Critical scales for the destabilization of the toroidal ion-temperature-gradient instability in magnetically confined toroidal plasmas

Summary of work done in collaboration with
J. W. Connor\textsuperscript{1}, H. Doerk\textsuperscript{2}, P. Helander\textsuperscript{2} and P. Xanthopoulos\textsuperscript{2}
and presented by Alessandro Zocco\textsuperscript{2} at the First JPP Conference
"Frontiers in Plasma Physics",
Abazia della Santissima Trinità Spineta
Maggio 2017

\textsuperscript{1}Culham Science Centre, Abingdon, Oxon, OX14 3DB, UK
\textsuperscript{2}Max-Planck-Institut für Plasmaphysik, D-17491, Greifswald, Germany

We present, for the first time, a complete mathematical treatment of the resonant limit of the ion-temperature-gradient (ITG) driven instability for magnetically confined fusion plasmas with arbitrary ion Larmor radii. The analysis is performed in the local kinetic limit, thus neglecting the variation of the eigenfunction along the magnetic field line, but is valid for arbitrary geometry (i). Analytical progress is made by introducing a Padé approximant (ii) for the ion response after integrating resonances (iii). These are treated by using a new series representation of the Owen $T-$function [Owen, D B (1956). "Tables for computing bivariate normal probabilities". Annals of Mathematical Statistics, 27, 1075–1090] which exploits the properties of the incomplete Euler gamma function [Tricomi F. G., Sulla funzione gamma incompleta, Annali di Matematica Pura ed Applicata, 31, 1, 263-279 (1950), Tricomi F. G., Asymptotische Eigenschaften der unvollständigen Gammafunktion, Mathematische Zeitschrift, Band 53, Heft 2, S. 136-148 (1950)] (iv). No approximation for the velocity space dependence of the particles drifts is made. The critical threshold for resonant destabilization of the toroidal branch of the ITG is then calculated and compared to previous analytical results. Comparisons to direct numerical simulations performed with the GENE code in a simple geometrical setting are also reported (v). The full Larmor radius resonant theory predicts higher critical gradients than the long wavelength resonant theory. Like results from numerical simulations, it gives a critical gradient that depends linearly with $\tau = T_i/T_e$, where $T_i$ and $T_e$ are the ion and electron temperature, respectively, but it underestimates the numerical results. The discrepancy can be attributed to the fact that analytical theories predict a real frequency at marginality (vi) that does not depend on $\tau$ as opposed to the numerical findings (vii). This is true for all analytical theories here revisited and derived. The stability diagram in Fig. (2) of Biglari Diamond and Rosenbluth [Phys. of Fluids B 1, 109 (1989)] has also been reproduced, and gives virtually no dependence on $\tau$. The results from GENE

(i) Eq. (4.1) of Sec. (4)
(ii) See Eqs. (4.8)-(4.15) and (4.17) of Section (4)
(iii) See Eq. (4.8) of Section (4)
(iv) This is done in Appendix (B)
(v) See Fig. (1) of Sec. (1)
(vi) A discussion of the evaluation of the real frequency at marginality is in Appendix (D)
(vii) See Fig. (3) of Sec. (1)
simulations are somewhat consistent with the fitting formula of Jenko Dorland and Hammett (JDH) [Phys. Plasmas, 8, 9 (2001)] (viii) but do not seem to relate to the theory of Hahm and Tang [PPPL Report (1988)] (ix). In the limit in which the streaming term contribution is negligible in the JDH formula ($q \to \infty$, and/or $\hat{s} \to 0$, where $q$ and $\hat{s}$ are the safety factor and the global shear, respectively) the analysis of the velocity space structure of solutions (x) allows us to characterize the role of the trapped ions in the modification of the critical threshold for destabilization of the ITG. These generate a family of unstable trapped ion modes (TIM) below the critical threshold for destabilization of the ITG (xi). Such modes require high velocity space resolution to be adequately resolved. The TIM-modified critical threshold is studied as a function of $\tau$, for several density gradients (xii). For $\tau \gtrsim 1$, such threshold shows a dependence on $\tau$. For $\tau \lesssim 1$, the critical threshold of $R/L_T$ is proportional to the inverse density gradient scale $R/L_n$. A TIM-modified critical threshold is found also in the Stellarator Wendelstein 7X; here the TIM effect is even more pronounced (xiii). The eventuality of trapped ion turbulence below critical ITG turbulence must therefore be assessed in devices that allow comparable TIM and ITG levels.

(viii) Fig. (1), compare JDH to GENE curves
(ix) Fig. (1), compare “Romanelli” to “GENE at $q = 100$” curve
(x) See Figs. (7)-(8) of Sec. (1)
(xi) Fig. (4) of Sec. (1)
(xii) Fig. (8) of Sec. (1)
(xiii) See Fig. (9)
1. Results

1.1. Tokamak preliminaries

Figure (1) shows the critical gradient for resonant destabilisation of the ion-temperature-gradient (ITG) instability from the GENE code for several ion temperatures [cross-line in Fig. (1)]. Adiabatic electrons. Cyclon Base Case type parameters in $s - \alpha$ geometry: $\hat{s} = 0.786$, $q = 1.4$, $\epsilon = r/R = 0.18$ $R/L_n = 2$, $k_y \rho_i = 0.3$. The critical values are extrapolated from Fig. (2), which shows the growth rates for several $\tau = T_i/T_e$. Notice the dependence on $\tau$ in Fig. (3) (mode frequency). The linear fit of the numerical data gives 

$$R/L_T = (2.45 \pm 0.07) \tau + 2.07 \pm 0.06,$$

with a reduced $\chi^2 = 6.5 \times 10^{-4}$. The fitting JDH formula of Jenko et al. (2001) is

$$R/L_T = (1 + \tau) \left( \frac{4}{3} + 1.91 \frac{\hat{s}}{q} \right)$$

The curve derived by Romanelli (1989) is

$$R/L_T = (1 + \tau) \left( \frac{4}{3} \right),$$

which is evaluated by replacing $v_\perp^2/2 + v_\parallel^2 \rightarrow 4/3(v_\perp^2 + v_\parallel^2)$ in the definition of the particles magnetic drift [see Eq. (4.4)] in order to fit the numerical solution of Eq. (4.1) at $R/L_n \ll 1$, for $k_y \rho_S = \sqrt{0.1} \approx 0.32$, where $\rho_S = \rho_i / \sqrt{2}$. It should not match the numerical results for $R/L_n = 2$, $R/L_T = O(1)$, even if $q \rightarrow \infty$ [in which case the streaming term effect measured by the factor $\hat{s}/q$ in the JDH formula should be negligible, as it is neglected in Eq. (4.1)]. The condition for destabilization of Biglari et al. (1989) (BDR) [Eq. (1.1), rederived in two different ways in Appendix (D)] would give $R/L_T = 2.7$, with virtually no dependence on $\tau$. We must stress that we reproduced the stability diagram of Fig. (5) of Biglari et al. (1989). Their result, $R/L_T = 2.7$, can be obtained only if we include an extra multiplicative factor in the definition of the magnetic drift, e.g. $\omega_c = 2\omega_{\nu_1}(L_n/R)$ [however, in this work we follow Hastie & Hesketh (1981) and use $\omega_c = \omega_{\nu_1}(L_n/R)$], and the frequency at marginal
stability is evaluated by solving the equation

\[
\left\{1 - \frac{\omega_{sj}}{\omega_r} + \eta_i \frac{\omega_{sj}}{\omega_r} \left(1 - 2 \frac{\omega_r}{\omega_{\kappa}}\right)\right\} \Re \left[\sqrt{\frac{\omega}{\omega_{\kappa}}}ight] - \eta_i \frac{\omega_{sj}}{\sqrt{\omega_r \omega_{\kappa}}} \omega_r = 0. \tag{1.1}
\]

More details are in Appendix (D).
Resonant destabilization of ITG mode

Figure 3. Real frequency as a function of the temperature gradient for different $\tau$. CBC parameters.

Figure 4. Imaginary part of eigenvalue as a function of the temperature gradient for different $\tau$. Parameters like in Fig 2 but $\hat{s} = 0.1$, $\epsilon = 0.03$. The pedestal at marginality is evident.
Figure 5. Real part of eigenvalue as a function of the temperature gradient for different $\tau$. Parameters like in Fig. 3 but $\hat{s} = 0.1$, $\epsilon = 0.03$. A jump is evident.

Figure 6. Ion distribution function in phase space. $\tau = 1$, $R/L_T = 3$, “weak” mode that belongs to the “pedestal” of Fig. 4. Resonance occurs inside the trapped-passing boundary.
Resonant destabilization of ITG mode

Figure 7. Ion distribution function in phase space. $\tau = 1$, $R/L_T = 5$, “strong” mode. Resonance occurs outside the trapped-passing boundary.

Figure 8. Critical threshold from GENE data of Fig. 4 for several density gradients $R/L_n$. 

$R/L_n = 1.5$ $R/L_n = 2$

While analysing the data that generated Fig. (1), and trying to recover an asymptotic limit where the local theory would be valid, we discovered a family of unstable modes below the critical threshold of the ITG [See Fig. (6)]. These modes are present at small global shear \( \hat{s} \), therefore we can conjecture they could also be present in Stellarators. Numerical simulations show that this is indeed the case [see Fig. (9)]. The role of such modes for turbulence in complex geometries has perhaps been overlooked.

2. Essential literature and preliminary discussion

The first study of the resonant toroidal ITG was presented by Terry et al. (1982). The full dispersion relation is studied numerically. Our integral \( I_i \) is presented in Eq. (A9), but the authors say “While this form is exact, it is cumbersome analytically”. The authors then focus their analysis on the so-called “grad-B drift model” and on the \( R/L_n \gg 1 \) limit. Romanelli (1989), instead, focuses on the more relevant flat density limit \( R/L_n \ll 1 \). In 1988, Romanelli also used the “grad-B drift model” in order to derive the mode frequency at marginality [Eq. (25) of Ref. (Romanelli 1989)], which in turns determines the critical gradient length for destabilization. The same year, Biglari et al. (1989) (BDR), however, anticipated that such frequency can only be evaluated numerically. We agree with BDR, as we derived analytically from our closed form the \( \Im[D] = 0 \) equation of page 11 of Ref. Biglari et al. (1989) [See Appendices C and D]. The impossibility of determining analytically the frequency at marginality can be traced to the use of the full expression of the magnetic drifts. In all cases, the frequency at marginal stability does not depend on the temperature ratio \( \tau \), at odds with the numerical results here reported. Therefore,
the connection of the fitting formula for the critical gradient of Jenko et al. (2001) to previous local resonant theories seems to be weakened by our results. The key original result of this study is the perhaps surprising presence of a family of unstable trapped-ion-modes below the critical threshold for ITG destabilisation. These modes modify what one would think is the threshold of marginal microstability, especially for low global shear. The same effect manifests in stellarators, shifting the critical threshold to values up to a factor of two lower than then in the ITG case.

3. Open Discussion

Questions

Answers
4. Complementary material: local kinetic limit formulation

We study the following dispersion relation

\[ 1 + \tau = I_i \equiv \int d^3v \frac{\omega - \omega_i^E}{\omega - \omega_i} J_0^2 (k_{\perp}^2 \rho_i^2 \hat{v}_i^2) \ F_{0i} \frac{v}{n_0} , \]  

(4.1)

where \( \tau \equiv T_i/T_e \), \( \omega \) is the mode complex frequency, \( k_{\perp} \) is the wave vector, \( \hat{v}_i^2 = (v_{\parallel}^2 + v_{\perp}^2) = \sqrt{2T_i/m_i} \) the ion thermal speed, and \( n_i \) and \( n_0 \) the ion mass and density, respectively. The equilibrium distribution function is taken to be Maxwellian,

\[ F_{0i} = \frac{n_0}{(\pi \nu_{hi}^2)^{3/2}} e^{-\nu_{hi}^2} . \]

(4.2)

Two characteristic frequencies are present in Eq. (4.1)

\[ \omega_i^E = \omega_i [1 + \eta_i (\hat{v}_i^2 - 3/2)] , \]

with \( \omega_i = 0.5k_{\parallel}v_{thi}/L_N \), \( \eta_i = L_N/L_T \), and

\[ \omega_d = \left( \omega_i \hat{v}_0^2 + \omega_B \hat{v}_0^2/2 \right) , \]

(4.4)

where \( \rho_i = v_{thi}/\Omega_i \) is the ion Larmor radius, \( \omega_i = \omega_B = L_N^{-1} = n_0^{-1} dn_0/dx \), and \( L_T^{-1} = T_i^{-1} dT_i/dx \). Equation (4.1) [explain]. We proceed with our analysis.

Some preliminary manipulations are in order. We add and subtract \( \omega_i \hat{v}_0^2 + \omega_B \hat{v}_0^2 /2 \) at the numerator of \( I_i \) to obtain

\[ I_i = 2 \int_0^\infty d\hat{v}_{\perp} \hat{v}_{\perp} J_0^2 (k_{\perp}^2 \rho_i^2 \hat{v}_i^2) e^{-\nu_{hi}^2} \]

\[ \frac{\omega_{\parallel} \omega_i}{\omega} \left( 1 - \frac{3}{2} \eta_i \right) J_{(0)} \]

\[ + \frac{\omega_i}{\omega} \omega_i \eta_i \frac{J_{(2)}}{\omega} \]

\[ + \frac{\omega_B/2 - \omega_i \eta_i \kappa_{\parallel}^{(2)}}{\omega} \]

(4.5)

with

\[ J_{(a)} = 2 \int_0^\infty d\hat{v}_{\perp} \hat{v}_{\perp} \int_{-\infty}^\infty d\hat{v}_{\parallel} \frac{J_0^2 (k_{\perp}^2 \rho_i^2 \hat{v}_i^2)}{1 - \hat{v}_0^2 \hat{v}_{\perp}^2 / \pi} e^{-(\hat{v}_0^2 + \hat{v}_{\perp}^2)} . \]

(4.6)

The first line of Eq. (4.5) gives the well known result of gyrokinetic theory

\[ 2 \int_0^\infty d\hat{v}_{\perp} \hat{v}_{\perp} J_0^2 (k_{\perp}^2 \rho_i^2 \hat{v}_i^2) e^{-\nu_{hi}^2} = \Gamma_0(b) , \]

(4.7)

where \( b = k_{\perp}^2 \rho_i^2 /2 \), and \( \Gamma_0(b) = I_0(b) \text{exp}(-b) \), with \( I_0(b) \) the modified Bessel function. We now integrate the resonances in a way similar to that presented by Biglari Diamond and Rosenbluth (Biglari et al. 1989) (BDR in the following). For \( J_{(0)} \) we have
Resonant destabilization of ITG mode

\[ J^{(0)} = -2i \int_0^\infty \tilde{d} v \tilde{v} J_0^2 \int_0^\infty \tilde{d} v \tilde{v} \int_0^\infty \tilde{d} v e^{i \lambda \left[ 1 - (\tilde{v}^2 + \tilde{v}^2/2) - (\tilde{v}^2 + \tilde{v}^2) \right]} \]

\[ = -2i \int_0^\infty \lambda^\lambda \int_0^\infty \tilde{d} v \tilde{v} e^{-\tilde{v}^2 (1 + \lambda \omega A/ \omega B)} e^{-\tilde{v}^2 (1 + \lambda \omega A/ \omega B)} \]

\[ = -i \int_0^\infty \lambda^\lambda \left( \frac{\tilde{b}}{1 + i \lambda \omega B/ \omega A} \right) \frac{1}{1 + i \lambda \omega B/ \omega A}, \]

with \( \tilde{b} = b/[1 + i \lambda \omega B/(2 \omega)] \). We need

\[ \Im[\lambda \omega / \omega] < 1, \] (4.9)

\[ \Im[\lambda \omega B / 2 \omega] < 1, \] (4.10)

and

\[ \Im[\lambda] > 0 \] (4.11)

in order to guarantee convergence of the velocity-space and \( d \lambda \) integrals, respectively.

Conditions (4.9)-(4.11) thus define an abscissa of convergence in the complex \( \lambda \)-plane

\[ \Re[\lambda] > \alpha_c \equiv \Im[\omega]^{-1} (\Im[\lambda] \Re[\omega] - |\omega|^2 / \omega A). \] (4.12)

Here we are using \( \omega_B \equiv \omega_A \) (valid in the electrostatic case) and taking the most restrictive of conditions (4.9)-(4.11). Similarly, we obtain

\[ J^{(2)}_\parallel = -i \int_0^\infty \lambda^\lambda \frac{e^{i \lambda}}{(1 + i \lambda \omega A/ \omega B)^{3/2}} \frac{\Gamma_0 \left( \tilde{b} \right)}{1 + i \lambda \omega B/ \omega A}, \] (4.13)

and

\[ J^{(2)}_\perp = -i \int_0^\infty \lambda^\lambda \frac{e^{i \lambda}}{(1 + i \lambda \omega A/ \omega B)^{1/2}} \frac{\Gamma_0 \left( \tilde{b} \right)}{1 + i \lambda \omega B/ \omega A} \left[ \Gamma_1 \left( \tilde{b} \right) - \Gamma_0 \left( \tilde{b} \right) \right]. \] (4.14)

We now introduce the Padé approximants

\[ \Gamma_0 \left( \tilde{b} \right) \approx \frac{1}{1 + \frac{b}{1 + i \lambda \omega A/ \omega B}}, \] (4.15)

and

\[ \Gamma_0 \left( \tilde{b} \right) + \tilde{b} \left[ \Gamma_1 \left( \tilde{b} \right) - \Gamma_0 \left( \tilde{b} \right) \right] \approx \frac{1}{1 + \frac{b}{1 + i \lambda \omega A/ \omega B}} \left( \Gamma_0 \left( \tilde{b} \right) + \tilde{b} \right), \] (4.16)

which will allow us to perform the \( d \lambda \) integration. The integral \( J^{(0)} \) becomes

\[ J^{(0)} = -i \int_0^\infty \lambda^\lambda \frac{e^{i \lambda}}{(1 + i \lambda \omega A/ \omega B)^{1/2}} \frac{1}{1 + i \lambda \omega B/ \omega A + b}, \] (4.17)

For \( \lambda \to \lambda \omega / \omega_A, b \to 0, \) and \( \omega_B \equiv \omega_A, \)
A. Zocco, et al.

\[
\lim_{b \to 0} J^{(0)} = -i \omega \int_0^\infty d\lambda \frac{e^{i\Omega \lambda}}{(1 + i\lambda)^{1/2}} \frac{1}{1 + i\frac{\omega}{\omega_c}}
\]

\[= \frac{\omega}{\omega_c} F_{1,1}, \tag{4.18}\]

with \(F_{1,1}\) defined in Eq. (A2) of BDR. We notice that the change of variables \(\lambda \to \lambda \omega/\omega_c\) requires \((\Re[\lambda]3[\omega] + 3[\lambda]3[\omega]) / \omega_c > 0\) for the convergence of the \(d\lambda\) integral. At marginal stability, for \(\omega_c < 0\) and \(\omega < 0\), this is condition (4.11).

5. Arbitrary Larmor radii solution

For \(b \neq 0\), the analytical technique used by BDR to solve for \(F_{1,1}\) cannot be used to perform the integration. We prefer to proceed in a different way.

Let us consider

\[
R = -i \int_0^\infty d\lambda \frac{e^{i\Omega \lambda}}{(1 + i\lambda)^{1/2}} \frac{1}{1 + i\frac{\omega}{\omega_c} \frac{\omega_c}{\omega} + b}, \tag{5.1}\]

with \(\Omega = \omega/\omega_c\). We change variables

\[i\lambda = t^2 - 1, \tag{5.2}\]

and obtain

\[
R = -4 \frac{\omega_c}{\omega_B} \int_1^{\frac{\pi}{2}} dt \frac{e^{\Omega(t^2 - 1)}}{2 \omega_B (1 + b) - 1 + t^2}
\]

\[= -4 \frac{\omega_c}{\omega_B} e^{-\Omega} \int_{1/\sqrt{g}}^{\sqrt{g}} \frac{e^{\Omega u^2}}{\sqrt{g} \sqrt{1 + u^2}}, \tag{5.3}\]

with

\[g = 2 \frac{\omega_c}{\omega_B} (1 + b) - 1. \tag{5.4}\]

In the case \(\omega_c = \omega_B\), \(g = 1 + 2b > 0 \ \forall \ b\). We split the domain of integration and define

\[
R = -4 \frac{\omega_c}{\omega_B} e^{-\Omega} \left( \int_0^{\sqrt{g}} du \frac{e^{\Omega u^2}}{\sqrt{g} \sqrt{1 + u^2}} \right.
\]

\[\left. + \int_0^{1/\sqrt{g}} du \frac{e^{\Omega u^2}}{\sqrt{g} \sqrt{1 + u^2}} \right) \left( \frac{e^{\Omega u^2}}{1 + u^2} \right)
\]

\[\equiv -4 \frac{\omega_c}{\omega_B} e^{-\Omega} (I_{\infty} - I_g). \tag{5.5}\]

In appendix A and B, we show that

\[
R = -4 \frac{\omega_c}{\omega_B} e^{-\Omega} \left\{ \frac{i}{2} \sqrt{\pi} Z \left( \sqrt{g} \Omega \right) - 2 \pi e^{-\sqrt{g}\Omega} T \left[ i \sqrt{2g\Omega}, \frac{1}{\sqrt{g}} \right] \right\}, \tag{5.6}\]

where \(Z\) is the plasma dispersion function (Fried & Conte 1961), and \(T\) the Owen \(T\)-function (Owen 1956). By direct comparison of Eqs. (4.17) and (5.1), we have

\[J^{(0)} = \frac{\omega}{\omega_c} R. \tag{5.7}\]
The integral $J^{(2)}_\perp$ now becomes
\[
J^{(2)}_\perp = -i \int_0^\infty d\lambda \frac{e^{i\lambda}}{(1 + i\lambda \frac{\omega_0}{\omega})^{1/2}} \left( \frac{1}{1 + i\lambda \frac{\omega_0}{\omega} + b} \right)^2
\]
\[
= -\frac{d}{db} J^{(0)}_\perp. \tag{5.8}
\]

To evaluate the integral $J^{(2)}_\parallel$, it is convenient to consider a simple generalization of the auxiliary integral $R$, namely
\[
R_\nu = -i \int_0^\infty d\lambda e^{i\lambda} \frac{e^{i\nu \Omega}}{(\nu + i\lambda)^{1/2}} \left( \frac{1 + i\lambda \omega B}{1 + i\lambda \frac{\omega_0}{\omega} + b} \right)^2 \tag{5.9}
\]
Then
\[
J^{(2)}_\parallel = -\lim_{\nu \to 1} \Omega \frac{d}{d\nu} R_\nu. \tag{5.10}
\]
The integral $R_\nu$ can be related to $R$ after some straightforward algebra. We find
\[
R_\nu = -\frac{\omega_\kappa}{\omega_B} e^{-\nu \Omega} \int_{\sqrt{\nu/g_\nu}}^{\infty} \frac{d\nu}{\sqrt{2\nu}} \frac{\sqrt{2\nu + 1 + u^2}}{1 + u^2} \tag{5.11}
\]
with
\[
g_\nu = \frac{2\omega_B}{\omega_\kappa} (1 + b) - \nu. \tag{5.12}
\]

Summarizing, the local kinetic dispersion relation is
\[
1 + \tau = \Gamma_0(b) - \frac{\omega_s}{\omega} \left( 1 - \frac{3}{2} \eta \right) J^{(0)}_\perp + \frac{\omega_s}{\omega} \left( \frac{\omega_\kappa}{\omega_s} - \eta \right) J^{(2)}_\perp + \frac{\omega_s}{\omega} \left( \frac{\omega_B}{2\omega_s} - \eta \right) J^{(2)}_\parallel, \tag{5.13}
\]
with
\[
J^{(0)}_\perp = \lim_{\nu \to 1} \frac{\omega}{\omega_\kappa} R_\nu, \tag{5.14}
\]
\[
J^{(2)}_\perp = -\frac{d}{db} J^{(0)}_\perp, \tag{5.15}
\]
\[
J^{(2)}_\parallel = -\lim_{\nu \to 1} \Omega \frac{d}{d\nu} R_\nu, \tag{5.16}
\]
with $R_\nu$ defined in Eq. (5.11), $Z$ is the plasma dispersion function, $\Omega = \omega/\omega_\kappa$, $g_\nu = 2(1 + b)\omega_\kappa/\omega_B - \nu$, and
\[
T \left[ i \sqrt{2g_\nu \frac{\omega_\kappa \omega}{\omega_\kappa} \sqrt{\nu}} \right] = e^{g_\nu \frac{\omega}{\omega_\kappa}} \frac{\sqrt{\nu}}{2\pi \sqrt{g_\nu}} \sum_{n=0}^{\infty} \frac{(-\nu/g_\nu)^n}{(2n + 1)} \sum_{m=0}^{n} \frac{(-g_\nu \frac{\omega}{\omega_\kappa})^m}{m!}. \tag{5.17}
\]
This expression for the Owen $T-$function is derived in Appendix (B).
Appendix A. The integral $I_\infty$

For $I_\infty$, we change variables to

$$g\Omega u^2 \to -\xi^2,$$

(A 1)

so that, when $u \to e^{i\pi}_\infty$, $\xi \to e^{(\frac{3\pi}{4} + \pi/2)}_\infty$. A closed form of the integral can be found for an unstable mode $\arg(\Omega g) = \arg \left\{ \omega \frac{[2\omega \kappa (1 + b) - \omega B]}{\omega B} \right\} \approx \pi/2$, for $\Re[\omega] \to 0$. Then, $\xi \to e^{(\frac{3\pi}{4} + \pi/2)}_\infty \approx e^{i(\frac{3\pi}{4} + \pi/4)}_\infty = -\infty$, and $I_\infty$ becomes

$$I_\infty = -i\sqrt{\Omega} \int_{-\infty}^{\infty} d\xi \frac{e^{-\xi^2}}{(\sqrt{g}\xi)^2 - \xi^2}$$

$$= i\frac{\sqrt{\Omega}}{2} \int_{-\infty}^{\infty} d\xi \frac{e^{-\xi^2}}{2\sqrt{g}\xi} \left\{ \frac{1}{\xi + \sqrt{\Omega}} - \frac{1}{\xi - \sqrt{\Omega}} \right\}$$

$$= -\frac{i}{2} \frac{\sqrt{\pi}}{\sqrt{g}} Z (\sqrt{g}\Omega),$$

(A 2)

where $Z$ is the plasma dispersion function (Fried & Conte 1961). Analytic continuation is needed for damped modes.

Appendix B. The integral $I_g$ and the Owen $T$–function

In the case of $I_g$, we have

$$I_g = \int_0^{1/\sqrt{g}} du \frac{e^{\Omega g u^2}}{\sqrt{g} 1 + u^2}. \quad (B 1)$$

This integral can be written in terms of the Owen $T$–function (Owen 1956).

$$I_g = \int_0^{1/\sqrt{g}} du \frac{e^{\Omega g u^2}}{\sqrt{g} 1 + u^2}$$

$$= e^{-g\Omega} \int_0^{1/\sqrt{g}} du \frac{e^{-(i\sqrt{g}\Omega) t} \left( \frac{\xi^2}{2} + \frac{1}{2} \right)}{\sqrt{g} 1 + u^2}$$

$$= 2\pi e^{-g\Omega} T \left[ i\sqrt{2g\Omega}; \frac{1}{\sqrt{g}} \right].$$

For analytical manipulations and numerical implementation, the most convenient way of writing the Owen function is perhaps the following. Since

$$T \left[ i\sqrt{2g\omega}; \frac{\nu}{\omega\kappa}; \frac{\nu}{g} \right] = \frac{1}{2\pi} \int_0^{\nu/g} dt \frac{e^{g\nu} \left( 1 + t^2 \right)}{1 + t^2}, \quad (B 2)$$
if we rewrite
\[
T \left[ i \sqrt{2g_\nu \frac{\omega}{\omega_\kappa}}; \frac{\sqrt{\nu}}{g_\nu} \right] = \frac{1}{2\pi} \int_0^\infty dt e^{\frac{g_\nu}{\omega_\kappa} (1+t^2)}
\]
\[
= \frac{1}{2\pi} \int_0^\infty d\xi \int_0^\infty e^{\xi (1+t^2)} dt e^{\xi (1+t^2)}
\]
\[
= \frac{1}{2\pi} \int_0^\infty d\xi \int_0^\infty e^{\xi (1+t^2)} dt e^{\xi (1+t^2)}
\]
we are then able to introduce the series representation of the Error function, to obtain
\[
T \left[ i \sqrt{2g_\nu \frac{\omega}{\omega_\kappa}}; \frac{\sqrt{\nu}}{g_\nu} \right] = \frac{1}{2\pi} \int_0^\infty d\xi e^{-\xi} \sum_{n=0}^\infty \frac{(-\nu/g_\nu)^n}{n!(2n+1)} \left( \xi - g_\nu \frac{\omega}{\omega_\kappa} \right)^n
\]
\[
= \frac{1}{2\pi} \int_0^\infty d\xi e^{-\xi} \sum_{n=0}^\infty \frac{(-\nu/g_\nu)^n}{n!(2n+1)} \left( \xi - g_\nu \frac{\omega}{\omega_\kappa} \right)^n
\]
\[
= \frac{1}{2\pi} \int_0^\infty d\xi e^{-\xi} \sum_{n=0}^\infty \frac{(-\nu/g_\nu)^n}{n!(2n+1)} \Gamma (n+1) - \gamma (n+1, -g_\nu \frac{\omega}{\omega_\kappa})
\]
where
\[
\gamma (n+1, -g_\nu \frac{\omega}{\omega_\kappa})
\]
is the incomplete gamma function (Tricomi 1950b,a). Tricomi shows that, for \( n \) integer
\[
\gamma (n+1, -g_\nu \frac{\omega}{\omega_\kappa}) = n! \left[ 1 - e^{\nu/g_\nu} \sum_{m=0}^{n} \frac{(-g_\nu \frac{\omega}{\omega_\kappa})^m}{m!} \right],
\]
therefore, our series representation of the Owen function is
\[
T \left[ i \sqrt{2g_\nu \frac{\omega}{\omega_\kappa}}; \frac{\sqrt{\nu}}{g_\nu} \right] = \frac{1}{2\pi} \int_0^\infty d\xi e^{-\xi} \sum_{n=0}^\infty \frac{(-\nu/g_\nu)^n}{n!(2n+1)} \sum_{m=0}^n \frac{(-g_\nu \frac{\omega}{\omega_\kappa})^m}{m!}.
\]

**Appendix C. Proof of some identities**

We derived the following result
\[
R = -4 \frac{\omega_\kappa e^{-\Omega}}{\omega_B \sqrt{g}} \left\{ - \frac{i}{2} \sqrt{\pi} Z \left( \sqrt{g} \Omega \right) - 2\pi e^{-\Omega} T \left[ i \sqrt{2g} \Omega; \frac{1}{\sqrt{g}} \right] \right\},
\]
which is valid \( \forall b \). We now prove that, for \( \omega_\kappa \equiv \omega_B \),
\[
\lim_{b \to 0} R = F_{1,1},
\]
where

\[ F_{1,1}^{1/2} = e^{-\Omega} \int_{-\infty}^{\Omega} dz \frac{e^{-z}}{z^{1/2}} \]  \hspace{1cm} (C3)

is the result found by BDR (Biglari et al. 1989).

Since for \( b \to 0 \), \( g \to 1 \), then we have

\[
\lim_{b \to 0} R = 2i\sqrt{\pi}e^{-\Omega}Z\left(\sqrt{\Omega}\right) + 4\pi e^{-2\Omega}\Phi\left(i\sqrt{2\Omega}\right) \left[1 - \Phi\left(i\sqrt{2\Omega}\right)\right],
\]  \hspace{1cm} (C4)

where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dt e^{-t^2/2}. \]  \hspace{1cm} (C5)

It is easy to see that

\[ \Phi(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz e^{-z} = \frac{i}{2\sqrt{\pi}} F_{1,1}^{1/2} \]  \hspace{1cm} (C6)

Then

\[
\lim_{b \to 0} R = 2i\sqrt{\pi}e^{-\Omega}Z\left(\sqrt{\Omega}\right) + 4\pi e^{-2\Omega}\Phi\left(i\sqrt{2\Omega}\right) \left[1 - \Phi\left(i\sqrt{2\Omega}\right)\right],
\]  \hspace{1cm} (C7)

However, it is also true that

\[ F_{1,1}^{1/2} = e^{-\Omega} \int_{-\infty}^{\Omega} dz \frac{e^{-z}}{z^{1/2}} \]  \hspace{1cm} (C8)

then the first two terms in equation (C7) cancel, leaving us with

\[
\lim_{b \to 0} R = F_{1,1},
\]  \hspace{1cm} (C9)

which is what we wanted to prove.

**Appendix D. Real frequency at marginality, long wavelength limit**

\( b \to 0 \)

In the light of the identities just proved in Appendix C, we are in the position to say that, for \( b \to 0 \),

\[
D \equiv -(1 + \tau) + I_i = -\left(1 + \tau\right) + \left(1 - \frac{\omega r}{\omega}\right) \Omega Z^2 \left(\sqrt{\Omega}\right) + \eta \frac{\omega r}{\omega} \left\{(1 - 2\Omega) \Omega Z^2 \left(\sqrt{\Omega}\right) - 2\Omega^{3/2} Z \left(\sqrt{\Omega}\right)\right\},
\]  \hspace{1cm} (D1)

which corresponds to \( D_0 \) in Eq. (3) of BDR (Biglari et al. 1989). The real frequency at marginality is evaluated by solving \( \Im[D(\omega_r)] \equiv D_i(\omega_r) = 0 \), where we are setting
\( \omega = \omega_r + i\gamma \), and taking \( \gamma \to 0 \). This comes directly from Eq. (D1)

\[
\left\{ 1 - \frac{\omega_{ix}}{\omega_r} + \eta_i \frac{\omega_{ix}}{\omega_r} \left( 1 - 2 \frac{\omega_i}{\omega_r} \right) \right\} \Re \left[ \frac{Z}{\sqrt{\omega_r/\omega_i}} \right] = \eta_i \frac{\omega_{ix}}{\sqrt{\omega_i/\omega_r}} \tag{D2}
\]

The solution of this equation cannot possibly be given Eq. (25) or Romanelli (1989), which was evaluated by using the “grad-B drift” model. Indeed, Eq. (D2) is the third equation on the second column of page 11 of BDR (Biglari et al. 1989). Since the condition that determines destabilization is \( \Im[D(\omega_r)] > 0 \), we see how important is to obtain the frequency correctly. As a sanity check, one can set \( J_0^2 \to 1 \) in the definition of \( I_i \) and integrate the resonances in yet another way. It shall be

\[
I_i = 2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{v}_1^2} \int_{-\infty}^\infty d\hat{\nu}_1 e^{-\hat{\nu}_1^2} I \Omega - \Omega_1 \left[ 1 + \eta_i \left( \frac{\hat{v}_1^2}{\hat{\nu}_1^2} + \hat{\nu}_1^2 - 3/2 \right) \right]
\]

\[
= -2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{v}_1^2} \left\{ \Omega - \Omega_{\perp} \left( 1 - \frac{3}{2} \eta_i \right) Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) \right\} + \eta_i \Omega_{\perp} \left\{ 1 + 2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{v}_1^2} Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) \right\} + \eta_i \Omega_{\perp} 2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{v}_1^2} \sqrt{\Omega - \hat{v}_1^2}/2Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right). \tag{D3}
\]

It is easy to see that

\[
2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{\nu}_1^2} Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) = -4i e^{-2\Omega} \int_{\Omega^{1/2}}^\infty d\zeta e^{-2\zeta^2} Z(i\zeta) \equiv I_1, \tag{D4}
\]

\[
2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{\nu}_1^2} Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) = 2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 \left( \frac{\hat{v}_1^2}{2} - \hat{\nu}_1^2 \right) e^{-\hat{\nu}_1^2} Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) \equiv 2\Omega I_1 + 2(-4i) e^{-2\Omega} \int_{\Omega^{1/2}}^\infty d\zeta e^{-2\zeta^2} Z(i\zeta) \equiv 2\Omega I_1 + 2I_2, \tag{D5}
\]

and

\[
2 \int_0^\infty d\hat{\nu}_1 \hat{v}_1 e^{-\hat{\nu}_1^2} \sqrt{\Omega - \hat{v}_1^2}/2Z \left( \frac{\sqrt{\Omega - \hat{v}_1^2}/2}{\sqrt{\Omega - \hat{\nu}_1^2}/2} \right) = -I_2. \tag{D6}
\]

Then

\[
1 + \tau - \eta_i \Omega_{\perp} = - \left\{ \Omega - \Omega_{\perp} \left[ 1 - \frac{3}{2} \eta_i \left( 1 - \frac{4}{3} \Omega \right) \right] \right\} I_1 - \eta_i \Omega_{\perp} I_2. \tag{D7}
\]

By writing the plasma dispersion function as an Error function of imaginary argument, one sees that the integrals \( I_1 \) and \( I_2 \) can be performed analytically to obtain

\[
1 + \tau = \left\{ - (\Omega - \Omega_{\perp}) - \eta_i \Omega_{\perp} (1 - 2\Omega) \right\} e^{-2\Omega} \left[ 1 - Erfi \left( \sqrt{\Omega} \right) - 2i Erfi \left( \sqrt{\Omega} \right) \right] + 2\eta_i \Omega_{\perp} \sqrt{\pi} \Omega e^{-\Omega} \left[ Erfi \left( \sqrt{\Omega} \right) + i \right], \tag{D8}
\]
whose imaginary part set to zero gives Eq. (D2).

REFERENCES


