaction of an electron Maxwellian distribution shifted with respect to the ion function by an amount **u** will have the simple form $R^0 = -(m_e n_e/\tau_e)\mathbf{u}$ (without numerical coefficients). Actually, if any force, say an electric field, produces an electron velocity **u** directed along **B** (or if B = 0) the electron distribution function is not a Maxwellian simply shifted as a whole by an amount **u**. This results from the fact that the Coulomb cross section diminishes with increasing electron energy ($\tau \sim v^3$); hence, a Coulomb force shifts the faster electrons more (with respect to the ions) than the slow electrons. The distribution function is then distorted in such a way that the mean velocity **u**, i.e., transport of electric current, depends more on the fast electrons so that the friction coefficient is smaller than for a true shifted Maxwellian. This effect would vanish if electron-electron collisions, which tend to establish a Maxwellian distribution, were to occur much more frequently than the electron-ion collisions, which distort the distribution. Since $\tau_{ee} \sim \tau_e$, however, an "effect of order unity" is obtained, that is to say, the distortion of the Maxwellian is of the same order as the shift. For example when Z = 1 the friction coefficient is reduced by a factor 0.51. The friction coefficient is reduced still more for higher values of Z, where electron-ion collisions are relatively more important than electron-electron collisions.

In the motion of electrons with respect to ions across a strong magnetic field ($\mathbf{u} = \mathbf{u}_\perp$) the correction to the shifted Maxwellian is of order $(\omega_e\tau_e)^{-1}$; thus, when $\omega_e\tau_e \gg 1$ this correction can be neglected so that the transverse frictional force is simply $R_\perp = -(m_e n_e/\tau_e)\mathbf{u}_\perp$. In a strong magnetic field the coefficient of friction between electrons and ions is then smaller for a longitudinal current than for a transverse current, that is to say, the longitudinal electric conductivity $\sigma_\parallel$ is greater than the transverse conductivity $\sigma_\perp$. When Z = 1 we find $\sigma_\parallel \approx 2\sigma_\perp$.*

Thermal Force R_T. Let us assume that the electrons and ions are at rest on the average ($V_e = V_i = 0$); then, the number of electrons moving from left to right and from right to left per unit time will be exactly the same through any cross section, say $x = x_0$. The order of magnitude of these two compensating fluxes is $n_e v_e$. As a result of electron-ion collisions these fluxes experience frictional forces R_+ and R_- of order $m_e n_e v_e/\tau_e$; in a completely homogeneous situation these frictional forces balance exactly and there is no resultant force. Collisions of electrons with ions can, however, produce a resultant force if the velocity distribution of the electrons coming from the left is different from the distribution characterizing electrons coming from the right, in which case the forces R_+ and

*The presence of runaway electrons increases the ratio $\sigma_\parallel/\sigma_\perp$.

R_- do not cancel. For example, if electrons coming from the right have higher average energies than those coming from the left the force acting on the fast "right" electrons will be less than the force acting on the slower "left" electrons (since $\tau \sim v^3$); as a result a force directed to the left is produced.

Let us assume that there is a temperature gradient along the x axis (Fig. 1) and no magnetic field (or that there is a magnetic field along ΔT). At the point $x = x_0$ collisions will be experienced by electrons that have come from the right and from the left and that have traversed distances of the order of the mean free path $l \sim v_T$; thus, electrons coming from the right come from regions in which the temperature is approximately $l\partial T_e/\partial x$ greater than in the regions from which the electrons from the left originate. The unbalanced part of the forces R_+ and R_- will be of order

$$R_T \sim \frac{l}{T_e}\frac{\partial T_e}{\partial x}\frac{m_e n_e v_e}{\tau_e} \sim \frac{m_e v_e^2}{T_e} n_e \frac{\partial T_e}{\partial x} \sim n_e \frac{\partial T_e}{\partial x}$$

and will be directed to the left, that is to say, in the opposite direction to the temperature gradient [minus sign in Eq. (2.9)]. As in the case of the longitudinal friction force R_u (and for the same reason) the size of this effect increases with increasing Z (cf. Table 1). It should be emphasized that the thermal force arises specifically as a consequence of collisions; hence its magnitude and sign depend on the actual velocity dependence of the collision frequency (in the present case $\tau \sim v^3$) even though the thermal force const $n\partial T_e/\partial x$ does not contain τ explicitly.

Let us now investigate the case in which there is a strong magnetic field along the z axis while the temperature gradient is still along the x axis (Fig. 2). In a strong magnetic field ($\omega_e\tau_e \gg 1$) the electrons gyrate in circles of radius $r_e \sim v_e/\omega_e$; at the point $x = x_0$ there will be electrons

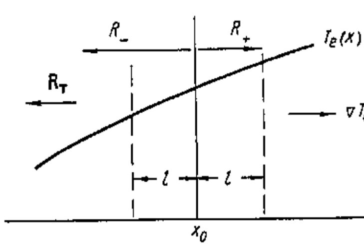

Fig. 1

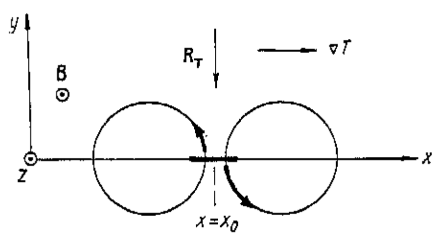

Fig. 2

that come from the right and from the left and that have traversed distances of order r_e. These electrons "carry" a temperature difference of order $r_e \partial T_e/\partial x$ and produce, as is evident from Fig. 2, an unbalance in the friction forces for fluxes directed along the y axis. On the other hand, the fluxes along the x axis at the point $x = x_0$ are due to electrons that come from regions where $x = x_0$, so that the frictional forces are balanced in this case. As a result of collisions with ions there then arises a thermal force directed perpendicularly to both **B** and ΔT_e, i. e., along the y axis; the order of magnitude of this quantity is

$$R_T \sim \frac{r_e}{T_e}\frac{\partial T_e}{\partial x}\frac{m_e n_e v_e}{\tau_e} \sim \frac{n_e}{\omega_e \tau_e}\frac{\partial T_e}{\partial x}.$$

It is easy to verify that the sign of the thermal force (minus) is precisely the same as in Eq. (2.9).

We may note that the effect of a magnetic field on the thermal force is directly analogous to the well-known phenomenon in metals, where it is known as the Nernst effect [19].

Electron Heat Flux q_u^e. The existence of thermal forces is intimately related to the presence of terms proportional to the relative velocity **u** in the expression for the electron heat flux. Starting from the general principles of the thermodynamics of irreversible processes (the so-called principle of symmetry of the kinetic coefficients, or the Onsager principle) it can be shown that a knowledge of the terms in the frictional force which are proportional to ∇T_e can be used to find the terms in the heat flux that are proportional to **u**. This is actually done in detail in §4. For the present purposes, however, the qualitative significance of these terms can be stated as follows. As we have shown above, because $\tau \sim v^3$ the current along the magnetic field (or the current with no magnetic field) is carried predominantly by the faster electrons. Thus, in the coordi-

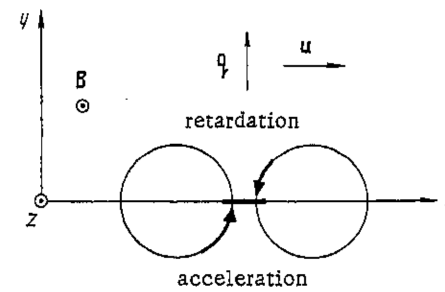

Fig. 3

nate system in which $V_e = 0$, more fast electrons move in the direction of **u** and more slow electrons move in the direction $-$**u**. Although the electron fluxes are balanced in this system, the energy fluxes are not and heat flows in the **u** direction. It is clear from the considerations used in the analysis of the friction force that this effect is of "order unity" so that the corresponding heat flow will be of order $\sim n_e T_e u$. Like the longitudinal thermal force, this heat flow is due to collisions although it does not contain τ explicitly.

Heat also flows when a current flows across a strong magnetic field $\mathbf{u} = \mathbf{u}_\perp$, but for a different reason. The friction force exerted on the electrons by the ions accelerates the electrons over one half cycle of rotation of the electrons, and retards them over the other (Fig. 3). Hence an area in the plane defined by **u** and **B** is intersected by accelerated electrons and by retarded electrons. The difference in the energies of these two electron groups is of order $(m_e u/\tau_e) r_e$. As a result, when $u = u_x$, $B = B_z$, a heat flux of the following magnitude arises

$$q_y \sim \frac{m_e u_x}{\tau_e} r_e n_e v_e \sim \frac{m_e u_x}{\tau_e}\frac{v_e^2}{\omega_e} n_e \sim \frac{n_e T_e}{\omega_e \tau_e} u_x.$$

Gas Kinetic Approximations (Kinetic Theory). Before analyzing the remaining effects we wish to review some simple approximations that should be familiar from the elementary kinetic theory of gases; these approximations are used to determine the order of magnitude of the diffusion coefficient, the thermal conductivity, and the viscosity of gases. There is a rather general analogy between these processes, in which matter, energy, and momentum are transported.

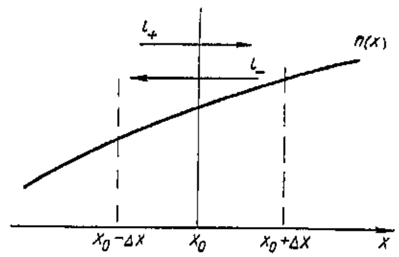

Fig. 4

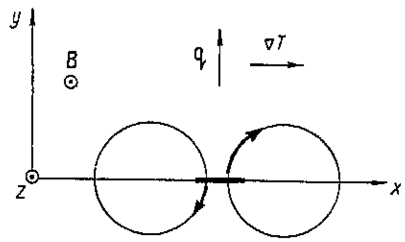

Fig. 5

Let us first consider diffusion. Diffusion occurs in a specified medium which we will assume to be fixed and not affected by the particles that diffuse through it.

Assume that the particle density is $n(x)$ (Fig. 4) and that each particle is displaced through a distance Δx, with equal probability for motion to the right or to the left, in the time τ between two successive collisions. In unit time, the plane $x = x_0$ is traversed in the positive direction (from the left) by half of the particles which experience collisions in the layer between $x_0 - \Delta x$ and x_0; the other half of the particles move to the left as a result of collisions. Assuming that $n(x)$ does not change greatly over a distance Δx so that

$$n(x) = n(x_0) + \left.\frac{\partial n}{\partial x}\right|_{x=x_0}(x - x_0),$$

we find that the unidirectional flux from the left is

$$i_+ = \frac{1}{2}\int_{x_0-\Delta x}^{x_0} \frac{1}{\tau} n(x)\,dx = \frac{1}{2}\left[n(x_0) - \frac{\partial n}{\partial x}\frac{\Delta x}{2}\right]\frac{\Delta x}{\tau}.$$

The diffusion flux is the difference between the flux moving to the left and the flux moving to the right $i = i_+ - i_-$ and is given by

$$i = -\frac{(\Delta x)^2}{2\tau}\frac{\partial n}{\partial x} = -D\frac{\partial n}{\partial x}; \qquad D = \frac{(\Delta x)^2}{2\tau}. \tag{3.1}$$

This relation can still be used to estimate the diffusion coefficient if Δx and τ are not constant but proper values must be used for Δx and τ.*

The heat and momentum fluxes can be estimated in similar fashion.

Suppose that there is no particle flux. The unidirectional heat flux, for example from the left to the right, will be of order $q_+ \sim (\Delta x/\tau)nT$. Because of the presence of a temperature gradient a relative fraction of order $(\Delta x/T)(\partial T/\partial x)$ of the unidirectional fluxes is not balanced and there arises a heat flux q equal to

$$q = -\varkappa\frac{\partial T}{\partial x}; \qquad \varkappa \sim \frac{n(\Delta x)^2}{\tau} \sim nD. \tag{3.2}$$

Now assume that the velocity V_y varies with x; in precisely the same way there will be a flux π_{yx} of y momentum along the x axis because of the

*We note that different particles can have different Δx and τ; if Δx and τ depend on velocity, Eq. (3.1) describes particles with a given velocity and the total flux is obtained by summing (or integrating) the particle fluxes for all velocity classes:

$$i = -\frac{\partial}{\partial x}\int\left[\frac{(\Delta x)^2}{2\tau}\right]_v f(v)\,dv = -\frac{\partial}{\partial x}\left\{\left\langle\frac{(\Delta x)^2}{2\tau}\right\rangle n\right\}, \tag{3.1'}$$

where the angle brackets denote averages over the particles at point x_0. For example if $\Delta x = v_x\tau$ and τ = const,

$$(\Delta x)^2 = v_x^2\tau^2; \quad < v_x^2 > = \frac{T}{m},$$
$$i = -\frac{\partial}{\partial x}\left\{\frac{\tau}{2} < v_x^2 > n\right\} = -\frac{\tau}{2m}\frac{\partial nT}{\partial x} = -\frac{\tau}{2m}\frac{\partial p}{\partial x}. \tag{3.1''}$$

However, if τ depends on v the flux will contain a term proportional to ∇p in addition to the term proportional to ∇T. This effect is called thermal diffusion. In moving through the medium the particles experience a friction force of order mi/τ. It is evident from Eq. (3.1) that diffusion of particles can be regarded as motion with friction under the effect of a force ∇p. The thermal diffusion can be regarded as motion with friction under the effect of a corresponding thermal force.

lack of cancellation between the two unidirectional momentum fluxes, each of which is of order $(\Delta x/\tau)\,nmV_y$:

$$\pi_{yx} = -\eta \frac{\partial V_y}{\partial x}, \quad \eta \sim \frac{mn(\Delta x)^2}{\tau} \sim mnD. \tag{3.3}$$

Equations (3.2) and (3.3) give the connection between the thermal conductivity $\varkappa$, the viscosity η, and the diffusion coefficient.

If a particle moves freely between collisions we note that $\Delta x \sim l \sim v\tau$ and Eq. (3.1) gives the usual expression $D \sim lv$ found in texbooks on the kinetic theory of gases. However Eq. (3.1) is more general than $D \sim lv$ since the approximation $D \sim (\Delta x)^2/\tau$ applies in those cases in which the displacement of the particle between collisions is not equal to the mean free path. The same considerations apply to Eqs. (3.2) and (3.3); thus, these expressions can be used to estimate the transport coefficients in the presence of a magnetic field.

Thermal Conductivity. The thermal conductivities appearing in the expressions for the electron and ion heat fluxes parallel and perpendicular to a magnetic field can be easily estimated using the kinetic-theory relation in (3.2). Here we need only take account of the fact that in motion across a strong magnetic field ($\omega\tau \gg 1$) a particle is displaced by a distance of the order of the Larmor radius (between collisions) rather than the usual mean free path $(\Delta x)_\perp \sim r \sim v/\omega$ so that $\varkappa_\perp \sim nr^2/\tau \sim nT/m\omega^2\tau$; on the other hand the particle moves freely along the field $(\Delta x)_\parallel \sim l \sim v\tau$ so that $\varkappa_\parallel \sim nl^2/\tau \sim nT\tau/m$. Thus, $\varkappa_\parallel/\varkappa_\perp \sim (\omega\tau)^2$. These estimates apply for both ions and electrons so that the subscripts i and e can be omitted and we need only use the velocities, temperatures, etc., appropriate to the species at hand. We note that if $T_e \sim T_i$ the electron thermal conductivity in direction of the field is greater than the ion conductivity $\varkappa_\parallel^e/\varkappa_\parallel^i \sim (m_i/m_e)^{1/2}$; on the other hand, the ion thermal conductivity is greater in the transverse direction: $\varkappa_\perp^e/\varkappa_\perp^i \sim (m_e/m_i)^{1/2}$.

The relations in (2.11) and (2.14) also contain the "transverse" heat fluxes that are perpendicular to both **B** and ∇T. These fluxes arise because an area lying in the plane of B and ∇T (Fig. 5) will, on the average, be traversed by more fast particles from one side than from the other; if the unidirectional particle fluxes $\sim nv$ are balanced, the unidirectional energy fluxes, of order nTv, will have an unbalanced part of order $(r/T)\,\partial T/\partial x$. As a result there is produced a heat flux

$$q_y \sim nvr \frac{\partial T}{\partial x} \sim \frac{cnT}{eB} \frac{\partial T}{\partial x}.$$

These fluxes are of opposite sign for the ions and electrons. The transverse fluxes carry heat along isotherms and do not cool the plasma or increase its entropy.

Viscosity. The viscosity of a plasma in a magnetic field is complicated because it is a tensor quantity. Expressions for the stress tensor when $\omega\tau \gg 1$ are given in §2; expressions for arbitrary $\omega\tau$ are given in §4. These expressions show that the viscous stress is not a simple function of the velocity derivatives $\partial V_\alpha/\partial x_\beta$ either with or without a magnetic field; rather, it depends on combinations of these derivatives in a way given by the so-called rate-of-strain tensor

$$W_{\alpha\beta} = \frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\,\mathrm{div}\,\mathbf{V}.$$

It is easily shown that this tensor vanishes if the plasma rotates as a rigid body $\mathbf{V} = [\mathbf{\Omega r}]$ or if it undergoes a uniform isotropic compression, $\mathbf{V} = \mathrm{const}\,\mathbf{r}$; that is to say, this tensor vanishes if the volume elements of the plasma are not deformed. The tensors $W_{\alpha\beta}$ and $\pi_{\alpha\beta}$ are symmetric and have zero trace: $W_{\alpha\alpha} = 0$.

In the absence of a magnetic field the relation between $\pi_{\alpha\beta}$ and $W_{\alpha\beta}$ is the simple one, $\pi_{\alpha\beta} = -\eta_0 W_{\alpha\beta}$. The magnitude of the viscosity coefficient can be estimated from the usual kinetic-theory formula (3.3) by substituting $\Delta x \sim l \sim v\tau$; this procedure yields $\eta_0 \sim nT\tau$.

The presence of a magnetic field leads to significant differences between momentum transfer along the magnetic field and across the field; moreover, the direction of the transported momentum itself becomes important. In this case the relation between $\pi_{\alpha\beta}$ and $W_{\alpha\beta}$ is much more complicated and contains five independent viscosity coefficients. Since a symmetric tensor with zero trace has five independent components the most general linear homogeneous dependence requires precisely five independent coefficients of proportionality.

We now consider several simple cases involving plasma viscosity in a strong magnetic field $\omega\tau \gg 1$. In all of these cases it is assumed that the magnetic field is along the z axis.

First assume that the velocity is along the z axis and exhibits a derivative in this same direction; the momentum flux that arises is of the same order as in the absence of the field $\pi_{zz} \sim -\eta_0 \partial V_z/\partial z$ and the viscous stress is $F_z = (\partial/\partial z)\,\eta_0(\partial V_z/\partial z)$ since the longitudinal momentum is freely transported along the magnetic field.

Now suppose that the velocity V_z varies in a direction perpendicular to the field direction, say along the x axis. In this case momentum is transported across the magnetic field and in estimating the viscosity coefficient from Eq. (3.3) we must substitute $\Delta x \sim r$; this relation gives the following momentum flux and viscous stress:

$$\pi_{zx} \sim -\eta_\perp \frac{\partial V_z}{\partial x}, \quad F_z \sim \frac{\partial}{\partial x} \eta_\perp \frac{\partial V_z}{\partial x},$$

where

$$\eta_\perp \sim \frac{\eta_0}{(\omega\tau)^2} \sim \eta_1 \sim \eta_2$$

An analogous reduction of the viscosity by a factor of $(\omega\tau)^2$ obtains in the case in which the transverse velocity V_y varies along the x axis. In this case

$$\pi_{xy} \sim -\eta_\perp \frac{\partial V_y}{\partial x}, \quad F_y \sim \frac{\partial}{\partial x} \eta_\perp \frac{\partial V_y}{\partial x}.$$

The transport of transverse momentum is inhibited by the magnetic field even when this momentum is transported in the field direction. For example, if V_x varies in the z direction, i.e., if $\partial V_x/\partial z \neq 0$, the rate-of-strain tensor $W_{\alpha\beta}$ turns out to be the same as for $\partial V_z/\partial x \neq 0$ so that $\pi_{\alpha\beta}$ will also be the same: $\pi_{xz} \sim -\eta_\perp \partial V_x/\partial z$. It might be said that as a consequence of its gyration the particle loses its "memory" of transverse ordered velocity in a time $\sim \omega^{-1}$, during which it can only be displaced by a distance of the order of the Larmor radius.

As is evident from Eq. (2.21), the stress tensor also contains terms that are only reduced by a factor $\omega\tau$ (rather than $(\omega\tau)^2$) compared with the case B = 0. When $\partial V_y/\partial x \neq 0$, for example, there still are viscous fluxes and stresses given by

$$\pi_{xx} = -\pi_{yy} \sim \frac{nT}{\omega}\frac{\partial V_y}{\partial x} \sim \frac{\eta_0}{\omega\tau}\frac{\partial V_y}{\partial x}, \quad F_x \sim \frac{\partial}{\partial x}\frac{nT}{\omega}\frac{\partial V_y}{\partial x}.$$

When $\partial V_x/\partial z \neq 0$, there is a flux and stress

$$\pi_{yz} \sim \frac{nT}{\omega}\frac{\partial V_x}{\partial z} \sim \frac{\eta_0}{\omega\tau}\frac{\partial V_x}{\partial z}, \quad F_y \sim \frac{\partial}{\partial z}\frac{nT}{\omega}\frac{\partial V_x}{\partial z}.$$

These stresses are perpendicular to the velocity and do not result in dissipation of energy. The terms in the momentum flux that are independent of τ are analogous to the transverse terms in the heat flux $\sim (nT/m\omega)[\mathbf{h}\nabla T]$; we shall not interpret these terms in detail, but refer the reader to a paper by Kaufmann[22] which contains a lucid discussion of this point.

Finally, we consider compression of a plasma in a direction perpendicular to a strong magnetic field, which leads to a correction in the scalar pressure because of a completely different mechanism. For example, assume that $V = V_x$ changes in the x direction so that $\operatorname{div}\mathbf{V} = \partial V_x/\partial x = -\dot{n}/n \neq 0$; the magnetic lines of force are also compressed and the magnetic field increases: $B \sim n$. In the growing field the transverse energy of the particles is increased and the energy is distributed over all three degrees of freedom as a consequence of collisions. However, equipartition of the energy does not occur in zero time; as a result the transverse pressure is found to be greater than the longitudinal pressure by a fractional amount of order $\tau\dot{B}/B = -\tau\dot{n}/n$, giving rise to a stress.

$$\pi_{xx} = \pi_{yy} \sim -p\frac{\tau\dot{n}}{n} \sim -\eta_0\frac{\partial V_x}{\partial x}, \quad \pi_{zz} \sim \eta_0\frac{\partial V_x}{\partial x}.$$

Thus, for motion characterized by $\operatorname{div}\mathbf{V} \neq 0$ the viscosity coefficient in a strong magnetic field is of the same order as with no field. The establishment of equilibrium is an irreversible process which, as is well known, always implies the dissipation of energy in the form of heat. In the present case the heat generated is

$$Q_{\text{vis}} = -\pi_{\alpha\beta}\frac{\partial V_\alpha}{\partial x_\beta} = -\pi_{xx}\frac{\partial V_x}{\partial x} \sim \eta_0\left(\frac{\partial V_x}{\partial x}\right)^2 \sim \eta_0\left(\frac{\dot{B}}{B}\right)^2.$$

Plasma heating based on this process is sometimes called gyrorelaxational heating.

In contrast with the case of thermal conductivity, where the electron conductivity is greater along the field (for $\omega_i\tau_i \gg 1$), when $T_e \sim T_i$ the ion viscosity is always much larger than the electron viscosity:

$$\eta_0^i \sim \left(\frac{m_i}{m_e}\right)^{1/2}\eta_0^e, \quad \frac{\eta_0^i}{\omega_i\tau_i} \sim \frac{m_i}{m_e}\frac{\eta_0^e}{\omega_e\tau_e}, \quad \frac{\eta_0^i}{\omega_i^2\tau_i^2} \sim \left(\frac{m_i}{m_e}\right)^{3/2}\frac{\eta_0^e}{\omega_e^2\tau_e^2}.$$

For this reason the viscosity of a plasma is determined essentially by the ions.

We may also note that the existence of thermal fluxes can also lead to the transfer of momentum and the production of viscous stresses even when $V = 0$. These stresses are always small but, in a strong magnetic field, can in principle be of the same order as the terms in Eq. (2.21) that contain the viscosity coefficient reduced by the factor $(\omega\tau)^2$. These stress terms can be estimated roughly by adding to the tensor $W_{\alpha\beta}$ a similar tensor composed of the derivatives of the vector $\mathbf{q}/nT$. A more rigorous quantitative calculation of these terms is given in [16, 22a].

Heat Generation. We assume first that the ion mass is infinite and that the ions are at rest on the average: $V_i = 0$. In this case collisions of electrons with ions occur without the exchange of energy. The electron velocities are randomized in the collisions so that the energy associated with the ordered velocity $\mathbf{u} = \mathbf{V}_e - \mathbf{V}_i$ is converted into heat. The ion energy is not changed. In this case the heat generated in the electron gas is equal to the work resulting from the frictional force exerted on the electrons by the ions $-\mathbf{Ru}$. We now assume that the ratio m_i/m_e is large, although finite, and that $u = 0$. If $T_e = T_i$ the ions and electrons are in thermal equilibrium and no heat is transferred between them. However, if $T_e > T_i$ heat is transferred from the electrons to the ions. It is well known, that when a light particle collides with a heavy fixed particle the order of magnitude of the transferred energy is given by the mass ratio m_1/m_2. For example, the mean transferred fractional energy in isotropic scattering is $2m_1/m_2$. Thus the energy exchanged per unit time between electrons and ions Q_Δ is roughly

$$Q_\Delta \sim \frac{n_e}{\tau_e}\frac{2m_e}{m_i}\frac{3}{2}(T_e - T_i).$$

The calculation of Q_Δ using a collision term was first carried out by Landau [11]. Landau showed that when τ_e is chosen in the form given in Eq. (2.5e) this relation becomes exact.

If $u \neq 0$ and $T_e - T_i \neq 0$ simultaneously, neglecting the fraction ($\sim m_e/m_i$) of $-\mathbf{Ru}$ acquired by the ions and the $\sim m_e u^2/T_e$ corrections, we can simply add both of these effects so that

$$Q_i = Q_\Delta, \quad Q_e = -\mathbf{Ru} - Q_\Delta = -\mathbf{R}_u\mathbf{u} - \mathbf{R}_T\mathbf{u} - Q_\Delta.$$

The term $-\mathbf{R}_T\mathbf{u}$ is the Joule heat, which can be written in the more familiar form

$$Q_J = \frac{j_\parallel^2}{\sigma_\parallel} + \frac{j_\perp^2}{\sigma_\perp}.$$

This term $(\mathbf{R}_T\mathbf{u})$ changes sign when either the direction of current flow or the temperature gradient are reversed and represents a reversible generation of heat. The analogous effect in metals is called the Thomson effect [4].

In formulating the heat balance in a plasma at high temperatures it is necessary to take account of bremsstrahlung and electron synchrotron radiation as well as the heat generated by thermonuclear reactions. Under these conditions Q_e and Q_i must be modified by the addition of appropriate terms.

Conditions of Applicability. The "fluxes" $\mathbf{q}$, $\pi_{\alpha\beta}$, $\mathbf{R}$, and Q that appear in the transport equations are defined under the assumption that the relaxation process, which forces the distribution function to approach a Maxwellian, is not inhibited; thus, these equations apply only when certain requirements are satisfied: essentially, the requirements are that all average quantities in the plasma must change slowly in time and space. The distribution function becomes a Maxwellian in a time of the order of the collision time; hence, if the transport equations are to be used all plasma quantities must not change significantly in a time τ, characteristic of the collisions, or over distances comparable to that traversed by the particles between collisions. The requirement that the time variations must be slow can be written

$$\frac{d}{dt} \ll \frac{1}{\tau}. \quad (3.4)$$

The requirement that the spatial variations must be slow (for the case in which there is no magnetic field or when the field is weak, i.e., $\omega\tau \lesssim 1$) can be written

$$L \gg l, \quad (3.5)$$

where L is the characteristic scale length over which all quantities change significantly; $\nabla \sim 1/L$. These two conditions must also be satisfied when the usual kinetic-theory (gas dynamics) equations are used.

In a strong magnetic field ($\omega\tau \gg 1$) the first requirement still applies but the second becomes somewhat more complicated. The motion of particles across the magnetic field is bounded by the Larmor radius r, which is smaller than the mean free path by a factor $\omega\tau$. Thus, in many cases the conditions of applicability are relaxed, becoming

$$L_\perp \gg r, \quad L_\parallel \gg l, \quad (3.6)$$

where $L_\perp$ and $L_\parallel$ are the characteristic distances in the directions perpendicular and parallel to the magnetic field: $\nabla_\perp \sim 1/L_\perp$; $\nabla_\parallel \sim 1/L_\parallel$. However, the requirements are relaxed in this way only if the system is highly elongated along the magnetic field and has the required symmetry; typical examples are a long axisymmetric plasma cylinder of radius $\sim L_\perp$ with symmetric magnetic field, or a torus obtained by closing such a cylinder upon itself using a very large radius of curvature R. In an inhomogeneous magnetic field, the particles execute a drift motion with velocity of order $V_c \sim vr|\nabla B/B|$ in addition to gyrating around the Larmor circle. This drift represents particle displacement between collisions; if the drift trajectories of various particles pass through regions with different temperature there will be an additional transport of heat and a resulting deviation of the distribution function from a Maxwellian. The condition $L_\perp \gg r$ applies only when this "mixing" mechanism does not operate. In a symmetric system in which the curvature of the lines of force is small (for example a torus with a large radius of curvature $R \gg L_\perp$), the mixing occurs with a characteristic velocity of order vr/R; hence a particle can be displaced over a distance $\sim v\tau r/R \sim l r/R$ in one collision time and the requirement $L_\perp \gg l r/R$ must be satisfied in addition to $L_\perp \gg r$. If this special kind of symmetry does not obtain mixing occurs with velocities of the order of $vr/L_\perp$ and the applicability requirement becomes

$$L_\perp \gg \sqrt{lr}, \quad L_\parallel \gg l. \tag{3.7}$$

In computing the transport coefficients we have used the collision term in the Landau form, in which case the effect of the magnetic field on the collision itself is neglected. This approach is valid if the Larmor radius is large compared with the effective dimensions of the region in which the Coulomb interaction occurs, that is to say, if the Larmor radius is large compared with the Debye radius $\delta_D = (T/4\pi e^2 n)^{1/2}$: $r \gg \delta_D$ or $B^2 \ll 8\pi n_e m_e c^2$.

The transport coefficients are changed to some extent if this condition is not satisfied; in practice the change is usually less than one order of magnitude because the importance of collisions characterized by impact parameters smaller than r can at most be $[\ln(\delta_D/r)]^{-1}$ times that of collisions having large impact parameters. The effect of a very strong magnetic field ($r \ll r_D$) on the transport coefficients has been considered in [19].

Let us now consider some numerical examples.

Take $B = 10^4$ G, $m_i = m_p$. Then $\omega_e = 1.8 \cdot 10^{11}$ sec^{-1}; $\omega_i = 10^8$ sec^{-1}.

With $n = 10^{14}$ cm^{-3} and $T_e = T_i = 30$ eV, using the notation $v = (2T/m)^{1/2}$ we find: $r_e = v_e/\omega_e = 1.8 \cdot 10^{-3}$ cm; $r_i = v_i/\omega_i = 7.7 \cdot 10^{-2}$ cm; $\delta_D = 4.1 \cdot 10^{-4}$ cm. The Coulomb logarithm $\lambda = 11$; $l_e \approx l_i \approx 20$ cm; $\tau_e = 5 \cdot 10^{-8}$ sec; $\tau_i = 3 \cdot 10^{-6}$ sec; $\omega_e \tau_e = 10^4$; $\omega_i \tau_i = 3 \cdot 10^2$.

When $n = 10^{17}$ cm^{-3}, $T_e = T_i = 10^2$ eV, we find $r_e = 3 \cdot 10^{-3}$ cm; $r_i = 1.4 \cdot 10^{-1}$ cm; $\delta_D = 2.4 \cdot 10^{-5}$ cm; $\lambda = 10$; $l_e \approx l_i \approx 0.25$ cm; $\tau_e = 3.5 \cdot 10^{-10}$ sec; $\tau_i = 2.1 \cdot 10^{-8}$ sec; $\omega_e \tau_e = 63$; $\omega_i \tau_i = 2$.

There is one other factor that can limit the applicability of the transport equations that have been used here. This is the presence of any instability in the plasma. An unstable plasma can generate random fluctuating fields which, in turn, can produce strong mixing with a significant enhancement of the transport coefficients. This effect is analogous to turbulence in hydrodynamics. For example, it is well known, that the flow in an ordinary water pipe cannot be computed by means of the stationary solutions of the Navier-Stokes equations because of turbulence effects.

At the present time the theory of plasma turbulence is only in its infancy. This subject will, in fact, be treated in subsequent volumes of the present review series. However, it may be appropriate at this point to make a simple estimate showing the extent to which the transport mechanisms can be enhanced in a magnitized plasma in a turbulent state.

Let us assume that the plasma generates fluctuating electric fields of amplitude $\sim E'$ which become uncorrelated at points separated by distances greater than l'. These fields cause particle drifts with characteristic velocities $V'_c \sim cE'/B$ which change direction randomly after the particle has drifted a distance of order l'. The effective diffusion coefficient appropriate to this mechanism can be estimated from Eq. (3.1) and we find $D_{turb} \sim l' V'_c \sim cE' l'/B$.

We now make the reasonable assumption that the amplitudes of the fluctuating fields are such that the corresponding energy is of the same order as the thermal energy of the particles, i.e., $eE'l' \sim T$. While this relation has obviously not been rigorously justified it can serve as a rough guide. We then have

$$D_{turb} \sim \frac{cT}{eB}. \tag{3.8}$$

Similar results are obtained for the other transport coefficients through the use of Eqs. (3.2) and (3.3). This estimate was first proposed by Bohm, who was one of the first investigators to note the possibility of a strong enhancement of the diffusion coefficient in a plasma as a result of turbulence and the associated fluctuating fields. Bohm published the expression $D = cT_e/16eB$ without formal derivation [23] and this diffusion coefficient is sometimes called the Bohm diffusion coefficient (cf. also [23a]). Comparing this coefficient with the "classical" transverse diffusion coefficient we find

$$\frac{D_{\text{turb}}}{r^2/\tau} \sim \omega\tau. \tag{3.9}$$

Thus, when $\omega\tau \gg 1$ turbulence in a plasma can, in principle, cause strong enhancement of all perpendicular (to the magnetic field) transport processes.

§ 4. Kinetics of a Simple Plasma (Quantitative Analysis)

The local distribution functions for ions and electrons can be determined by a successive-approximation method that is described, for example, in the well-known monograph of Chapman and Cowling [1]. This approach can be described roughly as follows. The distribution function is assumed to be approximately a Maxwellian f^0 with parameters n, **V**, and T, that are slowly varying functions of the coordinates and time, and is expanded in the form

$$f = f^0 + f^1 + f^2 + \cdots \tag{4.1}$$

The important terms in the kinetic equation are assumed to be the collision term and the magnetic term. The other terms, which contain space and time derivatives and the electric field, are assumed to be small. The magnetic term $[\mathbf{v}\boldsymbol{\omega}]\nabla_v f$ vanishes for any spherically symmetric velocity function.

If small terms are neglected the solution will be the function f^0, which is obtained by also setting equal to zero the collisional term and the magnetic term.*

In the next approximation, substituting $f = f^0 + f^1$ in the kinetic equation we only take account of f^0 in the small terms, neglecting f^1; in the

*This statement is not completely accurate and will be modified appropriately below.

collision term $C(f, f)$ we only consider the part that is linear in f^1, i.e., $C(f^0, f^1) + C(f^1, f^0)$; terms quadratic in f^1 are neglected. In the small terms the derivatives over coordinates and time appear only as a consequence of differentiation of the parameters n, **V**, and T; by means of the transport equations, (cf. §1) the time derivatives can then be expressed with the desired accuracy in terms of the coordinate derivatives at a given instant of time. This procedure results in a linear integro-differential equation for the function f^1 in velocity space. Having solved this equation we then find the function $f^1(\mathbf{v})$, which will be a linear function of both the parameters and of the factors that disturb the Maxwellian distribution: ∇T, $\partial V_\alpha/\partial x_\beta$, etc.

This procedure can be extended to take account of second-order perturbation terms in order to find f^2; however, this step leads to extremely complicated calculations. By substituting f^1 in the expressions for the heat flux, momentum flux, etc., it is possible to find these fluxes so that the chain of transport equations can be closed. This procedure requires that the neglected terms must be small compared with those that have been considered in determining the local distribution function, that is to say, the series in (4.1) must converge sufficiently rapidly. To determine the condition of applicability for the first approximation rigorously one should really find the corrections associated with the second approximation f^2 to be convinced that they are in fact small; however, we shall limit ourselves to the qualitative considerations given in §3.

Simplification of the Cross Terms in the Collision Integral. In the further analysis it will be found convenient to make a substitution in the kinetic equation (1.1); we shall replace the velocity **v** by the random velocity $\mathbf{v}_a = \mathbf{v} - \mathbf{V}_a(t, \mathbf{r})$. The function $f_a(t, \mathbf{r}, \mathbf{v}_a)$ is then described by the equation

$$\frac{d_a f_a}{dt} + \mathbf{v}_a \nabla f_a + \left(\frac{e_a}{m_a}\mathbf{E}_a^* - \frac{d_a \mathbf{V}_a}{dt}\right)\nabla_v f_a - \frac{\partial V_{a\gamma}}{\partial x_\beta} v_{a\beta}\frac{\partial f_a}{\partial v_{a\gamma}} + \frac{e_a}{m_a c}[\mathbf{v}_a \mathbf{B}]\nabla_v f_a = \sum_b C_{ab}(f_a, f_b), \tag{4.2}$$

where ∇_v is the gradient in velocity space;

$$\frac{d_a}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_a \nabla), \quad \mathbf{E}_a^* = \mathbf{E} + \frac{1}{c}[\mathbf{V}_a \mathbf{B}].$$

In deriving Eq. (4.2) from (1.1) we have also taken account of the fact that $\nabla_v \mathbf{F} = 0$.

The collisional term is taken in the Landau form [11]:

$$C_{ab}(f_a, f_b) = -\frac{2\pi\lambda e_a^2 e_b^2}{m_a}\frac{\partial}{\partial v_\beta}\int\left\{\frac{f_a(\mathbf{v})}{m_b}\cdot\frac{\partial f_b(\mathbf{v}')}{\partial v'_\gamma} - \frac{f_b(\mathbf{v}')}{m_a}\frac{\partial f_a(\mathbf{v})}{\partial v_\gamma}\right\}U_{\beta\gamma}d\mathbf{v}';$$

$$U_{\beta\gamma} = \frac{1}{u^3}(u^2\delta_{\beta\gamma} - u_\beta u_\gamma);\quad u_\beta = v_\beta - v'_\beta. \tag{4.3}$$

In collisions between particles it is the relative velocity that is important $\mathbf{v} - \mathbf{v}'$; hence, Eq. (4.3) retains its form in any coordinate system, but obviously the distribution functions in Eq. (4.3) must be expressed in the same coordinate system.

The Coulomb logarithm λ in Eq. (4.3) is equal to the logarithm of the ratio of maximum to minimum impact parameters $\lambda = \ln(p_{max}/p_{min})$. The lower parameter will henceforth be taken to be the impact parameter characterizing a deflection through an angle of $\pi/2$ so that $p_{min} \approx e^2/m\langle v^2\rangle \approx e^2/3T$. The maximum impact parameter is defined in such a way that the Coulomb field of the plasma particles is screened at distances of the order of the Debye length $p_{max} \approx \delta_D$, where $\delta_D = (T_e/4\pi e^2 n)^{1/2}$. At large velocities, in which case $e^2/\hbar v < 1$, where h is Planck's constant (i.e., $v/c < 1/137$), it is necessary to use a smaller value for the maximum impact parameter; specifically, we use the distance for which the scattering angle is of the same order as the quantum uncertainty, in which case $p_{max} \approx \delta_D e^2/\hbar v$. The effect of the magnetic field on the collisions themselves is not considered in Eq. (4.3); this procedure is justified as the fields are weak enough so that the radius of curvature of the particle trajectory is large compared with the Debye length.

The solution of the ion and electron kinetic equations can be simplified by exploiting the fact that the mass ratio of these particles is small. When this is done the cross terms in the collision integral C_{ei} and C_{ie} can be simplified and the equations in (4.2) can be solved separately. This simplification results from the fact that the relative velocity is essentially equal to the electron velocity since the electron velocities are much greater than the ion velocities. Thus, to a high degree of accuracy the collisional cross term $C_{ei}(f_e, f_i)$ is independent of the detailed form of the ion distribution function and can be determined from a knowledge of the mean quantities n_i, $\mathbf{V}_i$, and T_i.

The tensor $U_{\alpha\beta} = (u^2\delta_{\alpha\beta} - u_\alpha u_\beta)u^{-3}$ that appears in C_{ei} depends on the difference between the electron velocity $\mathbf{v}$ and the ion velocity $\mathbf{v}'$ since $\mathbf{u} = \mathbf{v} - \mathbf{v}'$. Let us expand $U_{\alpha\beta}$ in powers of the ion velocity

$$U_{\alpha\beta} = V_{\alpha\beta} - \frac{\partial V_{\alpha\beta}}{\partial v_\gamma}v'_\gamma + \frac{\partial V_{\alpha\beta}}{\partial v_\gamma \partial v_\delta}\frac{v'_\gamma v'_\delta}{2} + \dots,$$

where

$$V_{\alpha\beta} = U_{\alpha\beta}|_{v'=0} = \frac{1}{v^3}(v^2\delta_{\alpha\beta} - v_\alpha v_\beta),$$

and integrate over $\mathbf{v}'$; an approximate expression is then obtained for C_{ei}. This calculation is convenient in the coordinate system in which the mean ion velocity is zero. As a result we find

$$C_{ei} = \frac{3\sqrt{\pi}}{8}\frac{1}{\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2}\frac{\partial}{\partial v_\alpha}\left\{V_{\alpha\beta}\frac{\partial f_e}{\partial v_\beta} + \frac{m_e}{m_i}\left(\frac{2v_\alpha}{v^3}f_e + \frac{T_i}{m_e}\frac{3v_\alpha v_\beta - v^2\delta_{\alpha\beta}}{v^5}\frac{\partial f_e}{\partial v_\beta}\right)\right\}. \tag{4.4}$$

Here

$$\tau_e = \frac{3\sqrt{m_e}\,T_e^{3/2}}{4\sqrt{2\pi}\,\lambda e^2 e_i^2 n_i}. \tag{4.5}$$

The electron velocity in Eq. (4.4) is computed from the mean ion velocity $\mathbf{V}_i$ (rather than $\mathbf{V}_e$). The first (principal) term in Eq. (4.4) will be called C'_{ei} below; it is completely independent of the ion distribution function:

$$C'_{ei} = \frac{3\sqrt{\pi}}{8}\frac{1}{\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2}\frac{\partial}{\partial v_\alpha}\left(V_{\alpha\beta}\frac{\partial f_e}{\partial v_\beta}\right). \tag{4.4'}$$

In the integration over $\mathbf{v}'$ in the second term $C''_{ei} \sim m_e/m_i$ we have neglected the difference between the ion pressure tensor and the scalar pressure $n_i T_i$ (i.e., we have neglected $\pi_{i\alpha\beta}$). We compute the friction force $\mathbf{R}^0$ exerted on the electrons by the ions when the electrons have a Maxwellian distribution shifted with respect to the ion distribution by an amount* $\mathbf{U} = \mathbf{V}_e - \mathbf{V}_i$. Assuming that the displacement is small compared with the electron thermal velocity and expanding in U we can write this electron distribution approximately in the coordinate system in which $V_i = 0$:

*Here, in contrast with §2 and 3, the difference of the mean velocities is denoted by a capital letter.

$$f_e = f_e^0\left(1 + \frac{m_e}{T_e}\mathbf{U}\mathbf{v}\right), \text{where } f_e^0 = \frac{n_e}{(2\pi T_e/m_e)^{3/2}}\exp\left(-\frac{m_e v^2}{2T_e}\right).$$

Substituting this expression in Eq. (4.4), using Eq. (1.18), and neglecting m_e/m_i terms we find

$$\mathbf{R}^0 = \frac{3\sqrt{\pi}}{8}\frac{1}{\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2}\int m_e \mathbf{v}\frac{\partial}{\partial v_\alpha}\left\{V_{\alpha\beta}\frac{m_e}{T_e}U_\beta f_e^0\right\}d\mathbf{v} = -\frac{m_e n_e}{\tau_e}\mathbf{U}. \tag{4.6}$$

Here we have made use of the following property of the tensor $V_{\alpha\beta}: v_\alpha V_{\alpha\beta} = V_{\alpha\beta}v_\beta = 0$; thus, C'_{ei} vanishes for any spherically symmetric electron distribution function. The property $\partial V_{\alpha\beta}/\partial v_\alpha = -2v_\beta/v^3$ and $\overline{v_\alpha v_\beta} = (v^2/3)\delta_{\alpha\beta}$ has also been used (the bar denotes an average over direction).

The quantity τ_e derived in Eq. (4.4), which represents the characteristic time between electron-ion collisions, is chosen in such a way that Eq. (4.6) (for $\mathbf{R}^0$) will be of simple form.

The ion-electron collision integral $C_{ie}(f_i, f_e)$ can also be simplified by expanding the tensor $U_{\alpha\beta}$ in powers of the ratio of ion velocity $\mathbf{v}$ to electron velocity $\mathbf{v}'$:

$$U_{\alpha\beta} = V'_{\alpha\beta} - \frac{\partial V'_{\alpha\beta}}{\partial v'_\gamma}v_\gamma + \ldots, \quad V'_{\alpha\beta} = \frac{1}{v'^3}\left(v'^2\delta_{\alpha\beta} - v'_\alpha v'_\beta\right).$$

Here, however, the electron distribution function must be known, in order to actually carry out the integration over electron velocity $\mathbf{v}'$. Let us assume that the electron distribution function is essentially a Maxwellian f_e^0, i. e., assume the form $f_e = f_e^0 + f_e^1$, where f_e^1 is a small correction, in which the difference in mean velocities $\mathbf{U} = \mathbf{V}_e - \mathbf{V}_i$ is small compared with the characteristic electron velocity. Some simple calculations yield the approximate relation

$$C_{ie} = \frac{m_e n_e}{m_i n_i}\frac{1}{\tau_e}\frac{\partial}{\partial v_\alpha}\left(v_\alpha f_i + \frac{T_e}{m_i}\frac{\partial f_e}{\partial v_\alpha}\right) + \frac{1}{m_i n_i}\mathbf{R}_i \nabla_v f_i. \tag{4.7}$$

Here the ion velocity is computed from the mean ion velocity $\mathbf{V}_i$. In accordance with Eq. (1.18) we have used the notation $\mathbf{R}_i = \int m_i \mathbf{v} C_{ie}\,d\mathbf{v} = -\mathbf{R}$ ($\mathbf{R}$ without a subscript denotes $\mathbf{R}_e$). The calculation required for the derivation of Eq. (4.7) can be carried out conveniently in the coordinate system in which the mean electron velocity is zero ($V_e = 0$); we then convert to a system in which the mean ion velocity vanishes ($V_i = 0$) for which $\mathbf{v}$

is replaced by $\mathbf{v} - \mathbf{U}$ and Eq. (4.6) (for $\mathbf{R}^0$) is used. In computing small-correction collisions due to f_e^1 we need consider only the leading term $V'_{\alpha\beta}$ in $U_{\alpha\beta}$.

As is to be expected, Eq. (4.7) is of the same form as the Fokker-Planck collisional term that describes Brownian motion of particles in a moving medium with temperature T_e.

The collisional heat exchange between electrons and ions can be computed neglecting the small deviations from a Maxwellian in the distribution functions. Substituting Eq. (4.7) in Eq. (1.22), for a Maxwellian ion function we find $Q_i = Q_\Delta$ where

$$Q_\Delta = \frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_e - T_i). \tag{4.8}$$

Similarly, using Eq. (4.5) and taking account of m_e/m_i terms we obtain $Q_e = -Q_\Delta$ if it is assumed that the electrons have a Maxwellian distribution with $\mathbf{V}_e = \mathbf{V}_i$. In the general case it is easiest to compute Q_e using conservation of energy and momentum (1.24) for the collisions: $Q_e + Q_i = -\mathbf{RU}$, whence

$$Q_e = -\mathbf{RU} - Q_\Delta. \tag{4.9}$$

In the remainder of this section we shall only use the variables $\mathbf{v}_a = \mathbf{v} - \mathbf{V}_a(t, \mathbf{r})$, i. e., the random velocities; for reasons of brevity the subscript a will be omitted.

<u>Correction Equations.</u> We now derive the equations for the electron distribution functions. The electron kinetic equation (4.2) can be written

$$\begin{gathered} C_{ee}(f_e, f_e) + C'_{ei}(f_e, f'_i) - [\mathbf{v}\boldsymbol{\omega}_e]\nabla_v f_e = \\ = \frac{d_e f_e}{dt} + \mathbf{v}\nabla f_e + \left(\frac{e_e}{m_e}E_e^* - \frac{d_e \mathbf{V}_e}{dt}\right)\nabla_v f_e - \frac{\partial V_{e\alpha}}{\partial x_\beta}v_\beta\frac{\partial f_e}{\partial v_\alpha} - \\ - C'_{ei}(f_e, f_i - f'_i) - C''_{ei}(f_e, f_i). \end{gathered} \tag{4.10}$$

Here $\boldsymbol{\omega}_e = (e_e/m_e c)\mathbf{B}$ is a vector whose magnitude is equal to the cyclotron frequency of the electrons and whose direction is antiparallel to the magnetic field since $e_e = -e$.

The terms on the right side of this equation will be small if the gradients are small, if the time variations are slow, and if the shift between the mean velocities of the electrons and ions is small.

In Eq. (4.10) we have added and subtracted the term $C'_{ei}(f_e, f'_i)$ where f'_i is the ion distribution function shifted in such a way that the mean ion velocity coincides with the mean electron velocity. Hence $C'_{ei}(f_e, f'_i)$ represents the quantity C'_{ei} [in accordance with Eq. (4.4)] but with the electron velocity computed from V_e [as is the case for all terms in Eq. (4.10)]. The term $C'_{ei}(f_e, f_i - f'_i)$ on the right side of the equation is small compared with $C'_{ei}(f_e, f'_i)$ if the relative macroscopic velocity of the electrons and ions $\mathbf{U} = \mathbf{V}_e - \mathbf{V}_i$ is small compared with the thermal velocity of the electrons, a condition that has been assumed.

The zeroth approximation satisfies the equation with the right side set equal to zero. Its solution is a Maxwellian distribution with mean velocity $\mathbf{V}_e$ and arbitrary density and temperature. We take the parameters of this distribution to be the density and temperature of the electrons at a given point in space.

Consider Eq. (4.10); if the entire cross-collisional integral C_{ei} on the left is retained, the solution of the equation without the right side (with the corresponding equation for the ions) is a Maxwellian distribution with $T_e = T_i$ and $\mathbf{V}_e = \mathbf{V}_i$. This is the approach used in the monograph of Chapman and Cowling. However, this approach does not exploit the fact that the ratio m_e/m_i is small. The regrouping in C_{ei} used in Eq. (4.10), in which only $C'_{ei}(f_e, f'_i)$ remains as a principal term is necessary specifically in order to eliminate the effect of small terms on the choice of the zeroth approximation. This feature makes it possible to obtain separate transport equations for the electrons and ions with different temperatures (and different velocities) and to uncouple the electron and ion kinetic equations.

Let us write the electron distribution function in the form $f_e = f_e^0(1 + \Phi)$ where Φ is a small correction. By substituting this expression in Eq. (4.10) and neglecting second-order terms we can obtain an equation for the correction term. As a result of this linearization procedure the left side becomes

$$I_{ee}(\Phi) + I_{ei}(\Phi) - f_e^0 [\mathbf{v}\,\omega_e]\,\nabla_v \Phi,$$

where

$$I_{ee}(\Phi) = C_{ee}(f_e^0,\ f_e^0\Phi) + C_{ee}(f_e^0\Phi,\ f_e^0), \qquad (4.11)$$

$$I_{ei}(\Phi) = C'_{ei}(f_e^0\Phi,\ f'_i).$$

It is valid to substitute f^0 on the right side of Eq. (4.10) and to omit m_e/m_i terms; then, expanding the integral $C'_{ei}(f_e^0, f_i - f'_i)$ in powers of $U(m_e/T_e)^{1/2}$, it is valid to neglect all terms beyond the first term of the expansion. The time derivatives of n_e, $\mathbf{V}_e$, and $\mathbf{T}_e$ on the right side can be replaced by their zeroth approximations. With the right side set equal to zero the equation has the solutions* $\Phi = 1$, v^2; hence, if the equation is to be solved the right side must be orthogonal to this solution.

Multiplying the correction equation by 1, $\mathbf{v}$, and $m_e v^2/2$ and integrating over velocity we obtain an expression for the zeroth time derivatives of n_e, $\mathbf{V}_e$, and $\mathbf{T}_e$ which are to be substituted in the right side. These expressions are the same as those derived from the transport equations in the zeroth approximation, i.e., neglecting viscosity, heat flow, etc. The first-approximation correction Φ is thus given by

$$\begin{aligned} I_{ee}(\Phi) + I_{ei}(\Phi) - f_e^0 [\mathbf{v}\,\omega_e]\,\nabla_v\Phi = f_e^0 \Big\{ &\left(\frac{m_e v^2}{2T_e} - \frac{5}{2}\right) \mathbf{v}\nabla \ln T_e + \\ &+ \left[\frac{3\sqrt{\pi}}{\sqrt{2}}\,\frac{(T_e/m_e)^{3/2}}{v^3} - 1\right] \frac{m_e}{T_e\tau_e}\mathbf{U}\mathbf{v} + \frac{1}{n_e T_e}\mathbf{R}^1\mathbf{v} + \\ &+ \frac{m_e}{2T_e}\left(v_\alpha v_\beta - \frac{v^2}{3}\delta_{\alpha\beta}\right) W_{e\alpha\beta} \Big\}, \end{aligned} \qquad (4.12)$$

where

$$\mathbf{R}^1 = \int m_e \mathbf{v} I_{ei}(\Phi)\, d\mathbf{v}. \qquad (4.13)$$

We note that the right side of Eq. (4.12) does not contain terms proportional to ∇n and $\mathbf{g}_e = (e_e \mathbf{E}_e^*/m_e) - d_e\mathbf{V}_e/dt$. This results from the fact that the $f_e^0\, \mathbf{v}\nabla \ln n_e$ term is combined with $\mathbf{g}_e \nabla_v f_e^0 = -f_e^0 (m_e/T_e)(\mathbf{v}\mathbf{g}_e)$ and that the sum of these gives terms proportional to ∇T_e and $\mathbf{R} = \mathbf{R}^0 + \mathbf{R}^1 = -(m_e n_e/\tau_e)\mathbf{U} + \mathbf{R}^1$ as follows from the equation of motion $-m_e n_e \mathbf{g}_e = \nabla n_e T_e + \mathbf{R}$.

We now symmetrize the last term on the right side of Eq. (4.12) forming the symmetric tensor with zero trace $W_{\alpha\beta}$

$$W_{\alpha\beta} = \frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\,\mathrm{div}\,\mathbf{V}, \qquad (4.14)$$

which is called the rate-of-strain tensor.

*This follows immediately from the fact that the left side of Eq. (4.10) vanishes for a Maxwellian distribution with arbitrary n and t.

The ion kinetic equation is transformed in similar fashion with the difference that the cross-collisional term C_{ie} (as can be shown by simple estimates) is small compared with the "self" term C_{ii} so that the former is grouped with the small terms and transferred to the right side. The zeroth approximation, which satisfies the equation without the right side, is the Maxwell distribution f_i^0. The ion distribution function is written in the form $f_i = f_i^0(1 + \Phi)$ where the small correction is given by the equation:

$$I_{ii}(\Phi) - f_i^0[\mathbf{v}\boldsymbol{\omega}_i]\nabla_v\Phi =$$
$$= f_i^0\left\{\left(\frac{m_i v^2}{2T_i} - \frac{5}{2}\right)\mathbf{v}\nabla \ln T_i + \frac{m_i}{2T_i}\left(v_\alpha v_\beta - \frac{v^2}{3}\delta_{\alpha\beta}\right)W_{i\alpha\beta}\right\}. \tag{4.15}$$

The terms on the right associated with electron-ion collisions cancel if C_{ie} is written in the form corresponding to Eq. (4.7). Thus, Eq. (4.15) has the same form as for a single-component gas (rather than a mixture). In this approximation the form of the ion distribution function is determined exclusively by ion-ion collisions. On the other hand, the form of the electron distribution function is determined both by self-collisions (electron-electron) and cross-collisions (electron-ion), as follows from Eqs. (4.12) and (4.13).

Equation (4.15) determines the correction Φ to terms of order $c_0 + \mathbf{c}_1\cdot\mathbf{v} + c_2v^2$, which causes the left side to vanish. Since the zeroth approximation gives the correct value of the density, mean velocity, and mean energy of the ions, these terms are determined from the requirement that the correction must not change the values of these parameters, i.e.,

$$\int f^0\Phi d\mathbf{v} = 0, \quad \int \mathbf{v} f^0\Phi d\mathbf{v} = 0, \quad \int v^2 f^0\Phi d\mathbf{v} = 0. \tag{4.16}$$

The same conditions must be satisfied by the correction to the electron distribution function. This requirement can obviously be satisfied: the left side of Eq. (4.12) vanishes as an expression of the form $c_0 + c_2v^2$, while the right side contains a term proportional to the unspecified magnitude $\mathbf{R}^1$ so that a solution can be sought in a form that will satisfy the condition $\int \mathbf{v} f_e^0 \Phi d\mathbf{v} = 0$.

<u>Solution of Eqs. (4.12) and (4.15).</u> Equations (4.12) and (4.15) are linear; this means that the solutions can be written as a sum of terms, each of which corresponds to some single perturbing factor – the temperature gradient ∇T, the velocity shift $\mathbf{U}$, the inhomogeneity in the velocities $W_{\alpha\beta}$.

Considerations of tensor invariance indicate the following form of the solution:

$$\Phi(\mathbf{v}) = \Phi_\alpha(v^2)\,v_\alpha + \Phi_{\alpha\beta}(v^2)\left(v_\alpha v_\beta - \frac{v^2}{3}\delta_{\alpha\beta}\right). \tag{4.17}$$

Here, the first (vector) term corresponds to the vector perturbations ∇T and $\mathbf{U}$ while the second (tensor) term corresponds to $W_{\alpha\beta}$. The first and second terms are obviously orthogonal since averaging over angle in velocity space gives $\overline{v_\alpha} = 0$, $\overline{v_\alpha v_\beta v_\gamma} = 0$. The angular dependence of the first and second terms in velocity space is expressed by spherical functions of first and second order respectively. The heat flux $\mathbf{q}$ and the momentum transfer due to collisions $\mathbf{R}^1$ are specified exclusively by the vector Φ while the viscosity $\pi_{\alpha\beta}$ is determined only by the tensor $\Phi_{\alpha\beta}$.

As an example, let us consider how to determine the correction $\Phi_\alpha v_\alpha$ for the electrons connected with ∇T_e and the appropriate parts of $\mathbf{q}_e$ and $\mathbf{R}^1$. The collision integrals are isotropic and do not depend on a specified direction. Hence, in the absence of a magnetic field, the symmetry of the problem indicates that the dependence of the vector $\Phi(v^2)$ on ∇T_e must be of the form $\Phi(v^2) = A(v^2)\nabla \ln T_e$, where A is a scalar function. In a magnetic field this dependence is of the form

$$\Phi(v^2) = A\nabla_\parallel \ln T_e + A'\nabla_\perp \ln T_e + A''[\boldsymbol{\omega}_e\nabla \ln T_e], \tag{4.18}$$

where $\nabla_\parallel \ln T_e$ and $\nabla_\perp \ln T_e$ are the components of the vector $\nabla \ln T_e$ parallel and perpendicular to the magnetic field. Evidently it is sufficient to consider the case of a transverse gradient since $A(v^2)$ is obtained from $A'(v^2)$ by writing $\omega_e = 0$.

The equation that describes the part of the correction arising from $\nabla_\perp \ln T_e$ is

$$I_{ee}(\Phi) + I_{ei}(\Phi) - f_e^0[\mathbf{v}\boldsymbol{\omega}_e]\nabla_v\Phi = f_e^0\left\{\left(\frac{m_e v^2}{2T_e} - \frac{5}{2}\right)\mathbf{v}\nabla_\perp \ln T_e + \right.$$
$$\left. + \frac{1}{n_e T_e}\mathbf{R}_T^1\mathbf{v}\right\}. \tag{4.19}$$

The thermal force $\mathbf{R}_T^1$ can be written in the form

$$\mathbf{R}_T^1 = n_e T_e\left(K'\nabla_\perp \ln T_e + K''[\boldsymbol{\omega}_e\nabla \ln T_e]\right),$$

where K' and K" are functions as yet unknown. Substituting Eqs. (4.18) and

(4.20) and setting the coefficients of $\nabla_\perp \ln T_e$ and $[\omega_e \nabla \ln T_e]$ equal to zero, we obtain two equations for determining A' and A". Introducing the complex quantities

$$A = A' + i(\omega_e \mathbf{h})A'', \quad K = K' + i(\omega_e \mathbf{h})K'', \qquad (4.20)$$

we can reduce these to a single equation for A:

$$I_{ee}(A\mathbf{v}) + I_{ei}(A\mathbf{v}) - i(\omega_e \mathbf{h}) f_e^0 A\mathbf{v} = f_e^0 \left\{ \frac{m_e v^2}{2T_e} - \frac{5}{2} + K \right\} \mathbf{v}. \qquad (4.21)$$

In order to avoid the need for numerical solution of this integral equation, we proceed as in reference [1]. The quantity $A(v^2)$ is expanded in terms of orthogonal functions; in the present case it is convenient to use the Sonine polynomials (sometimes called Laguerre polynomials). These polynomials $L_p^{(m)}(x)$ have the following generating function:

$$(1-\xi)^{-m-1} \exp\left(-\frac{x\xi}{1-\xi}\right) = \sum_{p=0}^{\infty} \xi^p L_p^{(m)}(x). \qquad (4.22)$$

The polynomials are orthogonal over the interval 0 to ∞ with respect to the weighting factor $x^m e^{-x}$:

$$\int_0^\infty x^m e^{-x} L_p^{(m)}(x) L_q^{(m)}(x)\, dx = \frac{(p+m)!}{p!} \delta_{pq}. \qquad (4.23)$$

The first two polynomials are $L_0^{(m)} = 1$; $L_1^{(m)} = m + 1 - x$.

We expand $A(v^2)$ in the form

$$A(v^2) = \tau_e \sum_{k=1}^{\infty} a_k L_k^{(3/2)}(x), \quad x = \frac{m_e v^2}{2T_e}. \qquad (4.24)$$

The expansion starts with the k = 1 term rather than the k = 0 term in order to satisfy the condition $\int \mathbf{v} f_e^0 \Phi d\mathbf{v} = 0$. Multiplying Eq. (4.21) by

$$-\frac{4}{15}\frac{1}{n_e}\frac{m_e}{2T_e} \mathbf{v} L_k^{(3/2)} \left(\frac{m_e v^2}{2T_e}\right) d\mathbf{v},$$

integrating over velocity, and using Eq. (4.23), we now find that the integral equation is replaced by an infinite system of algebraic equations for

the expansion coefficients:

$$\sum_{l=1}^{\infty} (\alpha_{kl} + \alpha'_{kl}) a_l + i(\omega_e \mathbf{h}) \tau_e \frac{(k+3/2)!}{k!(5/2)!} a_k = \delta_{1k}, \quad k = 1, 2, \ldots, \qquad (4.25)$$

where α_{kl} and α'_{kl} are the dimensionless matrices:

$$\alpha_{kl} = -\frac{4\tau_e}{15 n_e}\frac{m_e}{2T_e} \int L_k^{(3/2)}(x) v_\beta I_{ee}\left(L_l^{(3/2)}(x) v_\beta\right) d\mathbf{v}; \quad x = \frac{m_e v^2}{2T_e};$$

$$\alpha'_{kl} = -\frac{4\tau_e}{15 n_e}\frac{m_e}{2T_e} \int L_k^{(3/2)}(x) v_\beta I_{ei}\left(L_l^{(3/2)}(x) v_\beta\right) d\mathbf{v}. \qquad (4.26)$$

Equations (1.18), (1.21), (4.23) and (4.26) can be used to write the heat flux $\mathbf{q}_T$ and the thermal force $\mathbf{R}_T$ in terms of the expansion coefficients in (4.24):

$$\mathbf{q}_T = -\frac{5}{2}\frac{n_e T_e \tau_e}{m_e}\left(a'_1 \nabla_\perp T_e + a''_1 [\omega_e \nabla T_e]\right), \qquad (4.27)$$

$$\mathbf{R}_T = -\frac{5}{2} n_e \sum_{k=1}^{\infty} \alpha'_{0k}\left(a'_k \nabla_\perp T_e + a''_k [\omega_e \nabla T_e]\right), \qquad (4.28)$$

where, by analogy with Eq. (4.20), we write $a_k = a'_k + i(\omega_e \mathbf{h}) a''_k$. If the expansion of $A(v^2)$ is now limited to the first few terms in the series (4.24) a corresponding cutoff can be introduced in (4.25). Solving the resulting finite system for the first few coefficients we obtain approximate expressions for the heat flux and thermal force by dividing a_k into real and imaginary parts and using Eqs. (4.27) and (4.28).

In completely analogous fashion we can find the contributions to the distribution functions and the contributions to $\mathbf{q}_e$ and $\mathbf{R}^1$ due to the relative velocity $\mathbf{U} = \mathbf{V}_e - \mathbf{V}_i$ as well as the correction to the ion distribution function due to ∇T_i; the ion heat flux can also be found. The appropriate system of equations analogous to (4.25) and the coefficient matrices α_{kl} and α'_{kl} are given in [17].

The calculation of $\Phi_{\alpha\beta}$ and the viscosity tensor proceeds in analogous fashion but the division of the perturbation into independent parts is somewhat more complicated. The tensor $W_{\alpha\beta}$ is divided into three independent parts $W_{\alpha\beta} = W_{0\alpha\beta} + W_{1\alpha\beta} + W_{2\alpha\beta}$ and two new tensors $W_{3\alpha\beta}$ and $W_{4\alpha\beta}$ made up of components of $W_{\alpha\beta}$ are introduced. The correction to the

Maxwellian distribution due to the perturbation $W_{\alpha\beta}$ is written in the form

$$\Phi_{\alpha\beta} v_{\alpha\beta} = -\sum_{p=0}^{4} B_p(v^2)\, W_{p\alpha\beta} v_{\alpha\beta}, \text{ where } v_{\alpha\beta} = v_\alpha v_\beta - \frac{v^2}{3}\,\delta_{\alpha\beta} \tag{4.29}$$

The tensors $W_{p\alpha\beta}$ can be chosen so that the magnetic operator $[\mathbf{vh}]\nabla_v$ causes the term $W_{0\alpha\beta}v_{\alpha\beta}$ to vanish and $[\mathbf{vh}]\nabla_v W_{1\alpha\beta}v_{\alpha\beta} = 2W_{3\alpha\beta}v_{\alpha\beta}$; $[\mathbf{vh}]\nabla_v W_{3\alpha\beta}v_{\alpha\beta} = -2W_{1\alpha\beta}v_{\alpha\beta}$; $[\mathbf{vh}]\nabla_v W_{2\alpha\beta}v_{\alpha\beta} = W_{4\alpha\beta}v_{\alpha\beta}$; $[\mathbf{vh}]\nabla_v W_{4\alpha\beta}v_{\alpha\beta} = -W_{2\alpha\beta}v_{\alpha\beta}$. Thus, there are three independent kinds of motion all of which have different effects on the viscosity. The corrections associated with each of these $\Phi_{\alpha\beta}$ can be found independently. The equations for B_1, B_3 and for B_2, B_4 can be combined in pairs by the introduction of appropriate complex quantities. The function $B(v^2)$ is found by the same approximation method as $A(v^2)$. The function $B(v^2)$ can be represented conveniently in a series in the polynomials $L_k^{(5/2)}(mv^2/2T)$ in which an appropriate cutoff is introduced in the chain of algebraic equations for the expansion coefficients. This system of equations is given in [17]. The viscosity tensor is found by means of Eq. (1.17). It depends only on the coefficient of $L_0^{(5/2)}$.

The larger the number of polynomials N used to approximate the correction to the distribution function the more exact the transport coefficients that are obtained by the approximate method we have described. Comparison of the results obtained with different values of N shows that the error in certain coefficients can be comparable with the coefficient itself when N = 1; however, the accuracy increases sharply if N = 2. Further increases in N do not increase the accuracy significantly but do increase the complexity of the expressions greatly. The transport coefficients obtained with two approximation polynomials [17]* are given below.

The results of calculations of the electron fluxes in which a large number of polynomials have been used (up to N = 6) for $0 \le \omega_e\tau_e \le 6$ are given in [21]. "Exact" values of the transport coefficients for $\omega_e\tau_e = 0$ have been obtained in [14] by numerical integration of the correction equations.

An accuracy of several percent is obtained when N = 2 and $\omega\tau = 0$. The asymptotic behavior of the transport coefficients for $\omega\tau \to \infty$ is determined by numerical coefficients which are given below in the form of simple rational fractions. These coefficients have been obtained exactly [18]. The largest error (10-20%) in the transport coefficients occurs in the intermediate region $\omega\tau \sim 1$.

*We wish to note certain typographical errors in [17]: In Eq. (3.18) for $\mathbf{q}_u$ the minus sign in front of the curly brackets should be replaced by a plus sign. In Eq. (4.14) the quantity b" should read −b", since −b" is a positive number, as follows from Eq. (4.13).

<u>Results.</u> The transfer of momentum from the ions to the electrons in collisions $\mathbf{R} = \mathbf{R}^0 + \mathbf{R}^1$ is made up of the frictional force $\mathbf{R}_u = \mathbf{R}_u^0 + \mathbf{R}_u^1$ and the thermal force $\mathbf{R}_T = \mathbf{R}_T^1$:

$$\mathbf{R}_u = -\alpha_\parallel \mathbf{u}_\parallel - \alpha_\perp \mathbf{u}_\perp + \alpha_\wedge [\mathbf{hu}], \tag{4.30}$$

$$\mathbf{R}_T = -\beta_\parallel^{uT}\nabla_\parallel T_e - \beta_\perp^{uT}\nabla_\perp T_e - \beta_\wedge^{uT}[\mathbf{h}\nabla T_e]. \tag{4.31}$$

The electron heat flux $\mathbf{q}_e = \mathbf{q}_u^e + \mathbf{q}_T^e$ consists of two analogous parts:

$$\mathbf{q}_u^e = \beta_\parallel^{Tu}\mathbf{u}_\parallel + \beta_\perp^{Tu}\mathbf{u}_\perp + \beta_\wedge^{Tu}[\mathbf{hu}], \tag{4.32}$$

$$\mathbf{q}_T^e = -\varkappa_\parallel^e \nabla_\parallel T_e - \varkappa_\perp^e \nabla_\perp T_e - \varkappa_\wedge^e [\mathbf{h}\nabla T_e]. \tag{4.33}$$

Here

$$\left.\begin{aligned} \alpha_\parallel &= \frac{m_e n_e}{\tau_e}\alpha_0, \quad \alpha_\perp = \frac{m_e n_e}{\tau_e}\left(1 - \frac{\alpha_1' x^2 + \alpha_0'}{\Delta}\right), \\ \alpha_\wedge &= \frac{m_e n_e}{\tau_e}\,\frac{x\left(\alpha_1'' x^2 + \alpha_0''\right)}{\Delta}, \end{aligned}\right\} \tag{4.34}$$

$$\beta_\parallel^{uT} = n_e\beta_0, \quad \beta_\perp^{uT} = n_e\frac{\beta_1' x^2 + \beta_0'}{\Delta}, \quad \beta_\wedge^{uT} = n_e\frac{x\left(\beta_1'' x^2 + \beta_0''\right)}{\Delta}, \tag{4.35}$$

$$\beta_\parallel^{Tu} = \beta_\parallel^{uT} T_e, \quad \beta_\perp^{Tu} = \beta_\perp^{uT} T_e, \quad \beta_\wedge^{Tu} = \beta_\wedge^{uT} T_e, \tag{4.36}$$

$$\varkappa_\parallel^e = \frac{n_e T_e \tau_e}{m_e}\gamma_0, \quad \varkappa_\perp^e = \frac{n_e T_e \tau_e}{m_e}\,\frac{\left(\gamma_1' x^2 + \gamma_0'\right)}{\Delta}, \quad \varkappa_\wedge^e = \frac{n_e T_e \tau_e}{m_e}\,\frac{x\left(\gamma_1'' x^2 + \gamma_0''\right)}{\Delta}, \tag{4.37}$$

where

$$x = \omega_e\tau_e, \quad \Delta = x^4 + \delta_1 x^2 + \delta_0. \tag{4.38}$$

The coefficients α, β, γ, and δ are given in Table 2 for various values of Z.

The exact values of the coefficients as obtained by direct numerical solution of the integral equation for B = 0 and Z = 1 are as follows [14]: $\alpha_0 = 0.5063$; $\beta_0 = 0.7033$; $\gamma_0 = 3.203$. For Z = ∞ an exact solution of the correction equation indicates that

$$\alpha_0 = \frac{3\pi}{32} = 0.2945, \quad \beta_0 = \frac{3}{2}, \quad \gamma_0 = \frac{128}{3\pi} = 13.58.$$

The ion heat flux is

$$\mathbf{q}_i = -\varkappa^i_\parallel \nabla_\parallel T_i - \varkappa^i_\perp \nabla_\perp T_i + \varkappa^i_\wedge [\mathbf{h}\nabla T_i], \tag{4.39}$$

$$\left.\begin{aligned} \varkappa^i_\parallel &= 3.906 n_i T_i \tau_i / m_i, \\ \varkappa^i_\perp &= (n_i T_i \tau_i / m_i)(2x^2 + 2.645)/\Delta, \\ \varkappa^i_\wedge &= (n_i T_i \tau_i / m_i)\, x \left(\frac{5}{2} x^2 + 4.65\right)/\Delta, \end{aligned}\right\} \tag{4.40}$$

where

$$x = \omega_i \tau_i, \quad \Delta = x^4 + 2.70x^2 + 0.677.$$

The stress tensor for particles of a given species (the symbols i and e are omitted) is expressed in terms of the corresponding tensor $W_{\alpha\beta}$ [cf. Eq. (4.14)] by means of the five viscosity coefficients:

$$\pi_{\alpha\beta} = -\eta_0 W_{0\alpha\beta} - \eta_1 W_{1\alpha\beta} - \eta_2 W_{2\alpha\beta} + \eta_3 W_{3\alpha\beta} + \eta_4 W_{4\alpha\beta}, \tag{4.41}$$

where $W_{\alpha\beta} = W_{0\alpha\beta} + W_{1\alpha\beta} + W_{2\alpha\beta}$.

Here

$$\left.\begin{aligned} W_{0\alpha\beta} &= \frac{3}{2}\left(h_\alpha h_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)\left(h_\mu h_\nu - \frac{1}{3}\delta_{\mu\nu}\right) W_{\mu\nu}, \\ W_{1\alpha\beta} &= \left(\delta^\perp_{\alpha\mu}\delta^\perp_{\beta\nu} + \frac{1}{2}\delta^\perp_{\alpha\beta} h_\mu h_\nu\right) W_{\mu\nu}, \\ W_{2\alpha\beta} &= \left(\delta^\perp_{\alpha\mu} h_\beta h_\nu + \delta^\perp_{\beta\nu} h_\alpha h_\mu\right) W_{\mu\nu}, \\ W_{3\alpha\beta} &= \frac{1}{2}\left(\delta^\perp_{\alpha\mu}\varepsilon_{\beta\gamma\nu} + \delta^\perp_{\beta\nu}\varepsilon_{\alpha\gamma\mu}\right) h_\gamma W_{\mu\nu}, \\ W_{4\alpha\beta} &= \left(h_\alpha h_\mu \varepsilon_{\beta\gamma\nu} + h_\beta h_\nu \varepsilon_{\alpha\gamma\mu}\right) h_\gamma W_{\mu\nu}, \end{aligned}\right\} \tag{4.42}$$

where $\delta^\perp_{\alpha\beta} = \delta_{\alpha\beta} - h_\alpha h_\beta$; $\varepsilon_{\alpha\beta\gamma}$ is an antisymmetric unit tensor.

TABLE 2

	$Z=1$	$Z=2$	$Z=3$	$Z=4$	$Z\to\infty$
$\alpha_0 = 1 - (\alpha_0'/\delta_0)$	0.5129	0.4408	0.3965	0.3752	0.2949
$\beta_0 = \beta_0'/\delta_0$	0.7110	0.9052	1.016	1.090	1.521
$\gamma_0 = \gamma_0'/\delta_0$	3.1616	4.890	6.064	6.920	12.471
δ_0	3.7703	1.0465	0.5814	0.4106	0.0961
δ_1	14.79	10.80	9.618	9.055	7.482
α_1'	6.416	5.523	5.226	5.077	4.63
α_0'	1.837	0.5956	0.3515	0.2566	0.0678
α_1''	1.704	1.704	1.704	1.704	1.704
α_0''	0.7796	0.3439	0.2400	0.1957	0.0940
β_1'	5.101	4.450	4.233	4.124	3.798
β_0'	2.681	0.9473	0.5905	0.4478	0.1461
β_1''	3/2	3/2	3/2	3/2	3/2
β_0''	3.053	1.784	1.442	1.285	0.877
γ_1'	4.664	3.957	3.721	3.604	3.25
γ_0'	11.92	5.118	3.525	2.841	1.20
γ_1''	5/2	5/2	5/2	5/2	5/2
γ_0''	21.67	15.37	13.53	12.65	10.23

In the coordinate system in which the z axis is along the magnetic field (x, y, z → 1, 2, 3):

$$h = (0,\ 0,\ 1),\ \delta^{\perp}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \varepsilon_{\alpha\gamma\beta} h_\gamma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the tensors $W_{p\alpha\beta}$ are:

$$W_{0\alpha\beta} = \begin{Bmatrix} \frac{1}{2}(W_{xx}+W_{yy}) & 0 & 0 \\ 0 & \frac{1}{2}(W_{xx}+W_{yy}) & 0 \\ 0 & 0 & W_{zz} \end{Bmatrix}$$

$$W_{1\alpha\beta} = \begin{Bmatrix} \frac{1}{2}(W_{xx}-W_{yy}) & W_{xy} & 0 \\ W_{yx} & \frac{1}{2}(W_{yy}-W_{xx}) & 0 \\ 0 & 0 & 0 \end{Bmatrix} W_{2\alpha\beta} = \begin{Bmatrix} 0 & 0 & W_{xz} \\ 0 & 0 & W_{yz} \\ W_{zx} & W_{zy} & 0 \end{Bmatrix}$$

$$W_{3\alpha\beta} = \begin{Bmatrix} -W_{xy} & \frac{1}{2}(W_{xx}-W_{yy}) & 0 \\ \frac{1}{2}(W_{xx}-W_{yy}) & W_{xy} & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad W_{4\alpha\beta} = \begin{Bmatrix} 0 & 0 & -W_{yz} \\ 0 & 0 & W_{xz} \\ -W_{zy} & W_{zx} & 0 \end{Bmatrix}$$

It is easily shown that the following orthogonality relation holds:

$$W_{p\alpha\beta} W_{q\alpha\beta} = 0, \text{ when } \quad p \neq q. \tag{4.43}$$

The ion viscosity coefficients are

$$\left.\begin{aligned} \eta_0^i &= 0.96 n_i T_i \tau_i, \\ \eta_2^i &= n_i T_i \tau_i \left(\frac{6}{5} x^2 + 2.23\right)/\Delta, \\ \eta_4^i &= n_i T_i \tau_i x (x^2 + 2.38)/\Delta, \end{aligned}\right\} \tag{4.44}$$

where

$$x = \omega_i \tau_i, \quad \Delta = x^4 + 4.03x^2 + 2.33.$$

The coefficients η_1^i and η_3^i are obtained from η_2^i and η_4^i by replacing ω_i by $2\omega_i$:

$$\eta_1^i = \eta_2^i(2x), \quad \eta_3^i = \eta_4^i(2x).$$

The electron viscosity coefficients are (for Z = 1):

$$\left.\begin{aligned} \eta_0^e &= 0{,}733 n_e T_e \tau_e, \\ \eta_2^e &= n_e T_e \tau_e (2.05x^2 + 8.50)/\Delta, \quad \eta_1^e = \eta_2^e(2x), \\ \eta_4^e &= -n_e T_e \tau_e x (x^2 + 7.91)/\Delta, \quad \eta_3^e = \eta_4^e(2x), \end{aligned}\right\} \tag{4.45}$$

where

$$x = \omega_e \tau_e, \qquad \Delta = x^4 + 13.8x^2 + 11.6.$$

Symmetry of the Kinetic Coefficients. We now wish to review briefly certain terminology and results of the thermodynamics of irreversible processes (these are discussed in greater detail, for example, in [9]). The various agencies giving rise to deviations from thermal equilibrium X_m (for example ∇T, $W_{\alpha\beta}$, etc.) are called thermodynamic forces. These produce corresponding fluxes I_m (for example **q**, $\pi_{\alpha\beta}$, etc.). For small deviations from equilibrium the fluxes and forces are related linearly:

$$I_m = \sum_n L_{mn} X_n. \tag{4.46}$$

The irreversible increase of entropy in a nonequilibrium system is called entropy production and is denoted by θ. According to the second law of thermodynamics it is always true that θ > 0. A flux I_m and a force X_n are "conjugate" if the entropy production can be expressed in the form

$$\theta = \sum_m I_m X_m. \tag{4.47}$$

One of the important theorems in the thermodynamics of irreversible processes is the so-called principle of symmetry of the kinetic coefficients, or the Onsager principle. Assume that the fluxes and forces are chosen in such a way that Eq. (4.47) is satisfied. The kinetic coefficients relating these fluxes and forces then satisfy the condition

$$L_{mn}(\mathbf{B}) = L_{nm}(-\mathbf{B}), \tag{4.48}$$

if both forces X_m and X_n are even functions of the particle velocities (for example ∇T) or if both functions are odd in the particles velocities (for example $W_{\alpha\beta}$). However, if one force is odd and the other is even, then the following relation holds:

$$L_{mn}(\mathbf{B}) = -L_{nm}(-\mathbf{B}). \tag{4.48'}$$

The entropy balance for the electrons can be obtained easily through the use of the heat-balance equation (2.3e) and the equation of continuity (2.1e). The entropy per electron is

$$s_e = \frac{3}{2}\ln T_e - \ln n_e + \text{const}. \tag{4.49}$$

The entropy balance is written in the form

$$\frac{\partial n_e s_e}{\partial t} + \operatorname{div}\left(s_e n_e \mathbf{V}_e + \frac{\mathbf{q}_e}{T_e}\right) + \frac{Q_\Delta}{T_e} = \theta_e, \tag{4.50}$$

where θ_e is the entropy production per unit volume:

$$T_e\theta_e = -\mathbf{q}_e\nabla\ln T_e - \mathbf{R}\mathbf{u} - \frac{1}{2}\pi_{e\alpha\beta}W_{e\alpha\beta}. \tag{4.51}$$

The left side of Eq. (4.50) contains the change of entropy in time and the loss of entropy into other regions of space and to the ions.

It is evident from Eq. (4.51) that the fluxes $\mathbf{q}_e$, $\mathbf{R}$, $\pi_{\alpha\beta}$, and Q_Δ are conjugate to the forces $\nabla\ln T_e$, $\mathbf{u}$, $1/2\,W_{\alpha\beta}$, and $(T_e - T_i)/T_eT_i$.

Let us now examine the relations (4.30) - (4.33) between the fluxes and forces. Since **B** is an axial vector while **q**, **R**, ∇T, and **u** are polar vectors the coefficients α, β, and γ must be even functions of **B**. The Onsager principle then leads to the following nontrivial relation for the "cross" effects—the dependence of the thermal force and heat flux on the relative velocity:

$$T_e\beta_\parallel^{uT} = \beta_\parallel^{Tu}, \quad T_e\beta_\perp^{uT} = \beta_\perp^{Tu}, \quad T_e\beta_\wedge^{uT} = \beta_\wedge^{Tu}. \tag{4.52}$$

These relations are satisfied automatically when the transport coefficients are derived from the kinetic equation [cf. Eq. (4.36)]. The Onsager principle does not yield a nontrivial relation for the viscosity or for the ion transport coefficients.

Taking account of the symmetry of the transport coefficients and the orthogonality condition (4.43) we can write the entropy production in the form

$$T_e\theta_e = \frac{\varkappa_\parallel^e}{T_e}(\nabla_\parallel T_e)^2 + \frac{\varkappa_\perp^e}{T_e}(\nabla_\perp T_e)^2 + \frac{j_\parallel^2}{\sigma_\parallel} + \frac{j_\perp^2}{\sigma_\perp} + \\ + \frac{1}{2}\left\{\eta_0^e W_{0\alpha\beta}^2 + \eta_1^e W_{1\alpha\beta}^2 + \eta_2^e W_{2\alpha\beta}^2\right\}. \tag{4.53}$$

Similarly, the entropy production for the ions can be written

$$T_i\theta_i = \frac{\varkappa_\parallel^i}{T_i}(\nabla_\parallel T_i)^2 + \frac{\varkappa_\perp^i}{T_i}(\nabla_\perp T_i)^2 + \frac{1}{2}\sum_{p=0}^{2}\eta_p^i W_{p\alpha\beta}^2. \tag{4.54}$$

The entropy balance for the entire plasma is

$$\frac{\partial S}{\partial t} + \operatorname{div}\left\{s_e n_e \mathbf{V}_e + s_i n_i \mathbf{V}_i + \frac{q_e}{T_e} + \frac{q_i}{T_i}\right\} = \theta_e + \theta_i + \theta_{ei}, \tag{4.55}$$

where $S = s_e n_e + s_i n_i$ is the plasma entropy per unit volume:

$$\theta_{ei} = Q_\Delta\left(\frac{1}{T_i} - \frac{1}{T_e}\right) = \frac{3m_e}{m_i}\cdot\frac{n_e}{\tau_e}\cdot\frac{(T_e - T_i)^2}{T_eT_i}. \tag{4.56}$$

§ 5. Certain Paradoxes

The direct application of the transport equations to a magnetized plasma in which $\omega\tau \gg 1$ frequently leads to apparent contradictions with what might be expected from a cursory examination of individual particle motion in a magnetic field (drift theory). Some of these paradoxes have been analyzed in [6, 26, 27, 28] and a few particular cases are considered in this section.

Let us assume that the electric field and the gradients of all quantities are perpendicular to the magnetic field. We shall also be interested in processes that are so slow that the electron and ion inertia terms can be neglected; similarly, it is assumed that all quantities do not change greatly in the time between collisions. From these conditions and the equations of motion we can obtain explicit expressions for the transverse velocities of the ions and electrons in terms of the gradient. If the ion and electron equations of motion are added (neglecting viscosity and inertia) the plasma equilibrium relation can be written in the form

$$-\nabla(p_e + p_i) + \frac{1}{c}[\mathbf{jB}] = 0.$$

This expression then yields the transverse electric current

$$\mathbf{j}_\perp = -en_e(\mathbf{V}_e - \mathbf{V}_i)_\perp = \frac{c}{B}[\mathbf{h}\nabla(p_e + p_i)] \quad (5.1)$$

Substituting Eq. (5.1) in the expressions for the force **R**, (2.6) and (2.9), and invoking the condition $-e_e n_e = e_i n_i = en$, we have

$$\mathbf{V}_e = \frac{c}{B}[\mathbf{Eh}] - \frac{c}{enB}[\mathbf{h}\nabla p_e] + \mathbf{V}_D, \quad (5.2e)$$

$$\mathbf{V}_i = \frac{c}{B}[\mathbf{Eh}] + \frac{c}{enB}[\mathbf{h}\nabla p_i] + \mathbf{V}_D, \quad (5.2i)$$

where

$$\mathbf{V}_D = -\frac{c^2}{\sigma_\perp B^2}\left\{\nabla(p_e + p_i) - \frac{3}{2}n_e\nabla T_e\right\} =$$
$$= -\frac{m_e c^2}{e^2\tau_e B^2}\left\{(T_e + T_i)\frac{\nabla n}{n} + \nabla T_i - \frac{1}{2}\nabla T_e\right\}. \quad (5.3)$$

Taking account of viscosity would lead to the appearance of terms of order B^{-4}.

Now let us temporarily neglect the collision term in Eq. (5.2) $\mathbf{V}_D$, and compare the remaining terms with those that would be obtained from single-particle motion.

In a strong magnetic field the motion of a charged particle (without collisions) can be described as gyration around a circle whose center

(the so-called guiding center) moves with velocity $\mathbf{V}_c$ given by (cf. for example [8, 38])

$$\mathbf{V}_c = \frac{c}{B}[\mathbf{Eh}] + \frac{mv_\perp^2 c}{2eB}\left[\mathbf{h}\frac{\nabla B}{B}\right] + \frac{mv_\parallel^2 c}{eB}[\mathbf{h}(\mathbf{h}\nabla)\mathbf{h}] +$$
$$+ \frac{mv_\perp^2 c}{2eB}\mathbf{h}(\mathbf{h}\,\mathrm{rot}\,\mathbf{h}) + v_\parallel \mathbf{h}, \quad (5.4)$$

where $\mathbf{h} = \mathbf{B}/B$; $v_\parallel$ and $v_\perp$ are the projections of the particle velocity (averaged over the gyration) in the direction of the magnetic field and perpendicular to the magnetic field at the location of the guiding center. The first term in Eq. (5.4) is usually called the electric drift, the second term the magnetic drift, and the third the centrifugal drift. If Eq. (5.4) is averaged over a velocity distribution that is approximately Maxwellian we find

$$\langle \mathbf{V}_c \rangle = \frac{c}{B}[\mathbf{Eh}] + \frac{cT}{eB}\left[\mathbf{h}, \frac{\nabla B}{B} + (\mathbf{h}\nabla)\mathbf{h}\right] +$$
$$+ \frac{cT}{eB}\mathbf{h}(\mathbf{h}\cdot\mathrm{rot}\,\mathbf{h}) + V_\parallel \mathbf{h}. \quad (5.5)$$

Here $m\langle v_\parallel^2\rangle = T$; $m\langle v_\perp^2\rangle = 2T$; $\langle v_\parallel\rangle = V_\parallel = \mathbf{V}\cdot\mathbf{h}$.

The quantity $n\langle\mathbf{V}_c\rangle$ is the flux density of guiding centers, and $\int_S n\langle\mathbf{V}_c\rangle d\mathbf{S}$ is the flux of centers through a surface S, while $n\mathbf{V}$ and $\int_S n\mathbf{V}d\mathbf{S}$ represent the flux density and flux of the particles themselves. In general the particle flux can differ from the guiding-center flux and certain paradoxes arise when these two quantities are confused.

Let us compare Eqs. (5.2) and (5.5).

The first term in Eq. (5.2) is easily interpreted – it is the electric drift.

The second term in Eq. (5.2), which we shall call the Larmor term, is associated with the fact that particles intersecting an area in opposite directions arrive from regions characterized by different densities and temperatures, as a result of which the unidirectional fluxes do not balance each other. Particles arrive from a distance $\sim r = mvc/eB$ and "carry" a flux $\sim nv$ so that the resulting difference in flux is of order $(mc/eB)\nabla nv^2 \sim (c/eB)\nabla p$. At first glance the fact that Eq. (5.2) does not contain terms corresponding to the magnetic and centrifugal drifts, i. e., terms explicitly exhibiting the spatial derivatives of the magnetic field, appears to be paradoxical. Actually, however, the absence of such terms is completely natural since the magnetic field, whether it is uniform or not, does not disturb the Maxwellian distribution: $[\mathbf{v}\omega]\nabla_v f^0 = 0$. Hence, if the particle density and temperature are independent of coordinates the particle flux within the plasma (5.2) vanishes although the guiding-center flux (5.5) does not vanish if the magnetic field is inhomogeneous. In this case the magnetic and centrifugal drifts appear as edge effects which produce surface particle fluxes at the interface with the region of constant density and temperature. This is easily shown by simple examples but can also be shown in general form. Introducing the identities

$$\left.\begin{aligned}(\mathbf{h}\nabla)\,\mathbf{h} &= -[\mathbf{h}\,\mathrm{rot}\,\mathbf{h}],\\ [\mathbf{h}\cdot(\mathbf{h}\nabla)\,\mathbf{h}] &= \mathrm{rot}\,\mathbf{h} - \mathbf{h}\,(\mathbf{h}\cdot\mathrm{rot}\,\mathbf{h}),\end{aligned}\right\} \qquad (5.6)$$

we can write Eq. (5.5) as follows:

$$\langle \mathbf{V}_c \rangle = \frac{c}{B}\,[\mathbf{E}\mathbf{h}] + \frac{cT}{e}\,\mathrm{rot}\,\frac{\mathbf{h}}{B} + V_{\parallel}\,\mathbf{h}. \qquad (5.7)$$

Now, comparing the guiding-center flux with the particle flux (neglecting collisions) for particles of any sort we find

$$n\mathbf{V} = n\langle \mathbf{V}_c \rangle - \mathrm{rot}\left(\frac{cnT}{eB}\,\mathbf{h}\right) \qquad (5.8)$$

or

$$\int n\mathbf{V}\cdot d\mathbf{S} = \int n\langle \mathbf{V}_c \rangle \cdot d\mathbf{S} - \oint \frac{cnT}{eB}\,(\mathbf{h}\cdot d\mathbf{l}). \qquad (5.8')$$

It is evident that the difference between the particle flux and the guiding-center flux through any area as a whole is determined by the values of the quantities at the boundary of the area. This difference arises for the following reason: near the edge of the area in question particles enter and leave whose guiding centers are outside the area. The magnitude of the flux associated with these particles is cnT/eB per unit length of edge along **h**; opposite edges of the area are intersected in opposite directions.

Thus, a magnetized plasma can be regarded as consisting of "quasi-particles" – or "circlets" that move with the drift velocity. It is easily shown that the magnetic moment of a circlet is $\boldsymbol{\mu} = -(mv_\perp^2/2B)\,\mathbf{h}$; hence, if collisions are neglected the plasma magnetization per unit volume is

$$\mathbf{M} = -\sum_a n_a \frac{m_a \langle v_\perp^2 \rangle_a}{2B}\,\mathbf{h}. \qquad (5.9)$$

The total current density obtained in this representation is the sum of the drift (convection) current and the magnetization current:

$$\mathbf{j} = \mathbf{j}_c + c\,\mathrm{rot}\,\mathbf{M} = \sum_a e_a n_a \langle V_c \rangle_a - \mathrm{rot}\sum_a \frac{cn_a \langle v_\perp^2 \rangle_a}{2B}\,\mathbf{h}. \qquad (5.10)$$

When $m_a \langle v_\perp^2 \rangle_a = 2T_a$ Eq. (5.1) is obtained exactly. As is usually done in macroscopic electrodynamics, we can introduce $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ in addition to **B** and in this case Maxwell's equations are written in the form

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c}\,\mathbf{j}_c + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \mathrm{div}\,\mathbf{B} = 0. \qquad (5.11)$$

In practice, however, in plasma problems it is usually more convenient to write all currents in explicit form without separating the drift and magnetization currents. When all currents are written explicitly $\mathbf{B} \equiv \mathbf{H}$ and Maxwell's equations become

$$\mathrm{rot}\,\mathbf{B} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \mathrm{div}\,\mathbf{B} = 0. \qquad (5.11')$$

The $\mathbf{V}_D$ terms in Eq. (5.2) stem from particle collisions, specifically, from collisions of electrons with ions. These terms might well be called diffusion terms. They are exactly the same for the ions and electrons and depend only on the gradients of density and temperature but are independent of electric field. At first glance both of these results appear to be paradoxical for the following reason. The diffusion coefficient for diffusion of a charged particle across a magnetic field is approximately $D_\perp \sim r^2/\tau$. For ions, r and τ are larger by a factor of $(m_i/m_e)^{1/2}$ than for electrons. Hence it would appear that the ion diffusion coefficient should be $(m_i/m_e)^{1/2}$ times greater than the electron diffusion coefficient. Actually, however, this is not the case. For example, assume an ion-density gradient along the x axis and take the magnetic field along the z axis. There then arises a Larmor flux of ions along the y axis characterized by the velocity $V_y = (cT/enB)\,dn/dx$. In this case, however, the formula $D \sim r^2/\tau$ cannot be applied directly because the diffusion is taking place in a moving medium and in the collisions the ion obtains some momentum along the y axis (on the average). We then transform to a coordinate system in which $V_y = 0$. In this system there is an electric field $E'_x = (V_y/c)B = (T/e)\, d\ln n/dx$ in which the ions are described by a Boltzmann distribution and in which the ion flux is zero since the flux produced by the electric field compensates the diffusion flux. Thus, collisions between like particles can not produce diffusion across the magnetic field. On the other hand, collisions between electrons and ions do cause diffusion because the Larmor currents of electrons and ions are in opposite directions. The flux along the x axis that results can be regarded as coming from the drift produced by the effect of a frictional force between the electrons and ions which is along the y axis. Since $R_i = -R_e$ the velocity is exactly the same for both kinds of particles.

Now let us consider the role of the electric field. Assume an electric field along the x axis. This field produces a particle drift (both charge signs) along the y axis with velocity $V_y = -cE/B$. In the coordinate system in which $V_y = 0$, however, the electric field $E' = 0$; hence there is no flux along the x axis, that is to say, there is no flux in the direction of the applied electric field. In this connection it is sometimes said (erroneously) that the plasma conductivity across the magnetic field is zero.

Now let us consider the heat transport equation. The quantity $v^2_\perp/B$ is conserved for a magnetic field that changes slowly in time, i.e., the energy associated with the transverse motion, $\varepsilon_\perp = mv^2_\perp/2$, is proportional to the field—this is the so-called betatron effect. The heat transport equation does not contain a term proportional to $\partial B/\partial t$. Nevertheless, using some simple examples we can easily show that the equation does take account of the betatron effect.

Let us consider a uniform magnetic field along the z axis that increases in time. Let the plasma occupy a cylindrical volume of infinite length along the z axis. We assume that the plasma density and temperature are constant over the volume (so there is no heat flux); also, for reasons of simplicity we neglect collisions of electrons with ions and the consequent Joule heating. We can also neglect the screening of the external magnetic field by the plasma currents. Under these conditions the induction electric field $E = E_\varphi = -\dot{B}r/2c$. The electric drift leads to plasma compression at a rate $V_r = -\dot{B}r/2B$ so that $\operatorname{div}\mathbf{V} = -\dot{B}/B$. The heat transport equation becomes

$$\frac{3}{2} n \frac{dT}{dt} = -nT \operatorname{div} \mathbf{V} = \frac{nT}{B} \frac{dB}{dt}. \qquad (5.12)$$

This expression is a statement of the betatron effect; actually in betatron heating it is only the energy associated with the transverse motion $d\varepsilon = \varepsilon_\perp dB/B$ that increases directly. Collisions then establish an equipartition of energy over the degrees of freedom so that $\varepsilon_\perp = (2/3)\varepsilon$ and $(3/2)\, d\varepsilon/dt = (\varepsilon/B)\, dB/dt$. The betatron effect then appears as heating by virtue of adiabatic compression of the plasma. This reversible (in the thermodynamic sense) heating should not be confused with the irreversible gyrorelaxational heating mentioned in §3 which arises as a consequence of an irreversible process: the equipartition of energy over the degrees of freedom. When the magnetic field is reduced to its original value (with the corresponding expansion of the plasma) the adiabatic cooling associated with the expansion is equal to the heating that took place in compression, in accordance with Eq. (5.12); on the other hand, the heat generated in gyrorelaxational heating remains in the plasma since it is proportional to $(\dot{B}/B)^2$.

We now consider the case in which electron-ion collisions precisely equilibrate the electric drift: $cE_\varphi/B + V_D = 0$; in this case the plasma remains immobile. Under these conditions heat fluxes will arise in the plasma and we must consider the total increase in energy over the entire plasma volume. If the ion and electron heat transport equations are added and integrated over the volume of the plasma cylinder (unit length along the z axis) we find

$$\frac{3}{2} \cdot \frac{d}{dt} \int (n_e T_e + n_i T_i)\, 2\pi r\, dr = \int E_\varphi j_\varphi 2\pi r\, dr.$$

Substituting $E_\varphi = -\dot{B}r/2c$, $j_\varphi = (c/B)\,\partial p/\partial r$, $p = n_e T_e + n_i T_i$, and integrating on the right by parts we have $(3/2)\, d\varepsilon/dt = (\varepsilon/B)\, dB/dt$. In this case the betatron effect appears as the generation of Joule heat.

There is at least one difference between the two examples we have just considered. In the first case the ions and electrons are heated uniformly by the compression. In the second case the heat is generated directly in the electron gas and is then transferred to the ions by means of collisions. It would appear that the ions should obtain as much heat as the electrons in betatron heating. In the absence of ion current a radial electric field arises in the plasma; the magnitude of this field is determined by the ion equilibrium condition $e_i n_i E_r = -\partial p_i/\partial r$. This field causes an ion drift in the azimuthal direction in opposition to the induced electric field. It is easy to show that the work done by this field (negative) on the drift precisely compensates the betatron heating of the ions. The electrons also drift in the radial field and acquire exactly as much energy as is lost by the ions.

The transport of heat (like the transport of particles) can also be interpreted in terms of the motion of guiding centers. If collisions are neglected a formula analogous to Eq. (5.8) is obtained. The total flux density of internal energy is $\mathbf{q}_{total} = (5/2)\, nT\mathbf{V} + \mathbf{q}$, as follows from Eq. (1.20). We use Eq. (5.8) taking account of the $\mathbf{q}_\wedge$ term in the heat flux, which is independent of collisions; this term is given by Eqs. (2.11) and (2.14): $\mathbf{q}_\wedge = (5/2)(cnT/eB)[\mathbf{h}\nabla T]$. The expression for $\mathbf{q}_{total}$ can then be written

$$\mathbf{q}_{total} = \frac{5}{2}\, nT\mathbf{V} + \mathbf{q}_\wedge = \frac{5}{2}\, nT < V_c > - \operatorname{rot}\left(\frac{5}{2}\frac{cnT^2}{eB}\mathbf{h}\right). \tag{5.13}$$

The remarks made above in connection with the derivation of Eq. (5.8) also apply to the derivation of this formula.

§ 6. Hydrodynamic Description of a Plasma

The transport equations correspond to a plasma model consisting of interpenetrating charged gases—the ion gas (one or more species) and the electron gas. It is frequently more convenient to use a single-fluid model for the plasma. In this case, the two equations of motion (for the ions and electrons) are replaced by a single equation of motion for the plasma as a whole; this equation represents an extension of the equation of motion of conventional hydrodynamics while the expression for the electric current is essentially a generalization of the familiar Ohm's law. The single-fluid hydrodynamic model is found to be most useful for the description of low-frequency phenomena because it is then valid to neglect electron inertia and to assume that the plasma remains neutral.

We shall first treat the single-fluid gas dynamic model for a simple plasma; in §7 we discuss certain characteristic features of multicomponent plasmas.

Equations of Continuity and Quasi-Neutrality. We first introduce the mass density ρ and the hydrodynamic velocity $\mathbf{V}$ (the velocity of the mass):

$$\varrho = \sum_a m_a n_a, \tag{6.1}$$

$$\mathbf{V} = \frac{1}{\varrho}\sum_a m_a n_a \mathbf{V}_a. \tag{6.2}$$

Neglecting the electron mass compared with the ion mass we can now make the approximation

$$\varrho = m_i n_i, \tag{6.3}$$

$$\mathbf{V} = \mathbf{V}_i. \tag{6.4}$$

The equation of continuity for the ions is rewritten in the form of a mass conservation relation (it is simply called the equation of continuity):

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{V}) = 0. \tag{6.5}$$

Equation (6.5) also holds for the exact definitions (6.1) and (6.2).

The density of electrical charge ρ_e and the density of electric current $\mathbf{j}$ are (we use the notation $\mathbf{u} = \mathbf{V}_e - \mathbf{V}_i$):

$$\varrho_e = \sum_a e_a n_a = e\,(Zn_i - n_e), \tag{6.6}$$

$$\mathbf{j} = \sum_a e_a n_a \mathbf{V}_a = \varrho_e \mathbf{V}_i - en_e\mathbf{u}. \tag{6.7}$$

The equations of continuity for the electrons and ions yield an equation for the conservation of electric charge:

$$\frac{\partial \varrho_e}{\partial t} + \operatorname{div}\mathbf{j} = 0. \tag{6.8}$$

We shall assume hereinafter that the plasma is quasi-neutral. This does not mean zero space charge in the plasma; rather it means that the space charge is small compared with the quantity en_e so that the difference $Zn_i - n_e$ can be neglected compared with $n = n_e$. The current density is then expressed in the form

$$\mathbf{j} = -en\mathbf{u}. \tag{6.9}$$

We will assume that all processes are slow (in electrodynamics these are called quasi-stationary processes) so that $\partial\rho_e/\partial t$ can be neglected in Eq. (6.8) and the displacement current can be neglected in Maxwell's equations. Under these conditions Eq. (6.8) and Maxwell's equations become

$$\operatorname{div} \mathbf{j} = 0, \tag{6.10}$$

$$\operatorname{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial B}{\partial t}, \tag{6.11}$$

$$\operatorname{rot} \mathbf{B} = \frac{4\pi}{c} \mathbf{j}; \ \operatorname{div} \mathbf{B} = 0. \tag{6.12}$$

The neutrality condition provides one relation for the quantities that describe the plasma: $Zn_i = n_e$. Consequently one equation must be omitted from the system of plasma equations; specifically, this is the Poisson equation, in which the space charge appears explicitly:

$$\operatorname{div} \mathbf{E} = 4\pi \varrho_e. \tag{6.13}$$

The quantity $\rho_e = e(Zn_i - n_e)$ is neglected in the remaining equations and no requirement is imposed on div **E**. In this case the rotational electric fields are determined from Eq. (6.11) while the irrotational fields arising from the small differences in positive and negative space charge (although the fields themselves are not small !) are determined from Eq. (6.10) in conjunction with the equations of motion. In other words, the irrotational electric fields in the plasma are automatically chosen in such a way as to avoid too strong a charge separation $Zn_i - n_e$. In this case Poisson's equation only serves as a means of determining ρ_e once the field **E** is known.

It will now be useful to estimate certain quantities. In order-of-magnitude terms, we find from Eq. (6.13) that $\rho_e \sim E/4\pi L$. In a static or slowly moving plasma or in a plasma with no magnetic field we usually find $enE \sim \nabla p$ or $E \sim T/eL$, whence $\rho_e/en \sim \delta_D^2/L^2$ where $\delta_D = (T/4\pi en)^{1/2}$ is the Debye length; the Debye length is always small compared with the characteristic dimensions of the plasma (if this condition is not satisfied the ionized gas cannot properly be called a plasma). The neutrality condition can obviously be violated in layers of thickness $\sim \delta_D$. These departures from neutrality usually occur near the boundaries of the plasma or in high-frequency oscillations. It is also possible for an induction field $E \sim VB/c$ to arise in a plasma moving across a magnetic field. Let us assume that we are dealing with a fast process (see below) in which the velocity of a plasma is determined by its inertia and by the magnetic force. Then $\omega\rho V \sim (1/c)jB$ where ω is the characteristic frequency of the process. In this case the charge density is of order $\rho_e \sim E/4\pi L \sim (c_A^2/c^2)j/\omega L$ where $c_A = B/(4\pi\rho)^{1/2}$ is the so-called Alfvén velocity and the first term in

Eq. (6.8) is of order $\omega\rho_e/(j/L) \sim c_A^2/c^2$ so that Eq. (6.10) can be used if $c_A^2/c^2 = B^2/4\pi\rho c^2$ is small. It will be assumed that this condition is satisfied everywhere below. For example, take $n_i = 10^{14}$ cm^{-3}, $m_i = 1.6 \cdot 10^{-24}$ g, $B = 10^4$ G, in which case $c_A^2/c^2 \sim 10^{-4}$.

The neutrality condition can also be violated in a low density plasma and in a relativistic plasma, in which $V \sim c$ or $u \sim c$.

Equation of Motion. Adding the ion and electron equations of motion and neglecting the electron inertia, we obtain the plasma equation of motion

$$\varrho \frac{d\mathbf{V}}{dt} = -\nabla p + \frac{1}{c}[\mathbf{jB}] + \mathbf{F}, \tag{6.14}$$

where $\mathbf{V} = \mathbf{V}_i$ $d/dt = \partial/\partial t + (\mathbf{V}\nabla)$; p is the total pressure,

$$p = p_e + p_i. \tag{6.14'}$$

In a magnetized plasma under laboratory conditions the principal forces are the pressure gradient and the magnetic force. The term **F** represents the sum of the remaining forces acting on a unit volume of plasma. These include the following: the viscous force $F_\alpha^\pi = -\partial\pi_{\alpha\beta}/\partial x_\beta$, where $\pi_{\alpha\beta} = \pi_{i\alpha\beta} + \pi_{e\alpha\beta} \approx \pi_{i\alpha\beta}$ is the stress tensor; the gravitational force $\mathbf{F}_g = \rho\mathbf{g}$, which is important in many astrophysical problems, where **g** is the gravitational acceleration. The electric force $\mathbf{F}_E = \rho_e\mathbf{E}$ is usually very small compared with the others.

For greater clarity the magnetic force is frequently expressed in the terms of the Maxwell stress tensor

$$F_\alpha^B = \frac{1}{c}[\mathbf{jB}]_\alpha = \frac{\partial T_{\alpha\beta}^B}{\partial x_\beta}, \ T_{\alpha\beta}^B = \frac{1}{4\pi}\left(B_\alpha B_\beta - \frac{1}{2}B^2\delta_{\alpha\beta}\right). \tag{6.15}$$

This expression is easily obtained from Eq. (6.12). The tensor $T_{\alpha\beta}^B$ corresponds to the pressure $B^2/8\pi$ across the magnetic lines and the tension along the lines, i.e., this tensor gives the isotropic pressure $B^2/8\pi$ and the longitudinal tension $B^2/4\pi$. For example, if there is a tangential field **B** at the plasma boundary and this field is shielded by currents in the surface layer the field pressure $B^2/8\pi$ is transferred to the plasma which shields it. The field normal to the plasma surface cannot transfer tension to the plasma since the magnetic lines of force cannot be cut off (div **B** = 0) but continue into the plasma. The tension of the magnetic lines can be transferred to the plasma if currents flow in the plasma in such a way as to distort the lines of force. A "straightening" force is produced under these conditions; this force is again across the field lines. If **n** is the principal normal to the line of force and R the radius of curvature,

$$\mathbf{F}_B = -\nabla \frac{B^2}{8\pi} - \frac{B^2}{4\pi} \cdot \frac{\mathbf{n}}{R}. \qquad (6.15')$$

A magnetic pressure $B^2/8\pi$ equal to 1 kg/cm^2 requires a field $B = 5 \cdot 10^3$G.

The electrical force can also be represented in terms of Maxwell stresses. If rot **E** = 0 we have from Eq. (6.13): *

$$F_\alpha^E = \varrho_e E_\alpha = \frac{\partial T_{\alpha\beta}^E}{\partial x_\beta}, \quad T_{\alpha\beta}^E = \frac{1}{4\pi}\left(E_\alpha E_\beta - \frac{1}{2} E^2 \delta_{\alpha\beta}\right). \qquad (6.16)$$

Only the normal component of the electric field can change sharply near the plasma boundary. In this case the lines of force terminate on charges close to the surface and the plasma is not subject to pressure that acts to contain it, as in the case of a magnetic field, but rather is subject to a tension (negative pressure) $E^2/8\pi$. A tension of 1 kg/cm^2 corresponds to $E = 1.5 \cdot 10^6$ V/cm.

If $enE \sim \nabla p$ we find $|\nabla E^2/8\pi| \,/\, |\nabla p| \sim \delta_D^2/L^2$ so that the force F_E can only be large in thin layers. If $E \sim VB/c$ then $E^2/B^2 \sim V^2/c^2$. It is then obvious that the electrical forces in a relativistic plasma can be of the same order as the magnetic forces.

The effect of inertia can be used to classify plasma phenomena as fast or slow. In fast phenomena the inertia term in Eq. (6.14) is of the same order as the other terms—these are phenomena characterized by relatively high frequencies [of order c_S/L or c_A/L where $c_S \sim (p/\rho)^{1/2}$ and

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*In the general case, taking account of the displacement current and rot **E** ≠ 0 we find

$$F_\alpha^B + F_\alpha^E = \frac{1}{c}[jB]_\alpha + \varrho_e E_\alpha = \frac{\partial T_{\alpha\beta}^B}{\partial x_\beta} + \frac{\partial T_{\alpha\beta}^E}{\partial x_\beta} - \frac{1}{c^2} \cdot \frac{\partial S_\alpha}{\partial t},$$

where $\mathbf{S} = (c/4\pi)[\mathbf{EB}]$ is the Poynting vector and $\mathbf{S}/c^2$ is the electromagnetic momentum density.

$c_A = (B^2/4\pi\rho)^{1/2}$]. In slow phenomena the inertia term can be neglected as a first approximation—either the plasma is at rest or it moves so slowly that the forces acting upon it are approximately in equilibrium at all times. Fast phenomena include various short-lived and transient processes; for a long-lived plasma magnetohydrodynamic waves are regarded as fast phenomena. Slow phenomena are those in which equilibrium is established in characteristic times appreciably greater than L/c_S. A typical example is the compression of a plasma by a rapidly applied magnetic field—either externally produced or produced by current flowing through the plasma (fast pinch). The characteristic compression velocity in such cases is of order c_A.

To analyze slow phenomena we need retain only the principal terms in Eq. (6.14); thus,

$$\nabla p = \frac{1}{c}[\mathbf{jB}]. \qquad (6.17)$$

This equation, together with Eq. (6.12), defines so-called equilibrium magnetohydrodynamic configurations. It is evident from Eq. (6.17) that $\mathbf{B}\nabla p = 0$ and $\mathbf{j}\nabla p = 0$. Consequently, the magnetic lines of force and the current flow lines lie on surfaces of constant pressure which are called magnetic surfaces. A plasma confined by a magnetic field (equilibrium configuration) can be regarded as a series of magnetic surfaces nested within each other.

An important dimensionless parameter that characterizes the effectiveness of a magnetic field for plasma containment is the ratio $8\pi p/B^2$, where p and B are the pressure and field, respectively. In actual laboratory devices used to study slow phenomena this quantity is generally much smaller than unity since the plasma usually loses heat very easily.

Our classification of phenomena as being fast or slow is obviously an arbitrary one and does not exhaust all possibilities. In this review, however, we shall only be interested in making rough estimates for purposes of orientation rather than in the detailed classification of plasma phenomena. In actual problems a more detailed analysis would obviously be required.

Ohm's Law. As it is conventionally stated Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ relates the current density to the electric field at a given instant of time. However, the electric field is actually responsible for the acceleration of electrons rather than their velocity so that in the general case no such relation obtains. In processes in which all quantities vary slowly in time (no significant changes in one electron-ion collision time) the electron inertia is unimportant and the effect of the electric field is balanced by friction due to collisions of electrons with ions, $\mathbf{u} = -\mathbf{j}/en$. This equilibrium condition for an electron gas is called Ohm's law. An Ohm's law can be derived for a plasma in similar fashion.

It is convenient to express the electric field in terms of the current rather than vice versa. In addition to simplifying the formulas, this method of description gives a more accurate picture of the qualitative nature of the effect: in a highly conducting plasma, where the reactance is greater than the real resistance, the current is usually determined by the external conditions while the electric field is determined from the current by Ohm's law.

Neglecting the electron inertia and viscosity in the equations of motion and using Eqs. (2.6) and (2.9) or Eqs. (4.30) and (4.31) for the force $\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T$ we have

$$\mathbf{E}' = \frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp} + \frac{1}{en_e c}[\mathbf{jB}], \qquad (6.18)$$

where $\mathbf{E}'$ is the effective field, given by

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}[\mathbf{VB}] + \frac{1}{en_e}(\nabla p_e - \mathbf{R}_T). \qquad (6.19)$$

If $\mathbf{j}$ is expressed in terms of $\mathbf{E}'$ then

$$\mathbf{j} = \sigma_\parallel \mathbf{E}'_\parallel + \frac{\sigma_\perp}{1+\omega_e^2\tau_e^2}\{\mathbf{E}'_\perp + \omega_e\tau_e[\mathbf{hE}']\}. \qquad (6.20)$$

The quantity $\mathbf{E}'$ contains the electric field $\mathbf{E}^*$ in the coordinate system moving with the matter (with the ions):

$$\mathbf{E}^* = \mathbf{E} + \frac{1}{c}[\mathbf{VB}]. \qquad (6.21)$$

Furthermore, $\mathbf{E}'$ contains the thermoelectric force $-\mathbf{R}_T/en_e$ and the electron pressure term $\nabla p_e/en_e$. The latter is not important in ordinary metal conductors because the electron pressure is uniform. In a plasma, however, the electron pressure can vary sharply and this term can be of great importance.

If there is no magnetic field $\mathbf{E}' = \mathbf{j}/\sigma_\parallel$. In a strong magnetic field ($\omega_e\tau_e \gg 1$) the same relation holds for the components along the field

$$E'_\parallel = j_\parallel/\sigma_\parallel, \qquad (6.22a)$$

but the transverse components are modified significantly (Fig. 6). The effective field $\mathbf{E}'_\perp$ is essentially perpendicular to the current $\mathbf{j}_\perp$. The projection of the field $\mathbf{E}'_\perp$ on the current is related to $\mathbf{j}_\perp$ by

$$(E'_\perp)_j = j_\perp/\sigma_\perp, \qquad (6.22b)$$

and is not very different from Eq. (6.22a). The magnetic field does not have much effect on the friction produced by electron-ion collisions. For example $\sigma_\perp \approx \sigma_\parallel/2$ when $Z = 1$. However, flow of current across the magnetic field requires a component $\mathbf{E}'$ perpendicular to both it and the magnetic field, the so-called Hall field. This field equilibrates the force acting on the electron $(1/c)[\mathbf{jB}]$ and is given by

$$\mathbf{E}'_{\text{Hall}} = \frac{1}{en_e c}[\mathbf{jB}] = \frac{\omega_e\tau_e}{\sigma_\perp}[\mathbf{jh}]. \qquad (6.22c)$$

Frequently $\mathbf{E}'_{\text{Hall}}$ arises automatically in a plasma as a consequence of the small charge separation allowed within the framework of quasineutrality, while the external field, which must be applied to the plasma, is determined by Eqs. (6.22a) and (6.22b). In this connection it is sometimes said that the magnetic field does not affect the conductivity of the plasma. This statement is to be understood within the context we have indicated here.

Ohm's law for a plasma can be written in several equivalent forms. Frequently it is convenient to replace the electron equation of motion by the ion equation of motion and to introduce a new effective field $\mathbf{E}''$ defined by

$$\mathbf{E}'' = \mathbf{E} + \frac{1}{c}[\mathbf{VB}] - \frac{1}{en_e}(\nabla p_i + \mathbf{R}_T) - \frac{m_i}{Ze}\frac{d\mathbf{V}}{dt} + \frac{1}{en_e}\mathbf{F}. \qquad (6.23)$$

In this case the resulting expression does not contain the Hall term:

$$\mathbf{E}'' = \frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp}. \qquad (6.24)$$

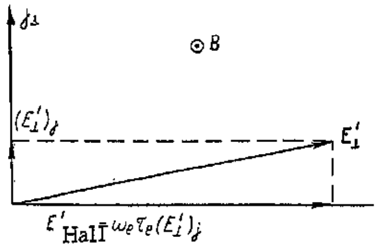

Fig. 6

This expression is also obtained if the $(1/c)$[**jB**] term is eliminated from Eq. (6.18) by means of the equation of motion (6.14). Schlüter has proposed that Ohm's law for a plasma should be written in the form given in (6.24) without the Hall term [24]. This form of the equation is especially convenient in cases in which $d\mathbf{V}/dt$ can be determined easily, in particular, in slow phenomena where this term is small and can be neglected to first approximation.

Although Eqs. (6.18) and (6.24) appear to be different they are actually the same when the equation of motion is taken into account. A peculiar inversion of the equation occurs in a magnetized plasma confined by a magnetic field. The transverse component of the current can be determined from Eq. (6.14). If viscosity is neglected this quantity is

$$\mathbf{j}_\perp = \frac{c}{B}[\mathbf{h}\nabla p]. \tag{6.25}$$

On the other hand, Ohm's law determines the plasma velocity across the magnetic field. Substituting Eq. (6.25) in Eq. (6.24) and omitting $\partial\pi_{\alpha\beta}/\partial x_\beta$ and $d\mathbf{V}/dt$ (neglecting $\mathbf{R}_T$ for simplicity), we find

$$\mathbf{V}_\perp = \frac{c}{B}\left[\left(\mathbf{E} - \frac{\nabla p}{en_e}\right)\mathbf{h}\right] - \frac{c^2}{\sigma_\perp B^2}\nabla_\perp p. \tag{6.26}$$

The second term in the square brackets in Eq. (6.26) is called the rate of diffusion across the magnetic field. We note that according to the generally accepted terminology diffusion means the relative motion of different components of a complex plasma; actually Eq. (6.26) expresses the hydrodynamic velocity of the plasma. For this reason it appears to be desirable to use a different terminology, the "leak" rate. If the current (6.25) is maintained by induction, which is expressed by the term $(1/c)$[**VB**], the

leak rate of the plasma across the field is given precisely by the second term in Eq. (6.26), which is proportional to the transverse resistance of the plasma $1/\sigma_\perp$.

The order of magnitude of the current and velocity $u = -j/en$ are determined by the equation of motion. According to Eq. (6.12), $B'/L \sim 4\pi j/c$ where B' is the field produced by the currents in the plasma. For fast phenomena, from $\rho V^2 \sim B'^2/4\pi$ we find $u^2/V^2 \sim 1/\Pi$; for slow phenomena from $p \sim B'^2/4\pi$ we find $u^2/c_S^2 \sim 1/\Pi$ where

$$\Pi = \frac{4\pi e^2 n_e^2 L^2}{\varrho c^2} = \frac{4\pi Z e^2 n_e L^2}{m_i c^2}. \tag{6.27}$$

The dimensionless number Π is proportional to the number of particles nL^2 per unit length of the system.

Thus, in a system containing a large number of particles the electron velocity is "tied" to the ion velocity by the self-consistent magnetic field as well as collisions.

The characteristic frequencies of fast phenomena are of order $c_A/L \sim \omega_i/\Pi^{1/2}$. If $\Pi \gg 1$ these frequencies are small compared with the ion cyclotron frequency.

Estimating the order of magnitude of the various terms in Ohm's law for fast phenomena with $\Pi \gg 1$, $\omega \ll \omega_i$, $L \gg r_i$ we find that the principal terms in the effective field are $\mathbf{E} + (1/c)$[**VB**]. When collisions can be neglected we can also neglect $\mathbf{j}/\sigma$, and to a first approximation Ohm's law can be written in the form

$$\mathbf{E} + \frac{1}{c}[\mathbf{VB}] = 0. \tag{6.28}$$

Eliminating the electric field by means of Eq. (6.28) we obtain the induction equation in the ideal magnetohydrodynamics approximation:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{rot}\,[\mathbf{VB}]. \tag{6.29}$$

This equation allows a simple interpretation: the magnetic lines of force behave as though tied to the matter and move with the matter at a velocity **V**. Equation (6.29) is frequently used in the analysis of plasma oscillations and plasma stability.

When all terms are retained the induction equation obtained from Eq. (6.18) becomes rather complicated:*

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,[\mathbf{VB}] - \text{rot}\left[\frac{\mathbf{j}}{en_e}\mathbf{B}\right] - \frac{c}{en}[\nabla n \nabla T] -$$

$$-\frac{c}{e}\,\text{rot}\,\frac{\mathbf{R}_T}{n_e} - c\,\text{rot}\left(\frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp}\right). \qquad (6.30)$$

In place of this equation one frequently makes use of the following simplified equation with isotropic uniform conductivity

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,[\mathbf{VB}] + \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B}. \qquad (6.31)$$

Here we have used Eq. (6.12) and the relation rot rot $\mathbf{B} = -\nabla^2\mathbf{B}$. This equation describes the skin effect; it takes account of the fact that the lines of force are not completely entrained but move through the matter with a diffusion coefficient D_m given by

$$D_m = \frac{c^2}{4\pi\sigma}. \qquad (6.32)$$

*Eliminating [jB] from Eq. (6.30) by means of the equation of motion or by using the relation (6.24), by means of the transformation rot $d\mathbf{V}/dt = \partial\,\text{rot}\,\mathbf{V}/\partial t - \text{rot}\,[\mathbf{V}\,\text{rot}\,\mathbf{V}]$ we can write the induction equation in the form

$$\frac{\partial}{\partial t}\left(\mathbf{B} + \frac{m_i c}{Ze}\text{rot}\,\mathbf{V}\right) = \text{rot}\left[\mathbf{V}\left(\mathbf{B} + \frac{m_i c}{Ze}\text{rot}\,\mathbf{V}\right)\right] +$$

$$+\frac{c}{Zen}[\nabla n \nabla T_i] - (c/en_e)\,\text{rot}\left(\mathbf{F}_i^\pi + \mathbf{R}_T\right) - c\,\text{rot}\,(\mathbf{j}_\parallel/\sigma_\parallel + \mathbf{j}_\perp/\sigma_\perp), \qquad (6.30')$$

where

$$F_{i\alpha}^\pi = -\,\partial\pi_{i\alpha\beta}/\partial x_\beta.$$

If the last three terms on the right can be neglected the lines $\mathbf{B} + (m_i c/e_i)$ rot $\mathbf{V}$ are tied to the matter; if, however, $\Pi \gg 1$, $\omega \ll \omega_i$ we find $|\text{rot}\,\mathbf{V}| \ll e_i B/m_i c$ and it may be assumed that the lines $\mathbf{B}$ are tied to the matter. Thus, when $B = 0$ we obtain the familiar theorem on attachment of the circulation lines of conventional hydrodynamics [3].

The same phenomenon, penetration of plasma through the magnetic field, is described by the last term in Eq. (6.26). When $p \sim B^2/4\pi$ the rate of penetration is of order D_m/L. It is evident from Eq. (6.26) that the rate of penetration is of order $(4\pi p/B^2)\,D_m/L$ when $p \ll B^2/4\pi$. This same result is obviously obtained from Eq. (6.32) since the relative magnitude of the field gradient is of order $B'/LB \sim 4\pi p/B^2 L$ where B' is the difference in fields outside and inside the plasma as determined by the plasma equilibrium condition $BB'/4\pi \sim p$; thus, the rate of penetration of the field is $\sim D_m B'/LB \sim (4\pi p/B^2)\,D_m/L$.

If the effective value of $\sigma_\perp$ is reduced as a result of turbulence Eq. (6.26) shows that the penetration of the plasma through the magnetic field is much more rapid; this is the so-called "anomalous diffusion" effect.

Equations of Energy and Heat Transport. When the electron and ion temperatures are very different the individual heat-balance equations given in the preceding sections are used. However, if the thermal coupling between the electrons and ions is strong, the relative temperature difference is small $|T_e - T_i| \ll T$ and one can use the combined equation, writing $T_e = T_i = T$.

The temperatures of the electron and ion fluids can be very different if much more heat is generated in one than in the other. For example, Joule heat is generated in the electron gas. If Q_{Joule} is equated roughly with the heat transferred to the ions Q_Δ we can obtain the difference of temperatures responsible for this transfer. A rough estimate yields

$$\frac{(T_e - T_i)_{\text{Joule}}}{T} \sim \frac{j^2 m_i}{e^2 n^2 T} \sim \frac{u^2}{v_i^2} \quad \text{or} \quad \frac{(T_e - T_i)_{\text{Joule}}}{T} \sim \frac{1}{\Pi}\frac{B'^2}{4\pi p}.$$

Here we have used the relation $(4\pi/c)\,j \sim B'/L$ where the magnetic field B' is produced by the currents in the plasma. In slow phenomena, in which the main source of heat is Joule heating, the electron temperature tends to exceed the ion temperature.

On the other hand, heat generation by virtue of viscosity effects occurs primarily in the ion gas since the ion viscosity is much greater than the electron viscosity. Equating roughly the quantities $Q_{\text{vis}} \sim \eta V^2/L^2 \sim Q_\Delta$ we obtain the appropriate temperature difference. If the "longitudinal" viscosity η_0 is important

$$\frac{(T_i - T_e)_{\text{vis}}}{T} \sim \frac{m_i}{m_e}\frac{V^2\tau_i\tau_e}{L_\parallel^2} \sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{V\tau_i}{L_\parallel}\right)^2 \sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{l}{L_\parallel}\frac{V}{v_i}\right)^2,$$

and if the "transverse" viscosity is important $\eta_{1,2} \sim \eta_0/\omega_i^2\tau_i^2$

$$\frac{(T_i - T_e)_{\text{vis}}}{T} \sim \frac{m_i\tau_e}{m_e\tau_i}\left(\frac{m_iVc}{eBL_\perp}\right)^2 \sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{m_iVc}{eBL_\perp}\right)^2 \sim$$
$$\sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{r_i}{L_\perp}\frac{V}{v_i}\right)^2.$$

The velocity of strong shock waves in a plasma is greater than the ion thermal velocity but usually smaller than the electron thermal velocity; hence strong heating due to shock waves is experienced only by the ions. When the primary source of heat is the dissipation of energy associated with plasma motion (shock waves, intense plasma oscillations resulting from instabilities, etc.) most of the heat is generated in the ion gas and the ion temperature can be higher than the electron temperature.

If $T_e = T_i = T$ (to the required accuracy) the individual energy equations for the electrons and ions can conveniently be replaced by a general energy balance equation. Combining the energy transport equations for the ions and electrons (1.20), taking account of Eq. (1.24), and neglecting electron inertia, we obtain the plasma energy transport equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho V^2 + \frac{3}{2}p\right) + \frac{\partial}{\partial x_\beta}\left\{\left(\frac{1}{2}\varrho V^2 + \frac{5}{2}p\right)V_\beta + \right.$$
$$\left. + \pi_{\alpha\beta}V_\alpha + q_\beta\right\} = \mathbf{E}\mathbf{j}. \tag{6.33}$$

Here $p = p_e + p_i$,

$$\mathbf{q} = \mathbf{q}_e + \mathbf{q}_i + \frac{5}{2}p_e\mathbf{u}. \tag{6.33'}$$

On the other hand, Maxwell's equations (6.11) and (6.12) provide an energy relation for the field (the Poynting theorem in the approximation $E \ll B$):

$$\frac{\partial}{\partial t}\frac{B^2}{8\pi} + \operatorname{div}\mathbf{S} = -\mathbf{E}\mathbf{j}, \tag{6.34}$$

where $\mathbf{S} = (c/4\pi)[\mathbf{EB}]$ is the Poynting vector.

Combining Eq. (6.33) and Eq. (6.34) we obtain a conservation law for the total energy:

$$\frac{\partial \varepsilon_{\text{total}}}{\partial t} + \operatorname{div}\mathbf{q}_{\text{total}} = 0,$$
$$\varepsilon_{\text{total}} = \frac{1}{2}\varrho V^2 + \frac{3}{2}p + \frac{B^2}{8\pi}, \tag{6.35}$$
$$q_\beta^{\text{total}} = \left(\frac{1}{2}\varrho V^2 + \frac{5}{2}p\right)V_\beta + \pi_{\alpha\beta}V_\alpha + q_\beta + S_\beta.$$

Adding Eq. (1.23) for the electrons and ions, taking account of Eq. (1.24) and the relation $\operatorname{div}(n_e\mathbf{u}) = 0$, or eliminating the kinetic energy from Eq. (6.33) and using Ohm's law (6.18), we obtain the plasma heat balance equation

$$\frac{3}{2}\frac{\partial p}{\partial t} + \operatorname{div}\left(\frac{3}{2}p\mathbf{V}\right) + p\operatorname{div}\mathbf{V} = -\operatorname{div}\mathbf{q} + \mathbf{u}\nabla p_e + \Sigma Q. \tag{6.36}$$

The right side of this equation contains all heat sources, including the heating due to viscosity: $\Sigma Q = -\mathbf{Ru} + Q_{\text{vis}}$.

Energy is transferred by macroscopic mechanisms (transport with velocity $\mathbf{V}$ and work due to the pressure) and by microscopic mechanisms (thermal conductivity, viscosity, etc.). The microscopic mechanisms and the corresponding terms in the energy and heat equations are called dissipative. These mechanisms increase the entropy of the plasma and result in the conversion of mechanical energy into heat. The entropy of the plasma (per particle), as for any monatomic gas, is equal to $\ln(T^{3/2}/n)$ or $\ln(p^{3/2}/\rho^{5/2})$ (to within an unimportant constant). It is evident from Eq. (6.36) that the entropy is conserved in the absence of dissipative processes.

We note that the left side of Eq. (6.36) can be written in the following form if the equation of continuity is used:

$$\frac{3}{2}\frac{dp}{dt} - \frac{5}{2}\frac{p}{\varrho}\frac{d\varrho}{dt} = \frac{3}{2}p\frac{d}{dt}\ln(p/\varrho^\gamma),$$

where $\gamma = 5/3$ is the adiabaticity index for the monatomic gas.

The relative importance of dissipative processes becomes smaller as the dimensions of the system become larger because the energy transport due to these processes is of a diffusional nature. If the characteristic time of the dissipative processes (L^2/D_m for the electrical resistance, $L^2n/\varkappa$ for the thermal conductivity, $L^2\rho/\eta$ for the viscosity, etc.) is large compared with the reciprocal frequency of the plasma motion $1/\omega$ or L/V, the dissipative terms are small. Under these conditions we can assume that the process is adiabatic and write as a first approximation

$$\frac{d}{dt}\frac{p}{\varrho^{\gamma}} = 0. \tag{6.37}$$

Dissipative processes increase the entropy and result in the damping of any macroscopic motion of the plasma, magnetohydrodynamic waves, and so on.

The plasma entropy balance equation can be obtained by adding Eq. (1.23') for the electrons and ions:

$$\frac{\partial S}{\partial t} + \operatorname{div}\left\{S\mathbf{V} + S_e\mathbf{u} + \frac{\mathbf{q}_e + \mathbf{q}_i}{T}\right\} = \theta, \tag{6.38}$$

$$T\theta = -(\mathbf{q}_e + \mathbf{q}_i)\nabla \ln T + \Sigma Q. \tag{6.39}$$

Here, $S = S_e + S_i = n_e \ln(T^{3/2}/n_e) + n_i \ln(T^{3/2}/n_i)$ is the plasma entropy per unit volume. The quantity θ is called the entropy production. It is easily shown that this quantity is always positive, that is to say, dissipative processes always cause a monotonic increase in entropy. The quantity $T\theta$ contains the Joule heat $j_{\parallel}^2/\sigma_{\parallel} + j_{\perp}^2/\sigma_{\perp}$ and the viscous heat $-{}^1/_2\pi_{\alpha\beta}W_{\alpha\beta}$, which are always positive.

§ 7. Multicomponent Plasma

Diffusion in a Three-Component Mixture. A simple model of a three-component mixture is an incompletely ionized gas that contains electrons, one ion species, and one species of neutral particles. The motion of these components can be specified by the three velocities $\mathbf{V}_e$, $\mathbf{V}_i$, and $\mathbf{V}_n$ or by a common hydrodynamic velocity $\mathbf{V}$ which is approximately

$$\mathbf{V} = \frac{1}{\varrho}(m_i n_i \mathbf{V} + m_n n_n \mathbf{V}_n), \tag{7.1}$$

and two relative velocities, which can be taken as

$$\mathbf{u} = \mathbf{V}_e - \mathbf{V}_i, \quad \mathbf{w} = \mathbf{V}_i - \mathbf{V}_n. \tag{7.2}$$

In a simple plasma there is one equation for the relative velocity, Ohm's law; in a three-component mixture two equations are required: one for the diffusion velocity $\mathbf{w}$ and one for the velocity associated with the current $\mathbf{u} = -\mathbf{j}/en_e$.

The determination of the diffusion velocities requires the solution of a system of three kinetic equations and the determination of the local distribution functions for all the components [12]. However, approximate results can be obtained by a simpler technique if the equations of motion for the individual components are used (1.14). The friction force is then obtained from the interaction of particles of one species with the remaining species: $\mathbf{R}_a = \sum_b \mathbf{R}_{ab}$. The forces $\mathbf{R}_{ab} = -\mathbf{R}_{ba}$ can be computed approximately under the assumption that the components a and b have Maxwellian distributions with velocities $\mathbf{V}_a$ and $\mathbf{V}_b$. As a result we find

$$\mathbf{R}_{ab} = -\alpha_{ab}(\mathbf{V}_a - \mathbf{V}_b), \quad \alpha_{ab} = \alpha_{ba}. \tag{7.3}$$

The coefficient of friction α_{ab} is obviously proportional to $n_a n_b$, the reduced mass of the colliding particles $m_{ab} = m_a m_b/(m_a + m_b)$, and the coefficient α'_{ab}, which is of order $v\sigma$ where v and σ are the characteristic values of the relative velocity and the effective cross sections for the colliding particles:

$$\alpha_{ab} = n_a n_b m_{ab} \alpha'_{ab}. \tag{7.4}$$

The calculation of α_{ab} is shown in the Appendix and can also be found in [1]. Suppose that $T_a = T_b = T$. If the scattering cross section [here the so-called transport cross section $\sigma' = \int(1 - \cos\vartheta)\,d\sigma(\vartheta)$ is the important factor] is inversely proportional to the relative velocity $\sigma'_{ab}(v) = \alpha'_{ab}/v$ Eq. (7.4) is obtained directly. For solid smooth spheres of radius r_a and r_b the cross section is $\sigma_{ab} = \pi(r_a + r_b)^2$. For this case

$$\alpha'_{ab} = \frac{4}{3}\sigma_{ab}\left(\frac{8}{\pi}\frac{T}{m_{ab}}\right)^{1/2}. \tag{7.5}$$

For ions of charge e_a and e_b [compare Eq. (2.5e)]

$$\alpha'_{ab} = \frac{4\sqrt{2\pi}\lambda e_a^2 e_b^2}{3\sqrt{m_{ab}}T^{3/2}}. \tag{7.6}$$

The relation in (7.3) does not take account of the possible anisotropy of the coefficient of friction in a magnetic field and also neglects the thermal force, so that it does not give thermal diffusion. In simple cases these effects can be estimated as in §3.

The component equations of motion, [using (7.2) and (7.3) and neglecting electron inertia] can be written in the form

$$-\nabla p_e - en_e\left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_e\mathbf{B}]\right) = -\alpha_e\frac{\mathbf{j}}{en_e} + \alpha_{en}\mathbf{w}, \quad (7.7e)$$

$$-m_i n_i\frac{d_i\mathbf{V}_i}{dt} - \nabla p_i + en_e\left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_i\mathbf{B}]\right) = \alpha_{ei}\frac{\mathbf{j}}{en_e} + \alpha_{in}\mathbf{w}, \quad (7.7i)$$

$$-m_n n_n\frac{d_n\mathbf{V}}{dt} - \nabla p_n = \alpha_{en}\frac{\mathbf{j}}{en_e} - \alpha_n\mathbf{w}. \quad (7.7n)$$

Here we use the substitution $\mathbf{u} = -\mathbf{j}/en_e$ and the notation

$$\alpha_e = \alpha_{ei} + \alpha_{en} = m_e n_e/\tau_e; \quad \tau_e = \left(\frac{1}{\tau_{ei}} + \frac{1}{\tau_{en}}\right)^{-1}, \quad (7.8)$$

$$\alpha_n = \alpha_{en} + \alpha_{in} = \alpha_{in}/(1-\varepsilon); \ \varepsilon = \alpha_{en}/\alpha_n. \quad (7.9)$$

Actually, we should take account of the viscosity in Eq. (7.7) writing $\partial P_{a\alpha\beta}/\partial x_\beta = \partial p_a/\partial x_\alpha + \partial\pi_{a\alpha\beta}/\partial x_\beta$ in place of $\nabla p_a (a = e, i, n)$. In the present analysis viscosity will be neglected.

The coefficient of friction between the plasma and neutral gas α_n depends primarily on the collisions between heavy particles because such collisions result in large momentum transfer; thus, one usually writes

$$\alpha_n \approx \alpha_{in}, \quad \varepsilon = \alpha_{en}/\alpha_n \approx (m_e/m_{in})(\alpha'_{en}/\alpha'_{in}) \ll 1.$$

Adding Eqs. (7.7e) and (7.7i) we obtain the equation of motion for the ionized particles; the electric field vanishes in the equation:

$$-m_i n_i\frac{d_i\mathbf{V}_i}{dt} - \nabla(p_e + p_i) + \frac{1}{c}[\mathbf{jB}] = -\alpha_{en}\frac{\mathbf{j}}{en_e} + \alpha_n\mathbf{w}. \quad (7.7p)$$

The sum of equations (7.7n) and (7.7p) gives the general equation of motion.

If the collision frequency is low it is more convenient to use the individual equations of motion for the neutral and ionized particles since the velocities are no longer "tied together" by collisions (strictly speaking the kinetic equation should be used in the low-collision case). On the other hand, the hydrodynamic description applies at high collision frequencies and it is more convenient to use the general equation of motion for $\mathbf{V}$ and the relative-velocity equations. We shall be interested in the second case. In the inertia terms we replace $d_i\mathbf{V}_i/dt$ and $d_n\mathbf{V}_n/dt$ by $d\mathbf{V}/dt$; this corresponds to neglecting terms of order $d\mathbf{w}/dt$ compared with terms of order $\mathbf{w}/\tau$ which are contained in the friction force. Viscosity is also neglected.

Various authors, for example [7, 25, 29], transform Eq. (7.7) by different methods and different expressions are obtained for the various relative velocities convenient for particular problems. The plasma diffusion rate with respect to the neutrals $\mathbf{w}$ can be determined, for example, from Eqs. (7.7n) and (7.7p). Eliminating the inertia term from these two equations we write $\mathbf{w}$ in the form

$$\mathbf{w} = \frac{1}{\alpha_n}\left\{-\mathbf{G} + \frac{\xi_n}{c}[\mathbf{jB}]\right\} + \frac{\varepsilon\mathbf{j}}{en_e}, \quad (7.10)$$

where

$$\mathbf{G} = \xi_n\nabla(p_e + p_i) - \xi_i\nabla p_n. \quad (7.11)$$

Here we have introduced the relative densities

$$\xi_n = m_n n_n/\varrho, \quad \xi_i = m_i n_i/\varrho, \quad \xi_n + \xi_i = 1.$$

If $T_i = T_n = T$, $m_i = m_n$ then $\mathbf{G} = \xi_n\nabla p_e + (p_i + p_n)\nabla\xi_i$; $p_i + p_n = T\rho/m_i$.

For example Ohm's law can be obtained by eliminating $\mathbf{w}$ from the electron equation of motion (7.7e) by means of Eq. (7.10):

$$\mathbf{E} + \frac{1}{c}[\mathbf{V}_i\mathbf{B}] + \frac{1}{en_e}(\nabla p_e - \varepsilon\mathbf{G}) = \frac{\mathbf{j}}{\sigma} + \frac{1-\varepsilon\xi_n}{en_e c}[\mathbf{jB}], \quad (7.12)$$

where

$$\sigma = e^2 n_e^2\left(\alpha_{ei} + \frac{\alpha_{en}\alpha_{in}}{\alpha_{en} + \alpha_{in}}\right)^{-1} \approx \frac{e^2 n_e\tau_e}{m_e}. \quad (7.13)$$

Equation (7.12) is very similar to Eq. (6.18) since $\varepsilon \ll 1$ but it contains the resistance $1/\sigma$, which is determined by the total electron collision frequency $1/\tau_e$. If the [jB] term is eliminated from Eq. (7.12) by means of the general equation of motion, Ohm's law is obtained in the form given by Schlüter [25].

Sometimes in writing Ohm's law it is convenient to retain the electric field $\mathbf{E}^* = \mathbf{E} + (1/c)[\mathbf{VB}]$ in the coordinate system moving with the velocity of the common mass $\mathbf{V}$. Using the relation $\mathbf{V}_i = \mathbf{V} + \xi_n \mathbf{w}$ and Eq. (7.10) we have from Eq. (7.12)

$$\mathbf{E} + \frac{1}{c}[\mathbf{VB}] + \frac{1}{en_e}(\nabla p_e - \varepsilon \mathbf{G}) - \frac{\xi_n}{\alpha_n c}[\mathbf{GB}] = \frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma^*_\perp} + \frac{1 - 2\varepsilon\xi_n}{en_e c}[\mathbf{jB}], \qquad (7.14)$$

where

$$\sigma_\parallel = \sigma, \quad \frac{1}{\sigma^*_\perp} = \frac{1}{\sigma} + \frac{\xi_n^2 B^2}{\alpha_n c^2}. \qquad (7.15)$$

When $m_i = m_n$, $\varepsilon \ll 1$, we find $\sigma/\sigma^*_\perp = 1 + 2\xi_n \omega_e \tau_e \omega_i \bar{\tau}_{in}$, where

$$1/\bar{\tau}_{in} = \alpha'_{in}(n_i + n_n).$$

Proceeding in the same way as in the derivation of Eq. (6.26) and using Eq. (7.12), we find that in the presence of neutrals the rate at which a plasma moves across the magnetic surfaces (the corresponding term in $\mathbf{V}_i$) is $(c^2/\sigma B^2)\nabla p$ where σ is expressed by Eq. (7.13) and p is the total pressure.

Ohm's law in the form given in (7.14) was derived by Cowling [7]. It contains the effective transverse resistance $1/\sigma^*_\perp$; in a strong magnetic field this can be much greater than the longitudinal resistance. This effect is explained by the fact that the motion of a plasma across the magnetic field means motion of the plasma with respect to the neutral gas; because of the large coefficient of friction between the ions and neutrals this implies a high rate of energy dissipation. It should be noted, however, that in certain cases (especially in slow phenomena, where the plasma pressure is almost always in equilibrium with the magnetic force) the terms in the curly brackets in Eq. (7.10) can almost balance each other; in such cases the last term on the left in Eq. (7.14) will almost balance the $j_\perp/\sigma^*_\perp$ term.

We wish to emphasize the great difference between Eq. (7.12) and Eq. (7.14) which contain $\mathbf{E} + (1/c)[\mathbf{V}_i\mathbf{B}]$ and $\mathbf{E} + (1/c)[\mathbf{VB}]$ respectively; in evaluating an electric field in the presence of a magnetic field it is of paramount importance to specify the coordinate system in which the electric field is being determined. A marked change in electric field occurs for even a small change in the velocity of the coordinate system.

We now consider the energy dissipation caused by friction in a mixture e, i, n. The total heat generated as a result of friction is $Q_{fr} = Q_e + Q_i + Q_n$. Taking account of the general relation (1.24) this expression can be written in the form $Q_{fr} = -\mathbf{R}_{ei}\mathbf{u} - \mathbf{R}_{en}(\mathbf{u} + \mathbf{w}) - \mathbf{R}_{in}\mathbf{w}$. Using Eq. (7.3) we find

$$Q_{fr} = \alpha_{ei}u^2 + \alpha_{en}(\mathbf{u} + \mathbf{w})^2 + \alpha_{in}w^2 = \alpha_e u^2 + \alpha_n w^2 + 2\alpha_{en}\mathbf{uw}. \qquad (7.16)$$

Using Eqs. (7.10) and (7.13) we now transform Eq. (7.16) as follows:

$$Q_{fr} = \frac{j^2}{\sigma} + \frac{1}{\alpha_n}\left(\frac{\xi_n}{c}[\mathbf{jB}] - \mathbf{G}\right)^2. \qquad (7.17)$$

If the current flows along the magnetic field (or B = 0), we find $w \sim \varepsilon u \ll u$ because of the high coefficient of friction for the heavy particles so that the dissipation due to heavy-particle friction is small: $\alpha_n w^2 \sim \varepsilon \alpha_e u^2$. However, if the current flows across the magnetic field the dissipation can be larger because of collisions between ions and neutrals.

For example, when $\mathbf{G}$ can be neglected, if the plasma is cold or weakly ionized we find from Eq. (7.17) $Q_{fr} = j^2_\parallel/\sigma + j^2_\perp/\sigma^*_\perp$.

A fully ionized plasma consisting of electrons (a = e) and two ion species (a = 1, 2) represents another important example of a three-component mixture. Typical mixtures of this kind are a hydrogen plasma containing impurity ions, an ionized mixture of deuterium and tritium, and so on.

In such a plasma the density and hydrodynamic velocity are approximately

$$\varrho = m_1 n_1 + m_2 n_2, \quad \mathbf{V} = \frac{1}{\varrho}(m_1 n_1 \mathbf{V}_1 + m_2 n_2 \mathbf{V}_2). \qquad (7.18)$$

The current density is expressed as the sum of the electron and ion components:

$$\mathbf{j} = -en_e\mathbf{u}_e + \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\varrho\xi_1\xi_2\mathbf{w}, \tag{7.19}$$

where we have introduced the notation

$$\mathbf{u}_e = \mathbf{V}_e - \mathbf{V}, \quad \mathbf{w} = \mathbf{V}_1 - \mathbf{V}_2\ , \tag{7.20}$$

and the relative densities $\xi_1 = m_1n_1/\rho$ and $\xi_2 = m_2n_2/\rho$ which appear in the expressions $\mathbf{V}_1 = \mathbf{V} + \xi_2\mathbf{w}$ and $\mathbf{V}_2 = \mathbf{V} - \xi_1\mathbf{w}$.

The component equations are

$$m_1n_1\frac{d\mathbf{V}}{dt} = -\nabla p_1 + e_1n_1\left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_1\mathbf{B}]\right) - \alpha_{12}\mathbf{w} - \alpha_{1e}(\xi_2\mathbf{w} - \mathbf{u}_e), \tag{7.21}$$

$$m_2n_2\frac{d\mathbf{V}}{dt} = -\nabla p_2 + e_2n_2\left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_2\mathbf{B}]\right) + \alpha_{12}\mathbf{w} + \alpha_{2e}(\xi_1\mathbf{w} + \mathbf{u}_e), \tag{7.22}$$

$$0 = -\nabla p_e - en_e\left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_e\mathbf{B}]\right) + (\alpha_{e1}\xi_2 - \alpha_{e2}\xi_1)\,\mathbf{w} - \alpha_e\mathbf{u}_e. \tag{7.23}$$

Here we have replaced $d_a\mathbf{V}_a/dt$ by $d\mathbf{V}/dt$, $\partial P_{a\alpha\beta}/\partial x_\beta$ by $\partial p_a/dx_\alpha$ and have introduced the notation

$$\alpha_e = \alpha_{e1} + \alpha_{e2} = m_en_e/\tau_e, \quad 1/\tau_e = 1/\tau_{e1} + 1/\tau_{e2}. \tag{7.24}$$

Using this system of equations we can find the relative velocity and obtain Ohm's law and an expression for $\mathbf{w}$ which is rather complicated in the general case [10a, 10b].

If T_i is not very much greater than T_e, we have $\alpha_{12} \gg \alpha_{e1}, \alpha_{e2}$. For example when $T_i \sim T_e$, $\alpha_{12} \sim (m_i/m_e)^{1/2}\alpha_e$.

When B = 0, because of the large value of α_{12} we find $w \ll u_e$ so that the current is transported primarily by electrons and Ohm's law can be written approximately in the form

$$\mathbf{E} + \frac{1}{en_e}\nabla p_e = \frac{\mathbf{j}}{\sigma}, \tag{7.25}$$

$$\sigma = \frac{e^2n_e\tau_e}{m_e}. \tag{7.26}$$

The same relation holds for the component along the magnetic field.

Because of their greater mass ions can move across a strong magnetic field much more easily than can electrons; hence the ion component of the current across the magnetic field can be larger, esspecially in fast phenomena. Let us consider certain relations that hold between the transverse components when $\omega_i\tau_i$ is large. In this case, to a first approximation, we neglect collisions and then introduce them as a small correction. For example one can write $\mathbf{w} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}$ where $\mathbf{w}^{(1)}$ can be found neglecting collisions. We divide Eq. (7.21) by e_1n_1 and Eq. (7.22) by e_2n_2 and subtract one from the other. Neglecting the friction forces and writing $\mathbf{w}$ we find:

$$\mathbf{w}^{(1)} = \frac{c}{B}\left[\mathbf{h}\left(\frac{\nabla p_1}{e_1n_1} - \frac{\nabla p_2}{e_2n_2}\right)\right] + \frac{c}{B}\left(\frac{m_1}{e_1} - \frac{m_2}{e_2}\right)\left[\mathbf{h}\,\frac{d\mathbf{V}}{dt}\right]. \tag{7.27}$$

The electric field then vanishes. The ⊥ symbol has been omitted. Now, taking account only of the friction between the ions and substituting Eq. (7.27) in the $\alpha_{12}\mathbf{w}$ term, we have

$$\mathbf{w}^{(2)} = -\frac{m_{12}c^2\alpha_{12}n_e}{e_1e_2B^2}\left\{\left(\frac{\nabla p_1}{Z_1n_1} - \frac{\nabla p_2}{Z_2n_2}\right) + \left(\frac{m_1}{Z_1} - \frac{m_2}{Z_2}\right)\frac{d\mathbf{V}}{dt}\right\}, \tag{7.28}$$

where $Z_1 = e_1/e$; $Z_2 = e_2/e$.

The quantity $\mathbf{u}_e = \mathbf{u}_e^{(1)} + \mathbf{u}_e^{(2)}$ can be expressed in similar fashion using Eq. (7.23) and eliminating the electric field $\mathbf{E}^* = \mathbf{E} + c^{-1}[\mathbf{VB}]$.

Let us consider a slow steady-state process in a plasma confined by a magnetic field; inertia effects will be neglected.

An estimate of the terms in Eq. (7.28) shows that the diffusion velocity $\mathbf{w}^{(2)}$ is $\sim(\alpha_{12}/\alpha_e) \sim (m_i/m_e)^{1/2}$ times greater than the penetration velocity for a simple plasma (6.26). The following question then arises: is the penetration velocity of a plasma with different ion species increased by a factor of $\sim(m_i/m_e)^{1/2}$ as compared with a simple plasma as a result of the friction between ions. In a steady state this process does not actually occur since the electron velocity across a magnetic surface $\mathbf{u}_e^{(2)}$ remains of order $c^2p/\sigma B^2L$, where σ is given by Eq. (7.26). The current across the magnetic surface vanishes so that $w^{(2)} \sim u_e^{(2)}$. Consequently the ion distribution that is established in the plasma must satisfy the condition

$$\frac{\nabla p_1}{e_1 n_1} - \frac{\nabla p_2}{e_2 n_2} \sim \frac{\alpha_e}{\alpha_{12}} \approx 0. \qquad (7.29)$$

The equilibration process proceeds at a rate $\sim(m_i/m_e)^{1/2}$ times faster than the penetration rate, after which $w^{(1)}$ becomes small $w^{(1)} \sim (\alpha_e/\alpha_{12})u_e^{(1)}$ and there is no large frictional force between ions. It is evident from Eq. (7.29) that in this case the ions with higher charge will be concentrated in plasma regions of higher density; for example with $T_1 = T_2 = T$ and $\nabla T = 0$ the Boltzmann distribution $(1/e_1)\nabla \ln n_1 = (1/e_2)\nabla \ln n_2$ obtains.

Diffusion in a Weakly Ionized Gas. We denote by $\mathbf{u}_a$ the diffusion velocity of the a component; then

$$\mathbf{u}_a = \mathbf{V}_a - \mathbf{V}, \ \sum_a m_a n_a \mathbf{u}_a = 0, \qquad (7.30)$$

where $\mathbf{V}$ is given by Eq. (6.2). Using Eq. (1.14), replacing $d_a\mathbf{V}_a/dt$ by dV/dt and $dP_{a\alpha\beta}/dx_\beta$ by $\partial p_a/\partial x_\alpha$, the equation of motion for the a component can be written in the form

$$m_a n_a \frac{dV}{dt} = -\nabla p_a + e_a n_a\left(\mathbf{E}^* + \frac{1}{c}[\mathbf{u}_a\mathbf{B}]\right) - \sum_b \alpha_{ab}(\mathbf{u}_a - \mathbf{u}_b). \qquad (7.31)$$

We shall consider the case in which the number of ionized particles is much smaller than the number of neutrals (weakly ionized plasma) so that the friction force is due to the $\mathbf{R}_{an}$ term associated with the neutral gas. The electron-ion collision cross section is much larger than the electron-neutral cross section (for example, in hydrogen at T_e = 1 eV we find $\alpha_{ei}/\alpha_{en} \sim 10^2 n_i/n_n$). Hence, electron-ion collisions can only be neglected when the neutral particle density is several orders of magnitude greater than the ion density (the exact values depend on the electron temperature and the nature of the gas). In this case the hydrodynamic velocity can be taken to be the velocity of the neutral gas $\mathbf{V} \approx \mathbf{V}_n$, $\mathbf{u}_n \approx 0$. Equation (7.31) will then be treated for particles of all species a in addition to the basic neutral component. The same statement holds for the summations over a used below. The velocity $\mathbf{V}$ is determined from the general equation of motion, with viscosity taken into account. The frictional force in Eq. (7.31) can be written in the form $\mathbf{R}_a = -\alpha_{an}\mathbf{u}_a$.

Let B = 0. Then from Eq. (7.31) we have

$$\mathbf{u}_a = b_a(e_a\mathbf{E} - \mathbf{G}_a/n_a) = b_a\left(e_a\mathbf{E} - m_a\frac{dV}{dt}\right) - D_a\nabla \ln p_a, \qquad (7.32)$$

where

$$\mathbf{G}_a = \nabla p_a + m_a n_a \frac{dV}{dt}. \qquad (7.33)$$

The mobility b_a and diffusion D_a are

$$b_a = \frac{n_a}{\alpha_{an}} = \frac{\tau_a}{m_{an}}, \ D_a = \frac{\tau_a}{m_{an}}T_a, \text{ where } \tau_a = \frac{\alpha'_{an}}{n_n}. \qquad (7.34)$$

These coefficients obey the well-known Einstein relation

$$D_a = b_a T_a. \qquad (7.34')$$

The last term in Eq. (7.32) is proportional to $\nabla n_a/n_a + \nabla T_a/T_a$ but the thermal force can affect the coefficient of ∇T_a (cf. footnote on page 227).

The current density $\mathbf{j} = \sum e_a n_a \mathbf{u}_a$, where the neutrality condition has been used. According to Eq. (7.32)

$$\mathbf{j} = \sigma\mathbf{E} - \sum_a e_a b_a \mathbf{G}_a, \qquad (7.35)$$

where

$$\sigma = \sum_a e_a^2 n_a b_a. \qquad (7.36)$$

In the presence of a magnetic field vector quantities parallel to $\mathbf{B}$, i.e., $\mathbf{u}_\parallel$, $\mathbf{j}_\parallel$, $\mathbf{E}_\parallel$, $\mathbf{G}_\parallel$ satisfy the same relations as for B = 0. The perpendicular components are obtained from Eq. (7.31):

$$\mathbf{u}_{a\perp} = b_{a\perp}(e_a\mathbf{E}^* - \mathbf{G}_a/n_a)_\perp + b_{a\wedge}[(e_a\mathbf{E}^* - \mathbf{G}_a/n_a)\mathbf{h}], \qquad (7.37)$$

where

$$b_{a\perp} = \frac{b_a}{1 + \omega_{an}^2\tau_a^2}, \ b_{a\wedge} = \omega_{an}\tau_a b_{a\perp}. \qquad (7.38)$$

Here we have used the notation

$$\omega_{an} = e_a B / m_{an} c, \quad \omega_{en} \approx -\omega_e.$$

The current density across the magnetic field is

$$\mathbf{j}_\perp = \sigma_1 \mathbf{E}^*_\perp + \sigma_2 [\mathbf{E}^* \mathbf{h}] - \sum_a e_a b_{a\perp} \mathbf{G}_{a\perp} - \sum_a e_a b_{a\wedge} [\mathbf{G}_a \mathbf{h}], \quad (7.39)$$

where

$$\sigma_1 = \sum_a e_a^2 n_a b_{a\perp}, \quad \sigma_2 = \sum_a e_a^2 n_a b_{a\wedge}. \quad (7.40)$$

In order to use Eq. (7.39) to express $\mathbf{E}^*_\perp = \mathbf{E}_\perp + (1/c)[\mathbf{VB}]$ in terms of $\mathbf{j}_\perp$ we multiply (7.39) by $\sigma_1/(\sigma_1^2 + \sigma_2^2)$, take the vector product with $\mathbf{h}\sigma_2/(\sigma_1^2+\sigma_2^2)$, and then add the results. In this way we obtain

$$\mathbf{E}^*_\perp = \frac{1}{\sigma_\perp} \mathbf{j}_\perp + \frac{1}{\sigma_\wedge} [\mathbf{j}\mathbf{h}] + \sum_a \beta_{a\perp} \mathbf{G}_{a\perp} + \sum_a \beta_{a\wedge} [\mathbf{G}_a \mathbf{h}], \quad (7.41)$$

where $1/\sigma_\perp$ is the perpendicular resistance; $\mathbf{h}/\sigma_\wedge$ is the so-called Hall vector:

$$\frac{1}{\sigma} = \frac{\sigma_1}{\sigma_1^2 + \sigma_2^2}, \quad \frac{1}{\sigma_\wedge} = -\frac{\sigma_2}{\sigma_1^2 + \sigma_2^2},$$
$$\beta_{a\perp} = e_a \left(\frac{b_{a\perp}}{\sigma_\perp} - \frac{b_{a\wedge}}{\sigma_\wedge} \right), \quad \beta_{a\wedge} = e_a \left(\frac{b_{a\wedge}}{\sigma_\perp} + \frac{b_{a\perp}}{\sigma_\wedge} \right). \quad (7.42)$$

If gravity is important, $d\mathbf{V}/dt$ must be replaced by $d\mathbf{V}/dt - \mathbf{g}$ in all equations. If necessary this term can be eliminated by means of the general equation of motion.

Equations (7.39) and (7.41) can be simplified if the $\mathbf{G}_a$ terms can be neglected. In this case the frictional heat can also be expressed simply: $Q_{fr} = \mathbf{E}^*\mathbf{j} = j_\parallel^2/\sigma_\parallel + j_\perp^2/\sigma_\perp$.

We now consider what is called ambipolar diffusion of a plasma. Assume that all gradients and the electric field are parallel and in the x direction, and let $j_x = 0$. This is the case, for example, of a plasma in a long tube with insulating walls; the axial gradients can be neglected and we shall be concerned with plasma diffusion in the radial direction

only (the role of x is played by r) with $j_r = 0$. This joint diffusion of electrons and ions is called ambipolar diffusion.

Suppose that the plasma contains ions of one species (for simplicity $Z = 1$) and let $\nabla T_e = \nabla T_i = \mathbf{V} = d\mathbf{V}/dt = 0$.

If $B = 0$, we have from Eq. (7.32)

$$nu_{ex} = -b_e enE - D_e \nabla n, \quad nu_{ix} = b_i enE - D_i \nabla n.$$

From $j_x = nu_{ex} - nu_{ix} = 0$ we have

$$E_x = -\frac{D_e - D_i}{b_e + b_i} \frac{1}{en} \frac{dn}{dx}.$$

Eliminating the electric field we can express the plasma flux $nu_{ex} = nu_{ix}$ in terms of one of the density gradients:

$$nu_{ex} = nu_{ix} = -D_A \nabla n, \quad (7.43)$$

where D_A is the so-called ambipolar diffusion coefficient

$$D_A = \frac{b_i D_e + b_e D_i}{b_e + b_i} = (T_e + T_i) \frac{b_e b_i}{b_e + b_i}. \quad (7.44)$$

This same expression can be obtained immediately from Eq. (7.10) if it is assumed that $\xi_n \approx 1$; $V \approx V_a = 0$; $w_x = u_{ix} = u_{ex}$; $G_x \approx (T_e + T_i)\, dn/dx$ and use is made of Eqs. (7.9) and (7.34).

If there is a magnetic field $B = B_z$, Eq. (7.43) becomes $nu_{ex} = nu_{ix} = -D_{A\perp} \nabla n$ where $D_{A\perp}$ is obtained from Eq. (7.44) by replacing b_a with $b_{a\perp}$:

$$D_{A\perp} = \frac{D_A}{1 + \omega_e \tau_e \omega_{in} \tau_i}. \quad (7.44')$$

This same result can be obtained from Eq. (7.10) if we write $V = 0$, express j_y by means of Eq. (7.14), $j_y = (\sigma^*_\perp \xi_n B_z / \alpha_n c)\, G_x$ and take $\xi_n \approx 1$.

Diffusion has been considered above using an approximate expression (7.3) for the frictional force. A more exact analysis of diffusion and the calculation of thermal conductivity require the use of the kinetic equation.

A kinetic analysis of diffusion and thermal conductivity of electrons in a weakly ionized gas in the presence of a magnetic field has been carried out by Davydov [30]. It is shown in this work that the electron distribution function can be approximated in the form $f(t, \mathbf{r}, \mathbf{v}) = f_0(t, \mathbf{r}, v) + \mathbf{f}_1(t, \mathbf{r}, v)\cdot\mathbf{v}/v$; then, averaging the kinetic equation (multiplied by 1 and $\mathbf{v}/v$) over angle in velocity space a system of equations is obtained for f_0 and $\mathbf{f}_1$:

$$\frac{\partial f_0}{\partial t} + \frac{v}{3}\,\operatorname{div}\mathbf{f}_1 + \frac{e}{3m_e v^2}\frac{\partial(v^2\mathbf{f}_1)}{\partial v}\cdot\mathbf{E} = C_{en}(\mathbf{f}_0), \qquad (7.45)$$

$$\frac{\partial \mathbf{f}_1}{\partial t} + v\nabla f_0 - \frac{e}{m_e}\mathbf{E}\frac{\partial f_0}{\partial v} - \frac{e}{m_e c}[\mathbf{f}_1\mathbf{B}] = -\frac{\mathbf{f}_1}{\tau_e}, \qquad (7.45')$$

where $1/\tau_e = \alpha'_{en} n_n$ and $C_{en}(f_0)$ is the collision term averaged over angle. Here we have written $V = 0$.

Since it is assumed that electrons collide with neutrals but not with themselves there is no reason to assume that the spherically symmetric part of the distribution function f_0 will be Maxwellian. If $\partial/\partial t \ll 1/\tau_e$, Eq. (7.45') can be used to express $\mathbf{f}_1$ in terms of f_0 so that an equation for f_0 can be obtained from Eq. (7.45). This equation has been solved in [30], for various $\tau_e(v)$; the function f_0 has been obtained together with the corresponding expressions for the particle and energy fluxes. Flux expressions for the case of a Maxwellian function f_0 are also given in [30].

Multicomponent Plasma. In the laboratory and under geophysical and astrophysical conditions one is frequently concerned with a multicomponent plasma. A fully ionized plasma may contain ions of various kinds while a gas that is not fully ionized can contain various molecules, atoms, excited atoms, etc. If collisions between particles are sufficiently frequent the hydrodynamic description can still be applied to such a plasma. The plasma density and hydrodynamic velocity are determined by Eqs. (6.1) and (6.2) where the summation is carried out over all components; as in the case of a simple plasma the electrons can be neglected. In addition to the mass conservation equations (6.5) we now require equations for the components that describe the change of state of the plasma. If the rate of production per unit volume of particles of species a is Γ_a, these equations can be written in the form

$$\frac{\partial n_a}{\partial t} + \operatorname{div}(n_a\mathbf{V}_a) = \Gamma_a, \qquad (7.46)$$

$$\varrho\frac{d\xi_a}{dt} + \operatorname{div}(m_a n_a \mathbf{u}_a) = m_a\Gamma_a,$$

where $\mathbf{u}_a = \mathbf{V}_a - \mathbf{V}$ is the diffusion velocity and $\xi_a = m_a n_a/\rho$ is the relative concentration of the a component. From conservation of mass and charge we have $\sum_a m_a\Gamma_a = 0$, $\sum_a e_a\Gamma_a = 0$.

The equation of motion of the plasma, which describes the total momentum balance, is obtained by adding the momentum transport equations (1.12) for all components, taking account of momentum conservation in collisions. Using the equation of continuity this is reduced to the form of (6.14) where $p = \sum_a p_a$, $\pi_{\alpha\beta} = \sum_a \pi_{a\alpha\beta}$*.

The diffusion velocities $\mathbf{u}_a$ can be determined approximately from Eq. (7.31) but in the general case this procedure leads to rather complicated expressions. Certain particular cases have been treated in the earlier sections.

In accordance with Eqs. (1.24) and (7.3), the total frictional heat is

$$Q_{\text{fr}} = \sum_a Q_a = \sum_{a,b} Q_{ab} = \sum_{a>b} \alpha_{ab}(u_a - u_b)^2. \qquad (7.47)$$

Multiplying Eq. (7.31) by $\mathbf{u}_a$ and summing over all components, taking account of Eq. (7.47), we have

$$Q_{\text{fr}} = \mathbf{E}^*\mathbf{j} - \sum_a \mathbf{u}_a\nabla p_a. \qquad (7.48)$$

*It should be noted that these expressions are obtained if we adopt new definitions of the quantities p_a and $\pi_{a\alpha\beta}$ which are somewhat different than those used in §1; these definitions are frequently used in the analysis of gas mixtures. The difference lies in the fact that here, in defining the temperature in terms of the random velocity of component a we take $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ rather then $\mathbf{v}_a = \mathbf{v} - \mathbf{V}_a$ as in §1. The "new" and "old" quantities are related by

$$T_a^{\text{new}} = T_a^{\text{old}} + \frac{m_a u_a^2}{3}, \quad p_a^{\text{new}} = p_a^{\text{old}} + m_a n_a u_a^2,$$

$$\pi_{a\beta\gamma}^{\text{new}} = \pi_{a\beta\gamma}^{\text{old}} + m_a n_a u_{a\beta} u_{a\gamma}.$$

This difference is not important because u_a is small when collisions dominate and the hydrodynamic description applies, in which case the quadratic term can be neglected. Taking account of the difference between the "old" and "new" quantities increases the accuracy of the hydrodynamic description.

The energy balance equation for a multicomponent plasma is obtained by summing the energy balances for all components and is formally similar to Eqs. (6.33) or (6.35), where the heat flux is*

$$\mathbf{q} = \Sigma \mathbf{q}_a + \Sigma \frac{5}{2} p_a \mathbf{u}_a.$$

It should be noted, however, that Eqs. (6.33) and (6.35) apply only for a plasma consisting of monatomic components in which case it may be assumed that the kinetic energy of all particles is associated with the translational motion if, for example, the plasma is fully ionized. The internal energy of the plasma (per unit volume) is then $\varepsilon = \Sigma \varepsilon_a = (3/2)\,p$. In the general case, $(3/2)\,p$ must be replaced by the internal energy ε in the expression for the energy while $(5/2)\,p$ must be replaced by $\varepsilon + p$ in the expression for the energy flux. For example, in a diatomic molecule, which has five effective degrees of freedom, $\varepsilon_a = (5/2)\,T_a n_a = (5/2)\,p_a$; $\varepsilon_a + p_a = (7/2)\,p_a$.

The heat balance equation (the internal energy transport equation) is obtained for a multicomponent plasma in the same way as Eq. (6.36) and is of the form

$$\frac{\partial \varepsilon}{\partial t} + \operatorname{div}(\varepsilon \mathbf{V}) + p \operatorname{div} \mathbf{V} = -\operatorname{div} \mathbf{q} + \sum_a u_a \nabla p_a + \Sigma Q =$$

$$= -\operatorname{div} \mathbf{q} + \mathbf{E}^* \mathbf{j} + Q_{vis}, \qquad (7.49)$$

where $\Sigma Q = Q_{fr} + Q_{vis}$. Other sources (loss) of heat can also appear.

If the plasma can be assumed to be approximately in local thermodynamic equilibrium the entropy balance equation is obtained in the general case by the methods of irreversible thermodynamics [9, 10a]. This equation is of the same form as Eq. (6.38) except that $\mathbf{q}_e + \mathbf{q}_i$ is replaced by $\Sigma \mathbf{q}_a$ and $S_e \mathbf{u}$ by $\Sigma S_a \mathbf{u}_a$.

The heat flux and the viscous stresses of a plasma consisting of monatomic components can be obtained in general form and relevant orders

*More precisely, $\mathbf{q} = \Sigma q_a^{new} + \Sigma (5/2) p_a \mathbf{u}_a$ where $q_{a\beta}^{new} = q_{a\beta}^{old} + (m_a/2) n_a u_a^2 u_{a\beta} + \pi_{a\beta\gamma}^{old} u_{a\gamma}$ but this difference is unimportant (see preceding footnote). The quantity Σq_a is sometimes called the reduced heat flux.

of magnitude can be estimated using the qualitative considerations of § 3 and the results for a simple plasma. In a weakly ionized plasma the thermal conductivity and the viscosity are determined primarily by the neutral gas. These can be estimated for a monatomic gas using the expressions given by Chapman and Cowling [1].

New effects appear in gases that have rotational or internal degrees of freedom (excitation, ionization, dissociation). For example, an additional heat flux arises if particles are ionized or dissociate at one point and if they recombine at another point, liberating a corresponding energy [10a]. The stress tensor for a gas with internal degrees of freedom contains terms of the form $\zeta \delta_{\alpha\beta} \operatorname{div} \mathbf{V}$ where ζ is the so-called second viscosity [3]. Radiation can also play a role in the heat flux.

The electron temperature is frequently very different from the temperatures of the heavy particles—ions and neutrals in a gas that is not fully ionized; for this reason the individual energy equations are generally used. If ionization is by electron impact, T_e is determined primarily by the ionization potential of the gas and usually stabilizes at a level corresponding to a small fraction of the ionization potential so that the fastest electrons are capable of ionization. In this case energy obtained by electrons is dissipated primarily in excitation radiation and in ionization by electron impact. The ionization rate is very sensitive to T_e which, on the other hand, is a relatively weak function of the various parameters and is of the order of electron volts. If the gas density is not very high the transfer of heat from the electrons to the ions or neutrals is strongly inhibited because the ratio m_e/m_i is so small. Thus, if the neutral gas is cooled and there is no other source of heat it is easy for a large temperature difference to arise between the electrons and heavy particles; this difference can reach two orders of magnitude. In gases that are not fully ionized it is frequently found that the important factor is not the plasma dynamical situation, but rather the maintenance of ionization, excitation of atoms, energy loss by radiation, interaction of the plasma with the walls, and so on. When an electric current flows through a gas a host of new characteristic effects appear; these are the subject of study of the physics of electrical discharges in gases. A very good elementary introduction to this field is contained in the small volume by Penning [10].

§ 8. Examples

Pinch Effect. The magnetic field produced by a current flowing through a plasma tends to constrict the plasma because the current filaments which comprise the total current tend to attract each other. This

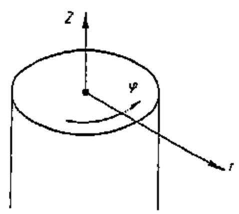

Fig. 7

phenomenon is called the pinch effect. For reasons of brevity a plasma contracted by its own magnetic field will simply be called a pinch.

As a simple example let us consider an infinitely long plasma cylinder contained by a magnetic field; this is the so-called linear pinch (Fig. 7). We assume that all quantities vary in the r direction only ($\partial/\partial z = \partial/\partial\varphi = 0$), that the plasma as a whole does not move along z, and that it does not rotate. Under these conditions the magnetic field and the current have only z and φ components and the magnetic surfaces are cylinders (r = const).

The equilibrium condition for the pinch is [cf. Eq. (6.17)]

$$-\frac{\partial p}{\partial r} = \frac{1}{c}(j_z B_\varphi - j_\varphi B_z) = B_\varphi \frac{\partial r B_\varphi}{r \partial r} + B_z \frac{\partial B_z}{\partial r}. \qquad (8.1)$$

We now multiply Eq. (8.1) by r^2 and integrate with respect to r from 0 to a, where a is the radius of the pinch. Integrating by parts and taking p(a) = 0, $B_\varphi(a) = 2J/ca$, where J is the total current, we obtain the equilibrium condition for the pinch in integral form [31]:

$$2c^2(N_e \overline{T}_e + N_i T_i) = J^2 + \frac{a^2c^2}{4}\left[B_z^2(a) - \overline{B_z^2}\right], \qquad (8.2)$$

where $\overline{B_z^2} = \int_0^a B_z^2 2\pi r\, dr/\pi a^2$; N_e and N_i are the numbers of electrons and ions per unit length of the pinch; $\overline{T}_e$ and $\overline{T}_i$ are the mean temperatures. The self-magnetic field of the current J always acts to constrict the plasma. The longitudinal magnetic field constricts the plasma if the external field $B_z(a)$ is greater than the internal field, and tends to expand it if the external field is smaller than the internal field.

Now let us consider the application of Ohm's law to the linear pinch. From Eqs. (6.18) and (6.19) we have

$$\mathbf{E}_\parallel = \mathbf{j}_\parallel/\sigma_\parallel, \quad \mathbf{E}_\wedge + \frac{1}{c}[\mathbf{VB}] - \frac{1}{en_e}\mathbf{R}_\perp = \mathbf{j}_\perp/\sigma_\perp, \qquad (8.3)$$

where $E_\wedge$ is the component of the electric field perpendicular to the magnetic field and tangent to the magnetic surface r = const. From Eq. (6.24) we find $E_r'' = 0$. If inertia and the radial viscous stress are neglected compared with $\partial p_i/\partial r$,

$$E_r = \frac{1}{en_e}\frac{\partial p_i}{\partial r}. \qquad (8.4)$$

At equilibrium the radial electric field automatically assumes the value given in (8.4) and balances the ion pressure; the ions exhibit a Boltzmann distribution in this field if T_i = const. Then the Hall field will automatically assume the required value in accordance with Eq. (6.18)

$$E_r' = E_r + \frac{1}{en_e}\frac{\partial p_e}{\partial r} = \frac{1}{c}[\mathbf{jB}]_r. \qquad (8.5)$$

In a fast pinch, in which ion inertia is important, Eq. (8.5) still holds but Eq. (8.4) does not.

The transverse current component $\mathbf{j}_\perp = (c/B)[\mathbf{h}\nabla p]$. Substituting this in the second equation in (8.3) we obtain an equation for the velocity:

$$V_r = \frac{c}{B}(h_z E_\varphi - h_\varphi E_z) - \frac{c^2}{\sigma_\perp B^2}\left(\frac{\partial p}{\partial r} - \frac{3}{2} n_e \frac{\partial T_e}{\partial r}\right). \qquad (8.6)$$

If the electric field corresponding to Eq. (8.4) is not established for any reason (for example if the pinch is of finite length and if the ends exert a strong effect) in accordance with Eq. (6.26) there will be a plasma velocity V_φ component

$$V_\varphi = \frac{c}{B}\left(E_r - \frac{1}{en_e}\frac{\partial p_i}{\partial r}\right) h_z.$$

The nonuniformity in V_φ means that the ions feel the effect of a φ projection of the viscous stress $F_\alpha^\pi = -\partial\pi_{i\alpha\beta}/\partial x_\beta$; in accordance with Ohm's law [cf. Eq. (6.24)] an additional term must now appear in the expression V_r,

$$V_{r\,\text{vis}} = \frac{c}{en_e B} F_\varphi^\pi h_z. \qquad (8.7)$$

This velocity is sometimes called the diffusion velocity due to ion collisions [20] although it is proportional to the third radial derivative of the density rather than the first. In order-of-magnitude terms $V_{r\,\text{vis}} \sim (r_i^4/\tau_i a^3) \sim 1/B^4$ and can be comparable with the Joule rate of penetration $\sim r_e^2/\tau_i a$ for $r_i^2/a^2 \sim (m_e/m_i)^{1/2}$.

If the plasma current or the external magnetic field increase very rapidly the equilibrium condition (8.2) is not satisfied and the magnetic field causes the plasma to become constricted rapidly. Because of the skin effect the magnetic field cannot penetrate to the inside in zero time, but compresses the plasma in piston-like fashion while moving in from the outer edge; thus, a strong shock wave moves from the outer edge to the axis. Because of its elasticity the pinch rebounds after contraction and expands again; in fact, it has been shown experimentally, that the pinch executes several oscillations before breaking up as a consequence of various instabilities.

A detailed analysis of the oscillations of a pinch requires the solution of a complex system of magnetohydrodynamic equations [34]; however, the time required for total compression can be estimated rather simply [32, 33]. For example, assume that $J = \dot{J}t$ and $B_z = 0$. The skin effect becomes important when the current in the pinch increases rapidly. The current flows along the surface of the pinch and the force due to the magnetic pressure acts in the surface layer. The layer thickness is determined by the specific resistance $1/\sigma^*_\perp$ associated with ion-neutral collisions [cf. Eq. (7.15)] which can be appreciably greater than the resistance due to electron-ion collisions $1/\sigma$; this effect is sometimes described as the movement of the magnetic lines with the ions [33]. Because of the skin effect and the formation of the strong shock wave initially the plasma "rakes" the field away from the edge and gradually more and more layers are accelerated. Let us assume that the entire mass of the gas in motion is concentrated at the point $a(t)$ and is equal to $\rho_0\pi(a_0^2 - a^2)$ where a_0 is the initial radius of the pinch and ρ_0 is the initial density; the equation of motion for this mass can be written in the form

$$\frac{d}{dt}\left\{\varrho_0\pi\left(a_0^2 - a^2\right)\frac{da}{dt}\right\} = -\frac{B_\varphi^2}{8\pi}2\pi a = -\frac{J^2}{c^2 a} = -\frac{\dot{J}^2 t^2}{c^2 a}. \quad (8.8)$$

We integrate this expression approximately, taking the value of $a(t)$ at the upper limit outside of the integral sign on the right in the first integration over time. As a result we obtain

$$a^2 = a_0^2\left(1 - t^2/t_0^2\right), \quad (8.9)$$

$$t_0 = (3\pi\varrho_0 c^2)^{1/4} a_0 \dot{J}^{-1/2}. \quad (8.10)$$

Total compression corresponds approximately to the time t_0 and current $J_0 = \dot{J}t_0$. It is evident that $1/t_0 \sim a_0/c_A$, where $c_A = B_0/(4\pi\rho_0)^{1/2}$ and $B_0 \sim J_0/ca_0$.

The compression time of a pinch compressed by a B_z field can be estimated in completely analogous fashion.

If small perturbations that disturb the equilibrium are produced rapidly in an equilibrium pinch the pinch will execute small oscillations. In the next section we shall analyze oscillations for the case of an infinite plasma; however, the order of magnitude of the quantities obtained in that analysis applies to the finite pinch.

If changes in an equilibrium pinch occur slowly (frequencies small compared with the characteristic frequencies of these magnetohydrodynamic oscillations) the pinch remains in a quasi-equilibrium state at all times and inertia does not play a role—this is in fact the description of a slow process.

All of the considerations given above will obviously hold only when the pinch is stable; however, stability will not be discussed in this review.

Magnetohydrodynamic Waves. We wish to consider small oscillations of a uniform plasma in a uniform magnetic field. A plasma is, in fact, capable of executing a large variety of oscillations. At this point, however, we shall be interested only in the relatively low-frequency and large-scale oscillations in which the motion of matter plays an important role and which can be described by a hydrodynamic analysis §6 (the neutrality conditions $c_A^2 \ll c^2$, etc., are satisfied). An analysis of these oscillations will give us an idea of the "elastic" properties of a plasma. Because an ordinary gas only exhibits longitudinal elasticity, it can support the propogation of only one kind of wave; this is the sound wave. As pointed out by Alfvén [4, 5], however, a conducting fluid in a magnetic field exhibits a peculiar kind of elasticity with respect to transverse displacements; this elasticity results from the fact that the magnetic lines of force behave as though they were stretched rubber bands. The resulting oscillations are called magnetohydrodynamic waves.

Let us write

$$\varrho = \varrho_0 + \varrho', \quad p = p_0 + p', \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}', \quad (8.11)$$

where the zeros denote unperturbed equilibrium values and the primes denote small perturbations. The velocity $\mathbf{V}$ is also regarded as a small quantity.

At the outset we neglect all dissipative processes, assuming that the plasma conductivity is very large and that the viscosity is small, etc. (more exact criteria will be given in the following section). The adiabaticity condition (6.37) then gives $p/\rho^\gamma = \text{const} = p_0/\rho_0^\gamma$ where $\gamma = 5/3$ is the adiabaticity index. Using Eq. (8.11) and expanding in terms of the small quantities we have

$$p = p_0\left(\frac{\varrho}{\varrho_0}\right)^\gamma = p_0 + \frac{\gamma p_0}{\varrho_0}\varrho' + \frac{p_0\gamma(\gamma-1)}{2}\left(\frac{\varrho'}{\varrho_0}\right)^2 + \dots =$$

$$= p_0 + c_s^2\varrho' + \frac{\gamma-1}{2\varrho_0}c_s^2\varrho'^2 + \dots, \quad (8.12)$$

where c_s is the velocity of sound, which is defined by the relation

$$c_s = \left(\frac{\partial p}{\partial \varrho}\right)_s^{1/2} = \left(\frac{\gamma p_0}{\varrho_0}\right)^{1/2}. \quad (8.13)$$

Neglecting all powers of the small perturbations higher than the first in the equations of continuity, motion, and induction we obtain the following linearized system:

$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \operatorname{div} \mathbf{V} = 0, \quad (8.14a)$$

$$\varrho_0\frac{\partial \mathbf{V}}{\partial t} = -c_s^2\nabla p' - \frac{1}{4\pi}[\mathbf{B}_0 \operatorname{rot} \mathbf{B}'], \quad (8.14b)$$

$$\frac{\partial \mathbf{B}'}{\partial t} = \operatorname{rot}[\mathbf{V}\mathbf{B}_0]. \quad (8.14c)$$

We have also neglected the Hall term in the induction equation in accordance with the estimate in §6, which applies when $\Pi \gg 1$. The characteristic dimension can be taken to be the wavelength of the oscillation λ or, better still, the reciprocal wave number $1/k = \lambda/2\pi$ so that Eq. (8.14c) is valid when

$$\Pi = \frac{4\pi e^2 n_e^2}{\varrho c^2 k^2} \gg 1. \quad (8.15)$$

We now seek a solution of Eq. (8.14) in which all perturbed quantities are proportional to $e^{i(\mathbf{kr}-\omega t)}$, that is to say, we seek plane wave solutions characterized by a frequency ω and wave vector $\mathbf{k}$. An arbitrary

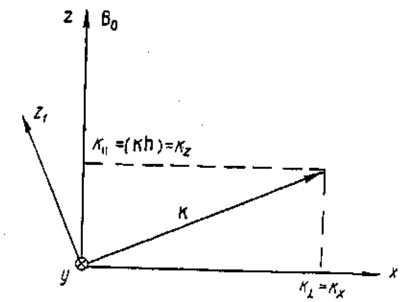

Fig. 8

perturbation can be expanded in a Fourier series and represented as a superposition of these component waves. Writing $\partial/\partial t \to -i\omega$, $\nabla \to i\mathbf{k}$ we find

$$-\omega\varrho'/\varrho_0 + (\mathbf{kV}) = 0, \quad (8.16a)$$

$$-\omega\mathbf{V} = -\mathbf{k}c_s^2\varrho'/\varrho_0 - [\mathbf{B}_0[\mathbf{kB}']]/4\pi\varrho_0, \quad (8.16b)$$

$$-\omega\mathbf{B}' = [\mathbf{k}[\mathbf{V}\mathbf{B}_0]]. \quad (8.16c)$$

The condition $\operatorname{div}\mathbf{B}' = 0$ indicates that the variable magnetic field is perpendicular to the wave vector. We choose the z axis along $\mathbf{B}_0 = B_{0z}\mathbf{h}$ and take the y axis to be perpendicular to both $\mathbf{B}_0$ and $\mathbf{k}$ (Fig. 8). Projecting Eq. (8.16) on these axes we see that the system of equations divides into two independent sets for the variables V_y, B'_y and for the variables ρ', V_x, V_z, B'_x, B'_z. The imaginary unit disappears in Eq. (8.16) indicating that ρ', $\mathbf{V}$, and $\mathbf{B}$ in each wave are in phase, that is to say, all are proportional, for example, to the factor $\cos(\mathbf{kr} - \omega t)$.

Let us consider the first set. Since $\mathbf{k}\cdot\mathbf{V} = 0$, then $\rho' = 0$ and consequently $p' = 0$. From Eq. (8.16) we find

$$-\omega V_y = k_\parallel B'_y B_0/4\pi\varrho_0, \quad (8.17a)$$

$$-\omega B'_y = k_\parallel V_y B_0. \quad (8.17b)$$

The condition that must be satisfied in order for this system to yield a nontrivial solution yields a relation between the frequency and the wave vector:

$$\omega^2 = c_A^2 k_\parallel^2 = \frac{(\mathbf{B}_0\mathbf{k})^2}{4\pi\varrho_0}, \qquad (8.18)$$

where c_A is the so-called Alfvén velocity:

$$c_A = \frac{B_0}{(4\pi\varrho_0)^{1/2}}. \qquad (8.19)$$

This wave is called the Alfvén wave. The velocity and variable field for this wave are perpendicular to $\mathbf{B}_0$ and $\mathbf{k}$ and are related by

$$V_y = B_y'(4\pi\varrho_0)^{-1/2}, \qquad (8.20)$$

while the density and pressure do not oscillate; hence the wave can propagate in either a compressable or an incompressible fluid.

The group velocity of the Alfvén waves $\partial\omega/\partial\mathbf{k} = \mathbf{B}_0(4\pi\rho_0)^{-1/2} = c_A\mathbf{h}$. This velocity is independent of $\mathbf{k}$, meaning that appropriate perturbations of any (but obviously not too small) scale size at any point of the plasma are propagated with the velocity c_A along the magnetic lines of force.

We now consider the second set of equations for $\mathbf{V}$ and $\mathbf{B}'$; these lie in the plane of $\mathbf{B}_0$ and $\mathbf{k}$. The magnetic field is perpendicular to $\mathbf{k}$ and is along the z_1 axis (cf. Fig. 8). We denote this projection by B'. Projecting Eq. (8.16c) along z_1 and Eq. (8.16b) along x and z we have

$$\omega\varrho'/\varrho_0 = k_\parallel V_z + k_\perp V_x, \qquad (8.21a)$$

$$\omega B'(4\pi\varrho_0)^{-1/2} = c_A k V_x, \qquad (8.21b)$$

$$\omega V_x = c_s^2 k_\perp \varrho'/\varrho_0 + c_A k B'(4\pi\varrho_0)^{-1/2}, \qquad (8.21c)$$

$$\omega V_z = c_s^2 k_\parallel \varrho'/\varrho_0. \qquad (8.21d)$$

Eliminating ρ' and B' we find

$$\left(\omega^2 - c_s^2 k_\perp^2 - c_A^2 k^2\right) V_x = c_s^2 k_\parallel k_\perp V_z, \qquad (8.22a)$$

$$\left(\omega^2 - c_s^2 k_\parallel^2\right) V_z = c_s^2 k_\parallel k_\perp V_x. \qquad (8.22b)$$

The condition that must be satisfied for this system to yield a nontrivial solution gives

$$\omega^4 - \omega^2\left(c_A^2 k^2 + c_s^2 k^2\right) + c_A^2 c_s^2 k^2 k_\parallel^2 = 0. \qquad (8.23)$$

Thus, we find the two roots:

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left(c_A^2 + c_s^2\right) \pm \left[\frac{1}{4}\left(c_A^2 + c_s^2\right)^2 - c_A^2 c_s^2 k_\parallel^2/k^2\right]^{1/2} \qquad (8.24)$$

These waves are called magnetoacoustic or magnetosonic waves: the plus sign corresponds to the fast magnetoacoustic wave. When $c_A \ll c_s$ one of these waves becomes the usual acoustic wave with frequency $\omega = c_s k$ while the other behaves like the Alfvén wave with frequency $\omega = c_A k$. An incompressible conducting fluid, for which $c_s \to \infty$, can thus support the propagation of two Alfvén waves with different polarizations.

Now let us consider the case $c_s \ll c_A$ in greater detail. The frequency of the fast wave is

$$\omega = c_A k. \qquad (8.25)$$

The group velocity is equal to the phase velocity and is in the direction of $\mathbf{k}$. In this wave the motion occurs primarily along the x axis and the density perturbation is small. From Eqs. (8.21) and (8.22) we find

$$V_x = \frac{B'}{(4\pi\varrho_0)^{1/2}}, \quad V_z = \frac{c_s^2}{c_A^2}\frac{k_\parallel k_\perp}{k^2}V_x, \quad \frac{\varrho'}{\varrho_0} = \frac{k_\perp}{k}\frac{V_x}{c_A}. \qquad (8.26)$$

When $c_s \ll c_A$ the slow wave represents the acoustic wave distorted by the magnetic field. The frequency of this wave is

$$\omega = c_s k_\parallel. \qquad (8.27)$$

The motion in this wave occurs primarily along the z axis and the perturbation of the magnetic field is small:

$$\frac{\varrho'}{\varrho_0} = \frac{V_z}{c_s}, \quad V_x = -\frac{c_s^2}{c_A^2}\frac{k_\parallel k_\perp}{k^2}V_z, \quad \frac{B'}{(4\pi\varrho_0)^{1/2}} = -\frac{c_s}{c_A}\frac{k_\perp}{k}V_z. \qquad (8.28)$$

The group velocity is also directed along the z axis and not along **k** as in ordinary sound waves: $\partial\omega/\partial\mathbf{k} = c_s\mathbf{h}$. These perturbations are carried along the lines of force as though the latter were rails.

The temperature oscillations can be expressed in terms of ρ':

$$\frac{T'}{T_0} = \frac{p'}{p_0} - \frac{\varrho'}{\varrho_0} = (\gamma - 1)\,\frac{\varrho'}{\varrho_0}. \tag{8.29}$$

The electric field is determined by Ohm's law. According to Eq. (6.28), $E = E_x = -V_yB_0/c$ in the Alfvén wave; in the fast wave $E = E_y = V_xB_0/c$. These expressions can be refined by using Eq. (6.18) without the dissipative terms and taking $p'_e/n_e = \gamma T_0\rho'/\rho_0$. Thus

$$\mathbf{E} = -\frac{1}{c}\,[\mathbf{VB}_0] + \frac{1}{en_ec}\,[\mathbf{jB}_0] + \frac{\nabla p_e}{en_e} =$$

$$= -\frac{1}{c}\,[\mathbf{VB}_0] + \frac{i\,[[\mathbf{kB}']\,\mathbf{B}_0]}{4\pi en_e} + \frac{ik\gamma T_0\varrho'}{e\varrho_0}. \tag{8.30}$$

Here, the second term (the Hall term) is smaller than the first by a factor of $\sim\Pi$. The third term (irrotational field) vanishes in the Alfvén wave but is of order $(c_s^2/c_A^2)/\Pi$ in the fast wave. In the acoustic wave the irrotational field can be smaller or greater than the induction field but does not affect the induction equation since rot **E** appears in this equation.

The energy in the wave can be found by substituting Eq. (8.11) in the general expression (6.35) and retaining second-order terms. Using the expansion in (8.12) we have

$$\varepsilon_{\text{total}} = \left(\frac{p_0}{\gamma - 1} + \frac{B_0^2}{8\pi}\right) + \left(\frac{c_s^2\varrho'}{\gamma - 1} + \frac{\mathbf{B}_0\mathbf{B}'}{4\pi}\right) + \\ + \left(\frac{\varrho_0V^2}{2} + \frac{c_s^2\varrho'^2}{2\varrho_0} + \frac{B'^2}{8\pi}\right) \tag{8.31}$$

The first bracket here represents the energy of the unperturbed plasma. The second bracket contains oscillating terms which vanish when integrated over the volume of the wave or when averaged in time. The third term gives the energy associated with the wave. The energy of the electric field is omitted since it is much smaller than the magnetic energy:

$$E^2/B'^2 \sim (VB_0/c)^2/4\pi\varrho_0V^2 \sim c_A^2/c^2,$$

and it is assumed that this ratio is small. The wave energy per unit volume is equal to the mean value of the third bracket:

$$\bar{\varepsilon} = \frac{\varrho_0\overline{V^2}}{2} + \frac{c_s^2\overline{\varrho'^2}}{2\varrho_0} + \frac{\overline{B'^2}}{8\pi}. \tag{8.32}$$

Using Eq. (8.14) it is easily shown that the quantity $\bar{\varepsilon}$ averaged over the volume of the wave is conserved in the oscillations. In accordance with Eq. (8.20), the energy of the Alfvén wave is

$$\bar{\varepsilon} = \varrho_0\overline{V_y^2} = \overline{B_y'^2}/4\pi. \tag{8.33}$$

The energy of the fast magnetoacoustic wave (when $c_s \ll c_A$) is approximately

$$\bar{\varepsilon} = \varrho_0\overline{V_x^2} = \overline{B'^2}/4\pi. \tag{8.34}$$

Here we have omitted the energy associated with the pressure and V_z^2.

When $c_s \ll c_A$ the energy of the acoustic wave is approximately

$$\bar{\varepsilon} = \varrho_0\overline{V_z^2} = c_s^2\overline{\varrho'^2}/\varrho_0. \tag{8.35}$$

Here we have omitted the magnetic energy and V_x^2.

In the general case of arbitrary c_s/c_A, using Eq. (8.24) it is easily shown that the following relation holds:

$$\bar{\varepsilon} = \varrho_0\overline{V^2} = \overline{B'^2}/4\pi + c_s^2\overline{\varrho'^2}/\varrho_0. \tag{8.36}$$

Damping of Magnetohydrodynamic Waves. Dissipative effects, which we have neglected so far, cause wave damping. The energy associated with the waves diminishes in time and is converted into heat. For this reason ω becomes complex $\omega = \omega_1 - i\omega_2$, and the amplitude is damped in time in proportion to $e^{-\omega_2 t}$. Repeating the foregoing calculations taking account of dissipative effects would lead to extremely complicated expressions; in the case of greatest interest, in which the damping is small ($\omega_2 \ll \omega_1$), the damping can be found more directly and simply (cf. [3] §77).

We use the following notation: $\omega_1 = \omega$ is the real part of the frequency; $\omega_2 = \omega\delta$, where δ is the logarithmic damping decrement. The energy of the wave (integrated over the volume) is proportional to the square of the amplitude and, consequently with damping, $\bar{\varepsilon} = \bar{\varepsilon}_1 e^{-2\omega\delta t}$ where $\bar{\varepsilon}_1 = \bar{\varepsilon}(t=0)$. Because part of the plasma energy (the wave energy) is in "organized" form, the entropy of the plasma must have some negative part $\overline{\Delta S}$; in the course of time this negative entropy damps out together with the wave energy: $-\overline{\Delta S} = -(\overline{\Delta S})_1 e^{-2\omega\delta t}$. When all of the wave energy has been dissipated the plasma entropy has been increased by an amount $\bar{\varepsilon}_1/T_0$ so that $(\overline{\Delta S})_1 = \bar{\varepsilon}_1/T_0$. Using this relation it is easy to express the damping decrement in terms of $d\Delta S/dt$:

$$\delta = \frac{1}{2\omega}\,\frac{T_0}{\bar{\varepsilon}}\,\frac{d\overline{\Delta S}}{dt} = \frac{1}{2\omega\bar{\varepsilon}}\,T_0\theta. \tag{8.37}$$

The quantity $d\overline{\Delta S}/dt$ can easily be computed by means of Eq. (6.38). For the case of weak damping, as a first approximation we can use the expressions obtained for all wave quantities with damping neglected. Both the entropy production (6.39) and the damping decrement (8.37) will then be expressed in the form of a sum, each term of which gives the damping corresponding to a particular dissipative effect:

$$\delta = \delta_{\text{Joule}} + \delta_{\text{vis}} + \delta_{\text{ther}} + \delta_{\text{dif}} + \cdots \tag{8.38}$$

If these calculations are to apply and if the expressions for the frequency and polarization given above are to hold (neglecting dissipative effects) the condition $\delta \ll 1$ must be satisfied.

We first consider wave damping in a simple plasma.

Alfvén Wave. Taking $Q_{\text{Joule}} = j^2_{\parallel}\sigma_{\parallel} + j^2_{\perp}/\sigma_{\perp}$ and $\mathbf{j} = (c/4\pi)\, i[\mathbf{k}\mathbf{B}']$, for the Alfvén wave we have

$$\overline{Q}_{\text{Joule}} = (c^2/4\pi)\left(k_{\parallel}^2/\sigma_{\perp} + k_{\perp}^2\sigma_{\parallel}\right)\overline{B'^2_y}/4\pi.$$

Thus, using Eq. (8.37) and the expression $\bar{\varepsilon} = \overline{B'^2_y}/4\pi$ we find

$$2\omega\delta_{\text{Joule}} = \frac{c^2}{4\pi\sigma_{\parallel}}\,k_{\perp}^2 + \frac{c^2}{4\pi\sigma_{\perp}}\,k_{\parallel}^2. \tag{8.39}$$

In viscous damping, $Q_{\text{vis}} = (1/2)\,\pi_{\alpha\beta}W_{\alpha\beta} = (1/2)\sum_{p=0}^{2}\eta_p W^2_{p\alpha\beta}$; the following are important: $\partial V_y/\partial x = W_{xy} = ik_{\perp}V_y$; $\partial V_y/\partial z = W_{yz} = ik_{\parallel}V_y$. The tensors $W_{1\alpha\beta}$ and $W_{2\alpha\beta}$ are nonvanishing; only the transverse viscosity is of importance. Simple calculations made on the basis of Eqs. (4.42) or (2.21) yield:* $\overline{Q}_{\text{vis}} = (\eta_1 k^2_{\perp} + \eta_2 k^2_{\parallel})\overline{V^2_y}$.

Taking $\bar{\varepsilon} = \rho\overline{V^2_y}$, we find

$$2\omega\delta_{\text{vis}} = \frac{1}{\varrho_0}\left(\eta_1 k_{\perp}^2 + \eta_2 k_{\parallel}^2\right). \tag{8.40}$$

Since the density and temperature do not oscillate in the Alfvén wave, $\rho_{\text{ther}} = 0$.

Fast Magnetoacoustic Wave. In this wave the current is directed along the y axis (across the magnetic field) so that $Q_{\text{Joule}} = j^2/\sigma_{\perp} = (c^2/4\pi\sigma_{\perp})k^2(B'^2/4\pi)$. Thus, taking account of Eq. (8.34) we have

$$2\omega\delta_{\text{Joule}} = \frac{c^2}{4\pi\sigma_{\perp}}\,k^2. \tag{8.41}$$

The velocity in this wave is directed primarily along the x axis and has derivatives with respect to x and z so that all three viscosity coefficients are important: η_0, η_1, η_2. Computing $\overline{Q}_{\text{vis}}$ by means of Eqs. (8.34) and (8.37) we have

$$2\omega\delta_{\text{vis}} = \frac{1}{\varrho_0}\left[\left(\frac{\eta_0}{3} + \eta_1\right)k_{\perp}^2 + \eta_2 k_{\parallel}^2\right]. \tag{8.42}$$

The entropy production due to the thermal conductivity is

$$\bar{\theta}_{\text{ther}} = -\overline{\mathbf{q}\nabla T}/T_0^2 = Q_{\text{ther}}/T_0,$$

where

$$\overline{Q}_{\text{ther}} = \frac{\varkappa_{\parallel}^e}{T_0}\overline{(\nabla_{\parallel}T_e)^2} + \frac{\varkappa_{\perp}^e}{T_0}\overline{(\nabla_{\perp}T_e)^2} + \frac{\varkappa_{\parallel}^i}{T_0}\overline{(\nabla_{\parallel}T_i)^2} + \frac{\varkappa_{\perp}^i}{T_0}\overline{(\nabla_{\perp}T_i)^2}.$$

*Here, in computing the viscous dissipation we have used the notation of §4. It is also possible to use Eq. (2.21) directly but this procedure is not as convenient.

Taking $T'_e = T'_i = T'$, determining T'/T_0 from Eq. (8.29) and using Eqs. (8.26) and (8.34), (with the notation $\varkappa = \varkappa^e + \varkappa^i$) we find

$$2\omega\delta_{\text{ther}} = \frac{(\gamma-1)^2 T_0 k_\perp^2}{\varrho_0 c_A^2 k^2}\left(\varkappa_\parallel k_\parallel^2 + \varkappa_\perp k_\perp^2\right). \qquad (8.43)$$

Acoustic Wave. Using the same method, by means of Eqs. (8.28) and (8.35), taking $V \approx V_z$ we find

$$2\omega\delta_{\text{Joule}} = \frac{c^2}{4\pi\sigma_\perp} k_\perp^2 \frac{c_s^2}{c_A^2}, \qquad (8.44)$$

$$2\omega\delta_{\text{vis}} = \frac{1}{\varrho_0}\left(\frac{4}{3}\eta_0 k_\parallel^2 + \eta_2 k_\perp^2\right), \qquad (8.45)$$

$$2\omega\delta_{\text{ther}} = \frac{(\gamma-1)^2 T_0}{\varrho c_s^2}\left(\varkappa_\parallel k_\parallel^2 + \varkappa_\perp k_\perp^2\right). \qquad (8.46)$$

The temperature vanishes from Eq. (8.46) if we substitute $c_s^2 = \gamma p_0/\rho_0 = \gamma(Z+1)T_0/m_i$. If the thermal conductivity of the electrons is very large Eqs. (8.43) and (8.46) must be modified. When $\varkappa_e \to \infty$ the electron motion is isothermal rather than adiabatic hence we must assume that the electron adiabaticity index is $\gamma_e = 1$. In this case the electron terms vanish from Eqs. (8.43) and (8.46) and the acoustic velocity is modified: $c_s^2 = (p_e + \gamma p_i)/\rho = (Z+\gamma)T_0/m_i$.

Collisions with Neutrals. The presence of a neutral gas causes the heat of friction to increase because the electrons collide with neutrals as well as ions; what is more important, heat is generated because of the friction between the ions and neutrals. This effect has been considered in [29, 35, 36]. Furthermore, the expressions for δ_{vis} and δ_{ther} now contain the coefficients of viscosity and thermal conductivity modified appropriately to take account of the neutral gas; these are isotropic in weakly ionized plasmas.

Frictional damping in a three-component mixture e, i, and n can be computed using Eq. (7.17). In the expressions calculated above for δ_{Joule} we now substitute σ in accordance with Eq. (7.13). The damping decrement now contains an additional term which we shall call the diffusion term δ_{dif}. It results from collisions of ions with neutrals and arises as a result of the second term in Eq. (7.17):

$$Q_{\text{dif}} = \frac{1}{\alpha_n}\left\{\frac{\xi_n}{c}[\mathbf{jB}] - \mathbf{G}\right\}^2, \quad \mathbf{G} = \xi_n \nabla(p_e + p_i) - \xi_i \nabla p_n.$$

In the Alfvén wave $\mathbf{G} = 0$; in the fast magnetoacoustic wave the ratio of G to the first term in the curly brackets is of order $\xi_i c_s^2/c_A^2$. Neglecting G we find $Q_{\text{dif}} = (\xi_n^2 B^2/\alpha_n c^2) j_\perp^2$. This expression has the same form as the Joule heat; thus, without repeating the calculations we can immediately find the damping by means of Eqs. (8.39) and (8.41) simply by substituting $1/\sigma^*_\perp = 1/\sigma + \xi_n^2 B^2/\alpha_n c^2$ in place of $1/\sigma$. In this case collisions of ions with neutrals simply increase the effective perpendicular resistance.

We now write the diffusion damping for the Alfvén wave and for the fast magnetoacoustic wave

$$2\omega\delta_{\text{dif}} = \frac{c^2 k_\parallel^2}{4\pi}\frac{\xi_n^2 B^2}{\alpha_n c^2} \text{ (Alfvén wave)}, \qquad (8.47a)$$

$$2\omega\delta_{\text{dif}} = \frac{c^2 k^2}{4\pi}\frac{\xi_n^2 B^2}{\alpha_n c^2} \text{ (fast wave)}. \qquad (8.47b)$$

Let $m_i = m_n$ and assume that α_{en} can be neglected compared with α_{in}. The diffusion damping is then larger than the Joule damping by a factor $2\xi_n \omega_e \tau_e \omega_i \bar{\tau}_{in}$ where $1/\bar{\tau}_{in} = \alpha'_{in}(n_i + n_n)$. Both expressions (8.47a) and (8.47b) then reduce to the form

$$\delta_{\text{dif}} = \omega\bar{\tau}_{in}(n_n/n_i). \qquad (8.48)$$

Estimating the magnitude of the diffusion velocity we find $\omega/V \sim \omega\bar{\tau}_{in}\xi_i$ and this quantity must be small if the expressions we have obtained are to apply.

Both terms in Q_{dif} are of the same order for the acoustic wave. Let $m_i = m_n$, in which case $\mathbf{G} = \xi_n \nabla p_e$. Calculations made with Eq. (8.28) give $\nabla p_e = i\mathbf{k}(n_e/n_0)\rho_0 c_s V_z$, where $n_0 = n_e + n_i + n_n$; $[\mathbf{jB}]/c = i\mathbf{k}_\perp \rho_0 c_s V_z$ and, in the usual way, we find

$$2\omega\delta_{\text{dif}} = \frac{\varrho_0 c_s^2 \xi_n^2}{\alpha_n}\left\{k_\parallel^2 \frac{n_e^2}{n_0^2} + k_\perp^2 \frac{(n_i+n_n)^2}{n_0^2}\right\} \approx$$

$$\approx \omega^2 \bar{\tau}_{in}\frac{2n_n}{n_i}\left\{\frac{n_e^2}{n_0^2} + \frac{k_\perp^2 (n_i+n_n)^2}{k_\parallel^2 n_0^2}\right\}. \qquad (8.49)$$

In the second relation we have neglected α_{en}/α_{in} and substituted $\omega = c_s k_{\parallel}$.

In the presence of a neutral gas δ_{dif} frequently can make the largest contribution to the damping.

Collisions Between Different Ions. If the plasma contains ions with different e/m ratios, these ions move with somewhat different velocities, and friction between them can also cause wave damping.

Consider the case $\omega_i \tau_i = e_i B/m_i c \gg 1$. Here we can take Eqs. (7.27) for the diffusion velocity of the ions. For the Alfvén wave and the fast magnetoacoustic wave, neglecting the pressure we have

$$\mathbf{w} = \frac{c}{B}\left(\frac{m_1}{e_1} - \frac{m_2}{e_2}\right)\left[\mathbf{h}\,\frac{d\mathbf{V}}{dt}\right] = -\frac{i\omega}{B}\left(\frac{m_1}{e_1} - \frac{m_2}{e_2}\right)[\mathbf{h}\mathbf{V}].$$

Substituting the frictional heat $\overline{Q_{fr}} = \alpha_{12}\overline{w}^2$ (we only consider collisions between ions), for both of these waves we find

$$2\omega\delta_{dif} = \frac{\alpha_{12}}{\varrho_0}\frac{c^2}{B^2}\left(\frac{m_1}{e_1} - \frac{m_2}{e_2}\right)^2\omega^2 \qquad (8.50)$$

or, in order-of-magnitude terms, $\delta_{dif} \sim \omega/\omega_i^2\tau_i$. This damping is weaker than the damping due to collisions with neutrals because the primary ion velocity is the velocity due to the electric drift, which is the same for both species.

APPENDIX

The collision term for elastic collisions is of the form [1, 2, 3]

$$C_{ab}(f_a, f_b) = \int \left\{ f_a(\mathbf{v}') f_b(\mathbf{v}_b') - f_a(\mathbf{v}) f_b(\mathbf{v}_b) \right\} u\, d\sigma\, d\mathbf{v}_b. \qquad (A.1)$$

Here $d\sigma = \sigma(u, \vartheta)\, do$ is the differential cross section for scattering into the element of solid angle $do = \sin\vartheta\, d\vartheta\, d\varphi$ for collisions of particles with relative velocity $u = |\mathbf{v} - \mathbf{v}_b|$. Before particle a collides with particle b it has a velocity $\mathbf{v}$ while the b particle has a velocity $\mathbf{v}_b$. The post-collision velocities $\mathbf{v}'$ and $\mathbf{v}_b'$ are related to $\mathbf{v}$ and $\mathbf{v}_b$ by the laws of elastic collisions (conservation of momentum and energy). The second term in the curly brackets gives the loss of a particles out of the element of volume $d\mathbf{v}$ in velocity space around $\mathbf{v}$ resulting from collisions with b particles; the first term corresponds to the influx into this elementary volume. Collisions of a and b particles with velocities v_a and v_b can be analyzed most simply in the coordinate system in which the total momentum vanishes: $m_a\mathbf{v}_a + m_b\mathbf{v}_b = 0$. Introducing the relative velocity $\mathbf{u} = \mathbf{v}_a - \mathbf{v}_b$, in this system we have $\mathbf{v}_a = m_b(m_a + m_b)^{-1}\mathbf{u}$; $\mathbf{v}_b = -m_a(m_a + m_b)^{-1}\mathbf{u}$; $m_a v_a^2/2 + m_b v_b^2/2 = m_{ab}u^2/2$, where $m_{ab} = m_a m_b(m_a + m_b)^{-1}$ is the reduced mass. As a consequence of the conservation of energy the relative velocity cannot change in magnitude in the collision $u = |\mathbf{v}_a - \mathbf{v}_b| = |\mathbf{v}_a' - \mathbf{v}_b'|$ but can only be deflected through some angle ϑ.

Strictly speaking, the Boltzmann form of the collision term (A.1) does not apply for Coulomb collisions. This results from the fact that substitution of the Rutherford cross section in the place of $d\sigma$ leads to a divergent integral. However, if this integral is cut off, as indicated in §4, the expression in (A.1) gives the same results as the collision term in the Landau form.

We now compute the friction force $\mathbf{R}_{ab}$ experienced by a gas of a particles in collisions with b particles, assuming that both particle species have Maxwellian distributions at the same temperature but with different mean velocities $\mathbf{V}_a$ and $\mathbf{V}_b$. We use the notation $\mathbf{U} = \mathbf{V}_a - \mathbf{V}_b$ and assume that the velocity shift is small compared with the relative velocities of the particles $U \ll (T/m_{ab})^{1/2}$. The friction force is

$$\mathbf{R}_{ab} = \int m_a \mathbf{v} C_{ab}(f_a, f_b)\, d\mathbf{v}.$$

This force is independent of the coordinate system and will be computed in the system in which $\mathbf{V}_a = 0$, $\mathbf{V}_b = -\mathbf{U}$. In this coordinate system $f_a = f_a^0$, and f_b can be expanded in powers of U to give the expression

$$f_b = f_b^0 - (m_b/T)(\mathbf{U}\mathbf{v}_b) f_b^0.$$

Substitution of f_a^0 and f_b^0 in $\mathbf{R}_{ab}$ obviously gives zero, so that

$$R_{ab\alpha} = -\frac{m_a m_b}{T} U_\beta \int v_\alpha C_{ab}\left(f_a^0, f_b^0 v_{b\beta}\right) d\mathbf{v}$$

Here, C_{ab} is given by (A.1) and represents a vector that depends on $\mathbf{v}$ (on $\mathbf{v}_b$ in the integration). Since this vector does not contain any vector parameters other than the velocity $\mathbf{v}$, it is of the form $\mathbf{v}A(v)$; in component form: $v_\beta A(v)$ where $A(v)$ is a scalar function of the speed. Averaging over angle under the integral sign and using the relation $\overline{v_\alpha v_\beta} = (v^2/3)\,\delta_{\alpha\beta}$ we find

$$R_{ab} = -\frac{m_a m_b}{3T}\,\mathbf{U}\int v_\beta C_{ab}\left(f_a^0,\ f_b^0 v_{b\beta}\right)dv.$$

We now substitute C_{ab} in accordance with Eq. (A.1) and use the relation $f_a^0(v_a')f_b^0(v_b') = f_a^0(v_a)f_b^0(v_b)$ which follows from the conservation of energy in collisions (we write $\mathbf{v}_a$ in place of $\mathbf{v}$ to obtain a symmetric expression). Then

$$R_{ab} = -\frac{m_a m_b}{3T}\,\mathbf{U}\int f_a^0 f_b^0 v_{a\beta}\left\{v_{b\beta}' - v_{b\beta}\right\} u\,d\sigma\,d\mathbf{v}_a d\mathbf{v}_b.$$

To compute the integral we convert from the variables $\mathbf{v}_a$ and $\mathbf{v}_b$ to the velocity of the center of mass $\mathbf{v}_c$ and the relative velocity $\mathbf{u}$:

$$\mathbf{v}_a = \mathbf{v}_c + \frac{m_b}{m_a+m_b}\mathbf{u},\quad \mathbf{v}_b = \mathbf{v}_c - \frac{m_a}{m_a+m_b}\mathbf{u}. \tag{A.2}$$

It is easily shown that $d\mathbf{v}_a d\mathbf{v}_b = d\mathbf{v}_c d\mathbf{u}$ and that $f_a^0 f_b^0 = n_a n_b f_c^0 f_u^0$, where

$$f_c^0 = \left(\frac{m_a+m_b}{2\pi T}\right)^{3/2} e^{-\frac{(m_a+m_b)v_c^2}{2T}},\qquad f_u^0 = \left(\frac{m_{ab}}{2\pi T}\right)^{3/2} e^{-\frac{m_{ab}u^2}{2T}}. \tag{A.3}$$

The integration over $d\mathbf{v}_c$ is easily performed and the $\mathbf{v}_c$ term drops out of $\mathbf{v}_a$; integration of f_c^0 gives unity. Figure 9 shows directly that $u_\beta(u_\beta - u_\beta') = u^2(1-\cos\vartheta)$ and finally we have

$$R_{ab} = -n_a n_b m_{ab}\alpha_{ab}'\mathbf{U}, \tag{A.4}$$

where

$$\alpha_{ab}' = \frac{m_{ab}}{3T}\int u^3\sigma_{ab}' f_u^0 d\mathbf{u}, \tag{A.5}$$

$$\sigma_{ab}'(u) = \int(1-\cos\vartheta)\,d\sigma(u,\vartheta). \tag{A.6}$$

When $\sigma_{ab}' = \alpha_{ab}'/u$ we obtain Eq. (7.4); if $\sigma_{ab}' = \text{const}$ we obtain Eq. (7.5). Substituting the Rutherford cross section in (A.6) we obtain a divergent integral in which the artificial cutoff gives Eq. (7.6).

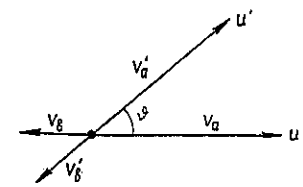

Fig. 9

Similarly, it is possible to compute the frictional heat Q_{ab} generated in a gas of a particles in collisions with b particles when both particle species have Maxwellian distributions at the same temperature but with small shifts $\mathbf{U}$. In the expression

$$Q_{ab} = \int\frac{m_a v_a^2}{2}C_{ab}\left(f_a^0,\ f_b\right)d\mathbf{v}_a$$

we carry out an expansion up to the quadratic terms in the shift:

$$f_b = f_b^0\left[1 - \frac{m_b}{T}(\mathbf{U}\mathbf{v}_b) - \frac{m_b}{2T}U^2 + \frac{m_b^2}{2T^2}(\mathbf{U}\mathbf{v}_b)^2\right].$$

Only the last term gives a nonvanishing contribution:

$$Q_{ab} = \frac{m_a m_b^2}{4T^2}U_\alpha U_\beta\int f_a^0 f_b^0 v_a^2\left\{v_{b\alpha}'v_{b\beta}' - v_{b\alpha}v_{b\beta}\right\}u\,d\sigma\,d\mathbf{v}_a d\mathbf{v}_b.$$

Carrying out the substitution $\mathbf{v}_a$, $\mathbf{v}_b \to \mathbf{v}_c$, $\mathbf{u}$, after some simple calculations we find

$$Q_{ab} = \frac{m_b}{m_a+m_b}n_a n_b m_{ab}\alpha_{ab}'U^2. \tag{A.7}$$

The expression for Q_{ba} is obtained by interchange of the subscripts. It is evident that $Q_{ab} + Q_{ba} = -\mathbf{R}_{ab}\mathbf{U}$, where the total heat generated by friction is distributed between the components in inverse proportion to their masses.

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