MHD Turbulence: A Biased Review

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This (self-contained and aspiring to be pedagogical) review of scaling theories of MHD turbulence aims to put the developments of the last few years in the context of the canonical time line (from Kolmogorov to Iroshnikov–Kraichnan to Goldreich–Sridhar to Boldyrev). It is argued that Beresnyak’s (valid) objection that Boldyrev’s alignment theory, at least in its original form, violates the RMHD rescaling symmetry can be reconciled with alignment if the latter is understood as an intermittency effect. Boldyrev’s scalings, a version of which can be recovered in this interpretation, and the concept of dynamic alignment (equivalently, local 3D anisotropy) are thus an example of a qualitative, physical theory of intermittency in a turbulent system. The emergence of aligned structures naturally brings into play reconnection physics and thus the theory of MHD turbulence becomes intertwined with the physics of tearing, current-sheet disruption and plasmoid formation. Recent work on these subjects by Mallet, Loureiro and their coworkers is reviewed and it is argued that we may, as a result, finally have a reasonably complete, if broad-brush, outline of the MHD turbulent cascade all the way to the dissipation scale. This picture appears to reconcile Beresnyak’s results advocating Kolmogorov scaling of the dissipation cutoff with Boldyrev’s aligned cascade in the inertial range. On the margins of this core narrative, standard weak-MHD-turbulence theory is argued to require revision—and a scheme for such a revision is proposed—to take proper account of the determining part that a spontaneously emergent 2D condensate plays in mediating the Alfvén-wave cascade from a weakly-interacting state to a strongly turbulent (critically balanced) one. This completes the picture of MHD cascade at large scales. A number of outstanding issues are surveyed, most of them concerning variants of MHD turbulence featuring some form of imbalance: between the two Elsasser fields or between velocity and magnetic field. Finally, it is argued that the natural direction of travel for current and future investigations is away from the fluid MHD theory and into kinetic territory—and then, possibly, back again. The review lays no claim to objectivity or completeness, focussing on topics and views that the author finds most appealing at the present moment and leaving fair and balanced coverage to more disinterested observers of the field.

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1. Introduction

Over the last decade, watching furious debates in the theory of MHD turbulence raging over increasingly technical and/or unfalsifiable issues, or working hard on minute refinements to existing results, one might have been forgiven for gradually losing interest. Is MHD turbulence to follow hydrodynamic (isotropic, homogeneous, Kolmogorov) turbulence and become a boutique field, ever more disconnected from the excitements of “real” physics? This perhaps is the fate of any successful theory (what more is there to be done?) or indeed of one that stalls for too long after initial breakthroughs (all the low-hanging fruit already picked?).

Most of the reasons for which I now find myself writing this piece with a degree of renewed enthusiasm have emerged or crystallised in 2016–17. Enough has happened recently for this text to be entirely different than it would have been had it been started only a year earlier; I do not think I could have said the same in any of the last 5 years, or perhaps even 10. The last (i.e., predating 2016–17) significant conceptual breakthrough in the subject had arguably been the dynamic-alignment theory of Boldyrev (2006) (see section 6.1), which updated the previous decade’s paradigm-changing theory of Goldreich & Sridhar (1995) (section 5) and was followed by a flurry of numerical activity, sustaining the field for nearly 10 years. Some of the furious debates alluded to above had to do with the validity of this work—but in the absence of a new idea as to what might be going on dynamically, the insistence in a series of papers by Beresnyak (2011, 2012, 2014b) that Boldyrev’s theory failed at small scales (meeting with casual dismissal from his opponents and amused indifference from the rest of the community) appeared doomed to be kicked into the long grass, waiting for ever bigger computers.¹

Simultaneously, the community has been showing increasing interest and investing increasing resources into studying the dissipation mechanisms in MHD turbulence—in particular, the role of spontaneously formed current sheets and associated local

¹Beresnyak (2011) did put forward an unassailable, if formal, theoretical objection, discussed in section 6.2, to Boldyrev’s original interpretation of dynamic alignment as an angular uncertainty associated with field-line wandering. This interpretation is not, however, essential for the dynamic alignment itself to remain a feasible feature of the turbulent cascade (Chandran et al. 2015; Mallet & Schekochihin 2017). I will put Beresnyak’s objection to good use in a revised model of the aligned cascade in section 6.3.
reconnection processes (this was pioneered a long time ago by Matthaeus & Lamkin 1986 and Politano et al. 1989 but has only recently bloomed into an active field; see references in section 7). The most intriguing question (which, however, remained mostly unasked—in print—until 2017) surely had to be this: if Boldyrev’s MHD turbulence consisted of structures that were ever more aligned and so ever more sheet-like at small scales, was a scale eventually to be reached, given a broad enough inertial range, where these sheets would become too thin to stay stable and the reconnection processes known to disrupt such sheets would kick in?

Like Boldyrev’s theory, the full/quantitative realisation that large-aspect-ratio current sheets cannot survive also dates back 10 years, if we accept that the trigger was the paper by Loureiro et al. (2007) on the plasmoid instability (see appendix C.3.2; as always, in retrospect, one can easily identify early precursors, notably Bulanov et al. 1978, 1979, Biskamp 1982, 1986 and Tajima & Shibata 1997). This, however, did not translate into a clear understanding of the disruption of dynamically forming sheets until the papers by Pucci & Velli (2014) and Uzdensky & Loureiro (2016) (which in fact had been around in preprint form since 2014, while PRL was undertaking its characteristically thorough deliberations on the potential impact of publishing it). Once this result was out, it did not take long (or, rather, it took surprisingly long) to apply it to Boldyrev’s aligning structures—it is this calculation (see section 7), published in the twin papers by Mallet et al. (2017b) and Loureiro & Boldyrev (2017b), that, in my view, has pushed the theory of MHD turbulence forward far enough that it is now both closer to a modicum of logical completeness and ripe for a new review. The outcome appears to be that the Beresnyak vs. Boldyrev controversy is resolved (both are right; see section 7.2.1), Kolmogorov’s dissipation scale is back, in a sense and in a somewhat unusual way (see section 7.2), reconnection and turbulence have joined hands, current-sheet-focused modellers have been vindicated and offered further scope for simulation, and we could perhaps step back, admire the complexity and simplicity of the Alfvénic cascade, tally up the various pieces of unfinished business that remain to be dealt with (some of them quite chunky; see section 8) and survey the broader vistas of the turbulent terrain—mostly of plasma-physical (kinetic) variety—that we might wish to declare the next frontier (section 9).

Because the subject of this review, if not exactly young, is certainly still an active one and no one narrative has been settled as definitive, my exposition will be chronological, rather than logical, viz., I will discuss ideas that proved to be wrong before getting to those that I believe to be right—not least because the latter were strongly influenced by, and would not have emerged without, the former. One day, there will be a much shorter story told in textbooks, with all intermediate steps forgotten. The literati who already know the history, are uninterested in my prose and just want to skim the essential/new bits can start by reading sections 6.3, 7 and perhaps 4.3.

Before proceeding, I would like, by way of a disclaimer, to stress the point that is already made in the title of this piece: this is a thoroughly biased review. Rather than merely peddling the truism that there is no such thing as an unbiased review, I am

2Whilst emphasising this development as conceptually the most exciting amongst the recent ones, I will also take the opportunity presented by this review to restate, in section 4, my objections to the standard version of weak Alfvén-wave turbulence theory (Schekochihin et al. 2012), which appear to have been vindicated recently (Meyrand et al. 2015, 2016); to summarise, in section 6.3, what I view as a set of rather pretty new ideas on the intermittency of aligned turbulence (Mallet et al. 2015, 2016; Chandran et al. 2015; Mallet & Schekochihin 2017); and to advocate (in section 9) various lines of further investigation focussing on plasma effects—some of which have started emerging with particular clarity during the last two years (Mallet et al. 2017a, Schekochihin et al. 2016, Squire et al. 2016, 2017b,a).
apologising here for this one drawing particularly heavily on published papers in which I myself participated. I hope that I might nonetheless be forgiven on the grounds that the lion’s share of the credit for those contributions in fact belongs to my coauthors. Leaving to more disengaged spectators the task of assigning to these works their true measure of significance, perhaps as minor flecks of colour on the vast canvas of MHD turbulence theory, I will instead present this subject as I see it, with those flecks in the foreground.

2. K41, IK and GS95

The basic starting point for this discussion is to imagine a static, homogeneous plasma or, more generally, a conducting continuous medium, threaded by a uniform magnetic field. We can think of this situation as describing some local patch of a larger system, in which the magnetic field and other equilibrium parameters (density, pressure, flow velocity) are large-scale and structured in some system-dependent way. We are not going to be concerned with the question of what this large-scale structure is or how it is brought about—locally, it always looks like our homogeneous patch. Within this patch, we will consider perturbations whose time and length scales are short compared to any length scales associated with that large-scale structure. Of course, such a local approximation is not entirely universal: we are putting aside the cases of strong shear, various stratified or rotating systems, etc.—or, to be precise, we are excluding from consideration perturbations that are sufficiently extended in space and/or time to “feel” these background gradients. Arguably, in an ideal asymptotic world inhabited by theoretical physicists, one can always go to scales small enough for this restriction to be justified, without hitting dissipation/microphysical scales first (in a real world, this is, regrettably, not always true, but let us understand the asymptotically idealised reality first). The only large-scale feature that does not thus go away at small scales is the magnetic field. This is what makes MHD turbulence a priori different from, for example, rotating or stratified turbulence, which, at small enough scales, always reverts to the universal Kolmogorov state (Nazarenko & Schekochihin 2011).

2.1. K41

Let us recall with maximum brevity what this Kolmogorov state is. Assume that energy is being pumped into the system at large scales and at some fixed rate $\varepsilon$. Then, in the inertial range (i.e., at small enough scales so the system is locally homogeneous but not small enough for viscosity or any other microphysics to matter yet), this same $\varepsilon$ is the constant energy flux from scale to scale. Assuming that the cascade (i.e., the passing of energy from scale to scale) is local, the energy spectrum is, by dimensional analysis,

$$E(k) \sim \varepsilon^{2/3} k^{-5/3},$$

(2.1)

the famous Kolmogorov spectrum (Kolmogorov 1941b; henceforth K41). The dimensions of the quantities involved are

$$[\varepsilon] = \frac{U^3}{L}, \quad \left[ \int \frac{dk}{dE(k)} \right] = U^2,$$

(2.2)

I hope, however, that, for those results and ideas that I do mention, I have given credit where credit is due. If I have made any omissions, I pledge lack of erudition and will be very grateful to be corrected, especially during the period when this paper is available in preprint form.
where $U$ is a unit of velocity and $L$ of length. As we will be dealing with an incompressible medium (which is always achievable by going to small enough scales and so to sufficiently subsonic motions), its density is an irrelevant constant, which we will ignore.

The dimensional result (2.1) can also be written in terms of typical velocity increments between points separated by a distance $\lambda$:

$$\delta u_\lambda \sim (\varepsilon \lambda)^{1/3}. \quad (2.3)$$

2.2. IK

It was Kraichnan (1965) who appears to have been the first to realise clearly the point made above about the irreducibility of the magnetic field. He therefore argued that, if this background uniform magnetic field $B_0$, which in velocity units is called the Alfvén speed,

$$v_A = \frac{B_0}{\sqrt{4\pi \rho_0}} \quad (2.4)$$

($\rho_0$ is the mass density of the conducting medium), was to have a persistent (at small scales) role in the energy transfer from scale to scale, then the energy spectrum in the inertial range must be, again by dimensional analysis,

$$E(k) \sim (\varepsilon v_A)^{1/2} k^{-3/2} \Leftrightarrow \delta u_\lambda \sim (\varepsilon v_A \lambda)^{1/4}. \quad (2.5)$$

This is known as the Iroshnikov–Kraichnan spectrum (henceforth IK; figure 1). The scaling exponent was fixed by the requirement, put forward with the trademark combination of deep insight and slightly murky agumentation that one often finds in Kraichnan’s papers, that the Alfvén time $\tau_A \sim 1/k v_A$ was the typical time during which interactions would occur (before build-up of correlations was arrested by perturbations propagating away from each other), so the energy flux had to be proportional to $\tau_A$ and, therefore, to $1/v_A$—thus requiring them to enter in the combination $\varepsilon v_A$. We shall see below how this argument was, in a certain subtle sense, revived 40 years later (section 6.1), but then ran into a contradiction with a more compelling restriction imposed by the symmetries of the problem (sections 6.2 and 6.3.1).

Kraichnan’s prediction was viewed as self-evidently correct for 30 years, then wrong for 10 years, then correct again (in a different sense) for another 10 years, and, as we will be discovering in section 7, is now looking to be in need of some further revision, at small enough scales. His own interpretation of it (which was also Iroshnikov’s, arrived at independently) was certainly wrong, as it was based on the assumption—natural for a true Kolmogorovian (who would be susceptible to Kolmogorov’s universalist notion of “restoration of symmetries” at small scales), but, in retrospect, illogical in the context of proclaiming the unwaning importance of $B_0$ as small scales—that turbulence sufficiently deep in the inertial range would be isotropic, i.e., that there is only one $k$ to be used in the dimensional analysis. In fact, one both can and should argue that, a priori, there is a $k_\| \parallel B_0$ and a $k_\perp$, reflecting variation of the turbulent fields along and across $B_0$, and they need not be the same. The presence of the dimensionless ratio $k_\|/k_\perp$ undermines the dimensional inevitability of (2.5) and opens up space for much theorising, inspired or otherwise.

Iroshnikov (1963) got the same result slightly earlier, by what one might view as an early weak turbulence calculation (before weak turbulence was invented), involving treatment of Alfvén waves as quasiparticles, opportune closure assumptions and, in the end, dimensional analysis. No one appears to have noticed his paper at the time and he disappeared into Soviet obscurity. In later years, he worked at the Institute of Oceanology and died in 1991, aged 54.
Intuitively, in a strong magnetic field, perturbations with \( k_\parallel \ll k_\perp \) should be more natural than isotropic ones, as the field is frozen into the motions but hard to bend. It turns out that MHD turbulence is indeed anisotropic in this way, at all scales, however small.\(^5\) Dynamically, the parallel variation (on scale \( l_\parallel \sim k_\parallel^{-1} \)) is associated with propagation of Alfvén waves, the wave period (or “propagation time”) being

\[
\tau_\Lambda = \frac{l_\parallel}{v_\Lambda},
\]  

(2.6)

and the perpendicular variation (on scale \( \lambda \sim k_\perp^{-1} \)) with nonlinear interactions, whose characteristic time is na"ively equal to

\[
\tau_{nl} \sim \frac{\lambda}{\delta u_\Lambda}
\]  

(2.7)

(we shall see in section 6 that this is a rash estimate). Here and below, we are using \( \delta u_\Lambda \) to represent the turbulent field on the grounds that, in Alfvénic perturbations, \( \delta u_\Lambda \sim \delta b_\Lambda \), where \( \delta b \) is the magnetic perturbation in velocity units (see section 3 for a discussion with equations). Declaring the two times comparable at all scales was an inspired conjecture

\(^5\)This was realised quite early on, when the first, very tentative, experimental and numerical evidence started to be looked at (Montgomery & Turner 1981; Shebalin et al. 1983), but, interestingly, it took over a decade after that for the IK theory to be properly revised.
by Goldreich & Sridhar (1995, 1997) (henceforth GS95; figure 1), which has come to be known as the critical balance (CB). I shall discuss the physical reasons for it properly in sections 4 and 5, but here let me simply postulate it. Then, naturally, the “cascade time” (i.e., the typical time to transfer energy from one perpendicular scale \( \lambda \) to the next) must be of the same order as either of the two other times:

\[
\tau_c \sim \tau_A \sim \tau_{nl}.
\] (2.8)

If (2.7) is used for \( \tau_{nl} \), then (2.8) obviates the magnetic field and returns us to the K41 scaling (2.3), viz.,

\[
\frac{\delta u^2}{\tau_c} \sim \varepsilon, \quad \tau_c \sim \tau_{nl} \sim \frac{\lambda}{\delta u_A} \quad \Rightarrow \quad \delta u_A \sim (\varepsilon \lambda)^{1/3} \quad \Rightarrow \quad E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}.
\] (2.9)

This anisotropic version of K41 is known as the Goldreich–Sridhar spectrum. Simultaneously, along the field, the velocity increments must satisfy

\[
\frac{\delta u^2}{\tau_c} \sim \varepsilon, \quad \tau_c \sim \tau_A \sim \frac{\lambda}{v_A} \quad \Rightarrow \quad \delta u_{l\|} \sim \left(\frac{\varepsilon l_{\|}}{v_A}\right)^{1/2}.
\] (2.10)

Thus, \( B_0 \)'s influence does persist, but its size enters only the parallel scaling relations, not the perpendicular ones. Formally speaking, (2.9) is just the K41 dimensional argument for the perpendicular scale \( \lambda \), with the CB conjecture used to justify not including \( v_A \) and \( l_{\|} \) amongst the local governing parameters. The assumption is that the sole role of \( B_0 \) is to set the value of \( l_{\|} \) for any given \( \lambda \): comparing (2.9) and (2.10), we get

\[
l_{\|} \sim v_A \varepsilon^{-1/3} \lambda^{2/3}.
\] (2.11)

Physically, this \( l_{\|} \) is the distance an Alfvénic pulse travels along the field, at speed \( v_A \), over the time \( \tau_{nl} \) [given by (2.7)] that it takes a turbulent perturbation of size \( \lambda \) to break up. It is natural to argue, by causality, that this is the maximum distance over which any perturbation can remain correlated (Boldyrev 2005; Nazarenko & Schekochihin 2011).

This narrative arc brings us approximately to the state of affairs in mid-1990s, although the GS95 theory did not really become mainstream until the early years of this century—and soon had to be revised. Before I move on to discussing this revision (section 6) and the modern state of the subject, I would like to put the discussion of what happens dynamically and how CB is achieved on a slightly less hand-waving basis than I have done so far. Indeed, why critical balance? Pace the causality argument, which sets the maximum \( l_{\|} \), why can’t \( l_{\|} \) be shorter? Is the nonlinear-time estimate (2.7), crucial for the scaling (2.9), justified? What happens dynamically?

From this point on, my exposition will be more sequential, I will avoid jumping ahead to the highlights and adopt a more systematic style, rederiving carefully some of the results reviewed in this section (an already well educated—or impatient—reader is welcome to skip forward at her own pace).

### 3. Reduced MHD

The theoretical assumption (or numerical/obserbational evidence) that MHD turbulence consists of perturbations that have \( k_\perp \gg k_{\|} \) but that their Alfvénic propagation...
remains important (so as to allow CB should the system want to be in it) leads to the following set of equations for these perturbations:

\[ \partial_t Z_{\perp}^{\pm} = v_A \nabla \parallel Z_{\parallel}^{\pm} + Z_{\perp}^{\pm} : \nabla_{\perp} Z_{\perp}^{\pm} = -\nabla_{\perp} p + \eta \nabla^2_{\perp} Z_{\perp}^{\pm} + f^{\pm}. \]  

These are evolution equations for the Elsasser (1950) fields \( Z_{\perp}^{\pm} = u_{\perp} \pm b_{\perp} \), where \( u_{\perp} \) is the fluid velocity perpendicular to the equilibrium field \( B_0 \), and \( b_{\perp} \) is the magnetic-field perturbation, also perpendicular to \( B_0 \) and expressed in velocity units, i.e., scaled to \( \sqrt{4\pi \rho_0} \). The total pressure \( p \) (which, physically, includes magnetic pressure) is determined by the condition that \( \nabla_{\perp} \cdot Z_{\perp}^{\pm} = 0 \), enforcing the solenoidality of the magnetic field and the incompressibility of the motions, the latter achieved at small enough scales by small enough perturbations. Namely, \( p \) is the solution of

\[ \nabla_{\perp}^2 p = -\nabla_{\perp} \cdot Z_{\perp}^{\pm} Z_{\perp}^{\pm}, \]  

which amounts to multiplying the nonlinear term on the left-hand side of (3.1) by a projection operator in Fourier space. We have, for simplicity, taken the kinematic viscosity and magnetic diffusivity \( \eta \) to be the same (we will relax this assumption in section 7, where dissipation will matter). The last term in equation (3.1), the body force \( f^{\pm} \), stands in for any energy-injection mechanism that this small-scale approximation might inherit from the nonuniversal large scales.

The Reduced MHD equations (3.1–3.2) (RMHD, first proposed by Strauss 1976, but, as often happened in those days, found independently and a little earlier in the Soviet Union, by Kadomtsev & Pogutse 1974) can be derived from the standard compressible MHD equations by ordering all perturbations of the equilibrium to be comparable to the Mach number and to \( k_{\parallel}/k_{\perp} \ll 1 \) and the rate of change of these perturbations to the Alfvén frequency \( k_{\parallel}v_A \).\(^8\) These equations (apart from the visco-resistive regularisation at small scales) are, in fact, more general than the collisional MHD approximation and apply also to collisionless perturbations near a gyrotropic equilibrium (Schekochihin et al. 2009; Kunz et al. 2015),\(^9\) which makes them applicable to the solar wind (notable for being thoroughly measurable) and many other, more remote, astrophysical plasmas (only measurable with difficulty, but endlessly fascinating to large numbers of curious researchers in gainful employment).

While, like any nonlinear equations of serious consequence, they are impossible to solve except in trivial special cases, the RMHD equations possess a number of remarkable properties that form the basis for all theories of their turbulent solutions.

(i) The perturbations described by them, known as Alfvénic, are nonlinear versions of (packets of) Alfvén waves: perturbations of velocity and magnetic field transverse to \( B_0 \) and propagating at speed \( v_A \) along it (\( Z_{\perp}^{\pm} \) in the \( B_0 \) direction, \( Z_{\parallel}^{\pm} \) in the \( -B_0 \) direction). They are entirely decoupled from all other perturbations (compressive in the case of fluid MHD, kinetic for a collisionless plasma; see Schekochihin et al. 2009 and Kunz et al. 2015) and can be considered in isolation from them.

(ii) Only counterpropagating fields interact, so the nonlinearity vanishes if either \( Z_{\perp}^{+} = 0 \) or \( Z_{\perp}^{-} = 0 \), giving rise to the so-called Elsasser states (\( u_{\perp} = \mp b_{\perp} \)), exact nonlinear

\(^8\)...or by a number of similar, if ever so subtly different, schemes; see recent review by Oughton et al. (2017) and references therein.

\(^9\)At high \( \beta \), the amplitudes of these perturbations have to be small enough in order not to run afoul of some rather interesting and only recently appreciated spoiler physics (Squire et al. 2016, 2017b, a). This will be discussed in section 9.3.
solutions that are arbitrary-amplitude, arbitrary-shape pulses travelling along $B_0$ at the velocity $\mp v_A$.

(iii) The energies of the two Elsasser fields are conserved individually (apart from any injection and dissipation terms), viz.,

$$\frac{\partial}{\partial t} \langle |Z_\perp^\pm|^2 \rangle = \epsilon^\pm - \eta \langle |\nabla_\perp Z_\perp^\pm|^2 \rangle. \quad (3.3)$$

The energy fluxes $\epsilon^\pm = \langle Z_\perp^\pm \cdot f^\pm \rangle$ need not be the same and their ratio $\epsilon^+ / \epsilon^-$ is, in general, a parameter of the problem—when it is different from unity, the turbulence is called *imbalanced* (section 8.1).

(iv) The amplitudes $Z_\perp^\pm$, time and the gradients can be arbitrarily but simultaneously rescaled: $\forall \epsilon$ and $a$,

$$Z_\perp^\pm \rightarrow \epsilon Z_\perp^\pm, \quad f^\pm \rightarrow \frac{\epsilon^2}{a} f^\pm, \quad \nabla_\perp \rightarrow \frac{1}{a} \nabla_\perp, \quad \nabla_\parallel \rightarrow \frac{\epsilon}{a} \nabla_\parallel, \quad t \rightarrow \frac{a}{\epsilon} t, \quad \eta \rightarrow \epsilon a \eta. \quad (3.4)$$

This means that $Z_\perp^\pm$ and $\nabla_\parallel$ are, formally speaking, infinitesimal compared to $v_A$ and $\nabla_\perp$, respectively (perpendicular and parallel distances in RMHD are measured in different units, as are the Alfvén speed and the perturbations). Any statistical scalings or heuristic theories must respect this symmetry (Beresnyak 2011, 2012)—this requirement will feature prominently in section 6.3.

(v) Defining field increments

$$\delta Z_\chi^\pm = Z_\perp^\pm(r + \lambda) - Z_\perp^\pm(r), \quad (3.5)$$

where $\lambda$ is a point separation vector in the perpendicular plane, assuming statistical isotropy in this plane and considering separations $\lambda$ belonging to the inertial range (i.e., smaller than the energy-injection scale but greater than the viscous/resistive scale), one finds, in a statistical steady state,

$$\langle \delta Z_\parallel^\pm \cdot \delta Z_\perp^\pm \rangle = -2 \epsilon^\pm \lambda, \quad (3.6)$$

where $\delta Z_\parallel^\pm = \delta Z_\perp^\pm \cdot \lambda / \lambda$ is the “longitudinal” increment. These exact laws are the RMHD version of the exact third-order laws that one always gets for turbulent systems with a convective nonlinearity, resembling the Kolmogorov (1941a) 4/5 law of hydrodynamic turbulence or (in fact, more closely) the Yaglom (1949) 4/3 law for a passive field (indeed, in RMHD, $Z_\perp^+$ advances $Z_\perp^-$ and vice versa). They were derived for incompressible MHD by Politano & Pouquet (1998a,b) assuming spatial isotropy and, isotropy having become untenable, adjusted to their RMHD form (3.6) by Boldyrev et al. (2009). They provide a useful (although not as restrictive as one might have hoped) analytical benchmark for any aspiring scaling theory of RMHD turbulence, weak or strong.

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10Another way of framing this result is by stating that RMHD has two invariants, total energy $\langle |u_\perp|^2 + |b_\perp|^2 \rangle / 2 = \langle |Z_\perp^+|^2 + |Z_\perp^-|^2 \rangle / 4$ and cross helicity $\langle u_\perp \cdot b_\perp \rangle = \langle |Z_\perp^+|^2 - |Z_\perp^-|^2 \rangle / 4$. The name of the second invariant has topological origins, alluding, in incompressible 3D MHD, to conservation of linkages between flux tubes and vortex tubes. In the context of small Alfvénic perturbations of a strong uniform mean field $B_0$, this does not appear to be a useful interpretation.

11Write an evolution equation for $\delta Z_\chi^\pm$ following directly from (3.1), take its scalar product with $\delta Z_\chi^\pm$ and average to get an evolution equation for the second-order structure function $\langle |\delta Z_\chi^\pm|^2 \rangle$, then throw out viscous/resistive terms, assume steady state ($\partial_t = 0$), homogeneity (correlation functions depend on $\lambda$ but not on $r$) and isotropy in the perpendicular plane (scalar averaged quantities depend on $\lambda = |\lambda|$ only), and, finally, integrate once with respect to $\lambda$. 

4. Weak Turbulence

The attraction of weak-turbulence (WT) approximation for wave-carrying systems, i.e., in the MHD case, a perturbation theory in the ratio $\tau_A/\tau_{nl}$ (assumed small), is that it is actually a theory, with equations, quantitative predictions for spectra and an appealing interpretation of the turbulent system as a gas of quasiparticles, or “quantised” waves (Zakharov et al. 1992; Tsytovich 1995; Nazarenko 2011). Putting aside the question of whether the conditions necessary for it to hold are commonly (or ever) satisfied by natural turbulent systems, it is still interesting—and, indeed, also a matter of due diligence—to inquire what the predictions for our system in such a regime are. “Such a regime” means small amplitudes—small enough for the nonlinear interactions to happen very slowly compared to wave motion. One can certainly imagine, at least in principle, driving an MHD system in such a way, very feebly.

4.1. WT is Irrelevant

On a broad-brush qualitative level, one can deal with this possibility as follows. Assume that in the energy-injection range, represented by some perpendicular scale $L_\perp$ and some parallel scale $L_\parallel = 2\pi/k_\parallel$, Alfvén waves are generated with amplitudes $Z^\pm$, so small that

$$\omega^\pm_k = \pm k_\parallel v_A = \frac{1}{\tau_A} \gg \frac{1}{\tau_{nl}} \sim \frac{Z^\pm}{L_\perp}. \tag{4.1}$$

If they are viewed as interacting quasiparticles ($+$ can only interact with $-$, and vice versa), the momentum and energy conservation in a three-wave interaction require

$$p + q = k, \quad \omega^\pm_p + \omega^\mp_p = \omega^\mp_k \quad \Rightarrow \quad p_\parallel - q_\parallel = k_\parallel; \quad p_\parallel = k_\parallel, \quad q_\parallel = 0. \tag{4.2}$$

Thus, three-wave interaction in fact only involves a wave ($p$) scattering off a 2D perturbation ($q_\parallel = 0$) and becoming a wave ($k$) with the same frequency ($k_\parallel = p_\parallel$) and a different perpendicular wave number ($k_\perp = p_\perp + q_\perp$). Intuitively, there will be a cascade of these waves to higher $k_\perp$. If the amplitude of the waves does not fall off with $k_\perp$ faster than $1/k_\perp^3$, which is equivalent to their spectrum being less steep than $k_\perp^{-3}$, then the nonlinear-interaction time will become shorter at larger $k_\perp$, even as the waves’ $k_\parallel$ and, therefore, their frequency stay the same. Eventually, at some perpendicular scale, which we shall call $\lambda_{CB}$ (or $1/k_\perp(CB)$), the condition $\tau_{nl} \gg \tau_A$ will be broken, so we end up with $\tau_{nl} \sim \tau_A$ and can return to considerations of the strong-turbulence regime, critical balance, etc.

A reader who is both convinced by this argument and regards it as grounds for dismissing the WT regime as asymptotically irrelevant, can at this point skip to section 5. The rest of this section is for those restless souls who would like to know what happens before the CB scale $\lambda_{CB}$ is reached.

4.2. Old WT

The first thing to realise from the above discussion is that the WT approximation is, in fact, not going to work for the turbulence of Alfvén waves, at least formally, because in every three-wave interaction, one of the three waves is not a wave at all, but a zero-frequency 2D perturbation, for which the nonlinear interactions are the dominant influence. If such $k_\parallel = 0$ perturbations are forbidden, one must consider four-wave interactions, which give rise to an apparently legitimate WT state (Sridhar & Goldreich 1994). There is no particular reason to think, however, that the four-wave interactions are thus privileged (Ng & Bhattacharjee 1996) or, even if one starts with
no energy at $k_{\parallel} = 0$, that such a state can be maintained, except in a box with some rather restrictive boundary conditions—we will see that, failing such restrictions, a 2D “condensate” must indeed emerge and does, in numerical simulations: see Boldyrev & Perez 2009; Wang et al. 2011; Meyrand et al. 2015, 2016). Returning to three-wave interactions then, the traditional approach has been to ignore the inapplicability of the WT approximation to the $k_{\parallel} = 0$ modes by means of a respectable-sounding conjecture of spectral continuity across $k_{\parallel} = 0$ (Galtier et al. 2000). One can then press on with putting MHD through the WT analytical grinder, find an evolution equation for the spectra and show that its steady-state, constant-flux solutions scale as $k_{\perp}^{-2\mu_{\pm}}$, where the spectral exponents of the “+” and “−” wave fields satisfy $\mu_{+} + \mu_{−} = 4$. This was done in the now-classic paper by Galtier et al. (2000). For balanced turbulence, obviously, $\mu_{+} = \mu_{−} = 2$. This scaling was indeed confirmed numerically—first tentatively, in early, semidirect simulations by Bhattacharjee & Ng (2001), and then definitively by Perez & Boldyrev (2008) and Boldyrev & Perez (2009), leading the community to tick this case off as done and dusted.

A very simple heuristic version of this WT calculation (Goldreich & Sridhar 1997)—a useful and physically transparent shortcut, which will also give us a point of comparison with what I consider a better theory, presented in section 4.3—goes as follows.

Imagine two counterpropagating Alfvénic structures of perpendicular size $\lambda$ and parallel coherence length $L_{\parallel}$ (which cannot change in WT, as per the argument in section 4.1) passing through each other and interacting weakly. Their transit time through each other is $\tau_{A} \sim L_{\parallel}/v_{A}$ and the change in their amplitudes during this time is

$$\Delta(\delta Z_{\lambda}^{\pm}) \sim \frac{\delta Z_{\lambda}^{\pm}}{\lambda} \tau_{A} \sim \frac{\lambda}{\tau_{\text{nl}}} \tau_{A}, \quad \tau_{\text{nl}} \equiv \frac{\lambda}{\delta Z_{\lambda}^{\pm}}. \tag{4.3}$$

If, as is assumed in WT, $\tau_{\text{nl}}^{\pm} \gg \tau_{A}$, then the amplitude change in any one interaction is small, $\Delta(\delta Z_{\lambda}^{\pm}) \ll \delta Z_{\lambda}^{\pm}$, and many such interactions are needed in order to change the amplitude $\delta Z_{\lambda}$ by an amount comparable to itself, i.e., to “cascade” the energy associated with scale $\lambda$ to smaller scales. Assuming that interactions occur all the time and that the kicks (4.3) accumulate as a random walk, we may estimate that if the amplitude change after time $t = N\tau_{A}$ is of order $\delta Z_{\lambda}^{\pm}$, then $t \sim \tau_{c}^{\pm}$, the cascade time:

$$\Delta(\delta Z_{\lambda}^{\pm})\sqrt{N} \sim \delta Z_{\lambda}^{\pm} \implies \frac{\tau_{A}}{\tau_{\text{nl}}} \sqrt{\frac{\tau_{c}^{\pm}}{\tau_{A}}} \sim 1 \implies \tau_{c}^{\pm} \sim \left(\frac{\tau_{\text{nl}}^{\pm}}{\tau_{A}}\right)^{2}. \tag{4.4}$$

The standard Kolmogorov constant-flux requirement gives

$$\varepsilon^{\pm} \sim \left(\frac{\delta Z_{\lambda}^{\pm}}{\tau_{c}^{\pm}}\right)^{2} \sim \left(\frac{\delta Z_{\lambda}^{\pm}}{\lambda}\right)^{2} \tau_{A}. \tag{4.5}$$

We see that this argument is incapable of accommodating an imbalanced case ($\varepsilon^{+} \neq \varepsilon^{-}$),

---

12Nazarenko (2007) argues, correctly, in my view, that this conjecture will certainly be false if the nonlinear broadening of the waves’ frequencies, which is of order $\tau_{\text{nl}}^{-1}$, is smaller than the linear frequency associated with the spacing of the $k_{\parallel}$ “grid” ($\sim 1/L_{\parallel}$, the inverse parallel “box” size)—i.e., if the Alfvénic perturbations at the longest parallel scales in the system are already in the WT limit (4.1), $v_{A}/L_{\parallel} \ll \tau_{\text{nl}}^{-1}$. He then proposes that in this situation, the Alfvén waves will be “enslaved” to the 2D condensate, which is true, in a manner of speaking, as I will show in section 4.3. He also argues that the conventional WT should survive in the opposite limit, viz., when $k_{\parallel}v_{A} \gg \tau_{\text{nl}}^{-1} \gg v_{A}/L_{\parallel}$. I do not see why this must be the case, given that $k_{\parallel} = 0$ modes still have a special role and are not waves. I will discuss this issue further in appendix A.
an embarrassment first noticed by Dobrowolny et al. (1980).\footnote{They were attempting an isotropic ($L_\parallel \sim \lambda$), imbalanced theory and concluded, incorrectly, that no imbalanced stationary state was possible except a pure Elsasser state.} Assuming $\varepsilon^+ \sim \varepsilon^-$ and, therefore, $\delta Z_\lambda^+ \sim \delta Z_\lambda^-$, we get the WT scaling

$$\delta Z_\lambda \sim \left( \frac{\varepsilon}{\tau_A} \right)^{1/4} \lambda^{1/2} \iff E(k_\perp) \sim \left( \frac{\varepsilon}{\tau_A} \right)^{1/2} k_\perp^{-2}. \quad (4.6)$$

Note that, with this scaling, the ratio of the time scales ceases to be small below a certain scale:

$$\frac{\tau_A}{\tau_{nl}} \sim \frac{\tau_A \delta Z_\lambda}{\lambda} \sim \frac{\tau_A^{3/4} \varepsilon^{1/4}}{\lambda^{1/2}} \ll 1 \iff \lambda \gg \varepsilon^{1/2} \tau_A^{3/2} \equiv \lambda_{CB}. \quad (4.7)$$

For $\lambda \lesssim \lambda_{CB}$, turbulence becomes strong and, presumably, critically balanced. Thus, the weak cascade, by transferring energy to smaller scales, where turnover times are shorter, saws the seeds of its own destruction.

The formal calculation (due to Galtier et al. 2000, 2002) backing up this argument is reproduced in appendix A. While it does not suffer from the inability to handle the imbalanced case noted above, it does suffer from being formally invalid (as does the hand-waving argument presented above), having to treat slow $k_\parallel = 0$ motions as high-frequency Alfvén waves. On a heuristic level, the WT argument can, however, be repaired in the following fashion (Schekochihin et al. 2012).

### 4.3. New WT

To construct a better WT scheme, it is a good idea to recognise that the RMHD equations (3.1) support two very different types of motion: (packets of) Alfvén waves (AWs) and 2D non-propagating modes. To separate them, let us Fourier-transform in the parallel direction and represent the solutions to (3.1) as

$$Z_\perp^\pm(t, x, y) = \sum_n Z_n^\pm(t, x, y) e^{i2\pi n(z \mp v_A t)/L_\parallel}. \quad (4.8)$$

It is notationally and conceptually convenient to treat the parallel wave numbers $k_\parallel = 2\pi n/L_\parallel$ as discrete and we will, for further convenience, imagine that forcing injects energy only into $n = \pm 1$ modes. For $n \neq 0$ modes, the fast Alfvénic propagation has already been included in (4.8), whereas the $n = 0$ (2D) modes do not propagate; the amplitudes $Z_\perp^\pm$ still have a time dependence, reflecting their evolution due to nonlinear interactions. We now write separately the equations of motion for the $n = 1$ and $n = 0$ components:

$$\partial_t Z_1^\pm + \hat{P} Z_0^\mp \cdot \nabla_\perp Z_1^\pm = - \hat{P} Z_0^\pm \cdot \nabla_\perp Z_0^\pm e^{\mp i2\omega_A t}$$

$$- \sum_{n \neq 0, 1} \hat{P} Z_n^\pm \cdot \nabla_\perp Z_1^\pm e^{\mp i2\omega_A t} + f^\pm e^{\mp i\omega_A t},$$

$$\approx f^\pm e^{\mp i\omega_A t}, \quad (4.9)$$

$$\partial_t Z_0^\pm + \hat{P} Z_0^\mp \cdot \nabla_\perp Z_0^\pm = - \sum_{n \neq 0} \hat{P} Z_n^\pm \cdot \nabla_\perp Z_n^\pm e^{\mp i2\omega_A t},$$

$$\approx - 2\text{Re} \left( \hat{P} Z_1^\mp \cdot \nabla_\perp Z_1^\pm e^{\mp i2\omega_A t} \right), \quad (4.10)$$
where $\omega_A = 2\pi/\tau_A$ is the Alfvén frequency for the $n = 1$ mode, $\hat{P}$ is the projection operator that takes care of the pressure term [cf. (3.2)] and has been introduced for brevity; dissipation terms have been dropped.

4.3.1. Scalings

The only nonlinear interaction in (4.9) that is not attenuated by fast oscillations is the advection of the AWs by the 2D modes. This is the only one that we shall keep, concluding that, in the inertial range,

$$
\varepsilon^\pm \sim \frac{(\delta Z_{1,\lambda}^\pm)^2}{\tau_c} \sim \frac{\delta Z_{0,\lambda} (\delta Z_{1,\lambda}^\pm)^2}{\lambda}, \quad \tau_c \sim \frac{\lambda}{\delta Z_{0,\lambda}}.
$$

(4.11)

In contrast, in (4.10), the fast-oscillating AW-AW interaction term on the right-hand side is the only source of energy for the 2D modes, whereas the nonlinear advection term can only redistribute this energy within the 2D condensate. We can view (4.10) as a kind of Langevin equation, with the 2D field $Z_{0,\lambda} \pm$ stirred by an oscillating force with frequency $2\omega_A$ and decorrelated by the 2D nonlinearity with characteristic time $\tau_c$. In view of (4.11), the oscillating force itself also has the decorrelation time $\tau_c$ and amplitude $\sim \delta Z_{1,\lambda}^+ \delta Z_{1,\lambda}^- / \lambda$. Assuming $\omega_A \tau_c \gg 1$, it follows that

$$
\delta Z_{0,\lambda} \sim \frac{\delta Z_{1,\lambda}^+ \delta Z_{1,\lambda}^-}{\lambda \omega_A}.
$$

(4.12)

Note that these amplitudes are the same for the $+ \text{ and } -$ 2D modes, which why we dropped the superscripts $\mp$ here and in (4.11), both for $\delta Z_{0,\lambda}$ and for $\tau_c$. Physically, the estimate (4.12) simply means that the typical amplitude of the 2D modes is a small surviving residue of the mostly averaged-away AW-AW interaction. In standard WT theory, this would be a second-order contribution, usually neglected, but here it must be kept as there is no other way of exciting the 2D modes (if they are excited directly, things get marginally more complicated; see Schekochihin et al. 2012).

Combining (4.12) with (4.11), we find

$$
\frac{(\delta Z_{1,\lambda}^+)^2}{(\delta Z_{1,\lambda}^-)^2} \sim \frac{\varepsilon^+}{\varepsilon^-}.
$$

(4.13)

and hence scalings and spectra for everything:

$$
\delta Z_{1,\lambda}^\pm \sim \frac{(\varepsilon^\pm)^{3/8}}{(\varepsilon^\mp)^{1/8}} \omega_A^{1/4} \lambda^{1/2} \quad \Rightarrow \quad E_1^\pm(k_\perp) \sim \frac{(\varepsilon^\pm)^{3/4}}{(\varepsilon^\mp)^{1/4}} \omega_A^{1/2} k_\perp^{-2},
$$

(4.14)

$$
\delta Z_{0,\lambda} \sim \frac{(\varepsilon^+ \varepsilon^-)^{1/4}}{\omega_A^{1/2}} \quad \Rightarrow \quad E_0(k_\perp) \sim \frac{(\varepsilon^+ \varepsilon^-)^{1/2}}{\omega_A} k_\perp^{-1}.
$$

(4.15)

The AW spectra (4.14) are the same as the spectra (4.6) under the old scheme, provided $\varepsilon^+ = \varepsilon^-$, thus rescuing to some degree the prevailing consensus—and, more importantly, avoiding a clash with the numerical evidence that underpins it (Perez & Boldyrev 2008; Boldyrev & Perez 2009). Note, however, that, unlike the old scheme, this new one has no problem with imbalanced WT, fixing the ratio of the spectra to be the same as the ratio of the fluxes, according to (4.13). Note also that the scaling exponents of the $+$ and $-$ spectra are the same (and equal to 2), unlike in the previous WT calculations, where, the scaling argument having failed, one must resort to formal (and formally invalid) perturbation theory, which fixes only their sum (to 4) and requires further constraints to determine non-intuitive and generally different exponents for the two components.
Figure 2. Weak-turbulence spectra (4.14), (4.15) and (4.17) and transition to critically balanced cascade at scale (4.18) (adapted from Schekochihin et al. 2012).

(Galtier et al. 2000; Lithwick & Goldreich 2003). The status of numerical evidence on this subject is unclear, as the WT regime is quite a bit more difficult to simulate that the strong-turbulence one, owing to long cascade times, a difficulty that becomes more acute at large imbalance.

4.3.2. Parallel Spectra

While we are at it, we may also determine the parallel spectra of our brand of WT. If we treat all the $n \neq \pm 1$ modes as unforced, they are somewhat similar to the $n = 0$ modes in that they can only receive energy via fast-oscillating coupling terms ultimately involving the dominant $n = \pm 1$ modes, while the dominant nonlinear interaction is the redistribution of this energy via advection by the 2D condensate:

$$\partial_t Z^\pm_n + \hat{P} Z^\mp_0 \cdot \nabla Z^\pm_n = -\hat{P} Z^\mp_m \cdot \nabla Z^\pm_{n-m} e^{+i2\omega_A t} - \sum_{m \neq 0,n} \hat{P} Z^\mp_m \cdot \nabla Z^\pm_{n-m} e^{+i2\omega_A t} \approx -\hat{P} Z^\mp_1 \cdot \nabla Z^\pm_{n-1} e^{+i2\omega_A t}. \quad (4.16)$$

That the $m = 1$ term is the main contribution to the mode-coupling sum is obvious both intuitively and by a posteriori verification. Note that the $m = n - 1$ contribution is of similar size but has a higher frequency, $2(n-1)\omega_A$, and so will be subdominant when $n \gg 1$.

So, just like for the 2D modes, we have a field, $Z^\pm_n$, decorrelated via passive 2D advection with characteristic time $\tau_c$ and forced by an oscillating force with frequency $2\omega_A \gg \tau_c^{-1}$ and an amplitude whose decorrelation time is also $\tau_c$. The resulting scalings, using (4.14), are, for $n \geq 2$, 14

$$\delta Z^\pm_{n,\lambda} \sim \delta Z^\mp_{1,\lambda} \delta Z^\pm_{n-1,\lambda} \sim (\delta Z^\mp_{1,\lambda})^n \sim (\varepsilon^\mp)^{3n/8} (\varepsilon^\pm)^{n/8} \omega_A^{3n/4} \lambda^{1-n/2}$$

$$\Leftrightarrow \quad E^\pm_{n}(k_\perp) \sim (\varepsilon^\mp)^{3n/4} (\varepsilon^\pm)^{n/4} \omega_A^{2-3n/2} k_\perp^{n-3}. \quad (4.17)$$

The $n = 2$ modes are somewhat similar to the $n = 0$ ones, whereas all $n \geq 3$ ones have even smaller amplitudes and ever steeper, upward sloping spectra (figure 2).

14For $n \leq -2$, replace $n \rightarrow -n$ everywhere.
4.3.3. Transition to Strong Turbulence

All the estimates in the above calculation required

$$\frac{1}{\tau_c \omega_A} \sim \frac{\delta Z_{0,\lambda}}{\lambda \omega_A} \sim \frac{(\epsilon^+ \epsilon^-)^{1/4}}{\omega_A^{3/2} \lambda} \ll 1 \quad \Leftrightarrow \quad \lambda \gg \lambda_{CB},$$

$$\frac{\delta Z_{1,\lambda}^\pm}{\delta Z_{0,\lambda}} \sim \frac{(\epsilon^\pm)^{1/8}}{(\epsilon^\mp)^{3/8} \omega_A^{3/4} \lambda^{1/2}} \gg 1 \quad \Leftrightarrow \quad \lambda \gg \lambda_{CB}^\pm.$$

For balanced turbulence, $\lambda_{CB} = \lambda_{CB}^\pm \sim (\epsilon^{1/2} \tau_A^{3/2})$ is the familiar scale (4.7) at which turbulence becomes strong and CB sets in. In our new WT scheme, rife with subdominant $n = 0, \pm 2, \pm 3, \ldots$ modes, this is also the scale at which the energy in all of these becomes comparable to the energy in the forced AWs ($n = \pm 1$), all their spectra meet and rearrange themselves according to the new constraints associated with a CB state. This kind of transition has indeed been diagnosed numerically by Meyrand et al. (2016), although they found the spectra for the subdominant modes to be roughly one power of $k_\perp$ steeper (see also Meyrand et al. 2015; in an earlier study, Bigot & Galtier 2011 had perhaps seen similar weak-turbulence spectra, as had Yousef & Schekochihin 2009). Whether the reason for this discrepancy is limited resolution (always a theoretician’s last best hope when numerical evidence disagrees) or a flaw in the theoretical argument presented above remains to be sorted out.

At any rate, the key conclusion of the WT theory that transition to a CB state occurs at $\lambda_{CB}$ given by (4.7)—arguably its only conclusion of real physical significance—remains robust; but it could of course have been anticipated on dimensional grounds, seeking a scale depending on $\epsilon$ and $\tau_A$ only.

In the case of imbalanced turbulence, with, say, $\epsilon^+ > \epsilon^-$, the spectrum of the 2D modes meets that of the smaller of the Elsasser fields at $\lambda_{CB}^-$, a larger scale than the scale $\lambda_{CB}$ where the linear and nonlinear times equalise. It is not necessarily obvious that this matters, but there may be interesting, or at least intricate, possibilities when the imbalance is large and the smallness of $\epsilon^-/\epsilon^+$ starts interfering with other approximations made above. We will, however, resist the temptation to pursue such arcane matters, pending evidence that the results would be relevant to anything. Another parenthetical complication arises if the 2D modes are directly forced and, as 2D turbulence would, exhibit an inverse cascade, giving rise to yet another scaling regime for the AWs passively advected by them (a case that was numerically probed by Alexakis 2011). However, given sufficient scale separations, this situation resolves itself, at small enough scales, back into the WT regime described above and thence to a CB state again (see Schekochihin et al. 2012, also for a brief discussion of other exotic things that might happen). Let me, therefore, put it aside, stop this prolonged digression into what are arguably matters of little conceptual impact and move on to the physics-rich core of the MHD-turbulence theory.

5. Critical Balance, Parallel Cascade and Local Anisotropy

5.1. Critical Balance

Section 4 can be viewed as one long protracted justificatory piece in favour of critical balance: even if an ensemble of high-frequency Alfvén waves is stirred up very gently ($\tau_{nl} \gg \tau_A$), it will, at small enough scales, get itself into the strong-turbulence regime ($\tau_{nl} \sim \tau_A$). An opposite limit, a 2D regime with $\tau_{nl} \ll \tau_A$, is unsustainable for the very simple reason of causality: as information in an RMHD system propagates along $B_0$ at
speed \( v_A \), no structure longer than \( l_\parallel \sim v_A \tau_{nl} \) can be kept coherent and so will break up (Boldyrev 2005; Nazarenko & Schekochihin 2011).

It is worth mentioning in passing that the CB turns out to be a very robust feature of the turbulence in following sense. With a certain appropriate definition of \( \tau_{nl} \) (which will be explained in section 6.1), the ratio \( \tau_A/\tau_{nl} \) has been found (numerically) by Mallet et al. (2015) to have a scale-invariant distribution (figure 4), a property that they dubbed refined critical balance (RCB). It gives a quantitative meaning to the somewhat vague statement \( \tau_A/\tau_{nl} \sim 1 \)—and becomes important in the (as it turns out, unavoidable) discussion of intermittency of MHD turbulence (section 6.3.2).

5.2. Parallel Cascade

The most straightforward—and the least controversial—consequence of CB is the scaling of parallel increments. I have already derived this result in (2.10), but let me now restate it using Elsasser fields. If it is the case that the nonlinear-interaction time and, therefore, the cascade time for \( Z_\perp^\pm \) are approximately the same as its propagation
time $\tau_A \sim l_\parallel/v_A$, then the parallel increments $\delta Z_{\parallel}^\pm$ satisfy
\[
\frac{(\delta Z_{\parallel}^\pm)^2}{\tau_A} \sim \varepsilon^\pm \quad \Rightarrow \quad \delta Z_{\parallel}^\pm \sim \left(\frac{\epsilon^\pm l_\parallel}{v_A}\right)^{1/2} \Leftrightarrow E^\pm(k_\parallel) \sim \frac{\epsilon^\pm}{v_A} k_\parallel^{-2}.
\] (5.1)

Beresnyak (2012, 2015) gives two rather elegant (and related) arguments, alongside robust numerical evidence in the latter paper, in favour of the scaling (5.1), independent of any explicit invocation of the CB conjecture or its justification based on WT and causality. First, he argues that the scaling relation (5.1) can be obtained by dimensional analysis because the RMHD equations (3.1) stay invariant if $v_A$ and $1/k_\parallel$ are scaled simultaneously [see (3.4)] and so these two quantities must always appear in the combination $k_\parallel v_A$ in scaling relations for any physical quantities—in the case of (5.1), energy, or field increment. Secondly, Beresnyak (2015) notes that following the structure of the fluctuating field (calculating its increments) along the field line (in the positive $B_0$ direction) is the MHD equivalent of following its time evolution forward (for $Z_{\perp}^+$) or backward (for $Z_{\perp}^-$) in time and it should, therefore, be possible to infer the parallel spectrum (5.1) from the Lagrangian frequency spectrum of the turbulence. Estimating the energy flux as the rate of change of energy in a fluid element in the Lagrangian frame (i.e., excluding trivial sweeping), one obtains (Landau & Lifshitz 1987; Corrsin 1963)
\[
\varepsilon^\pm \sim (\delta Z_\tau^\pm)^2 \tau^{-1} \Leftrightarrow E^\pm(\omega) \sim \varepsilon^\pm \omega^{-2},
\] (5.2)
where $\delta Z_\tau^\pm$ is the Lagrangian field increment over time interval $\tau$. Finally, (5.1) is recovered from (5.2) by changing variables $\omega = k_\parallel v_A$ and letting $E^\pm(\omega)d\omega = E^\pm(k_\parallel)dk_\parallel$.

Thus, the parallel cascade and the associated scaling (5.1) appear to be a very simple and solid property of MHD turbulence. What happens in the perpendicular direction, is a more complicated story.
5.3. Local, Scale-Dependent Anisotropy

Using instead of the parallel increments the perpendicular ones $\delta Z^\pm_\lambda$ and substituting the nonlinear time

$$\tau_{nl}^\pm \sim \frac{\lambda}{\delta Z^\pm_\lambda}$$  \hspace{1cm} (5.3)

for the cascade time, we recover (2.9):

$$\frac{(\delta Z^\pm_\lambda)^2}{\tau_{nl}^\pm} \sim \bar{\epsilon}^\pm \quad \Rightarrow \quad \frac{\delta Z^+_{\lambda}}{\delta Z^-_{\lambda}} \sim \frac{\bar{\epsilon}^+}{\bar{\epsilon}^-} , \quad \delta Z^\pm_{\lambda} \sim (\bar{\epsilon}^\pm \lambda)^{1/3} , \quad \bar{\epsilon}^\pm \equiv \frac{(\epsilon^\pm)^2}{\bar{\epsilon}^\mp}$$

$$\Rightarrow \quad E^\pm (k_{\perp}) \sim (\bar{\epsilon}^\pm)^{2/3} k_{\parallel}^{-5/3}. \hspace{1cm} (5.4)$$

Treating $\delta Z^\pm_\lambda$ and $\delta Z^\pm_{l_{\parallel}}$ as increments for the same structure, but measured across and along the field, and setting them equal to each other, we find a relationship between the parallel and perpendicular scales—the scale-dependent anisotropy:

$$l_{\parallel}^\pm \sim v_\Lambda (\bar{\epsilon}^\mp)^{-1/3} \lambda^{2/3}. \hspace{1cm} (5.5)$$

The fact of scale-dependent anisotropy of MHD turbulence [if, in retrospect, not with the same confidence the scaling (5.5)] was confirmed numerically by Cho & Vishniac (2000) and Maron & Goldreich (2001) and, in a rare triumph of theory correctly anticipating measurement, was observed in the solar wind by Horbury et al. (2008), followed by many others (e.g., Podesta 2009; Wicks et al. 2010; Luo & Wu 2010; Chen et al. 2011—a complete list is impossible here as this has now become an industry, as successful ideas do; see Chen 2016 for a recent review). Figure 5 shows some of the first of those results. An important nuance is that, in order to see scale-dependent anisotropy, one must measure the parallel correlations along the perturbed, rather than global mean magnetic field.\(^{16}\) The reason for this is as follows.

Both the causality argument (Boldyrev 2005; Nazarenko & Schekochihin 2011) and the Lagrangian-frequency one (Beresnyak 2015) that I invoked in sections 5.1 and 5.2 to justify long parallel coherence lengths of the MHD fluctuations rely on the property of Alfvénic perturbations to propagate along the magnetic field. Physically, a small such perturbation on any given scale does not know the difference between a larger perturbation on, say, a few times its scale, and the “true” mean field (whatever that is, outside the ideal world of periodic simulation boxes). Thus, it will propagate along the local field and so it is along the local field that the arguments based on this propagation will apply. What if we instead measure correlations along the global mean field or, more

\(^{15}\)Cf. Lithwick et al. (2007), the imbalanced version of the GS95 scalings. This and especially whether the parallel correlations obey (5.5) is by no means uncontroversial; see, e.g., Chandran (2008), Beresnyak & Lazarian (2008), Perez & Boldyrev (2009), Chandran et al. (2009), Podesta & Bhattacharjee (2010). A perceptive reader might have already spotted trouble in the failure of the ratio $\delta Z^+_{\lambda}/\delta Z^-_{\lambda}$ in equation (5.4) to match the ratio (4.13) with which the cascade comes in from the WT regime. I am going to discuss these things in section 8.1.3, but here I keep track of $\bar{\epsilon}^\pm$ purely for future convenience and invite the reader to substitute $\bar{\epsilon}^+ = \bar{\epsilon}^- = \bar{\epsilon}^\pm = \bar{\epsilon}$ whenever the thoughts of imbalance-related complications become too much to bear.

\(^{16}\)This detail was first understood by Cho & Vishniac (2000) and by Maron & Goldreich (2001), but still needed restating 10 years later (Chen et al. 2011) and, it seems, continues (or has until recently continued) to fail to be appreciated in some particularly die-hard sanctuaries where adherents of the old religion huddle for warmth before the dying fire of the isotropic IK paradigm (I will refrain from providing citations here—and will, in section 6, offer some comfort to the admirers of Robert Kraichnan, who was, in a certain weak sense, less wrong than it appeared in the early 2000s).
Figure 6. Measuring correlations along local vs. global mean field. True parallel correlations cannot be captured by a measurement along the global field $B_0$ if the distance $\Delta l_\perp$ [see (5.7)] by which the point-separation vector $l$ along $B_0$ “slips” off the exact field line $(B_0 + b_\perp)$ is greater than the perpendicular decorrelation length $\lambda$ between “neighbouring” field lines.

Generally, along some coarse-grained version of the exact field? Let that coarse-grained field to be the average over all perpendicular scales at and below some $L_\perp$ (to get the global mean field, let $L_\perp$ be the outer scale). Define Elsasser-field increments between pairs of points separated by a vector $l$,

$$\delta Z_l^\pm = Z_\perp^\pm(r + l) - Z_\perp^\pm(r),$$

and consider $l$ along the exact magnetic field vs. $l$ along our coarse-grained field. The perpendicular distance by which the latter vector will veer off the field line (figure 6) will be dominated by the magnetic perturbation at the largest scale that was not included in the coarse-grained field:

$$\Delta l_\perp \sim l \frac{\delta b_{L_\perp}}{v_A}. \quad (5.7)$$

If we are trying to capture parallel correlations corresponding to perturbations with perpendicular scale $\lambda \ll L_\perp$, then, using CB, $l/v_A \sim \tau_{nl}$, and (5.3) with $\delta Z_\perp^\pm \sim \delta b_\lambda$, we conclude that

$$\Delta l_\perp \sim \lambda \frac{\delta b_{L_\perp}}{\delta b_\lambda} \gg \lambda, \quad (5.8)$$

i.e., in such a measurement, the parallel correlations are swamped by perpendicular decorrelation, unless, in fact, $\lambda \sim L_\perp$ or larger (there is no such problem with measuring perpendicular correlations: small changes in a separtion vector $l$ taken perpendicular to the global vs. exact field make no difference to the perpendicular decorrelation).

Consequently, the easiest practical way to extract correlations along the local field, from both observed or numerically simulated turbulence (Chen et al. 2011), is to measure field increments (5.6) for many arbitrary separation vectors $l$ and also to calculate for each such increment the angle between $l$ and the “local mean field” $B_{loc}$ defined as the arithmetic mean of the magnetic field measured at the two points involved:

$$\cos \phi = \frac{l \cdot B_{loc}}{|l||B_{loc}|}, \quad B_{loc} = B_0 + \frac{b_\perp(r + l) + b_\perp(r)}{2}. \quad (5.9)$$

This amounts to coarse-graining the field always at the right scale for the correlations that are being probed. One can then measure (for example) perpendicular and parallel structure functions as conditional averages:

$$\langle (\delta Z_\perp^\pm)^n \rangle = \langle |\delta Z_l^\pm|^n | \phi = 90^\circ \rangle, \quad (5.10)$$

$$\langle (\delta Z_\parallel^\pm)^n \rangle = \langle |\delta Z_l^\pm|^n | \phi = 0 \rangle, \quad (5.11)$$

17 Alternatively, in simulations, one can simply follow field lines to get $\delta Z_{l_\parallel}^\pm$ (Cho & Vishniac 2000; Maron & Goldreich 2001) or, as was initially done in the solar wind, use local wavelet spectra (Horbury et al. 2008; Podesta 2009; Wicks et al. 2010).
and similarly for intermediate values of $\phi$.\textsuperscript{18} Thus, in general, one measures

$$\langle |\delta Z_{\pm}^{1 \pm n} | \phi \rangle \propto l^{\zeta_2(\phi)}.$$  \hspace{1cm} (5.12)

It turns out (see references cited above and innumerable others) that $\zeta_2(0) = 1$ quite robustly, and consistent with (5.1), whereas $\zeta_2(90^o)$ is typically between $2/3$ and $1/2$, i.e., between GS95 and IK, in the solar wind, and rather closer to $1/2$ in numerical simulations—although this, as we will discuss in sections 6.2 and 6.3, has been hotly disputed by Beresnyak (2011, 2012, 2014\textsuperscript{b}), who may have a point.

Thus, while little doubt remains about the reality of scale-dependent anisotropy [although not necessarily of the specific scaling (5.5)] and of the $k_{\parallel}^{-2}$ spectrum (5.1), both arising from the GS95 theory, the GS95 prediction for the perpendicular spectrum (5.4) has continued to be suspect and controversial.

6. Dynamic Alignment, Perpendicular Cascade and 3D Anisotropy

Whereas solar-wind turbulence observations were, for a period of time, viewed to be consistent with a $-5/3$ spectrum,\textsuperscript{19} leading ultimately to the GS95 revision of the IK paradigm, high-resolution numerical simulations of forced, incompressible MHD turbulence, starting with Maron & Goldreich (2001) and Müller \textit{et al.} (2003), have consistently shown scaling exponents closer to $-3/2$ (while strongly confirming the local anisotropy; see also Cho & Vishniac 2000; Cho \textit{et al.} 2002). This undermined somewhat the then still young GS95 consensus and stimulated hard questioning of the assumptions underlying its treatment of nonlinear interactions. This focused on whether the nonlinearity in MHD turbulence might be depleted in a scale-dependent way by some form of alignment between $Z^\parallel$ and $Z^\perp$ and/or, perhaps, between the magnetic and velocity fields. Maron & Goldreich (2001) commented in passing on field alignment in their simulations and Beresnyak & Lazarian (2006) focused on “polarisation alignment” explicitly, putting it on the table as a key effect requiring revision of GS95.\textsuperscript{20} The same possibility was mooted by Boldyrev (2005) and a year later (Boldyrev 2006), he came up with a very beautiful (if, as we will see in section 6.2, technically flawed) argument based on the idea of what he called “dynamic alignment”, which set the direction of the field for the subsequent 10 years and which I am now going to discuss.

6.1. Boldyrev’s Construction

The Alfvén wave being the basic elemental MHD motion (and an exact solution), including at finite amplitudes, it stands to reason that perturbations of a strong magnetic

\textsuperscript{18}As explained above, the difference between $B_0$ and $B_{\text{loc}}$ matters only for small $\phi$.

\textsuperscript{19}Matthaeus & Goldstein (1982) were possibly the first to come out with this claim; see the monumental review by Bruno & Carbone (2013) for an exhaustive bibliography and Chen (2016) for the current state of the art ($-3/2$ is back; solar wind and simulations seem more or less in agreement: see Boldyrev \textit{et al.} 2011). Interestingly, this $-5/3$ period intersected by more than 10 years with the undisputed reign of the IK theory, confirming that no amount of adverse evidence can dent a dominant theoretical paradigm—or, at any rate, it takes a while and a hungry new generation entering the field (Kuhn 1962). One wonders if, had simulations and observations showing a $-3/2$ spectrum been available at the time, IK might have survived forever.

\textsuperscript{20}The first inklings of correlations naturally arising between the two fields and affecting scalings in a significant way appear to be traceable to Dobrowolny \textit{et al.} (1980), Grappin \textit{et al.} (1982, 1983) and Pouquet \textit{et al.} (1986), although there was perhaps no explicit clarity about any physical distinction between alignment and (local) imbalance—and, of course, everybody was chained to the isotropic IK paradigm then.
Figure 7. Cartoon of a GS95 eddy (left) vs. a Boldyrev (2006) aligned eddy (right). The latter has three scales: $l_{||} \gg \xi \gg \lambda$ (along $B_0$, along $b_\perp$, across both). This picture is adapted from Boldyrev (2006). In the context of my discussion, the fluctuation direction should be thought of as along $Z_\mp$ (see figure 16).

field would “want” to resemble Alfvén waves as closely as possible—i.e., as consistent with sustaining a strong turbulent cascade. CB can be viewed as a manifestation of this principle: an Alfvénic perturbation decorrelates in roughly one wave period. Dynamic alignment is another such manifestation: in an Alfvén wave, $u_\perp$ and $b_\perp$ are the same, which is just a dynamical consequence of plasma flows dragging the field with them or the field accelerating the flows by relaxing under tension. Another angle at this is to think of the two Elsasser fields advecting each other and thus shearing each other into mutual alignment (Chandran et al. 2015).

However, were the two fields actually parallel to each other, there would be no nonlinearity at all: indeed, considering the nonlinear term in (3.1), we see that we ought to replace the estimate (5.3) of the nonlinear time with

$$\tau_{nl} \sim \frac{\xi}{\delta Z_\mp^\lambda} \sim \frac{\lambda}{\delta Z_\mp^\lambda \sin \theta_\lambda}.$$  \hspace{1cm} (6.1)

Here $\xi$ is the scale of variation of $Z_\mp^\lambda$ in the direction of $Z_\mp^\lambda$, taken at scale $\lambda$, where $\lambda$ is the scale of variation of $Z_\pm^\perp$ in the direction perpendicular both to itself and to $B_0$ (all interactions are still assumed local in scale). Then $\theta_\lambda$ is the angle between the two Elsasser fields taken at scale $\lambda$, or, equivalently, $\sin \theta_\lambda$ is the aspect ratio of the field structures in the perpendicular 2D plane, in which a local anisotropy is now posited (see figure 7):

$$\sin \theta_\lambda \sim \frac{\lambda}{\xi}.$$ \hspace{1cm} (6.2)

Thus, the fields must be misaligned in some minimal way in order to allow strong

---

21 Matthaeus et al. (2008) confirm numerically the fast dynamical tendency for the velocity and magnetic field to align locally, in patches, and discuss it in terms of the local evolution of the cross-helicity density $u_\perp \cdot b_\perp$, noting a formal analogy with velocity–vorticity alignment in hydrodynamic turbulence. I want to alert the reader here that alignment of $u_\perp$ and $b_\perp$, on which Matthaeus et al. (2008) or Boldyrev and his coworkers focused, is not, mathematically, the same thing as alignment of $Z_\perp^+$ and $Z_\perp^-$ advocated by Chandran et al. (2015), Mallet et al. (2015), and Mallet & Schekochihin (2017). In practice, both types of alignment occur (Mallet et al. 2016). I shall discuss these matters more carefully in section 8.1.2.
turbulence (i.e., in order for $\tau_{nl}$ to be finite). A version of Boldyrev’s argument\textsuperscript{22} is to conjecture that this minimal degree of misalignment would be set by a kind of uncertainty principle: since the direction of the local magnetic field along which these perturbations propagate can itself only be defined within a small angle $\sim \delta b / v_A$, the two Elsasser fields (or the velocity and the magnetic field) cannot be aligned any more precisely than this and so
\[
\sin \theta_\lambda \sim \theta_\lambda \sim \frac{\delta b_\lambda}{v_A} \ll 1.
\]
(6.3)

By being vague about which quantities might be different for the two different Elsasser fields, I have effectively put aside any attempt at a general argument valid for imbalanced, as well as balanced, turbulence. Since alignment and local imbalance can be related in a nontrivial way and there are several possibilities as to exactly how they are related, I do not wish to be distracted and so will postpone the discussion of that to section 8.1.2. Thus, we shall assume
\[
\varepsilon^+ \sim \varepsilon^- \quad \Rightarrow \quad \delta Z^+_\lambda \sim \delta Z^-_\lambda \sim \delta u_\lambda \sim \delta b_\lambda.
\]
(6.4)

This allows us to combine (6.3) with (6.1) to get:
\[
\tau_{nl} \sim \frac{\lambda v_A}{\delta Z^2_\lambda}.
\]
(6.5)

The constancy of flux then implies immediately\textsuperscript{23}
\[
\frac{\delta Z^2_\lambda}{\tau_{nl}} \sim \varepsilon \quad \Rightarrow \quad \delta Z_\lambda \sim (\varepsilon v_A \lambda)^{1/4} \iff E(k_\perp) \sim (\varepsilon v_A)^{1/2} k_\perp^{-3/2}.
\]
(6.6)

In dimensional terms, this has brought us back to the IK spectrum (2.5), except the wavenumber is now the perpendicular wavenumber\textsuperscript{24} and both anisotropy and CB are retained, although the relationship between the parallel and perpendicular scales changes:
\[
\tau_{nl} \sim \frac{l_\parallel}{v_A} \quad \Rightarrow \quad l_\parallel \sim v_A^{3/2} \varepsilon^{-1/2} \lambda^{1/2}.
\]
(6.7)

\textsuperscript{22}His actual original argument looked somewhat more complicated than this, but in the end amounted to the same thing. In later papers (Perez et al. 2012, 2014b), he does appear to embrace implicitly something more compatible with the line of thinking that I will advocate in section 6.3.1.

\textsuperscript{23}A perceptive reader might protest at this point that $\delta Z_\lambda \sim \lambda^{1/4}$ looks rather suspicious in view of the exact law (3.6), which seems to hint at a $\lambda^{1/3}$ scaling. In fact, there is no contradiction: since one of the three Elsasser increments in the exact law (3.6) is the \textit{longitudinal} one, the alignment angle successfully insinuates its way in and (3.6) should be viewed as saying that $\delta Z^\perp_\lambda (\delta Z^\perp_\lambda)^2 \sin \theta_\lambda \sim \varepsilon^{\pm} \lambda$ (Boldyrev et al. 2009). This tells us nothing new, other than that the estimate (6.1) for the nonlinear time is reasonable. Indeed, if the right way to turn (3.6) into a “twiddle” constant-flux relation of the form $(\delta Z^\perp_\lambda)^2 \sim \varepsilon^{\pm} \tau_{nl}^{-1}$ is by interpreting $\lambda$ in (3.6) as being in the direction perpendicular to $\delta Z^\perp_\lambda$ (the direction of the fastest variation of $\delta Z^\perp_\lambda$; see figure 16), then one might argue that the exact law (3.6) provides a good reason to view specifically the angle between the two Elsasser fields as the relevant quantity affecting $\tau_{nl}^{-1}$. Such an approach is different and, in general, not equivalent, to Boldyrev’s original focus on the alignment between the velocity and the magnetic perturbation. This difference will become important in the treatment of imbalanced turbulence—I will discuss it further in section 8.1.2 onwards.

\textsuperscript{24}If one embraces (6.6), one could argue that Kraichnan’s dimensional argument was actually right, but it should have been used with $k_\perp$, rather than with $k$, because $k_\parallel$ is not a “nonlinear” dimension. This is the style of reasoning that Kraichnan himself might have rather liked. We are about to see, however, that the result (6.6) again runs into serious trouble and needs revision.
Since CB remains in force, the parallel cascade stays the same as discussed in section 5.2. For imminent use in what follows, let us compute the extent of the inertial range that this aligned cascade is supposed to span. Comparing the nonlinear cascade time \( \tau_{nl} \) with the Ohmic diffusion time (we declare for convenience that the magnetic diffusivity \( \eta \) is either the same or larger than the kinematic viscosity of our MHD fluid), we find

\[
\tau_{nl} \sim \left( \frac{v_A \lambda}{\varepsilon} \right)^{1/2} \ll \tau_{\eta} \sim \frac{\lambda^2}{\eta} \quad \Leftrightarrow \quad \lambda \gg \eta^{2/3} \left( \frac{v_A \varepsilon}{\tau_{nl}} \right)^{1/3} \equiv \lambda_{\eta},
\]

(6.8)

where \( \lambda_{\eta} \) is the cutoff scale—the Kolmogorov scale for this turbulence. For comparison, let us note that the same calculation based on the GS95 scalings (5.3) and (5.4) gives

\[
\tau_{nl}^{\text{GS95}} \sim \varepsilon^{-1/3} \lambda^{2/3} \ll \tau_{\eta} \sim \frac{\lambda^2}{\eta} \quad \Leftrightarrow \quad \lambda \gg \eta^{2/3} \varepsilon^{-1/4} \equiv \lambda_{\eta}^{\text{GS95}},
\]

(6.9)

where \( \lambda_{\eta}^{\text{GS95}} \) is the classic Kolmogorov scale.

### 6.2. Plot Thickens

This is a very appealing theory, whose main conclusions were rapidly confirmed by a programme of numerical simulations undertaken by Boldyrev’s group—in particular, the angle between velocity and magnetic field, measured in a certain opportune way,\(^{25}\) was reported to scale according to \( \theta_{\lambda} \propto \lambda^{1/4} \), as implied by (6.6) and (6.3) (Mason et al. 2006, 2008, 2011, 2012; Perez et al. 2012, 2014b). The same papers confirmed the earlier numerical results on the spectrum of MHD turbulence indeed scaling as \( k_{\perp}^{-3/2} \) (figure 8a). However, the legitimacy of this conclusion was contested by Beresnyak (2011, 2012, 2014b), who disputed that those spectra were converged and argued that systematic convergence tests in fact favoured a trend towards a \( k_{\perp}^{-5/3} \) spectrum at small enough scales. His point was that convergence of spectra with increasing resolution ought to be checked from the dissipative end of the inertial interval and that rescaling the spectra in his simulations to the Kolmogorov scale (6.9) gave a better data collapse than rescaling them to Boldyrev’s cutoff scale (6.8) (figure 8b). Despite the sound and fury of the ensuing debate about the quality of the two competing sets of numerics (Perez et al. 2014a; Beresnyak 2013, 2014a), it would not necessarily be obvious to anyone who took a look at their papers that their raw numerical results themselves were in fact all that different—certainly not as different as their interpretation by their authors. Without dwelling on either, however, let me focus instead on a conceptual a wrinkle in Boldyrev’s original argument that Beresnyak (2011) spotted and that cannot be easily dismissed.

In the RMHD limit (whose applicability to MHD turbulence at sufficiently small scales we have no reason to doubt), \( \delta b_A / v_A \) is an arbitrarily small quantity and so must then be, according to (6.3), the alignment angle \( \sin \theta_{\lambda} \). Introducing such a large depletion of the nonlinearity into (3.1) would abolish it completely in the RMHD ordering and render

\(^{25}\)In all of their work, they focused on one particular measure of alignment, \( \sin \theta_{\lambda} = \langle |\delta u_{\lambda} \times \delta b_{\lambda}| / (|\delta u_{\lambda}| |\delta b_{\lambda}|) \rangle \), which indeed turns out to scale as \( \lambda^{1/4} \) in a certain range of scales. Obviously, one can invent other proxies for the alignment angle, involving different fields \( \delta Z_{\lambda} \) and different powers of the increments under the averages. This game produces many different scalings (Beresnyak & Lazarian 2009; Mallet et al. 2016) (some of which can be successfully theorised about: see Chandran et al. 2015) and it is not \textit{a priori} obvious which of these should be most representative of the “typical” alignment that figures in the “twiddle” theories of section 6.1 or 6.3.1. Perhaps a better handle on the scaling of the alignment is obtained when one studies the full distribution of the “RMHD ensemble” (see section 6.3.2 and Mallet & Schekochihin 2017) and/or the 3D-anisotropic statistics (see section 6.4 and papers by Chen et al. 2012a, Mallet et al. 2016 and Verdini et al. 2018).
The best-resolved currently available spectra of RMHD turbulence. (a) From simulations by Perez et al. (2012) (their figure 1), with Laplacian viscosity and resolution up to $2048 \times 512$. (b) From simulations by Beresnyak (2014b) (his figure 1), with Laplacian viscosity (top panel) and with 4th-order hyperviscosity (bottom panel); the resolution for the three spectra is $1024^3$, $2048^3$ and $4096^3$. His spectra are rescaled to Kolmogorov scale (6.9) (which he denotes $\eta$). He finds poorer convergence (see his figure 2) when he rescales to Boldyrev’s scale (6.21). Perez et al. (2012) appear to get a somewhat better outcome (see their figure 8) if they determine $\lambda_{CB}$ in each simulation as the normalisation constant in the scaling (6.18) of $\sin \theta_\lambda$ (in their analysis, however, this is the angle between velocity and magnetic perturbations, not the Elsasser fields).

The system linear. The only way to keep the nonlinearity while assuming a small angle $\theta_\lambda$ is to take the angle to be small but still ordered as unity in the RMHD ordering—in other words, it cannot scale with $\epsilon$ under the RMHD rescaling symmetry (3.4). The same rescaling symmetry implies that any physical scaling that involves $v_\Lambda$ and $l_\parallel$ (and no other scales) must involve them in the combination $l_\parallel / v_\Lambda$ (see section 5.2 and Beresnyak 2012), which (6.7) manifestly does not. All this flies in the face of the fact that a substantial body of numerical evidence supporting aligned MHD turbulence was obtained by means of RMHD simulations (Mason et al. 2011, 2012; Perez et al. 2012; Beresnyak 2012; Mallet et al. 2015, 2016)—complemented by explicit evidence that full MHD simulations...
produce quantitatively the same alignment—so the standard recourse to casting a cloud of suspicion on the validity of an asymptotic approximation is not available in this case.

In a further blow to the conjecture (6.3), it turns out that the alignment angle between the Elsasser fields at any given scale is anticorrelated with their amplitudes (Mallet et al. 2015), supporting the view that the dynamical alignment is indeed dynamical, being brought about by the mutual shearing of the Elsasser fields (Chandran et al. 2015), rather than by the uncertainty principle (6.3) (which would imply, presumably, a positive correlation between $\theta_\lambda$ and $\delta Z_\lambda$).

On the other hand, the (numerical) evidence of alignment is real. While numerical simulations at currently feasible resolutions cannot definitively verify or falsify Beresnyak’s expectation that it is but a transient feature that disappears at small scales, they certainly show aligned, locally 3D-anisotropic turbulence over a respectable inertial subrange at least one order of magnitude wide, and probably two. This is approaching the kind of scale separations that actually exist in Nature, e.g., in the solar wind, and we cannot be casually dismissive of a physical regime, even if transient, that occupies most of the phase space that we are able to measure!

6.3. Revised Model of Aligned MHD Turbulence

6.3.1. Dimensional and RMHD-Symmetry Constraints

Let me make the restrictions implied by Beresnyak’s objection more explicit. Under the RMHD rescaling symmetry (3.4),

$$\delta Z_\lambda \to \epsilon \delta Z_\lambda, \quad \varepsilon \to \frac{\varepsilon^3}{a}, \quad v_A \to v_A, \quad \lambda \to a \lambda. \quad (6.10)$$

Therefore, the scaling relation (6.6) becomes $\epsilon \delta Z_\lambda \sim \epsilon^{3/4}(\varepsilon v_A \lambda)^{1/4}$, which is obviously a contradiction. Indeed, trialling

$$\delta Z_\lambda \sim \varepsilon^\mu v_A^\nu \lambda^\gamma \quad (6.11)$$

and mandating both the symmetry (6.10) and dimensional consistency, we find that the GS95 solution (5.4), $\nu = 0$ and $\gamma = \mu = 1/3$, is the only possibility, which was Beresnyak’s point.

It seems obvious that the only way to rescue alignment is to allow another parameter—and the (almost) obvious choice is $L_\parallel$, the parallel outer scale, which transforms as

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26Just to make it all more confusing, the real (observational) evidence is far from conclusive: in the solar wind, Podesta et al. (2009) and Wicks et al. (2013a) see scale-dependent alignment, but only for fluctuations at large scales—larger that what is normally viewed as the outer scale/the start of the inertial range (in the solar wind, this shows up as a break between $f^{-1}$ and $f^{-5/3}$--$f^{-3/2}$ slopes in the frequency spectrum). Osman et al. (2011) also report alignment, on the outer scale as far as I can tell. Chen et al. (2012a) see alignment across the inertial range, but, to the best of their measurement, it is not scale-dependent. Most recently, Verdini et al. (2018) have managed to extract structure functions in three field-dependent directions (see section 6.4) that scale in a way that is consistent with scale-dependent alignment, but all measures of the alignment angle $\theta_\lambda$ that they tried had much shallower (but not flat!) scalings than $\lambda^{1/4}$. This appears to be the first time that scale-dependent alignment at small scales has (still quite timidly) shown itself in the solar wind. Theoreticians must live in hope that, as both instruments and analysis techniques become more refined, definite and universal scalings will eventually emerge from this sea of uncertainty—there is some recent history of this happening, e.g., with the turbulence spectra in the kinetic range (Alexandrova et al. 2009; Chen et al. 2012b, 2013; Sahraoui et al. 2013; Huang et al. 2014), so these hopes are perhaps not entirely foolish.
\[ L_\parallel \to (a/\epsilon)L_\parallel. \] Then

\[ \delta Z_\lambda \sim \epsilon^\mu v_\lambda^\nu \lambda^\gamma L_\parallel^\delta = \epsilon^{(1+\delta)/3} \left( \frac{L_\parallel}{v_\lambda} \right)^\delta \lambda^{(1-2\delta)/3}, \] (6.12)

where the second expression is the result of imposing on the first the RMHD symmetry (6.10) and dimensional correctness; \( \delta = 0 \) returns us GS95. The same argument applied to the scaling of \( l_\parallel \) with \( \epsilon, v_A, \lambda \) and \( L_\parallel \) gives

\[ l_\parallel \sim \epsilon^{(\sigma-1)/3} v_A^{-1-\sigma} L_\parallel^\sigma \lambda^{2(1-\sigma)/3}, \] (6.13)

with \( \sigma \) a free parameter. Note that both (6.12) and (6.13) manifestly contain the parallel scales and \( v_A \) in the solely allowed combinations \( l_\parallel/v_A \) and \( L_\parallel/v_A \). A reassuring consistency check is to ask what perpendicular scale \( \lambda = L_\perp \) corresponds to \( l_\parallel = L_\parallel \): this turns out to be

\[ L_\perp \sim \epsilon^{1/2} \left( \frac{L_\parallel}{v_A} \right)^{3/2} = \lambda_{\text{CB}}, \] (6.14)

the very same \( \lambda_{\text{CB}} \), given by (4.7) or (4.18), at which weak turbulence becomes strong—thus seamlessly connecting any strong-turbulence theory expressed by (6.12) and (6.13) with the WT cascade discussed in section 4.3.27 Notably, if we applied such a test to (6.7), we would find the price of consistency to be \( L_\perp = L_\parallel \), which is allowed but does not have to be the case in MHD and certainly cannot be the case in RMHD.

Finally, since the parallel-cascade scaling (5.1) remains beyond reasonable doubt and, as can be readily checked, respects the rescaling symmetry (3.4) (Beresnyak 2015), combining it with (6.13) and (6.12) fixes

\[ \sigma = 2\delta. \] (6.15)

Alas, CB does not help with determining \( \delta \) because, in aligned turbulence, the nonlinear time (6.1) contains the unknown scale \( \xi \), or, equivalently, the alignment angle \( \theta_\lambda \sim \lambda/\xi \). If we did know \( \delta \), CB would let us determine this angle:

\[ \frac{l_\parallel}{v_A} \sim \tau_{nl} \sim \frac{\lambda}{\delta Z_\lambda \sin \theta_\lambda} \Rightarrow \sin \theta_\lambda \sim \left( \frac{\lambda}{\lambda_{\text{CB}}} \right)^{2\delta}, \] (6.16)

where \( \lambda_{\text{CB}} \) is given by (6.14). The answer that we want to get—keeping Boldyrev’s scalings of everything with \( \lambda \) but not with \( \epsilon \) or \( v_A \)—requires

\[ \delta = \frac{1}{8}. \] (6.17)

Then, instead of (6.6), we end up with

\[ \delta Z_\lambda \sim \epsilon^{3/8} \left( \frac{L_\parallel}{v_A} \right)^{1/8} \lambda^{1/4}, \quad l_\parallel \sim \epsilon^{-1/4} v_A^{3/4} L_\parallel^{1/4} \lambda^{1/2}, \quad \sin \theta_\lambda \sim \epsilon^{-1/8} \left( \frac{v_A}{L_\parallel} \right)^{3/8} \lambda^{1/4}, \] (6.18)

---

27If we had included \( L_\perp \) with some unknown exponents into (6.12) and (6.13), we would have found that \( L_\perp \) had to satisfy (6.14) and so could not be treated as an independent quantity. What, might one ask, will then happen if I attempt to inject energy at some \( L_\perp \) that does not satisfy (6.14)? If this \( L_\perp > \lambda_{\text{CB}} \), then the cascade set off at the outer scale will be weak and transition to the strong-turbulence regime at \( \lambda_{\text{CB}} \) as described in section 4.3; if \( L_\perp < \lambda_{\text{CB}} \), then I am effectively forcing 2D motions, which should break up by the causality argument (section 5.1) and it is \( L_\parallel \) that will be determined by (6.14). Thus, \( \lambda_{\text{CB}} \) can be treated without loss of generality as the perpendicular outer scale of the CB cascade.
and the dissipation cutoff scale (6.8) is corrected as follows:\(^{28}\)

\[\tau_{\text{nl}} \sim \left( \frac{L_{\parallel}}{\varepsilon v_A} \right)^{1/4} \lambda^{1/2} \ll \tau_\eta \sim \frac{\lambda^2}{\eta} \quad \Leftrightarrow \quad \lambda \gg \eta^{2/3} \left( \frac{L_{\parallel}}{\varepsilon v_A} \right)^{1/6} \equiv \lambda_\eta. \quad (6.19)\]

For future convenience, let us recast all these scalings in a somewhat simpler form:

\[\delta Z_{\lambda} \sim \left( \frac{\varepsilon L_{\parallel}}{v_A} \right)^{1/2} \left( \frac{\lambda}{\lambda_{\text{CB}}} \right)^{1/4}, \quad l_{\parallel} \sim \left( \frac{\lambda}{\lambda_{\text{CB}}} \right)^{1/2}, \quad \sin \theta_{\lambda} \sim \left( \frac{\lambda}{\lambda_{\text{CB}}} \right)^{1/4}, \quad (6.20)\]

and, defining the magnetic Reynolds number based on the CB scale and the fluctuation amplitude on this scale,

\[\frac{\lambda_\eta}{\lambda_{\text{CB}}} \sim \text{Rm}^{-2/3}(1 + \text{Pm})^{2/3}, \quad \text{Rm} = \frac{\delta Z_{\lambda_{\text{CB}}} \lambda_{\text{CB}}}{\eta} \sim \frac{\varepsilon}{\eta} \left( \frac{L_{\parallel}}{v_A} \right)^2, \quad \text{Pm} = \frac{\nu}{\eta}. \quad (6.21)\]

We have restored the possibility that viscosity \(\nu\) might be larger than the magnetic diffusivity \(\eta\); if that is the case, we must replace the latter with the former in the calculation of the dissipative cutoff, hence the appearance of the magnetic Prandtl number \(\text{Pm}\); if \(\text{Pm} \lesssim 1\), it does not matter, hence the appearance of the combination \((1 + \text{Pm})\) in (6.21).

Yet another way to write the first of the scaling relations (6.18) is

\[\delta Z_{\lambda} \sim \varepsilon^{1/3} \lambda_{\text{CB}}^{1/12} \lambda^{1/4} \quad \Leftrightarrow \quad E(k_{\perp}) \sim \varepsilon^{2/3} \lambda_{\text{CB}}^{1/6} \lambda^{-3/2}. \quad (6.22)\]

This is effectively the prediction for the spectrum that Perez et al. (2012, 2014b) used in their numerical convergence studies. Thus, we are on the same page as to what the spectrum of aligned turbulence is expected to be, although the question remains why it should be that if Boldyrev’s uncertainty principle (6.3) can no longer be used.

A set of RMHD-compatible scalings (6.18), or (6.22), is also effectively what was deduced by Chandran et al. (2015) and by Mallet & Schekochihin (2017) from a set of plausible conjectures about the dynamics and statistics of RMHD turbulence (they did not explicitly discuss the issue of the RMHD rescaling symmetry, but used normalisations that enforced it automatically). The two papers differed in their strategy for determining the exponent \(\delta\); here I will base my exposition on Mallet & Schekochihin (2017) (henceforth MS17).

### 6.3.2. Intermittency Matters!

The premise of both Chandran et al. (2015) and MS17 is that in order to determine the scalings of everything, including the energy spectrum, one must have a working model of intermittency, i.e., of the way in which fluctuation amplitudes and their scales lengths in all three directions—\(\lambda, \xi\) and \(l_{\parallel}\)—are distributed in a turbulent MHD system. It may be disturbing to the reader, or viewed by her as a complication bound to be unnecessary, that we must involve “rare” events, which is what the theory of intermittency is ultimately about, in the mundane business of the scaling of the energy spectra, which are often viewed as made up from the more “typical” fluctuations. Perhaps these doubts might be alleviated by the following observation. The appearance of the outer scale \(L_{\parallel}\) in (6.12) suggests that the self-similarity is broken—this is somewhat analogous to what happens in hydrodynamic turbulence, where corrections to the K41 scaling (2.3) come in as powers of \(\lambda/L\) (Kolmogorov 1962; Frisch 1995). We may view \(\delta\) as just such a correction to the self-similar GS95 result and alignment as the physical mechanism whereby this intermittency

\(^{28}\)Since \(\lambda_\eta \propto \eta^{2/3}\) still, this does not address Beresnyak’s numerical evidence on the convergence of the spectra (section 6.2). This problem will be dealt with in section 7.
correction arises. The main difference with the hydrodynamic case is that δ is not all that small (MHD turbulence is “more intermittent” than the hydrodynamic one) and so we care.

I shall forgo a detailed discussion of the intermittency model that MS17 proposed; for my purposes here, a vulgarised version of their argument will suffice. They consider the turbulent field as an ensemble of structures, or fluctuations, each of which has some amplitude and three scales: parallel $l_\parallel$, perpendicular $\lambda$ and fluctuation-direction $\xi$ (they call this the “RMHD ensemble”). They then make certain conjectures about the joint probability distribution of these quantities, which then allow them to fix scalings. The most crucial (and perhaps also the most arbitrary) of these conjectures is, effectively, that for all structures, $l_\parallel \sim \lambda^\alpha$ with the same exponent $\alpha$, i.e., that the quantity $l_\parallel/\lambda^\alpha$ has a scale-invariant distribution (this appears to be confirmed by numerical evidence: see figure 9, MS17 and perhaps also Zhdankin et al. 2016b). They then determine the exponent $\alpha$ by considering “the most intense structures”\footnote{Often an object of particular importance in intermittency theories (e.g., She & Leveque 1994; Dubrulle 1994; She & Waymire 1995; Grauer et al. 1994; Müller & Biskamp 2000; Boldyrev 2002; Boldyrev et al. 2002).}—because it is possible to work out what the probability of encountering them is both as a function of $\lambda$ and as a function of $l_\parallel$.

They conjecture that the most intense structures in the RMHD ensemble are sheets transverse to the local perpendicular direction. Therefore, if one looks for their probability (filling fraction) in any perpendicular plane as a function of the perpendicular scale $\lambda$, one expects it to scale as

$$P \propto \lambda. \tag{6.23}$$

If, on the other hand, one is interested in their filling fraction in the plane locally tangent to a flux sheet (i.e., defined by the local mean field and the direction of the fluctuation
The next conjecture is the “refined critical balance” (RCB, already advertised in section 5.1), stating that not only is $\tau_{nl} \sim \tau_A$ in some vague “typical” sense, but the quantity

$$\chi = \frac{\delta Z l_{||}}{\xi v_A} \sim \frac{\tau_A}{\tau_{nl}}$$

(6.25)

has a scale-invariant distribution in the RMHD ensemble—this was discovered by Mallet et al. (2015) to be satisfied with truly remarkable accuracy in numerically simulated RMHD turbulence (figure 4). If this is true for all structures, it is true for the most intense ones—and a further assumption about those is that their amplitude $\delta Z_{\text{max}}$ is not a function of scale but is simply equal to some typical outer-scale value (i.e., the most intense structures are formed by the largest perturbations collapsing, or being sheared, into sheets without breaking up into smaller perturbations; see Chandran et al. 2015). This, together with (6.24), implies that for those structures,

$$\xi \sim l_{||} \frac{\delta Z_{\text{max}}}{v_A} \Rightarrow P \propto l_{||}^2.$$  

(6.26)

Comparing (6.26) with (6.23), we conclude that $l_{||} \propto \lambda^{1/2}$ for the most intense structures and, therefore, for everyone else—by the conjecture of scale invariance of $l_{||}/\lambda^\alpha$, where we now know that $\alpha = 1/2$. Comparing this with (6.13), we see that $\alpha = 2(1 - \sigma)/3$, whence

$$\sigma = \frac{1}{4} \quad \Rightarrow \quad \delta = \frac{1}{8},$$

(6.27)

the latter by virtue of (6.15). Q.E.D.: we now have the scalings (6.18).

I do not know if the reader will find this quasi-intuitive argument more (or less) convincing than the formal-looking conjectures and corollaries in MS17. There is no need to repeat their algebra here, but hopefully the above sheds some (flickering) light—if not, perhaps a better argument could be invented, but all I can recommend for now is reading their paper. Notably, in their more formal treatment, not just the energy spectrum but the two-point structure functions of all orders can be predicted—and turn out to be a decent fit to numerical data as it currently stands. The same is true about the model proposed in the earlier paper by Chandran et al. (2015). Their approach is based on a much more enthusiastic engagement with dynamics: a careful analysis of how aligned and non-aligned structures might form and interact. They get a slightly different value $\delta \approx 0.108$, which leads to $\delta Z_{\lambda} \propto \lambda^{0.26}$—not a great deal of difference with (6.18), considering that all of this is very far from being exact science. Their approach does have the distinction, however, of emphasising particularly strongly the dynamic nature of the dynamic alignment, which arises as Elsasser fields shear each other into sheet-like structures.

---

30 Note that it makes sense then that the alignment angle $\sin \theta_{\lambda} \sim \lambda/\xi$ should be anticorrelated with the fluctuation amplitude $\delta Z_{\lambda}$ at any given scale $\lambda$ (stronger fluctuations are more aligned—the strongest of them are the sheets being discussed here), as we mentioned in section 6.2 and as Mallet et al. (2015) indeed found.

31 The key tenet of their theory—a log-Poisson distribution of field increments, which follows from arguments essentially analogous to those advanced in the hydrodynamic-turbulence theory (She & Leveque 1994; Dubrulle 1994; She & Waymire 1995)—also appears to be at least consistent with numerical evidence (Zhdankin et al. 2016a; Mallet & Schekochihin 2017). I shall not review the log-Poisson model in detail and refer the reader to MS17 and references therein instead.
Figure 10. Locally 3D-anisotropic structures in the (a) solar wind and (b) numerical simulations (here \( l_\parallel \) is normalised to \( L_\parallel/2\pi \) and \( \lambda \) and \( \xi \) to \( L_\perp/2\pi \), hence apparent isotropy at the outer scale). These are surfaces of constant second-order structure function of the magnetic field (a) or one of the Elsasser fields (b). The three images correspond to successively smaller fluctuations and so successively smaller scales (only the last of the three is firmly in the universal inertial-range regime). In both cases, the emergence of statistics with \( l_\parallel \gg \xi \gg \lambda \) is manifest. In the solar wind, the route to this aligned state that turbulence takes appears to depend quite strongly on the solar-wind expansion, which distorts magnetic-field component in the radial direction compared to the azimuthal ones (Verdini & Grappin 2015; Vech & Chen 2016). The data shown in panel (a) was carefully selected to minimise this effect; without such selection, one sees structures most strongly elongated in the \( \xi \) direction at the larger scales (\( \xi > l_\parallel > \lambda \)), although they too tend to the universal aligned regime at smaller scales (Chen et al. 2012a).

6.4. 3D Anisotropy

Before moving on, I would like to re-emphasise the 3D anisotropy of the aligned MHD turbulence—and the fact that this anisotropy is local, associated at every point with the three directions that themselves depend on the fluctuating fields: parallel to the magnetic field (\( l_\parallel \)), along the vector direction of the perturbed field \( Z^\perp_+ \) that advects the field \( Z^\perp \) whose correlations we are measuring (\( \xi \)), and the third direction perpendicular to the other two (\( \lambda \)). This local 3D anisotropy is measurable\(^{32}\) and has indeed been observed both in the solar wind (Chen et al. 2012a; Verdini et al. 2018) and in numerical

\(^{32}\)A sophisticated reader interested in how this could be done, might wonder whether the prescription given in section 5.3 and based on defining the local field \( B_{\text{loc}} \) at each scale according to (5.9) is still valid for aligned turbulence: indeed, would the distance (5.7) by which the point-separation vector \( l \) veered off the exact field line not be \( \Delta l_\perp \gg \lambda \) even when the coarse-graining scale \( L_\perp \sim \lambda \) because in (5.7), \( l/v_A \sim \lambda/\delta b_\lambda \sin \theta_A \)? In fact, since \( \Delta l_\perp \) is clearly in the direction of \( b_\perp \), the fluctuation direction, all we need to do in order to preserve parallel correlations is to ensure \( \Delta l_\perp \ll \xi \). This is indeed marginally satisfied when \( L_\perp \sim \lambda \) because, in (5.7), \( l/v_A \sim \xi/\delta b_\lambda \). Chen et al. (2012a) and Verdini et al. (2018) observationally and Mallet et al. (2016) numerically used this prescription with apparent success.
simulations (Verdini & Grappin 2015; Mallet et al. 2016)—both are illustrated by figure 10. The main point of discrepancy between the true and virtual reality is the scale dependence of the anisotropy: confirmed solidly in simulations but only very tentatively in the solar wind (see footnote 26). We live in anticipation of future, better missions and of even more sophisticated analysis—for better missions, there is some, if currently waning, hope (Vaivads et al. 2016), and the recent paper by Verdini et al. (2018) is an example of analysis becoming more sophisticated.

The scaling of the energy spectrum in the parallel direction (section 5.2) was arguably the most robust and uncontroversial of the results that I have reviewed here. We then occupied ourselves with the scalings of the Elsasser-field increments and of \( l_\parallel \) vs. the perpendicular scale \( \lambda \), culminating in section 6.3 with a theory that we (hopefully) can believe in. The scalings with the fluctuation-direction distance \( \xi \) are very easy to obtain because the nonlinear time of the aligned cascade (6.1) has the same dependence on \( \xi \) as it did on \( \lambda \) in the unaligned, GS95 theory: see (5.3). Therefore,

\[
\delta Z_\xi \sim (\varepsilon \xi)^{1/3}, \quad \xi \sim \varepsilon^{1/8} \left( \frac{L_\parallel}{v_A} \right)^{3/8} \lambda^{3/4} \sim \lambda_{CB} \left( \frac{\lambda}{\lambda_{CB}} \right)^{3/4},
\]

with the latter formula following from (6.2) and CB,

\[
\xi \sim l_\parallel \frac{\delta Z_\lambda}{v_A} \sim l_\parallel \frac{\delta b_\lambda}{v_A},
\]

i.e., \( \xi \) is the typical displacement of a fluid element and a typical perpendicular distance a field line wanders within a structure coherent on the parallel scale \( l_\parallel \). Fluctuations must therefore preserve coherence in their own direction at least on the scale \( \xi \). They are not constrained in this way in the third direction \( \lambda \) and the fluctuation direction itself has an angular uncertainty of the order of the angle \( \theta_\lambda \) between the two fields, so it makes sense that the aspect ratio of the structures in the perpendicular plane should satisfy (6.2).

The dependence of the anisotropy on the local direction of the fluctuating fields makes the connection between anisotropy, alignment and intermittency more obvious: when we follow perturbed field lines to extract parallel correlations or measure one Elsasser field’s decorrelation along the direction of another Elsasser field, we are clearly not calculating second-order statistics in the strict sense—and so, in formal terms, local scale-dependent anisotropy always involves correlation functions of (all) higher orders.\(^{33}\) Thus, it makes a certain natural sense to speak of the alignment-induced departure of MHD-turbulence spectrum from the Kolmogorovian GS95 scaling and of the 3D anisotropy of the underlying fluctuation field as an intermittency effect, as we have done here.

6.5. Higher-Order Statistics

In several places (e.g., in sections 5.3 and 6.3.2), I have brushed against the more formal task of the intermittency theory—the calculation of the scaling exponents of higher-order structure functions or, equivalently, of the probability distributions of field increments (in the local, field-dependent frame, in terms of point separations \( \lambda, \xi, l_\parallel \)), as discussed in

\(^{33}\)It is easy to show that a Gaussian field cannot have scale-dependent alignment—although a solenoidal field will naturally have modest scale-independent one (Chen et al. 2012a; Mallet et al. 2016). Note also the paper by Matthaeus et al. (2012), where the role of higher-order statistics in locally parallel correlations is examined with great meticulousness.
Figure 11. Scaling exponents of the structure functions in RMHD turbulence simulated by Mallet et al. (2016) (the plot is from MS17). (a) Structure functions of the Elsasser-field increments (5.6): by definition, $\langle |\delta Z_l^+|^n \rangle \propto l^{\zeta_n}$ and $\zeta_n^{\text{fluc}}$, $\zeta_n^{\|}$ are exponents for $l = \lambda$, $\xi$, $l_{\parallel}$, respectively (i.e., all structure functions are conditional on point separations being in one of the three directions of local 3D anisotropy; see sections 5.3 and 6.4). Solid lines are for a 1024$^3$ simulation (with hyperviscosity), dashed ones for a 512$^3$ simulation, indicating how converged, or otherwise, the exponents are, and dotted lines, in both (a) and (b), are the theoretical model by MS17. (b) Similarly defined structure functions of the velocity (solid lines) and magnetic-field (dashed lines) increments from the same 1024$^3$ simulation. The magnetic field is “more intermittent” than the Elsasser fields and the latter more so than velocity.

section 6.4)—and recoiled every time, opting for “twiddle” algebra and statements about spectra. A fair amount of information on these matters is available from simulations and from the solar-wind measurements: what intermittency looks like in the former is illustrated by figure 11 (a survey of previous measurements of structure functions, both in simulations and in the solar wind, can be found in Chandran et al. 2015). Some of what is known is perhaps understood, but much remains a mystery: for example, we do not know why the higher-order scaling exponents are generally not the same for velocity, Elsasser and magnetic fields, with the latter “more intermittent” than the former (see figure 11).

Interesting as it is, I will leave the problem of higher-order statistics alone. We know from the (ongoing) history of hydrodynamic-turbulence theory that once this becomes the unsolved problem that everyone is interested in, the scope for further theorising expands to fill all available space (and time) while attention paid by the outside world diminishes. This said, I hasten to dispel any possible impression that I do not consider intermittency of MHD turbulence an important problem: in fact, as I have argued above, intermittency as a physical phenomenon appears to be so inextricably hard-wired into the structure of MHD turbulence that any workable theory of the latter has to be a theory of its intermittency. I do not, however, have much to add to what can be found in MS17.

Finally, let me jump ahead of myself and mention also that we know nothing whatsoever of the intermittency of “reconnecting turbulence”, which is about to be introduced (section 7), and almost nothing of the intermittency of the various kinds of imbalanced turbulence surveyed in sections 8.1 and 8.2.
7. MHD Turbulence Meets Reconnection

If we accept that MHD turbulence in the inertial range—or, at least, in some subrange of the inertial range immediately below the outer scale—has a tendency to organise itself into fettuccine-like structures whose aspect ratio in the 2D plane perpendicular to the mean magnetic field increases at smaller scales, we are opting for a state of affairs that is not sustainable at ever smaller scales. These structures are sheets and sheets in MHD tend to be tearing-unstable. Thus, just like WT, strong aligned turbulence too carries the seeds of its own destruction, making an eventual transition to some new state inevitable at sufficiently small scales.\footnote{That this transition can and, generally speaking, will happen within the inertial range is made obvious by the following rather apt observation due to Uzdensky & Boldyrev (2006). The aspect ratio of an aligned sheet-like structure at Boldyrev’s cutoff scale (6.21) is $\xi/\lambda \sim Rm^{1/6}(1+Pm)^{-1/6}$, using (6.28) for $\xi$ and setting $\lambda = \lambda_\eta$. The Lundquist number at this scale is $S_\xi = \delta Z_{\lambda_\eta} \xi/\eta \sim Rm^{1/3}(1+Pm)^{2/3}$. Therefore, $\xi/\lambda \sim S_\xi^{1/2}(1+Pm)^{-1/2}$. Apart from the $Pm$ dependence, this is the aspect ratio of a Sweet–Parker (SP) current sheet, which is $S_\xi^{1/2}(1+Pm)^{-1/4}$ (see appendix C.3.1). But, provided $S_\xi$ is large enough and $Pm$ not too large, such a sheet will be violently (i.e., high above threshold) unstable to plasmoid instability, which is a variety of tearing mode and has a growth rate that is much larger than the nonlinear time associated with the sheet (see appendix C.3.2). Therefore, tearing should muscle its way into turbulent dynamics already at some scale that is larger than $\lambda_\eta$.}

The notion that current sheets will spontaneously form in a turbulent MHD fluid is not new (Matthaeus & Lamkin 1986; Politano et al. 1989) and the phenomenology of these structures has been studied (numerically) quite extensively in the more recent years, most notably by Servidio et al. (2009, 2010, 2011a,b) in 2D and by Zhdankin et al. (2013, 2014, 2015, 2016b) in 3D (see also Wan et al. 2014), while solar-wind measurements (Retinò et al. 2007; Sundkvist et al. 2007; Osman et al. 2014; Greco et al. 2016) provided motivation and, perhaps, vindication. However, theoretical discussion of these results appeared to focus on the association between current sheets in MHD turbulence and its intermittent nature, identifying spontaneously forming current sheets as the archetypal intermittent events—and effectively segregating this topic from the traditional questions about the turbulence spectrum and the “typical” structures believed to be responsible for it, viz., Alfvénic perturbations, aligned or otherwise.

In fact, as we saw in sections 6.3.2 and 6.4, it is difficult and indeed unnatural to separate the physics of alignment from that of intermittency. Dynamic alignment produces sheet-like structures that measurably affect the energy spectrum but are also the intermittent fluctuations that can perhaps collapse into proper current sheets. The likelihood that they will do so—or, more generally, that aligned structures can survive at all—hinges on whether the nonlinear cascade time $\tau_{nl}$ at a given scale $\lambda$ is longer or shorter than the typical time scale on which a tearing mode can be triggered, leading to the break up of the forming sheets into islands (Uzdensky & Loureiro 2016). Since the growth rate of the tearing mode in resistive MHD is limited by resistivity and would be zero in the limit of infinitely small $\eta$, the aligned turbulent cascade should be safe from reconnection above a certain scale that must be proportional to some positive power of $\eta$. However, this need not be the same as the scaling (6.19) that arises from the competition between the cascade rate and vanilla Ohmic (or viscous) diffusion ($\tau_{nl}$ vs. $\tau_\eta$)—and so, at the very least, the cutoff scale of the aligned cascade may not be what you might have thought it was, and what happens below that scale might be more interesting than the usual dull exponential petering out of the energy spectrum.

This possibility was explored in the waning days of 2016 by Mallet et al. (2017b) and
by Loureiro & Boldyrev (2017b) (unaware of each other’s converging preoccupations), leading to a new scaling for the aligned cascade’s cutoff and to a model for the tail end of the MHD turbulence spectrum—mitigating some of the unsatisfactory features of the aligned-turbulence paradigm and thus providing a kind of glossy finish to the overall picture.\textsuperscript{35} While the key idea in the two papers is the same, their takes on its consequences for the “reconnecting turbulence” are different—here I will side with Mallet et al. (2017b), but present their results in a somewhat simpler, if less general, form.\textsuperscript{36}

7.1. Disruption by Tearing

The scale at which the aligned structures will be disrupted by tearing can be estimated very simply by comparing the nonlinear time (6.1) of the aligned cascade with the growth time of the fastest tearing mode that can be triggered in an MHD sheet of a given transverse scale $\lambda$. That this growth time is a good estimate for the time that reconnection needs to break up a sheet forming as a result of ideal-MHD dynamics is not quite as obvious as it might appear, but it is true and was carefully shown to be so by Uzdensky & Loureiro (2016). The maximum tearing growth rate is

$$\gamma \sim \frac{v_{Ay}}{\lambda} S_\lambda^{-1/2} (1 + \text{Pm})^{-1/4}, \quad S_\lambda = \frac{v_{Ay} \lambda}{\eta}, \quad \text{Pm} = \frac{\nu}{\eta}. \quad (7.1)$$

How to derive this is reviewed in appendix C.1 [see (C 30)]. Here $v_{Ay}$ is the Alfvén speed associated with the perturbed magnetic field at scale $\lambda$, $S_\lambda$ is the corresponding Lundquist number and Pm is the magnetic Prandtl number, which only matters if the viscosity $\nu$ is larger than the magnetic diffusivity $\eta$. In application to our aligned structures, we should replace $v_{Ay} \sim \delta Z_\lambda$. Then, the aligned cascade is faster than tearing as long as

$$\gamma \tau_{nl} \sim \frac{S_\lambda^{-1/2} (1 + \text{Pm})^{-1/4}}{\sin \theta_\lambda} \ll 1 \iff \lambda \gg \text{Rm}^{-4/7} (1 + \text{Pm})^{-2/7} \lambda_{CB} \equiv \lambda_D, \quad (7.2)$$

where we have used the scalings (6.20) and our definition of the magnetic Reynolds number at the CB scale, $\text{Rm} \sim S_{\lambda CB}$ [see (6.21)]. At scales $\lambda \lesssim \lambda_D$, aligned sheet-like structures can no longer retain their integrity against the onslaught of tearing.\textsuperscript{37}

The new disruption scale $\lambda_D$, upon comparison with the putative resistive cutoff (6.21)

\textsuperscript{35}Despite their rather esoteric nature, the two papers appear to have become instant classics: so much so as to merit logarithmic corrections being derived to their scaling predictions (Comisso et al. 2018).

\textsuperscript{36}Namely, I will ignore the nuance that, in an intermittent ensemble, fluctuations of different strengths that are always present even at the same scale will be affected by reconnection to a different degree and so more intense structures will be disrupted at larger scales than the less intense ones. This means that there is in fact not a single “disruption scale” but rather a “disruption range”. I will also not present scalings that follow from the theory of the aligned cascade by Chandran et al. (2015), focussing for simplicity exclusively on the MS17 model (which is similar to Boldyrev’s original theory if the latter is interpreted as explained in section 6.3). In this sense, my exposition in section 7.1 is closer in style to Loureiro & Boldyrev (2017b) than the paper by Mallet et al. (2017b) was. The material difference between the two arises in section 7.2 and concerns the spectrum of the reconnecting turbulence. This is now partially moot, however, as the follow-up paper by Boldyrev & Loureiro (2017) embraced the Mallet et al. (2017b) spectrum if not quite the physical model that led to it (see section 7.2.2).

\textsuperscript{37}This is equivalent to the idea of Pucci & Velli (2014) (see Tenerani et al. 2016 for a review of this and subsequent work on the subject by the same group) that one ought to determine the maximum aspect ratio of sheets in MHD by asking when the tearing time scale in the sheet becomes comparable to its ideal-MHD dynamical evolution time (see appendix C.4.1).
of the aligned cascade turns out to supercede it provided \( P_m \) is not too large:

\[
\frac{\lambda_D}{\lambda_\eta} \sim \left[ \frac{R_m}{(1 + P_m)^{16}} \right]^{2/21} \gg 1.
\] (7.3)

In view of the ridiculous exponents involved, this means that in a system with even moderately large \( P_m \) and/or not a truly huge \( R_m \), the aligned cascade will happily make it to the dissipation cutoff (6.21) and no further chapters are necessary in our story. However, we do want to tell this story in full and so will focus on situations in which the condition (7.3) is satisfied.

We shall turn to the question of what happens at scales below \( \lambda_D \) in section 7.2, but first let us examine what becomes of the aligned structures that are disrupted at \( \lambda_D \).

The tearing instability that disrupts them, the so called Coppi mode, or (the fastest-growing) resistive internal kink mode (Coppi et al. 1976), has the wave number [see (C 30)]

\[
k_* \sim \frac{1}{\lambda} S^{-1/4}_\lambda (1 + P_m)^{1/8} \sim \frac{1}{\lambda_{CB}} R_m^{-1/4} (1 + P_m)^{1/8} \left( \frac{\lambda}{\lambda_{CB}} \right)^{-21/16},
\] (7.4)

where we have used (6.20) for \( \delta Z_\lambda \) inside \( S_\lambda \). Therefore, at the disruption scale (\( \lambda = \lambda_D \)),

\[
k_* \sim \frac{1}{\lambda_{CB}} R_m^{1/2} (1 + P_m)^{1/2}.
\] (7.5)

If referred to the length of the sheet \( \xi_D \) [which depends on \( \lambda_D \) via (6.28)], this wave number gives us an estimate for the number of islands in the growing perturbation:

\[
N \sim k_* \xi_D \sim R_m^{1/14} (1 + P_m)^{2/7} \gg 1.
\] (7.6)

As this is always large, the mode fits comfortably into the sheet that it is trying to disrupt.\(^{38}\)

What happens to these islands? It turns out that when the tearing mode enters nonlinear regime, the island width is (see appendix C.2)

\[
w \sim k_* \lambda_D^2,
\] (7.7)

which is smaller than \( \lambda_D \) and so, technically speaking, the aligned structure need not be destroyed by these islands. Uzdensky & Loureiro (2016) (followed by Mallet et al. 2017b and by Loureiro & Boldyrev 2017b) argue that, after the tearing mode goes nonlinear, the \( X \)-points between the islands will collapse into current sheets on the same time scale (7.1) as the mode grew. The outcome is a set of \( N \) islands [see (7.6)], which can be assumed to have circularised. Their width is then \( (w k_*^{-1})^{1/2} \sim \lambda_D \) and so they do disrupt the aligned structure (ideal-MHD sheet) that spawned them. The argument leading to this conclusion (which is not specific to MHD turbulence) is rehearsed more carefully in appendices C.2 and C.4, but the key point for us here is that at the disruption scale, the aligned structures that cascade down from the inertial range are broken up by reconnection into flux ropes that are \textit{isotropic} in the perpendicular plane. This is a starting point for a new kind of cascade, which we shall now proceed to consider.

\(^{38}\)Based on (7.4), we see that this happens for all \( \lambda/\lambda_{CB} \lesssim R_m^{-4/9} (1 + P_m)^{2/9} \). At larger scales, the fastest tearing mode that fits into the sheet is the FKR mode (Furth et al. 1963) with \( \sim \) one growing island of size \( \sim \xi \) [see (C 32) and the discussion at the end of appendix C.1.3]. However, both this mode and the secular Rutherford (1973) evolution that succeeds it are always slower than the Coppi mode and, therefore, than the nonlinear ideal-MHD evolution of the sheet, so there is no chance of it disrupting anything at those scales.
7.2. Reconnecting Turbulence in the Disruption Range

If you accept the argument at the end of section 7.1 that the disruption by tearing of an aligned structure at the scale $\lambda_D$ leads to its break-up into a number of unaligned flux ropes, then the natural conclusion is that $\lambda_D$ now becomes a kind of “outer scale” for a new cascade (if you don’t accept this, see section 7.2.5). There need not be anything particularly different about this cascade compared to the standard aligned cascade except the alignment angle is now reset to being order unity. As these “disruption-range” structures interact with each other and break up into smaller structures, the latter should develop the same tendency to align as their inertial-range predecessors did. For a while, the structures in this new cascade are safe from tearing as their aspect ratio is not large enough, but eventually (i.e., at small enough scales), they too will become sufficiently aligned to be broken up by tearing modes. This leads to another disruption, another iteration of an aligned “mini-cascade”, and so on. Thus, if we rebaptise our critical-balance scale as $\lambda_{CB} = \lambda_0$, the disruption scale as $\lambda_D = \lambda_1$, and the subsequent disruption scales as $\lambda_n$, we can think of the MHD cascade as consisting of a sequence of aligned cascades interrupted by disruption episodes.

7.2.1. Dissipation Scale

Let us calculate the disruption scales $\lambda_n$, following Mallet et al. (2017b). Since the “mini-cascades” that connect them are just the same as the aligned cascade whose disruption we analysed in section 7.1, we can use (7.2) to deduce a recursion relation

$$\lambda_{n+1} \sim S_{\lambda_n}^{-4/7}(1 + Pm)^{-2/7}\lambda_n$$

(7.8)

(remembering that we defined Rm as the Lundquist number at scale $\lambda_{CB} = \lambda_0$). In working out the Lundquist number $S_{\lambda_n}$ at scale $\lambda_n$, we notice that there must be a downward jump in the amplitude of the turbulent fluctuations at any disruption scale: indeed, if the alignment angle $\theta$ just below $\lambda_n$ is reset to being order unity, the nonlinear time (6.1) shortens significantly compared to what it was in the aligned cascade just above $\lambda_n$, and the cascade accelerates. Since it still has to carry the same energy flux, we have, for amplitudes just below the disruption scale ($\lambda_n - \lambda$),

$$\frac{(\delta Z_{\lambda_n})^3}{\lambda_n} \sim \varepsilon \Rightarrow \delta Z_{\lambda_n} \sim (\varepsilon \lambda_n)^{1/3}.$$  

(7.9)

Therefore,

$$S_{\lambda_n} \sim \frac{\delta Z_{\lambda_n} \lambda_n}{\eta} \sim \frac{\varepsilon^{1/3} \lambda_n^{4/3}}{\eta} \sim \text{Rm} \left( \frac{\lambda_n}{\lambda_{CB}} \right)^{4/3}.$$  

(7.10)

In combination with (7.8), this gives us

$$\frac{\lambda_n}{\lambda_{CB}} \sim \left[ \text{Rm}^{-4/7}(1 + Pm)^{-2/7} \right]^{2/7} \left[ 1 - \left( \frac{5}{21} \right)^n \right] \rightarrow \text{Rm}^{-3/4}(1 + Pm)^{-3/8}, \quad n \rightarrow \infty.$$  

(7.11)

Apart from the Prandtl-number dependence, which we shall discuss in a moment, we are back to the Kolmogorov scale (6.9), where Achilles catches up with the turtle and the cascade terminates.

Let us confirm that this is indeed the final dissipation scale. For each “mini-cascade” starting at $\lambda_n$, we can calculate the Ohmic diffusion scale by replacing Rm with $S_{\lambda_n}$ in (6.21):

$$\lambda_{\eta, n} \sim S_{\lambda_n}^{-2/3}(1 + Pm)^{2/3} \lambda_n \sim \text{Rm}^{-\frac{3}{8} + \frac{1}{21}(\frac{5}{21})^n}(1 + Pm)^{\frac{5}{8} + \frac{1}{21}(\frac{5}{21})^n} \rightarrow \text{Rm}^{-3/4}(1 + Pm)^{5/8}.$$  

(7.12)
Thus, this too converges to the Kolmogorov scale as long as $P_m \lesssim 1$.

When $P_m \gg 1$, the situation is somewhat less clean. From (7.11) and (7.12), we see that $\lambda_\infty/\lambda_{n,\infty} \sim (1 + P_m)^{-1} \ll 1$, which means that even if the condition (7.3) is satisfied at the first disruption scale, it will break down at one of the subsequent ones and the cascade will terminate “prematurely”—simply because $P_m$ does not change in these iterations whereas the magnetic Reynolds (Lundquist) number $S_{\lambda_n}$ decreases according to (7.10). Leaving this feature in parentheses, we declare Kolmogorov’s scaling of the dissipative cutoff (partially) rehabilitated.

It is interesting to note that it is the Kolmogorov scaling at the dissipation scales that was the strongest claim made by Beresnyak (2011, 2012, 2014b) on the basis of a convergence study of his numerical spectra (see section 6.2 and figure 8b). While he inferred from that an interpretation of these spectra as showing a $-5/3$ scaling in the inertial range, it is their convergence at the dissipative end of the resolved range that appeared to be the least negotiable feature of his work. He may well have been right.

7.2.2. Spectrum in the Disruption Range

In the picture that I have described above, the disruption-range cascade looks like a ladder (figure 12), with amplitude dropping at each successive disruption scale as structures become unaligned. In between the disruption scales, there are aligned “mini-cascades” of the same kind as the original one discussed in section 6.3. This means that the overall scaling of the turbulent fluctuation amplitudes can be constrained between their scaling just below each disruption scale ($\lambda^n_\pm$) and just above it ($\lambda^n_\mp$). We already have the former: it is the Kolmogorov (or GS95) scaling (7.9). In this limited sense, Kolmogorov is back not just at the final cutoff scale (7.11) but also in the run up to it, at $\lambda_\infty \lesssim \lambda \lesssim \lambda_D$. The scaling of the amplitudes of the structures just before they get disrupted can be inferred from the fact that for these structures, the tearing growth rate (7.1) is the same as the nonlinear interaction (cascade) rate: letting $v_{\lambda y} \sim \delta Z_{\lambda^n_\mp}$ in (7.1), we get

$$\tau_{nl}^{-1} \sim \gamma \sim (\delta Z_{\lambda^n_\pm})^{1/2} \lambda_n^{-3/2} \eta^{1/2} (1 + P_m)^{-1/4}$$ (7.13)
and, therefore,\(^{39}\)

\[
\frac{(\delta Z_{\perp}^+)^2}{\tau_{nl}} \sim \varepsilon \quad \Rightarrow \quad \delta Z_{\perp}^+ \sim \varepsilon^{2/5} \eta^{-1/5} (1 + \text{Pm})^{1/10} \lambda_n^{3/5} \\
\sim \left(\frac{\varepsilon L_{\|}}{v_A}\right)^{1/2} \left(\frac{\lambda_D}{\lambda_{CB}}\right)^{1/4} \left(\frac{\lambda_n}{\lambda_D}\right)^{3/5}.
\]

The last expression puts this result explicitly in contact with the inertial-range scaling (6.20). Thus, the disruption-range spectrum is (Mallet et al. 2017b)

\[
\varepsilon^{2/3} k_{\perp}^{-5/3} \lesssim E(k_{\perp}) \lesssim \varepsilon^{4/5} \eta^{-2/5} (1 + \text{Pm})^{1/5} k_{\perp}^{-11/5}.
\]

Since the \(-11/5\) upper envelope is steeper than the \(-5/3\) lower one, the two converge and eventually meet at

\[
\lambda_{\infty} \sim \eta^{3/4} \varepsilon^{-1/4} (1 + \text{Pm})^{-3/8} \sim \lambda_D^{21/16} \lambda_{CB}^{-5/16},
\]

which is, of course, the final cutoff scale (7.11) obtained in the limit \(n \to \infty\).

While we are picturing the spectrum in the disruption region as a succession of “steps” representing the “mini-cascades” that connect the successive disruption scales (figure 12), the reality will almost certainly look more like some overall power-law spectrum with a slope for which the upper \(-11/5\) bound (7.15) seems to be a good estimate. Indeed, the tearing disruptions will be happening within intermittently distributed aligned structures of different amplitudes and sizes, on which the disruption scales will depend (Mallet et al. 2017b). Thus, each scale \(\lambda_n\) will in fact be smeared over some range and, as the successive intervals \((\lambda_n, \lambda_{n+1})\) become narrower, this smear can easily exceed their width. Pending a detailed theory of intermittency in the disruption range, perhaps the best way to think of the spectrum and other scalings in this range is, therefore, in a “coarse-grained” sense, focusing on the characteristic dependence of all interesting quantities on \(\lambda_n\), treated as a continuous variable.

### 7.2.3. Alignment in the Disruption Range

The structures corresponding to the lower (Kolmogorov) envelope (7.9) are unaligned, whereas the alignment corresponding to the upper envelope (7.14) is the tightest alignment

\(^{39}\)Boldyrev & Loureiro (2017) also use (7.13) to derive the \(-11/5\) spectrum (7.15). However, they have a different vision (or at least a different language) about what happens dynamically: they do not believe that inter-island X-points ever collapse (as I argued at the end of section 7.1) but that, rather, the tearing mode upsets alignment somewhat, changing the effective nonlinear cascade rate to the tearing rate (7.13). This is based on the observation (which makes a lot of sense) that the alignment angle at the disruption scale, \(\sin \theta_{\lambda_D} \sim S_{\lambda_D}^{-1/2} (1 + \text{Pm})^{-1/4}\), is the same (at least for \(\text{Pm} \leq 1\)) as the angular distortion of the field line caused by the tearing perturbation at the onset of the nonlinear regime: indeed, using (7.7) and (7.4) at \(\lambda = \lambda_D\), \(\theta_{\text{tearing}} \sim w_k \sim (k_\ast \lambda_D)^2 \sim S_{\lambda_D}^{-1/2} (1 + \text{Pm})^{1/4}\). In practice, their theory just amounts to postulating \(\tau_{nl} \sim \gamma^{-1}\) at each scale and so might be a different way of saying the same thing as section 7.2.5 does. This said, to the extent that their picture has specific dynamical implications, I am unconvinced: if there are no collapses, presumably aligned structures are not fully disrupted and there would just be a succession of secondary tearings, which do not make an efficient cascade (see appendix C.4.2); after those secondary tearings are done and the sheet is finally disrupted, we are back to the Mallet et al. (2017b) picture. In other words, I think that the collapse of the inter-island X-points is the way in which the distortion of alignment caused by tearing leads to faster nonlinear break-up of the aligned structures. It is not clear, however, how to convert these seemingly different dynamical interpretations of the tearing-assisted cascade into statements sufficiently quantitative as to be falsifiable.
sustainable in the disruption range and achieved by each aligned “mini-cascade” just before it is disrupted by tearing at the scale scale \( \lambda_n \). This is (cf. Boldyrev & Loureiro 2017)

\[
\sin \theta_{\lambda_n} \sim \frac{\lambda/\delta Z_{\lambda_n}}{\tau_{nl}} \sim \left( \frac{\lambda_D}{\lambda_{CB}} \right)^{1/4} \left( \frac{\lambda_n}{\lambda_D} \right)^{-4/5}.
\]

Equivalently, the fluctuation-direction coherence scale is

\[
\xi \sim \frac{\lambda_n}{\sin \theta_{\lambda_n}} \sim \lambda_{CB} \left( \frac{\lambda_D}{\lambda_{CB}} \right)^{3/4} \left( \frac{\lambda_n}{\lambda_D} \right)^{9/5}.
\]

The corresponding spectral exponent is again \(-5/3\), which is automatically the case given the definitions of \( \tau_{nl}, \theta_{\lambda} \) and \( \xi \) [see (6.1)].

Thus, the smallest possible alignment angle, having reached its minimum at \( \lambda_D \), gets larger through the disruption range, according to (7.17), until it finally becomes order unity at the cutoff scale (7.16). To the (doubtful) extent that existing numerical evidence can be considered to be probing this regime, perhaps we can take heart from the numerical papers by both Beresnyak and by Boldyrev’s group cited in section 6.2 all reporting that alignment fades away at the small-scale end of the inertial range—although this may also be just a banal effect of the numerical resolution cutoff.

7.2.4. Parallel Cascade in the Disruption Range

As ever, CB should be an enduring feature of our turbulence. This means that the parallel spectrum (5.1) will not notice the disruption scale and blithely extend all the way through the disruption range. Since the “isotropic” flux ropes produced in the wake of the disruption of aligned structures have a shorter decorrelation time than their aligned progenitors, they should break up in parallel direction. The resulting parallel coherence scale, the same as the scale (5.5) in the GS95 theory, is the lower bound on \( l_\parallel \) at each \( \lambda_n \). The upper bound can be inferred by equating the nonlinear time (7.13) at \( \lambda_n \) to the
Alfvén time $l_\parallel/v_A$. The result is

$$v_A\varepsilon^{-1/3}\lambda_n^{2/3} \lesssim l_\parallel \lesssim v_A\varepsilon^{-1/5}\eta^{-2/5}(1 + \text{Pm})^{1/5}\lambda_n^{6/5} \sim L_\parallel \left(\frac{\lambda_D}{\lambda_{CB}}\right)^{1/2} \left(\frac{\lambda_n}{\lambda_D}\right)^{6/5}. \quad (7.19)$$

Thus, the upper bound on the parallel anisotropy decreases with scale in this range. The lower and upper bounds meet at the final cutoff scale (7.16):

$$l_\parallel \sim v_A\varepsilon^{-1/2}\eta^{1/2}(1 + \text{Pm})^{-1/4} \sim L_\parallel\text{Rm}^{-1/2}(1 + \text{Pm})^{-1/4} \text{ at } \lambda \sim \lambda_\infty. \quad (7.20)$$

Dividing (7.19) and (7.20) by $v_A$ gives us the scaling and the cutoff value, respectively, for the nonlinear (cascade) time $\tau_{nl}$ (and so also the frequency cutoff).

7.2.5. Plasmoid Turbulence

At the end of section 7.1, I followed Uzdensky & Loureiro (2016), Mallet et al. (2017b) and Loureiro & Boldyrev (2017b) in invoking the collapse of the $X$-points separating the tearing-mode islands as a means of circularising these islands and thus consummating the disruption of the aligned structures. It is an interesting question whether this collapse might itself be disrupted by secondary tearing, producing more islands and more $X$-points, followed by an (also disrupted) collapse of those etc. Such a scenario is a distinct possibility, supported by a quick comparison of the secondary-tearing growth rate with the collapse rate: if the latter is the same as the primary-tearing growth rate (7.1), then the former is always greater (see appendix C.4.2, where we argue, however, that, while tearing can happen on many recursive levels before the aligned structure is fully disrupted, the smaller-scale islands that are produced in this process are not energetically relevant and so the cascade picture of Mallet et al. 2017b survives unscathed, modulo the arguments below about plasmoid interactions).

This conjures up the spectre of the kind of multi-scale plasmoid (island) turbulence that was proposed as a model of an unstable current sheet in the now-classic paper of Shibata & Tanuma (2001) and that Uzdensky et al. (2010) used to prove MHD reconnection to be fast (featuring some distinctive scaling laws for the plasmoid sizes and fluxes, around which a lot of subsequent discussion revolved). So far, such turbulence has been explored numerically only in the essentially 1D setting of a “stochastic plasmoid chain” (even if extended to 2D to capture the transverse structure of plasmoids and to 3D to identify them as flux ropes along the mean field; see references in appendix C.3.2). Should we worry that the unstable current sheets produced by the aligned cascade will give rise to a completely different type of turbulence, driven by reconnection and superceding—or at least coexisting with—the disruption-range cascade whose theory I lovingly honed in sections 7.2.1–7.2.4?\footnote{Its integrity in simulations is usually maintained via a particular numerical set up: by being established as an initial condition and/or driven by inflows/outflows from/to the boundaries of the domain.}

On the narrative level, the difference between the two stories is the difference between plasmoids of different sizes being hosted within one “global” current sheet, as they are seeded by tearing, grow, travel along the sheet with Alfvénic outflows, sometimes swallow each other on the way (possibly giving rise to transverse secondary current sheets and plasmoid chains in the process of coalescence: see Báráta et al. 2011)—and, alternatively, the “mother sheet” (aligned structure) breaking apart entirely, releasing the plasmoids (flux ropes) into the general turbulent wilderness, where they are free to interact with each other or with anything else that comes along and are thus no different from turbulent fluctuations of a particular size generically splashing around
in a large nonlinear system. While these might appear to be two distinct scenarios (or
perhaps the first could be interpreted as a local feature of the second), they might not
be all that different in terms of the scalings that can be predicted for them by simple
dimensional and qualitative considerations of the kind that I have endeavoured to use
everywhere in this review. Indeed, fundamentally, we are dealing with the fastest-growing
tearing perturbations of structures that feature magnetic fields of a particular scale-
dependent amplitude \(v_{A_y} \sim \delta Z_\lambda\) that reverse direction across a particular scale \(\lambda\).
The relevant tearing time scale is then always given by (7.1), or (7.13), which is also
the nonlinear-interaction time scale if it is short enough (and irrelevant if it is not).\(^{41}\)
Since the assumption that this is the nonlinear cascade time is the only assumption
that goes into the derivation (7.14) of the \(-11/5\) spectral envelope, this scaling may
well be the one towards which turbulence is pushed in current sheets that host multiple
plasmoids. Indeed, both Bártá et al. (2011) and Huang & Bhattacharjee (2016) report
spectra somewhat steeper than \(-2\), perhaps consistent with \(-11/5 = -2.2\). It makes
sense that this should be the upper bound on how steep a spectrum we might expect in
a larger turbulent system [see (7.15)] because the cascade within a current sheet must
be of a more vigorous kind than will be achieved on average across the system, whose
volume these sheets, being intrinsically intermittent structures, do not entirely fill (recall
section 6.3.2).

7.3. Is This the End of the Road?

It never quite is (see sections 8 and 9), but the story looks roughly complete for the
first time in years. The aligned cascade (section 6) gave one an impression of unfinished
business, both in the sense that it gave rise to a state that appeared unsustainable at
asymptotically small scales and in view of the objections, physical and numerical, raised
by Beresnyak (2011, 2012, 2014b). With the revised interpretation of alignment as an
intermittency effect (section 6.3) and with the disruption-range cascade connecting the
aligned inertial-range one to the Kolmogorov cutoff (7.16), these issues appear to be
satisfactorily resolved. In what is also an aesthetically pleasing development, tearing-
assisted accelerated cascade emerges as an ingenious way in which MHD turbulence
contrives to thermalise its energy, while shedding the excessive alignment that ideal-MHD
dynamics cannot help producing in the inertial range. More broadly, this development
joins together in a most definite way the physics of turbulence and reconnection—
arguably, this has been in the offing for some time.

Is this a falsifiable theory? Probably not any time soon.

Numerically, anything like a definite confirmation will require formidably large simula-
tions: the condition (7.3) requires at least \(Rm \sim 10^{542}\) and probably much larger if one is
to see the scaling of the disruption-range spectrum (7.15). However, as we mooted above,
an optimist might find cause for optimism in the evidence of the MHD turbulence cutoff
appearing to obey the Kolmogorov scaling (7.16) or in the alignment petering out at the
small-scale end of the inertial range, making section 7.2.3 seem at least reasonable. While
the trouble to which we have gone to keep track of the \(Pm\) dependence of the disruption-

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\(^{41}\)Note, however, the rather more involved argument in appendix C.4.2 as to why fast secondary
tearing that occurs before \(X\)-point collapse and full disruption of the MHD sheet is harmless.
In this context, the Mallet et al. (2017b) cascade describes what happens to the energy passed
to the dominant, \(\lambda_D\)-scale, plasmoids after they start interacting nonlinearly with each other,
within the mother sheet or outside it.

\(^{42}\)This is estimated via the slightly frivolous but basically sound principle that the smallest large
number is 3.
range quantities did not yield anything qualitatively spectacular, there is perhaps an opportunity here for numerical tests: e.g., can one obtain Boldyrev’s scaling (6.21) of the dissipation cutoff in the limit of large $P_m$?—which, in view of (7.3), is unlikely to need to be very large to take over and shut down reconnection.

Observationally, our best bet for fine measurements of turbulence is the solar wind and the terrestrial magnetosphere (e.g., the magnetosheath). However, these are collisionless environments, so, before any triumphs of observational confirmation can be celebrated, all the resistive reconnection physics on which the disruption-range cascade depends needs to be amended for the cornucopia of kinetic effects that await at the small-scale end of the cascade (see section 9.1). This is likely to keep us all busy for a while, with quite a lot of ripe, low-hanging fruit on offer (example: Mallet et al. 2017a; Loureiro & Boldyrev 2017a).

Pending all this validation and verification, the disruption-range reconnecting cascade remains a beautiful fantasy—but it is reassuring that after half a century of scrutiny, MHD turbulence still has such gifts to offer.

8. Known Unknowns: Imbalanced Regimes

The purpose of this section is to survey what has been done and what still remains to be done regarding the regimes in which there is an imbalance either between the energies of the two Elsasser fields or between the kinetic and magnetic energy. Such situations are manifestly present in the solar wind (see figure 13a), but, arguably, much less (or even less) well understood and explored than the nice (and somewhat imaginary) case in which one can just assume $\delta Z_+ \sim \delta Z_- \sim \delta b_\lambda \sim \delta u_\lambda$.

8.1. Elsasser Imbalance (Turbulence with Cross-Helicity)

Since both incompressible MHD and RMHD conserve two invariants—the total energy and the cross-helicity,—each of the two Elsasser fields $Z_{\perp}^\pm$ has its own conserved energy [see (3.3)]. The energy fluxes $\varepsilon^\pm$ of these fields are, therefore, independent parameters of MHD turbulence. Setting them equal to each other makes arguments simpler, but does not, in general, correspond to physical reality, for a number of reasons.

First, everyone’s favourite case of directly measurable MHD turbulence is the solar wind, where the Alfvénic perturbations propagating away from the Sun are launched from the Sun (Roberts et al. 1987), while the counterpropagating ones have to be supplied by some mechanism that is still under discussion and probably involves Alfvén-wave reflection as plasma density decreases outwards from the Sun (see Perez & Chandran 2013 and references therein). The counterpropagating component is usually energetically smaller, especially in the fast wind (Bruno & Carbone 2013).

Secondly—and, for a theoretical physicist interested in universality, more importantly—it is an intrinsic property of MHD turbulence to develop local imbalance. This can be understood dynamically as a tendency to evolve towards an Elsasser state, $Z_{\perp}^+ = 0$ or $Z_{\perp}^- = 0$, which is an exact solution of RMHD equations (a tendency confirmed in simulations of decaying MHD turbulence; see Maron & Goldreich 2001 and Cho et al. 2002), or statistically as a tendency for the local dissipation rates $\varepsilon^\pm$ to fluctuate in space—a mainstay of intermittency theories since the famous Landau objection (see Frisch 1995) to Kolmogorov (1941b) and the latter’s response in the form of the refined-similarity hypothesis, accepting a fluctuating $\varepsilon$ (Kolmogorov 1962). In this context, a complete intermittency theory for MHD turbulence must incorporate whatever local modification (if any) of the MHD cascade is caused by $\varepsilon^+ \neq \varepsilon^-$, something that no existing theory has as yet accomplished; there is no clarity as to what this modification is.
That an intimate connection must exist between any verifiable theory of MHD turbulence and local imbalance is well illustrated (in figure 13) by the following piece of observational analysis, rather noteworthy, in my (not impartial) view. Wicks et al. (2013b) took a series of measurements by Wind spacecraft of magnetic and velocity perturbations in fast solar wind and sorted them according to the amount of imbalance, both Elsasser and Alfvénic (section 8.2), at each scale. They then computed structure functions conditional on these imbalances. While the majority of perturbations were imbalanced one way or the other (or both), there was a sub-population with \( \delta Z^+ \sim \delta Z^- \sim \delta b \sim \delta u \). Interestingly, the structure function restricted to this population had what seemed to be a robust GS95 scaling (corresponding to a \( k^{5/3} \) spectrum), even though the structure functions of the imbalanced perturbations—and also of all perturbations averaged together—were consistent with Boldyrev’s \( k^{3/2} \) aligned-cascade scaling (and indeed exhibited some alignment, unlike the GS95 population; see Wicks et al. 2013a).\(^{43}\) It is important to recognise that imbalance and alignment do not automatically imply each other (see section 8.1.2), so balanced fluctuations are not absolutely required to be unaligned. However, as I argued in section 6.3.2, dynamical alignment is an intermittency effect and so there may be a correlation between the emergence and/or survival of imbalanced patches at ever smaller scales and Elsasser fields shearing each other into alignment (cf. Chandran et al. 2015).

Intuitively then, since patches of mild imbalance are locally ubiquitous even in globally balanced turbulence (Perez & Boldyrev 2009; see figure 14) and since our theories of balanced turbulence all incorporate intermittency effects in the form of alignment, we might expect that this allows for local imbalance—and, therefore, that mildly imbalanced

\(^{43}\)Podesta & Borovsky (2010) reported analogous results, conditioning on the presence of cross-helicity only.
turbulence might look largely similar to the balanced one. Indeed, how would perturbations in the middle of inertial range “know” that the local imbalance they “see” is local rather than global? Obviously, on average, there will not be an imbalance and so the results for $\delta Z$ that one derives for balanced turbulence (sections 6 and 7) are effectively averaged over the statistics of the stronger and weaker Elsasser fields—which of $\delta Z^+$ and $\delta Z^-$ is which depends on time and space.

If we now allow, without loss of generality, $\varepsilon^+ > \varepsilon^-$ on average, it becomes reasonable to expect $\delta Z^+ > \delta Z^-$ nearly everywhere or, at least, typically—unless $\varepsilon^+ / \varepsilon^-$ is close enough to unity that fluctuations of local imbalance overwhelm the overall global one. In the latter case, presumably imbalance does not matter—at any rate, in the balanced considerations of sections 6 and 7, we only ever required $\varepsilon^+ \sim \varepsilon^-$, rather than $\varepsilon^+ = \varepsilon^-$ exactly. What I am driving at here, perhaps with too much faffing about, is the rather obvious point that it is only the limit of strong imbalance, $\varepsilon^+ \gg \varepsilon^-$, that can be expected to be physically distinct, in a qualitative manner, from the balanced regime.

### 8.1.1. Numerical Evidence

As usual, it is this most interesting limit that is also the hardest to resolve numerically and so we have little definitive information as to what happens in such a regime. As in the case of the spectra of balanced turbulence, the debate about the numerical evidence regarding the imbalanced cascade and its correct theoretical interpretation has most recently been dominated by the antagonistic symbiosis of Beresnyak and Boldyrev, so it is from their papers (Beresnyak & Lazarian 2008, 2009, 2010; Perez & Boldyrev 2009, 2010a, b) that I derive most of the information reviewed below. Perez & Boldyrev (2010a, b) argue that large imbalances are unresolvable and refuse to simulate them. Beresnyak & Lazarian (2009, 2010) do not necessarily disagree with this, but believe that useful things can still be learned. Based on both groups’ simulations, imbalanced MHD turbulence appears to exhibit the following distinctive features.

(i) The stronger field has a steeper spectrum than the weaker one, with the former steeper and the latter shallower than the standard balanced-case spectra (figure 15a).\(^{44}\)

\(^{44}\)This appears to be consistent with structure functions measured in fast solar wind (Wicks et al. 2011), although, besides this, they also exhibit low Alfvén ratio (see section 8.2), which simulations do not, and a rather-hard-to-interpret (or, possibly, to trust) scale dependence of the anisotropy.
It is not, however, clear that these spectra are converged with resolution: as resolution
is increased, the tendency appears to be for the spectral slopes to get a little closer to
each other, both when the imbalance is weak (Perez & Boldyrev 2010a) and when it is
strong (Mallet & Schekochihin 2011). This led Perez & Boldyrev (2010a) to argue that
numerical evidence might be consistent with the two fields having the same spectral slope
in the asymptotic limit of infinite Reynolds numbers. There is no agreement as to whether
the two fields’ spectra are pinned to each other at the dissipation scale: yes in weakly
imbalanced simulations of Perez & Boldyrev (2010a), no in the strongly imbalanced ones
of Beresnyak & Lazarian (2009).45

(ii) The ratio of stronger to weaker fields’ energies, a crude outer-scale quantity that
Beresnyak & Lazarian (2008, 2009, 2010) argue (reasonably, in my view) to be more
likely to be numerically converged than inertial-range scalings, scales very strongly with
$\epsilon^+ / \epsilon^-$: it increases at least as fast as $(\langle |Z^+|^2 \rangle / \langle |Z^-|^2 \rangle) \sim (\epsilon^+ / \epsilon^-)^2$ and possibly faster
(which is inconsistent with the theory of Perez & Boldyrev 2009, another casus belli for
the two groups; see section 8.1.3).

(iii) The stronger field is less anisotropic than the weaker one, in the sense that $l^+ < l^-$
and $l^+_\lambda$ drops faster with $\lambda$ than $l^-\lambda$ (figure 15b).

8.1.2. Geometry and Types of Alignment

Let me first deal with a topic to which I have alluded several times but thus far avoided
discussing carefully: the formal relationship between imbalance and (various kinds of)
alignment. The first salient fact is purely geometric: the two alignment angles (defined

45Whereas the question of pinning may be subject to nontrivial discussion (Lithwick & Goldreich
2003; Chandran 2008) in application to MHD turbulence with viscous or resistive cutoff at small
scales, it would appear that it is more straightforward in a collisionless plasma, e.g., in the solar
wind. Indeed, there, the decoupling between the two Elsasser fields breaks down at the ion
Larmor scale, where they are allowed to exchange energy (Schekochihin et al. 2009; Kunz et al.
2015) and, presumably, cannot have very different typical amplitudes. Thus, an imbalanced
turbulence theory with pinning would probably be a desirable objective. If such an outcome
proves impossible, this might have interesting implications for Larmor-scale physics, e.g., energy
partition between ion and electron cascades.
Figure 16. Geometry of velocity, magnetic and Elsasser fields. All four fields are aligned: the angles $\theta$, $\theta^{ub}$, $\theta^\pm$ are all small (although they do not have to be). Also shown are the axes along which the $\lambda$ and $\xi$ scales in (6.1) are meant to be calculated (perpendicular to $Z_{\perp}^+$ and along $Z_{\perp}^-$, respectively). The angle between these axes is $\phi = \pi/2 - \theta$ and so $\cos \phi = \sin \theta$.

for a particular pair of field increments)

$$\sin^2 \theta = \frac{|\delta Z^+_\lambda \times \delta Z^-_\lambda|^2}{|\delta Z^+_\lambda|^2 |\delta Z^-_\lambda|^2}, \quad \sin^2 \theta^{ub} = \frac{|\delta u_\lambda \times \delta b_\lambda|^2}{|\delta u_\lambda|^2 |\delta b_\lambda|^2},$$

and the Elsasser and Alfvén ratios

$$R_E = \frac{|\delta Z^+_\lambda|^2}{|\delta Z^-_\lambda|^2}, \quad R_A = \frac{|\delta u_\lambda|^2}{|\delta b_\lambda|^2}$$

are related (see figure 16) by the following equations

$$\sin^2 \theta = \frac{\sin^2 \theta^{ub}}{\sin^2 \theta^{ub} + (1 - R_A)^2/4R_A}, \quad \sin^2 \theta^{ub} = \frac{\sin^2 \theta}{\sin^2 \theta + (1 - R_E)^2/4R_E},$$

so only two of these quantities are independent. Equivalently, in terms of the normalised local cross-helicity and residual energy

$$\sigma_c = \frac{|\delta Z^+_\lambda|^2 - |\delta Z^-_\lambda|^2}{|\delta Z^+_\lambda|^2 + |\delta Z^-_\lambda|^2} = \frac{R_E - 1}{R_E + 1}, \quad \sigma_r = \frac{|\delta u_\lambda|^2 - |\delta b_\lambda|^2}{|\delta u_\lambda|^2 + |\delta b_\lambda|^2} = \frac{R_A - 1}{R_A + 1},$$

we have

$$\cos \theta = \frac{\sigma_r}{\sqrt{1 - \sigma_c^2}}, \quad \cos \theta^{ub} = \frac{\sigma_c}{\sqrt{1 - \sigma_r^2}}.$$  

This means that, generally speaking, alignment between the velocity and magnetic field is not the same thing as alignment between the Elsasser variables, and it is a nontrivial decision which of these you think matters for the determination of $\tau^{\pm}_{nl}$. As was spelled out but not emphasised in section 6.1, my answer to this question is different from Boldyrev’s: I prefer the alignment of Elsasser variables, while he favours that of $u_{\perp}$ and $b_{\perp}$—hence my use of $\sin \theta$ in (6.1). In this, I follow Chandran et al. (2015) and Mallet et al. (2015), who think of alignment as a result of mutual shearing of Elsasser fields—and it makes sense that it is the alignment of $Z_{\perp}^+$ and $Z_{\perp}^-$ that leads to the depletion of the $Z_{\perp}^\pm \cdot \nabla_{\perp} Z_{\perp}^\pm$ nonlinearity in (3.1).

This approach is perhaps circumstantially supported by the “refined critical balance” (Mallet et al. 2015; see figure 4)—the remarkable self-similarity shown by the ratio $\tau_A/\tau^{\pm}_{nl}$, with $\tau^{\pm}_{nl}$ defined by (6.1), using the angle between the Elsasser fields. Arguably, this says that if it is this $\tau^{\pm}_{nl}$ that $\tau_A$ (and, therefore, $l^{\pm}_{nl}$) “knows” about, then it is this $\tau^{\pm}_{nl}$ that should be viewed as the cascade time of the decorrelating eddies. Another argument worth recalling that I wish to turn to my advantage in this context is one involving the exact law (3.6); see footnote 23.
This kind of alignment does not have to be directly related to the local dynamics enhancing the cross-helicity $u \perp \cdot b \perp$ (Matthaeus et al. 2008) or to the latter’s statistical tendency to cascade to small scales, together with the energy (Perez & Boldyrev 2009). Consider a strongly (locally) imbalanced situation, where $R_E \gg 1$, i.e., the cross-helicity is large ($\sigma_c \approx 1$). Then (8.3) gives us

$$\sin^2 \theta_{ub} \approx \frac{4\sin^2 \theta}{R_E} \ll 1, \quad (1 - R_A)^2 \approx \frac{16\cos^2 \theta}{R_E} \ll 1. \quad (8.6)$$

Thus, local imbalance implies that $u \perp$ and $b \perp$ are both closely aligned and have nearly the same amplitude (this is geometrically obvious from figure 16), but whether or not the Elsasser fields are aligned is up to the turbulence to decide. It does, it seems, decide to align them (Beresnyak & Lazarian 2009); hence the way in which I drew the field increments in figure 16.

There is an interesting piece of evidence from the solar wind, from a fairly strongly imbalanced case (Podesta & Borovsky 2010), that $\sigma_c$ (and, therefore, $R_E$) is independent of scale throughout the inertial range—according to the first relation in (8.6), this means that $\theta_{ub}$ and $\theta$ should have the same scaling. They appear to do so, approximately, also in balanced turbulence, which, of course, is patch-wise imbalanced and both types of alignment are again present (Mallet et al. 2016).

8.1.3. Theories of Imbalanced Turbulence: Two Strong Cascades

Let me stick with my use of the Elsasser-field alignment angle $\theta$ in the expression (6.1) for $\tau_{\pm n}$. This angle is obviously the same for both fields, so

$$\tau_{\pm n} \sim \frac{\lambda}{\delta Z_\lambda^+ \sin \theta} \Rightarrow \frac{\tau_{\pm n}}{\tau_{\pm n}} \sim \frac{\delta Z_\lambda^+}{\delta Z_\lambda^-} > 1, \quad (8.7)$$

i.e., the cascade of the stronger field is slower (because it is advected by the weaker field). Assuming nevertheless that both cascades are strong, we infer immediately

$$\frac{(\delta Z_\lambda^+)^2}{\tau_{\pm n}} \sim \varepsilon^+ \Rightarrow \frac{\delta Z_\lambda^+}{\delta Z_\lambda^-} \sim \frac{\varepsilon^+}{\varepsilon^-}. \quad (8.8)$$

Thus, the two fields’ increments have the same scaling with $\lambda$ and the ratio of their energies is $\sim (\varepsilon^+ / \varepsilon^-)^2$. Lithwick et al. (2007) arrive at the same conclusion: they consider unaligned GS95-style turbulence, but having no $\theta$ to worry about again gives (8.7) and, therefore, (8.8); we already had this result in (5.4). Things are, however, not as straightforward as they might appear. Lithwick et al. (2007) point out that it is, in fact, counterintuitive that the weaker $\delta Z_\lambda^-$ perturbation, which is distorted by $\delta Z_\lambda^+$ on a shorter time scale $\tau_{\pm n}$, can nevertheless coherently distort $\delta Z_\lambda^+$ for a longer time $\tau_{\pm n}$. Their solution to this is to argue that, while the weaker field is strongly distorted in space by the stronger one, it remains correlated in time for as long as the stronger field does. This is a bit awkward as, technically, after time $\tau_{\pm n}$, the allegedly still correlated perturbation $\delta Z_\lambda^-$ is not at scale $\lambda$ anymore. Their contention is, I think, that it continues to distort $\delta Z_\lambda^+$ in a coherent way and with the same rate of strain. I am unconvinced by this and am willing to contemplate instead the possibility that the cascade of $Z_\perp$ is weak—as are Chandran 2008 and Beresnyak & Lazarian 2008, despite Lithwick et al. 2007 warning their readers not to do so.

Perez & Boldyrev (2009) disagree with the entire approach leading to (8.8): they think that the two Elsasser fields should have two different alignment angles $\theta_{\lambda, \pm}$, both small,
and posit that those ought to be the angles that they make with the velocity field.\[^{47}\] Why that should be the case they do not explain, but if one takes their word for it, then (as is obvious from the geometry in figure 16)

\[
\delta Z^+ \theta^+_A \sim \delta Z^- \theta^-_A \quad \Rightarrow \quad \tau_{nl}^+ \sim \tau_{nl}^- \sim \frac{\lambda}{\delta Z^+ \theta^+_A} \quad \Rightarrow \quad \frac{\delta Z^+}{\delta Z^-} \sim \sqrt{\frac{\epsilon^+}{\epsilon^-}}. \quad (8.9)
\]

The last result follows from the first relation in (8.8) with \(\tau_{nl}^+ \sim \tau_{nl}^-\). The equality of cascade times also conveniently spares them having to deal with the issue, discussed above, of long-time correlatedness, or otherwise, of the weaker field [or with \(l_{||}^+ \neq l_{||}^-\); see arguments around (8.11)].

Perez & Boldyrev (2009, 2010\(^a,b\)) are not forthcoming with any detailed tests of this scheme (viz., either of the details of alignment or of the energy-ratio scaling), while Beresnyak & Lazarian (2010) present numerical results that contradict very strongly the expectation of the energy ratio scaling as \(\epsilon^+/\epsilon^-\) and possibly support \((\epsilon^+/\epsilon^-)^2\). Perez & Boldyrev (2010\(^b\)) reply that (8.9) should only be expected to hold for local fluctuating values of the amplitudes and of \(\epsilon^\pm\) and not for their box averages.\[^{48}\] That this could make a difference for cases of weak imbalance \((\epsilon^+/\epsilon^- \sim 1)\), with local fluctuations of energy fluxes superceding the overall imbalance, is not impossible, although I would argue that if it does, we are basically dealing with balanced turbulence anyway: I do not see any fluxes superceding the overall imbalance, is not impossible, although I would argue that a difference for cases of weak imbalance (\(\epsilon\) values of the amplitudes and of the expectation of the energy ratio scaling as \(\epsilon\))

Podesta (2011) shows, however, by collating both groups’ data, that the results of Perez & Boldyrev (2010\(^b\)) are in fact entirely compatible with Beresnyak & Lazarian (2010) and with \(\langle |Z^+_{||} |^2 \rangle / \langle |Z^-_{||} |^2 \rangle \approx (\epsilon^+/\epsilon^-)^2\).

\[^{47}\text{Podesta & Bhattacharjee (2010) base their theory on the same assumption (also unexplained), and have a different (to Perez & Boldyrev 2009) scheme for generalising Boldyrev’s aligned cascade to the imbalanced regime. Their picture of the geometric configuration of the fields assumes that }|\delta u_A| = |\delta b_A| \text{ and, consequently, that } \delta Z^+_A \text{ and } \delta Z^-_A \text{ are perpendicular to each other. I do not believe this is actually what happens, to the best of our knowledge, in either simulations or solar wind. Podesta & Bhattacharjee (2010) also inherit from Boldyrev’s original construction the incompatibility of their scalings with the RMHD symmetry (see section 6.3.1). There is one interesting angle in their paper though: they notice, in solar-wind observations, that the probabilities with which aligned or antialigned (in the sense of the sign of } \delta u_A \cdot \delta b_A) \text{ perturbations occur are independent of scale throughout the inertial range; they then use the ratio of these probabilities as an extra parameter in the theory. This is a step in the direction of incorporating patchy imbalance into the game—something that seems important and inevitable.}

\[^{48}\text{Podesta (2011) shows, however, by collating both groups’ data, that the results of Perez & Boldyrev (2010\(^b\)) are in fact entirely compatible with Beresnyak & Lazarian (2010) and with } \langle |Z^+_{||} |^2 \rangle / \langle |Z^-_{||} |^2 \rangle \approx (\epsilon^+/\epsilon^-)^2\text{.}
because $Z^+_{\perp}$ perturbations separated by distance $l^-_{\parallel}$ in the parallel direction are advected by completely spatially decorrelated $Z^-_{\perp}$ perturbations, which would then imprint their parallel coherence length on their stronger cousins. Note that if one accepts the Lithwick et al. (2007) argument, which I reproduced after (8.8), that the weaker field is distorted spatially faster than it is decorrelated temporally, then one has to justify the CB conjecture for $l^-_{\parallel}$ not by temporal (causal) decorrelation but as setting the parallel length beyond which the weaker field is distorted spatially beyond recognition, even if possibly remaining temporally coherent. Thus, both Elsasser fields have $\tau_A \sim l^-_{\parallel}/v_A$ that is smaller than their temporal correlation time $\tau_{nl}^+$ (even though the weaker field has a shorter spatial distortion time $\tau_{nl}^- \sim \tau_A$). I am not particularly happy with this. In any event, neither (8.10) nor (8.11) appear to be consistent with any of the cases reported by Beresnyak & Lazarian (2009), weakly or strongly imbalanced, which all have $l^+_{\parallel} < l^-_{\parallel}$ (see, e.g., figure 15b). No other numerical evidence on this is, as far as I know, available in print.

This and the following sections are unfinished. There will be more discussion of what is going on and possibly some new ideas on imbalanced turbulence.

8.2. Alfvénic Imbalance (Residual Energy)

The key papers here are Grappin et al. (1982, 1983); Müller & Grappin (2004, 2005); Grappin et al. (2016); Wang et al. (2011).

9. Next Frontier: Kinetic Turbulence

9.1. Sundry Microphysics at Low Collisionality

I ended the main part of this review with a proclamation in section 7.3 that the story of MHD turbulence looked reasonably complete. Since the main reason for this triumphalism was that MHD cascade finally made sense at the dissipation scales—and the key role in making it make sense belonged to reconnection, a dissipative phenomenon,—it is an inevitable complication that microphysics of dissipation will matter. The visco-resistive MHD description adopted above does apply to some natural plasmas, e.g., stellar convective zones or colder parts of accretion discs. These are mostly low-Pm environments. Despite my efforts to keep all results general and applicable to the high-Pm limit, it is, in fact, quite hard to find naturally occurring high-Pm plasmas for which the standard visco-resistive MHD equations are a good model: this would require the particles’ collision rate to be larger than their Larmor frequency, which rarely happens at high temperatures and low densities needed to achieve high Pm. In fact, most of the interesting (and observed) plasmas in this hot, rarefied category are either “dilute” (an apt term coined by Balbus 2004 to describe plasmas where turbulence is on scales larger than the mean free path, but the Larmor motion is on smaller scales that it—a good example is galaxy clusters; see, e.g., Melville et al. 2016 and references therein) or downright collisionless (i.e., everything happens on scales smaller than the mean free path; the most obvious example is the solar wind: see the mega-review by Bruno & Carbone 2013 or a human-sized one by Chen 2016). In either case, between the “ideal-MHD scales” and the resistive scale, there is a number of other scales at which the physics changes. These changes are of two distinct kinds.

The first is the appearance of dispersion in the wave physics: Alfvén waves become kinetic Alfvén waves (KAWs), with a different linear response and, therefore, a different variety of critically balanced cascade (Cho & Lazarian 2004; Schekochihin et al. 2009;
The culprits here are the ion inertial scale (at which the Hall effect comes in), ion sound scale (at which the electron pressure gradient becomes important in Ohm’s law) and the ion Larmor scale (at which the finite size of ion Larmor orbits starts playing a role). Which of these matters most depends on plasma beta and on the ratio of the ion and electron temperatures, but they all are essentially ion-electron decoupling effects and lead to more or less similar kinds of turbulence, at least in what concerns the KAW cascade.

The second important modification of MHD is that reconnection in a collisionless plasma need not be done by resistivity, but can also be due to other physics that breaks flux conservation, viz., electron inertia, electron finite Larmor radius (FLR) and, more generally, other kinetic features of the electron pressure tensor. Tearing modes are different in such plasmas, with a double ion-electron layer structure and a variety of scalings in a variety of parameter regimes.\textsuperscript{49} Since tearing is important for rounding off the MHD cascade, all these effects must be considered and appropriate modifications worked out for the theory of reconnecting turbulence described in section 7—this has been done by Mallet \textit{et al.} (2017a) and by Loureiro & Boldyrev (2017a). It is going to be interesting to find out whether, where and when any of this matters or if perhaps the aligned MHD cascade just segues directly into the KAW cascade (see, however, a discussion in a moment as to what that means). Since there are some mysteries still outstanding with regard to the scale at which the spectrum of solar-wind turbulence is supposed to have a spectral break between the inertial range and the “kinetic” (KAW) range (Chen \textit{et al.} 2014a; Boldyrev \textit{et al.} 2015), perhaps something interesting can be done here (is the break set by onset of reconnection, rather than by the Larmor scale?—see Vech \textit{et al.} 2018, fresh off the press).

Furthermore, KAW turbulence in the kinetic range and its relationship with reconnection is a topic that is rapidly becoming very popular with both numerical modellers (e.g., TenBarge & Howes 2013; Bañón Navarro \textit{et al.} 2016; Cerri & Califano 2017; Franci \textit{et al.} 2017) and observational space physicists (e.g., Greco \textit{et al.} 2016). There is a promise of interesting physics—interesting both conceptually and because it is eminently measurable in space. In the context of the prominent role that was given in section 7 to the break up of MHD sheets in setting up the tail end of the MHD cascade, I want to highlight an intriguing suggestion (implicitly) contained in the recent paper by Cerri & Califano (2017) and further fleshed out by Franci \textit{et al.} (2017). They look (numerically) at the formation of current sheets in kinetic turbulence and the disruption of these sheets by tearing (plasmoid) instabilities—and discover that it is precisely these processes that appear to seed the sub-Larmor-scale cascade with a steep (steeper than in the inertial range) energy spectrum usually associated with KAW turbulence. One might wonder then if such a KAW cascade is an entirely distinct phenomenon from a collisionless version of reconnecting turbulence in the disruption range. If we allow ourselves to get excited about this question, we might speculate that it rhymes nicely with the idea on which Boldyrev

\textsuperscript{49}Appendix B.3 of Zocco & Schekochihin (2011) has a review of standard results for collisionless and semicollisional tearing modes at low beta (using a convenient minimalist set of dynamical equations as a vehicle), as well as all the relevant references of which we were aware at the time. There is a huge literature on semicollisional and collisionless reconnection and, short of dedicating this review to name-checking it all (which would be a noble ambition, but a doomed one, as the literature is multiplying faster than one can keep track), I cannot give proper credit to everyone who deserves it. The most recent collections of appropriate pointers are reviews by Zweibel & Yamada (2016), Loureiro & Uzdensky (2016), Tenerani \textit{et al.} (2016) and Janvier (2017), written from different perspectives (or at least from within different tribes) and thus complementing each other.
A. A. Schekochihin & Perez (2012) relied to advocate a steeper \((-8/3)\) slope of KAW turbulence than the \(-7/3\) implied by the standard CB-based theory (Cho & Lazarian 2004; Schekochihin et al. 2009). They argued that the energetically dominant perturbations at each scale were concentrated in 2D structures, thus making turbulence non-volume-filling (and perhaps mono-fractal; cf. Kiyani et al. 2009 and Chen et al. 2014b). While Boldyrev & Perez (2012) did not appear to think of these 2D structures as reconnecting sheets, an interpretation of them as such does not seem \textit{a priori} unreasonable. So perhaps this is what happens in collisionless turbulence: sheet-like structures form in the usual (MHD) way, get disrupted by collisionless tearing and/or related instabilities and seed sub-Larmor turbulence,\footnote{See Mallet et al. (2017a) for a discussion of what else they seed and Loureiro & Boldyrev (2017a) for a bold argument that, if one abandons the KAW cascade entirely and instead declares the sub-Larmor turbulence to be of the reconnecting kind, one gets spectral slopes between \(-3\) and \(-8/3\), still consistent with observations and simulations.} which stays mostly concentrated in those sheets or their remnants, with an effectively 2D filling fraction.

What I have said about kinetic physics so far might not sound like a true conceptual leap: basically, at small scales, we have different linear physics and a zoo of possibilities, depending on parameter regimes; one could work productively on porting some of the basic ideas developed in the preceding sections to these situations. There are, however, ways in which kinetic physics does bring in something altogether new. I will mention two directions that I view as belonging to this category.

\section*{9.2. Phase-Space Turbulence}

What is turbulence? Some energy is injected into some part of the phase space of a nonlinear system (in fluid systems, that simply means position or wavenumber space), which is, generally speaking, not the part of the phase space where it can be efficiently thermalised. So turbulence is a process whereby this energy finds its way from where it is injected to where it can be dissipated and the means of doing this is nonlinear coupling, usually from large scales to small scales. What kind of coupling is possible and at what rate the energy can be transferred from scale to scale then determines such things as energy spectra in a stationary state with a constant flux of energy.

The same principle applies to kinetic turbulence, but now the phase space is 6D rather than 3D: the particle distribution depends on positions and velocities and energy transfer can be from large to small scales (or vice versa) in all six coordinates. The transfer of energy to small scales in velocity space, leading ultimately to activation of collisions, however small the collision rate, is known as “phase mixing”. It is not necessarily a nonlinear phenomenon: the simplest (although not necessarily very simple) phase-mixing process is the linear Landau (1946) damping. In a magnetised plasma, this is the \textit{parallel} (to \(B_0\)) phase mixing, whereas the \textit{perpendicular} phase mixing is nonlinear and has to do with particles on Larmor orbits experiencing different electromagnetic fields depending on the radius of the orbit (the Larmor radius is a kinetic variable, being proportional to \(v_\perp\)). The latter phenomenon leads to an interesting phase-space “entropy cascade” (Schekochihin et al. 2008, 2009; Tatsuno et al. 2009; Cerri et al. 2018), which is one of the more exotic phenomena that await a curious researcher at sub-Larmor scales. Its importance in the “grand scheme of things” is that it funnels turbulent energy into ion heat, while the KAW cascade heats electrons—the question of which dissipation channel is the more important one and when being both fundamental and “applied” (in the astrophysical sense of the word—e.g., to accretion flows: see Quataert & Gruzinov 1999). Understanding how energy is transferred between scales in phase space requires thinking
somewhat outside the standard turbulence paradigm and so perhaps counts as conceptual novelty. Not much of it has been done so far and it is worth doing more.

Returning to parallel phase mixing, this too turns out to be interesting in a nonlinear setting, even though it is a linear phenomenon itself. First theoretical (Schekochihin et al. 2016; Adkins & Schekochihin 2018) and numerical (Parker et al. 2016) analyses suggest that, in a turbulent system, parallel phase mixing is effectively suppressed by the stochastic plasma echo, perhaps rendering kinetic systems that are notionally subject to Landau damping effectively fluid, at least in terms of their energy-flow budgets. In the context of inertial-range MHD turbulence, this is relevant to the compressive (“slow-mode”) perturbations, which, in a collisionless plasma, are energetically decoupled from, and nonlinearly slaved to, the Alfvénic ones, while the latter are still governed by RMHD (Schekochihin et al. 2009; Kunz et al. 2015). Linearly, these compressive perturbations must be damped—but nonlinearly are perhaps not, thus accounting for them exhibiting a healthy power-law spectrum and other fluid features in the solar wind (Chen 2016; Verscharen et al. 2017). In this vein, one might also doubt whether the Landau damping of KAWs at sub-Larmor scales is efficient or even present at all (despite what Loureiro et al. 2013b, TenBarge & Howes 2013, Bañón Navarro et al. 2016 and Kobayashi et al. 2017 say). The broader question is whether there is generally Landau damping in turbulent systems and whether, therefore, to put it crudely, “all turbulence is fluid”. While it might be a little disappointing if it is, the way and the sense in which this seems to be achieved are surprising and pleasingly nontrivial—and possibly soon to be amenable to direct measurement if the first MMS results on velocity-space (Hermite) spectra in the Earth’s magnetosheath ( Servidio et al. 2017) are a good indication of the possibilities that are opening up.

9.3. Macro- and Microphysical Consequences of Pressure Anisotropy

The second line of inquiry pregnant with conceptual novelty concerns the effect of self-generated pressure anisotropy on MHD dynamics. Pressure anisotropies are generated in response to any motion in a magnetised collisionless or weakly collisional plasma as long as this motion leads to a change in the strength of the magnetic field. The conservation of the magnetic moment (proportional to the angular momentum of Larmor-gyrating particles) then causes positive (if the field grows) or negative (if it decreases) pressure anisotropy to arise (see, e.g., Schekochihin et al. 2010). This is usually quite small—in an Alfvén wave, it is of order \( (\delta b/v_A)^2 \)—but it becomes relevant at high beta, when even small anisotropies (of order \( 1/\beta \)) can have a dramatic effect, in two ways. Dynamically, pressure anisotropy supplies additional stress, which, when the anisotropy is negative \((p_\perp < p_\parallel)\), can cancel Maxwell’s stress and thus remove magnetic tension—the simplest way to think of this is in terms of the Alfvén speed being modified so:

\[
v_A \to \sqrt{v_A^2 + \frac{p_\perp - p_\parallel}{\rho}}.
\]

Kinetically, pressure anisotropy is a source of free energy and will trigger fast, small-scale instabilities, most notably mirror and firehose. The firehose corresponds to the Alfvén speed \( (9.1) \) turning imaginary; i.e., it is an instability caused by negative tension; the mirror is not quite as simple to explain, but is fundamentally a result of effective magnetic pressure going negative by means of some subtle resonant-particle dynamics (see Southwood & Kivelson 1993, Kunz et al. 2015 and references therein). These instabilities in turn can regulate the anisotropy by scattering particles or by subtler, more devious means (see Melville et al. 2016 and references therein).

In a recent investigation of the dynamics of a simple finite-amplitude Alfvén wave
in a collisionless, high-beta plasma, Squire et al. (2016, 2017b,a) showed that both of these effects did occur and altered the wave’s behaviour drastically: it first slows down to a near halt due to the removal of magnetic tension, transferring much of its kinetic energy into heat and then, having spawned a colony of particle-scattering Larmor-scale perturbations, dissipates as if it were propagating in a plasma with a large Braginskii (1965) parallel viscosity.

These effects occur provided the amplitude of the waves is above a certain limit that scales with plasma beta: this is because pressure anisotropy must be large enough to compete with tension in (9.1) and the amount of anisotropy that can be generated is of the order of the field-strength perturbation, which, for an Alfvén wave, is quadratic in the latter’s amplitude:

$$\left(\frac{v_A}{\delta b}\right)^2 \sim \frac{p_\perp - p_\parallel}{p} \gtrsim \frac{v_A^2}{p/\rho} \sim \frac{1}{\beta}.$$  

(9.2)

In formal terms, this means that in high-beta collisionless plasmas, the small-amplitude and high-beta limits do not commute. The picture of Alfvénic turbulence simply obeying RMHD equations, even in a collisionless plasma (Schekochihin et al. 2009), must then be seriously revised. We can probably live with the current theory for most instances of the solar wind, where $\beta \sim 1$, but a conventional Alfvénic picture for turbulence in galaxy clusters, for example, clearly needs a close examination—indeed it may turn out that much of the turbulent energy is efficiently converted into heat already at large scales (cf. Kunz et al. 2011).

This line of investigation is particularly rich in surprises because pressure-anisotropy stress undermines much of our basic intuition for ideal-MHD dynamics, not just modifies microscale plasma physics. This said, it is not entirely inconceivable that, at the end of the day (or rather of the decade), in some grossly coarse-grained sense, turbulent plasmas will just turn out to supply their own large effective collisionality even where Coulomb collisions are rare—and so astrophysicists, with their focus on large-scale motions, need not be too worried about the validity of fluid models. I hope life is not quite so boring, although, as a theoretical physicist and, therefore, a believer in universality, I should perhaps expect to be pleased by such an outcome.

10. Conclusion

Let us stop here. The story of MHD turbulence is a fascinating one: both the story of what happens and the story of how it has been understood. It is remarkable how long it takes to figure out simple things, obvious in retrospect. It is even more remarkable (and reassuring) that we get there after all, in finite time. This story now looks reasonably complete, at least in broad-brush outline (section 7.3) and modulo loose ends (section 8). Is this an illusion? Is it all wrong again? We shall know soon enough, but in the meanwhile, the siren call of kinetic physics is too strong to resist and the unexplored terrain seems vast and fertile (section 9). Is everything different there? Or will it all, in the end, turn out to be the same, with Nature proving itself a universalist bore and contriving to supply effective collisions where nominally there are few? Is turbulence always basically fluid or do subtle delights await us in phase space? Even if we are in danger of being disappointed by the answers to these questions, getting there is proving to be a journey of amusing twists and turns.

For a topic as broad as this, it is difficult to list all the people from whom I have learned what I know (or think I know) of this subject. The most important such influence has
been Steve Cowley. The views expressed in the main part of this paper (sections 5–7) were informed largely by my collaboration with Alfred Mallet and Ben Chandran on MHD turbulence and by conversations with Andrey Beresnyak, Nuno Loureiro and Dmitri Uzdensky. I have learned most of what I know of reconnection from Nuno and Dmitri and of the solar wind from Chris Chen, Tim Horbury and Rob Wicks. I owe the epigraph of this paper to Matt Kunz. I am grateful to him and to other participants of the 1st JPP Frontiers of Plasma Physics Conference at the Abbazia di Spineto for lively discussions of this paper (which started as an “opinion piece” written for that conference, then ballooned). I am also grateful to Marco Velli for some very useful comments on the draft, especially on appendix C. I am pleased to acknowledge the hospitality of the Wolfgang Pauli Institute, University of Vienna, where, in meetings held annually for the last 10 years, many key interactions took place. In the UK, my work was supported in part by grants from STFC and EPSRC (and earlier by the very enlightened Leverhulme Trust).

Appendix A. Conventional WT Theory

A.1. RMHD in Scalar Form

For the purposes of this calculation, it will be convenient to cast the RMHD equations (3.1) in terms of two scalar fields, so called Elsasser potentials \( \zeta^\pm \), which are the stream functions for the 2D-solenoidal fields \( Z^\pm_\perp \) (Schekochihin et al. 2009): namely,

\[
Z^\pm_\perp = \hat{z} \times \nabla_\perp \zeta^\pm, \tag{A 1}
\]

where \( \hat{z} = B_0 / B_0 \) and \( \zeta^\pm \) satisfy, as shown by taking the curl of (3.1) and using (A 1),

\[
\frac{\partial}{\partial t} \nabla_\perp^2 \zeta^\pm = -\frac{1}{2} \left\{ \left\{ \zeta^+, \nabla_\perp^2 \zeta^- \right\} + \left\{ \zeta^-, \nabla_\perp^2 \zeta^+ \right\} \right\} - \nabla_\perp \left\{ \zeta^+, \zeta^- \right\}, \tag{A 2}
\]

where we have dropped the dissipative terms and denoted

\[
\left\{ \zeta^+, \zeta^- \right\} = \frac{\partial \zeta^+}{\partial x} \frac{\partial \zeta^-}{\partial y} - \frac{\partial \zeta^+}{\partial y} \frac{\partial \zeta^-}{\partial x} \tag{A 3}
\]

(and similarly for all other nonlinear terms).

The rest of this section will contain a review of how WT is done, conventionally, for Alfvén-wave turbulence, more or less along the lines of Galtier et al. (2000, 2002). The purpose of this is to highlight what goes wrong with this derivation and also to set up the formalism that will come handy in the construction of a better theory. This is also a pedagogically useful recap of the basic WT scheme.

Appendix B. 2D Spectra of RMHD Turbulence

As we trade in \( k_\perp \) (or \( \lambda \)) and \( k_\parallel \) (or \( l_\parallel \)) scalings, it is only natural that we might wish to construct 2D spectra of RMHD turbulence, \( E_{2D}(k_\perp, k_\parallel) \). It is quite easy to do so, given the information we already have about the \( \lambda \) and \( l_\parallel \) scalings of the Elsasser increments.

Since, as I explained in section 5.3, the physically meaningful parallel correlations are along the local mean field, we should think of our Elsasser fields \( Z^\pm_\perp \) as being mapped on a grid of values of \( (r_\perp, r_\parallel) \), where \( r_\parallel \) is the distance measured along the exact field line (what matters here is not the parallel distances being slightly longer than their projection on the \( z \) axis—the difference is small in the RMHD ordering—but considering correlations along the exact field line rather than slipping off it; see figure 6). The Fourier
transform of $Z^\pm_\perp(r_\perp, r_\parallel)$ is a function of $k_\perp$ and $k_\parallel$, $Z^\pm_\perp(k_\perp, k_\parallel)$, and the 2D spectrum is defined to be

$$E_{2D}(k_\perp, k_\parallel) = 2\pi k_\perp \langle |Z^\pm_\perp(k_\perp, k_\parallel)|^2 \rangle.$$  \hfill (B 1)

Let us start with the premise that $E_{2D}(k_\perp, k_\parallel)$ will be a power law in both of its arguments and that these powers will be different depending on where in $(k_\perp, k_\parallel)$ space the spectrum is measured vis-à-vis the line of critical balance, which is another power-law relation, between $k_\perp$ and $k_\parallel$:

$$\tau_{nl} \sim \tau_A \iff \quad k_\parallel \sim k_\perp^\sigma.$$  \hfill (B 2)

We shall treat the wave numbers as dimensionless, $k_\parallel L_\parallel \to k_\parallel$, $k_\perp \lambda_{CB} \to k_\perp$; according to (6.20),

$$\sigma = \frac{1}{2}.$$  \hfill (B 3)

Thus, we expect

$$E_{2D}(k_\perp, k_\parallel) \sim \begin{cases} k_\parallel^{-\alpha} k_\perp^\beta, & k_\parallel \gtrsim k_\perp^\sigma, \\ k_\parallel^\delta k_\perp^{-\gamma}, & k_\parallel \lesssim k_\perp^\sigma. \end{cases}$$  \hfill (B 4)

The four exponents can be determined as follows (this argument is analogous to one proposed by Schekochihin et al. 2016 for drift-kinetic turbulence).

B.1. Determining $\delta$

At long parallel wavelengths, $k_\parallel \ll k_\perp^\sigma$, the $k_\parallel$ spectrum measures correlation between points along the field line that are separated by longer distances than an Alfvén wave can travel in one nonlinear time ($\tau_A \gg \tau_{nl}^\pm$) and, consequently, are causally disconnected (section 5.1). Therefore, their parallel correlation function is that of a white noise and the corresponding spectrum is flat:

$$\delta = 0.$$  \hfill (B 5)

B.2. Determining $\gamma$

Let us calculate the 1D $k_\perp$ spectrum: if we assume (and promise to check later) that $\alpha > 1$, then the $k_\parallel$ integral over $E_{2D}(k_\perp, k_\parallel)$ is dominated by the region $k_\parallel \lesssim k_\perp^\sigma$ and the 1D spectrum is essentially determined by the CB scales $k_\parallel \sim k_\perp^\sigma$ (as is indeed argued in the GS95 theory and its descendants reviewed in the main text):

$$E(k_\perp) \sim \int_0^{k_\perp^\sigma} dk_\parallel E_{2D}(k_\perp, k_\parallel) \sim k_\perp^{-\gamma + \sigma}.$$  \hfill (B 6)

On the other hand, since the amplitude of an Elsasser field at scale $\lambda = k_\perp^{-1}$ is

$$\delta Z^2_\lambda \sim \int_{k_\perp}^\infty dk_\perp' E(k_\perp') \sim k_\perp E(k_\perp) \sim k_\perp^{-\gamma + \sigma + 1},$$  \hfill (B 7)

assuming $\gamma - \sigma > 1$. The usual Kolmogorov constant-flux condition coupled with the CB gives us

$$\frac{\delta Z^2_\lambda}{\tau_{nl}} \sim \text{const}, \quad \tau_{nl}^{-1} \sim \tau_A^{-1} \propto k_\parallel \sim k_\perp^\sigma \quad \Rightarrow \quad E(k_\perp) \sim k_\perp^{-\sigma - 1}.$$  \hfill (B 8)

Comparing this with (B 6), we get

$$\gamma = 2\sigma + 1 = 2.$$  \hfill (B 9)
The 1D spectral exponent is then \(-\gamma + \sigma = -3/2\), as it should be [see (6.20)].

### B.3. Determining \(\beta\)

This calculation is essentially kinematic. Let us write the desired spectrum (B 1) as

\[
\langle |Z_{\perp}(k_\perp, k_\parallel)|^2 \rangle = \int \frac{d^2 r_{\perp}}{L_\perp^2} \int \frac{d^2 r_{\perp 2}}{L_\perp^2} e^{-i k_\perp \cdot (r_{\perp 1} - r_{\perp 2})} \langle Z_{\perp}^\pm(r_{\perp 1}, k_\parallel) \cdot Z_{\perp}^{\pm*}(r_{\perp 2}, k_\parallel) \rangle = \int \frac{d^2 r_{\perp}}{L_\perp^2} e^{-i k_\perp \cdot r_{\perp} C^\pm(r_{\perp}, k_\parallel)} = \frac{2\pi}{L_\perp^2} \int_0^\infty dr_{\perp} J_0(k_\perp r_{\perp}) C(r_{\perp}, k_\parallel),
\]

(B10)

where \(L_\perp\) is the perpendicular size of the system, \(r_{\perp} = r_{\perp 1} - r_{\perp 2}\), and \(C^\pm(r_{\perp}, k_\parallel)\) is the two-point correlation function of \(Z_{\perp}^\pm(r_{\perp}, k_\parallel)\). It is only a function of \(r_{\perp} = |r_{\perp 1} - r_{\perp 2}|\) because of statistical homogeneity and isotropy in the perpendicular plane. For any given \(k_\parallel\), we may assume that, by the CB conjecture, the correlation length of the field is \(\lambda \sim k_\parallel^{1/\sigma}\). The integral in (B 10) is then effectively restricted by \(C^\pm(r_{\perp}, k_\parallel)\) to \(r_{\perp} \lesssim \lambda\). If we now let \(k_\perp \lambda \ll 1\), then the Bessel function can be expanded in small argument: \(J_0(k_\perp r_{\perp}) = 1 - k_\perp^2 r_{\perp}^2/4 + \ldots\). The spectrum (B 1) is then

\[
E_{2D}(k_\perp, k_\parallel) = \frac{2\pi}{L_\perp^2} k_\perp (C_0 + C_2 k_\perp^2 + \ldots), \quad (B 11)
\]

\[
C_0 = 2\pi \int_0^\infty dr_{\perp} r_{\perp} C^\pm(r_{\perp}, k_\parallel), \quad C_2 = -\frac{\pi}{2} \int_0^\infty dr_{\perp} r_{\perp}^3 C^\pm(r_{\perp}, k_\parallel). \quad (B 12)
\]

But the first of these coefficients can be written simply is

\[
C_0 = \int d^2 r_{\perp} \langle Z_{\perp}^\pm(r_{\perp}, k_\parallel) \cdot Z_{\perp}^{\pm*}(0, k_\parallel) \rangle. \quad (B 13)
\]

It vanishes if \(\int d^2 r_{\perp} Z_{\perp}^\pm(r_{\perp}, k_\parallel) = 0\), which should be a safe assumption for a solenoidal field [see (A 1)] in, say, a periodic box. This leaves us with the series (B11) for \(E_{2D}\) starting at the second term and so \(E_{2D} \sim k_\perp^3\) to lowest order. Thus,

\[
\beta = 3. \quad (B 14)
\]

### B.4. Determining \(\alpha\)

Finally, \(\alpha\) is determined simply by the requirement that the 2D spectra match along the CB line: substituting \(k_\parallel \sim k_\perp^\sigma\) into (B 4) and equating powers of \(k_\perp\), we get

\[
\alpha = \frac{\beta + \gamma}{\sigma} - \delta = 10. \quad (B 15)
\]

This ridiculous exponent\(^{51}\) suggests that there is very little energy indeed in wave-like perturbations with \(\tau_A \ll \tau_{\text{nl}}\).

Note that the consistency of what I have done above can be checked by calculating the 1D \(k_\parallel\) spectrum:

\[
E(k_\parallel) = \int dk_\parallel E_{2D}(k_\perp, k_\parallel) \sim \int_0^{k_\parallel^{1/\sigma}} dk_\parallel k_\parallel^{-\alpha} k_\perp^\beta + \int_{k_\parallel^{1/\sigma}}^{\infty} dk_\parallel k_\parallel^\delta k_\perp^{-\gamma} \sim k_\parallel^{-\zeta}, \quad (B 16)
\]

\(^{51}\)Such a steep scaling is probably unmeasurable in practice. Indeed, one would need to follow the perturbed field line very precisely—much more precisely than is recommended in section 5.3—in order to detect the lack of energy at large \(k_\parallel\); slipping off a field line even slightly would access the perpendicular variation of the turbulent fields.
Figure 17. Sketch of the 2D spectra (B 18) of RMHD turbulence: (a) in the 2D wave-number plane, (b) at constant $k_\perp$, (c) at constant $k_\parallel$. Note that $k_\parallel$ here is measured along the perturbed field, not the $z$ axis (see discussion in section 5.3).

where

$$\zeta = \alpha - \frac{\beta + 1}{\sigma} = \frac{\gamma - 1}{\sigma} - \delta = 2,$$  \hspace{1cm}  (B 17)

as it should be (see section 5.2).

To summarise, the 2D spectrum (B 4) of critically balanced Alfvénic turbulence is

$$E_{2D}(k_\perp, k_\parallel) \sim \begin{cases} k_\parallel^{-10} k_\perp^3, & k_\parallel \gtrsim k_\perp^{1/2}, \\ k_\parallel^0 k_\perp^{-2}, & k_\parallel \lesssim k_\perp^{1/2}, \end{cases}$$  \hspace{1cm}  (B 18)

leading to 1D spectra $E(k_\perp) \sim k_\perp^{-3/2}$ and $E(k_\parallel) \sim k_\parallel^{-2}$. The spectra (B 18) are sketched in figure 17.

I leave it as an exercise for the reader to show that if the same scheme is applied to the reconnecting turbulence described in section 7.2, the exponents in (B 4) are

$$\sigma = \frac{6}{5}, \quad \delta = 0, \quad \gamma = \frac{17}{5}, \quad \beta = 3, \quad \alpha = \frac{16}{3}, \quad \zeta = 2.$$  \hspace{1cm}  (B 19)

Appendix C. A Reconnection Primer

Since reconnection phenomena are now clear to be essential in the theory MHD turbulence, it is perhaps useful to provide a series of shortcuts to the key results. I will not do any precise calculations of the kind that make the theory of resistive MHD instabilities such a mathematically accomplished subject (what better example on which
to hone one’s skills in solving ODEs with boundary layers than the many incarnations of the tearing mode!), but will instead go for relatively “quick and dirty” ways of getting at the right scalings.

When dealing with resistive MHD instabilities, it is convenient to write the RMHD equations in their original form (Strauss 1976), in terms of the stream (flux) functions for the velocity and magnetic fields:

\[
\mathbf{u}_\perp = \hat{z} \times \nabla_\perp \Phi, \quad \mathbf{b}_\perp = \hat{z} \times \nabla_\perp \Psi.
\] (C 1)

Since \( \zeta^\pm = \Phi \pm \Psi \), we can recover these equations from (A 2) or, indeed, use (C 1) and derive them directly from the momentum and induction equations of MHD (see Schekochihin et al. 2009, Oughton et al. 2017 and references therein):

\[
\frac{\partial}{\partial t} \nabla_\perp^2 \Phi + \{ \Phi, \nabla_\perp^2 \Phi \} = v_A \nabla_\par \nabla_\perp^2 \Psi + \nu \nabla_\perp^4 \Phi, \quad \frac{\partial}{\partial t} \Psi + \{ \Phi, \Psi \} = v_A \nabla_\par \Phi + \eta \nabla_\perp^2 \Phi;
\] (C 2)

where we have restored the difference between the Ohmic diffusivity \( \eta \) and viscosity \( \nu \).

C.1. Tearing Instability

Let us ignore parallel derivatives in (C 2–C 3) and consider small perturbations of a simple static equilibrium in which the in-plane magnetic field points in the \( y \) direction and reverses direction at \( x = 0 \):

\[
\Phi = \phi(x, y) e^{\gamma t}, \quad \Psi = \Psi_0(x) + \psi(x, y) e^{\gamma t} \Rightarrow \mathbf{b}_\perp = \hat{y} b_0(x) + \hat{z} \times \nabla_\perp \psi e^{\gamma t},
\] (C 4)

where \( b_0(x) = \Psi_0'(x) \) is an odd function (the equilibrium field reverses direction at \( x = 0 \)) and \( \gamma \) is the rate at which perturbations will grow (if they are interesting). If we now linearise our RMHD equations (C 2–C 3) and Fourier-transform them in the \( y \) direction, we get

\[
[\gamma - \nu (\partial_x^2 - k_y^2)] (\partial_x^2 - k_y^2) \phi = i k_y [b_0(x)(\partial_x^2 - k_y^2) - b_0'(x)] \psi,
\] (C 5)

\[
[\gamma - \eta (\partial_x^2 - k_y^2)] \psi = i k_y b_0(x) \phi.
\] (C 6)

When \( \eta \) is small, this system has a boundary layer around \( x = 0 \), of width \( \delta \), outside which the solution is an ideal-MHD one and inside which resistivity is important and reconnection occurs.

C.1.1. Outer Solution

If we assume that the outer-region solution has scale \( \lambda \) and, generally speaking,

\[
\tau_\eta^{-1} \equiv \frac{\eta}{\lambda^2} \sim \tau_\nu^{-1} \equiv \frac{\nu}{\lambda^2} \ll \gamma \ll \tau_{\lambda y} \equiv \frac{v_{Ay}}{\lambda}, \quad v_{Ay} \equiv \lambda b_0'(0),
\] (C 7)

then this outer solution satisfies

\[
\partial_x^2 \psi = \left[ k_y^2 + \frac{b_0'(x)}{b_0(x)} \right] \psi, \quad \phi = - \frac{i \gamma}{k_y b_0(x)} \psi.
\] (C 8)

Since the magnetic field \( b_y = \partial_x \psi \) must reverse direction at \( x = 0 \), \( \psi \) has a discontinuous derivative (figure 18). This corresponds to a singular current that is developed by the ideal-MHD solution as it approaches the boundary layer—with the singularity resolved inside the layer by resistivity. The solutions outside and inside the layer are matched to each other by equating the discontinuity in the former to the total change in \( \partial_x \psi \).
calculated from the latter:

$$\Delta' = \frac{[\partial_x \psi_{\text{out}}]^+}{\psi_{\text{out}}(0)} = \frac{2}{\delta_{\text{in}}} \int_0^\infty dX \frac{\partial^2 X \psi_{\text{in}}(X)}{\psi_{\text{in}}(0)},$$

(C9)

where $\psi_{\text{out}}(x) = \psi(x)$ is the outer solution, $\psi_{\text{in}}(X) = \psi(X \delta_{\text{in}})$ is the inner one, and $X = x/\delta_{\text{in}}$ is the “inner” variable, rescaled to the current layer width $\delta_{\text{in}}$.

To find $\Delta'$ from the outer solution, one must solve (C8) for some particular form of $b_0(x)$. For our purposes, all we need is the asymptotic behaviour of $\Delta'$ in the limit of $k_y \lambda \ll 1$, where $\lambda$ is the characteristic scale of $b_0(x)$. While in general this asymptotic depends on the functional form of $b_0(x)$, it is

$$\Delta' \sim \frac{1}{k_y \lambda^2}$$

(C10)

if one can assume that $b_0(x)$ varies faster at $|x| \lesssim \lambda$, in the region where it reverses direction, than at $|x| \gg \lambda$, where it might be approximately flat. An example of such a situation is the exactly solvable and ubiquitously useful Harris (1962) sheet $b_0(x) = v_A y \tanh(x/\lambda)$. This situation might be particularly relevant because in ideal MHD, field-reversing configurations of the kind that we need to support a tearing mode tend to be collapsing sheets, with $\lambda$ shrinking dynamically compared to the characteristic scales in the $y$ direction or indeed in the $x$ direction away from the field-reversal region (see further discussion in appendix C.4).52

A reader who is happy to accept (C10) can now skip to appendix C.1.2. For those who would like to see a more detailed derivation leading to (C10), let me put forward the following argument, which is adapted from Loureiro et al. (2007, 2013a).

Consider first $|x| \lesssim \lambda$. Since $b''_0/b_0 \sim 1/\lambda^2 \gg k_y^2$, we may neglect the $k_y^2$ term in (C8) and seek a solution in the form $\psi = b_0(x) \chi(x)$. This allows us to integrate the equation

52More generally, $\Delta' \sim 1/k_y^n \lambda^{n+1}$, where $n = 2$ would correspond to $b_0(x)$ decaying to zero at large $x$ on the same scale as it reverses direction around $x = 0$, e.g., for $b_0(x) = v_A y \tanh(x/\lambda)/\cosh^2(x/\lambda)$ (Porcelli et al. 2002). Generalising all the scalings derived here and in section 7 to general $n$ is an entirely straightforward exercise (Pucci et al. 2018; Loureiro & Boldyrev 2017a), which I have opted to forego, to avoid cumbersome $n$-dependent exponents everywhere. Admittedly, the above arguments notwithstanding, there is some space for discussion as to whether $n = 1$ or $n = 2$ is the best model for what happens in a typical MHD-turbulent structure.
directly, with the result

$$\psi = b_0(x) \left[ C_1^+ + C_2^+ \int_{x_0}^x \frac{dx'}{b_0^2(x')} \right], \quad (C11)$$

where $\pm$ refer to solutions at positive and negative $x$, respectively, $C_{1,2}^\pm$ are integration constants and $x_0 \sim \lambda$ is some integration limit, whose precise value does not matter (any difference that it makes can be absorbed into $C_1^\pm$). Since $b_0(x)$ is an odd function,

$$b_0(x) \approx \frac{x}{\lambda} v_{Ay} \quad \text{at} \quad |x| \ll \lambda. \quad (C12)$$

Taking $x \to 0$ in (C11), we can, therefore, fix the constant $C_2^\pm$ via

$$\psi(0) = -C_2^\pm \frac{\lambda}{v_{Ay}}. \quad (C13)$$

Considering now $|x| \gg \lambda$ and assuming that $b_0(x) \to \pm v_{Ay}^\infty$ as $x \to \pm \infty$, we find that the solution (C11) asymptotes to

$$\psi \approx \pm C_1^\pm v_{Ay}^\infty \mp \psi(0) \frac{v_{Ay} x}{v_{Ay}^\infty \lambda}. \quad (C14)$$

But in this limit $b_0''/b_0 \to 0$ by assumption, so we must solve (C8) neglecting the $b_0''/b_0$ terms while retaining $k_y^2$ and then match the resulting solution to (C14). The solution the vanishes at infinity is

$$\psi = C_3^\pm e^{\mp k_y x} \quad (C15)$$

and its $k_y x \ll 1$ asymptotic is

$$\psi \approx C_3^\pm \mp C_3^\pm k_y x. \quad (C16)$$

Demanding that this match (C14), we get

$$C_3^\pm = \frac{v_{Ay} \psi(0)}{v_{Ay}^\infty k_y \lambda}, \quad C_1^\pm = \pm \frac{C_3^\pm}{v_{Ay}^\infty}. \quad (C17)$$

Finally, returning to (C11) and using (C12), we obtain, for $k_y \lambda \ll 1$, 

$$\Delta' = \frac{\psi'(0) + \psi'(\infty)}{\psi(0)} \approx \frac{v_{Ay} C_1^+ - C_1^-}{\psi(0)} = 2 \left( \frac{v_{Ay}}{v_{Ay}^\infty} \right)^2 \frac{1}{k_y \lambda^2} \sim \frac{1}{k_y \lambda^2}, \quad \text{q.e.d.} \quad (C18)$$

Pending detailed insight into the functional form of the aligned fluctuations in MHD turbulence, we are going to treat this scaling of $\Delta'$ with $k_y$ and $\lambda$ as generic. Should a different scaling prove more compelling, all the results here and in section 7 would have to be adjusted—but it appears to be a straightforward adjustment.

C.1.2. Inner Solution

In the inner region, whose width is $\delta_{in}$, we can approximate the equilibrium magnetic field’s profile by (C12). Since $k_y \ll \partial_x \sim \delta_{in}^{-1}$, our equations (C5) and (C6) become

$$(\gamma - \nu \partial_x^2) \partial_x^2 \phi = ik_y \frac{x}{\lambda} v_{Ay} \partial_x^2 \psi, \quad (C19)$$

$$(\gamma - \eta \partial_x^2) \psi = ik_y \frac{x}{\lambda} v_{Ay} \phi. \quad (C20)$$
Combining them, we get
\[ \partial_x^2 \psi = -\left( \frac{\gamma \lambda}{k_y v_{A_y}} \right)^2 \frac{1}{x} \left( 1 - \nu \gamma \partial_x^2 \right) \partial_x^2 \frac{1}{x} \left( 1 - \frac{\eta}{\gamma} \partial_x^2 \right) \psi. \] (C 21)

This immediately tells us what the width of the boundary layer is:
\[ \frac{\nu}{\gamma \delta_{in}^2} \ll 1 \Rightarrow \left( \frac{\gamma \lambda}{k_y v_{A_y}} \right)^2 \frac{\eta}{\gamma \delta_{in}^2} \sim 1 \Rightarrow \frac{\delta_{in}}{\lambda} \sim \left( \frac{\gamma \tau_{A_y}}{\tau_{\eta}} \right)^{1/4} \frac{1}{(k_y \lambda)^{1/2}}, \] (C 22)
\[ \frac{\nu}{\gamma \delta_{in}^2} \gg 1 \Rightarrow \left( \frac{\gamma \lambda}{k_y v_{A_y}} \right)^2 \frac{\eta \nu}{\gamma^2 \delta_{in}^2} \sim 1 \Rightarrow \frac{\delta_{in}}{\lambda} \sim \left( \frac{\gamma \tau_{A_y}}{\tau_{\eta} \tau_{\nu}} \right)^{1/6} \frac{1}{(k_y \lambda)^{1/3}}. \] (C 23)

The latter case, in which viscosity is large, is a slightly less popular version of the tearing mode, but we can treat it together with the classic limit (C 22) at little extra cost.

Let us now rescale \( x = X \delta_{in} \) in (C 21). Then \( \psi_{in}(X) = \psi(X \delta_{in}) \) satisfies
\[ \frac{\nu}{\gamma \delta_{in}^2} \ll 1 \Rightarrow \partial_x^2 \psi_{in} = -\frac{1}{X} \partial_x^2 \frac{1}{X} \left( A - \partial_x^2 \right) \psi_{in}, \quad A = \left( \frac{\gamma \lambda}{k_y v_{A_y}} \right)^2 \frac{1}{\delta_{in}^2}, \] (C 24)
\[ \frac{\nu}{\gamma \delta_{in}^2} \gg 1 \Rightarrow \partial_x^2 \psi_{in} = \frac{1}{X} \partial_x^2 \frac{1}{X} \left( A - \partial_x^2 \right) \psi_{in}, \quad A = \left( \frac{\gamma \lambda}{k_y v_{A_y}} \right)^2 \frac{\nu}{\gamma \delta_{in}^2}. \] (C 25)

In both cases, the inner solution depends on a single dimensionless parameter \( A \) (the eigenvalue). In view of (C 22–C 23), this parameter is, in both cases, just the ratio of the growth rate of the mode to the rate of resistive diffusion across a layer of width \( \delta_{in} \), with the appropriate scaling of \( \delta_{in} \):
\[ A \sim \frac{\gamma \delta_{in}^2}{\eta} \sim \left\{ \begin{array}{ll}
\gamma^{3/2} \frac{\tau_{A_y}}{\tau_{\eta}} \frac{1}{\tau_{A_y}} & \sim \left( \frac{\gamma \tau_{A_y}}{k_y \lambda} \right)^{3/2} S_{\lambda}^{1/2}, \\
\gamma^{2/3} \frac{\tau_{\nu}}{\tau_{A_y}} & \sim \left( \frac{\gamma \tau_{A_y}}{k_y \lambda} \right)^{2/3} \left( S_{\lambda} Pm \right)^{1/2},
\end{array} \right. \]
\[ \frac{\nu}{\gamma \delta_{in}^2} \sim \frac{Pm}{A} \ll 1, \quad \frac{\nu}{\gamma \delta_{in}^2} \sim \frac{Pm}{A} \gg 1, \] (C 26)

where we have introduced the Lundquist number (associated with scale \( \lambda \)) and the magnetic Prandtl number:
\[ S_{\lambda} = \frac{\tau_{\eta}}{\tau_{A_y}} = \frac{v_{A_y} \lambda}{\eta}, \quad Pm = \frac{\tau_{\eta}}{\tau_{\nu}} = \frac{\nu}{\eta}. \] (C 27)

C.1.3. Peak Growth Rate and Wavenumber

Whatever the specific form of the solution of (C 24) (Coppi et al. 1976) or (C 25), \( \Delta' \) calculated from it according to (C 9) (and non-dimensionalised) must be a function only of \( A \):
\[ \Delta' \delta_{in} = f(A). \] (C 28)

Equating this to the the value (C 10) calculated from the outer solution, we arrive at an equation for \( A \):
\[ f(A) \sim \frac{\delta_{in}}{k_y \lambda^2} \sim \left\{ \begin{array}{ll}
\gamma^{1/4} \frac{1}{\tau_{A_y}} \frac{\tau_{\eta}}{k_y \lambda} & \sim A^{1/6} \left( k_y \lambda S_{\lambda}^{1/4} \right)^{-4/3}, \quad A \gg Pm, \\
\tau_{A_y} \left( \frac{\tau_{\eta}}{\tau_{\nu}} \right)^{-1/6} & \sim \left( k_y \lambda S_{\lambda}^{1/4} Pm^{-1/8} \right)^{-4/3}, \quad A \ll Pm.
\end{array} \right. \] (C 29)
Since the function \( f(\Lambda) \) does not depend on any parameters apart from \( \Lambda \), one might argue that the maximum growth of the tearing mode should occur at \( \Lambda \sim 1 \), when \( f(\Lambda) \sim 1 \). Then, using these estimates in (C 29) and (C 26), we find

\[
k_y \lambda \sim S_{\lambda}^{-1/4} (1 + \text{Pm})^{1/8} \equiv k_* \lambda \quad \Rightarrow \quad \gamma_{\Lambda y} \sim S_{\lambda}^{-1/2} (1 + \text{Pm})^{-1/4},
\]

(C 30)

where \( \text{Pm} \) only matters if it is large. Note that if \( S_\lambda \gg (1 + \text{Pm})^{1/2} \), the assumption \( k_y \lambda \ll 1 \) is confirmed. These are the maximum growth rate and the corresponding wave number of the tearing mode.\(^{53}\) Note that, for this solution, since \( f(\Lambda) \sim 1 \), we see from (C 29) that

\[
\delta_m \sim \frac{k_* \lambda}{\lambda}.
\]

(C 31)

If setting \( \Lambda \sim 1, f(\Lambda) \sim 1 \) does not feel inevitable to the reader, perhaps the following considerations will help solidify our case (a reader who is already convinced may skip to appendix C.4). Let us consider two physically meaningful limits that do not satisfy these assumptions.

First, let us consider what happens if \( \Lambda \ll 1 \). This means that the mode grows slowly compared to the Ohmic diffusion rate in the current layer, \( \gamma \ll \eta/\delta_{\text{in}}^2 \), a situation that corresponds, in a sense that is to be determined, to small \( \Delta' \). In this limit, we may expand \( f(\Lambda) \sim \Lambda \) to lowest order. Doing this in (C 29) and using (C 26) to unpack \( \Lambda \), we find

\[
\gamma_{\text{TA}_y} \sim \begin{cases} 
S_{\lambda}^{-3/5} (k_y \lambda)^{-2/5}, & k_y \lambda \ll S_{\lambda}^{-1/4} \text{Pm}^{-5/8}, \\
S_{\lambda}^{-2/3} \text{Pm}^{-1/6} (k_y \lambda)^{-2/3}, & k_y \lambda \gg S_{\lambda}^{-1/4} \text{Pm}^{-5/8}.
\end{cases}
\]

(C 32)

This is the famous FKR solution (Furth et al. 1963; see also Porcelli 1987 for the large-Pm case). Since to prove it, we assumed \( \Lambda \ll 1 \), substituting (C 32) into (C 26) tells us that the approximation is valid at wave numbers exceeding the wave number (C 30) of peak growth, \( k_y \gg k_* \). Note that this imposes an upper bound on \( \Delta' \):

\[
\Delta' \ll \frac{1}{k_y \lambda} \ll \frac{1}{k_* \lambda}.
\]

(C 33)

This is sometimes (perhaps misleadingly) called the “small-\( \Delta' \)” (or weakly driven) limit.

Let us now ask what happens in the limit opposite to (C 33), i.e., when \( \Delta' \) is very large and \( k_y \ll k_* \). In (C 29), this corresponds to \( f(\Lambda) \to \infty \) and we argue that this limit must be reached for some value \( \Lambda \sim 1 \) (it is not physically reasonable to expect that \( \Lambda \gg 1 \), i.e., that the growth rate of the mode can be much larger than than the Ohmic diffusion rate in the current layer; this thinking is confirmed by the exact solution—see Coppi et al. 1976). This implies, with the aid of (C 26),

\[
\gamma_{\text{TA}_y} \sim S_{\lambda}^{-1/3} (1 + \text{Pm})^{-1/3} (k_y \lambda)^{2/3}.
\]

(C 34)

This long-wavelength (“infinite-\( \Delta' \)”, or strongly driven) limit of the tearing mode was first derived by Coppi et al. (1976) (and by Porcelli 1987 for the large-Pm case).

We see that the long-wavelength asymptotic (C 34) is an ascending and the short-wavelength one (C 32) a descending function of \( k_y \). The wave number \( k_* \) of peak growth lies in between, where these two asymptotics meet, which is quite obviously the solution (C 30).

The applicability of this solution is subject to an important caveat. The Harris-like

\(^{53}\)I learned the general idea of this argument from J. B. Taylor (2010, private communication); it is a slight generalisation of his treatment of the tearing mode in Taylor & Newton (2015).
equilibrium that we used is a 1D configuration, implicitly assumed to extend as far in
the $y$ direction as the mode requires to develop. In reality, any sheet-like configuration
forming as a result of (ideal) MHD dynamics will have a length, as well as width: $\xi \gg \lambda$,
but still finite. The finiteness of $\xi$ will limit the wave numbers of the tearing perturbations
that can develop. The fastest-growing mode (C30) will only fit into the sheet if

$$ k_s \xi \gtrsim 1 \iff \frac{\xi}{\lambda} \gtrsim \frac{S_{1/4}^{1/4}(1 + \text{Pm})^{-1/8}}{\lambda}. \quad (C35) $$

If this condition fails to be satisfied, i.e., if the aspect ratio of the sheet is too small, the
fastest-growing mode will be the FKR mode (C32) with the smallest possible allowed
wavenumber $k_s \xi \sim 1$. Thus, low-aspect-ratio sheets will develop tearing perturbations
comprising just one or two islands whereas the high-aspect-ratio ones will spawn whole
chains of them, viz., $N \sim k_s \xi$ islands.

### C.2. Onset of Nonlinearity and Saturation of Tearing Mode

The tearing mode normally enters a nonlinear regime when the width $w$ of its islands
becomes comparable to $\delta_{in}$. The islands then grow secularly (Rutherford 1973) until
$w \Delta' \sim 1$. As we saw in appendix C.1.3, for the fastest-growing Coppi mode, $\Delta' \sim \delta_{in}^{-1}$,
so the secular-growth stage can be skipped. The width of the islands at the onset of the
nonlinear regime is, therefore,

$$ \frac{w}{\lambda} \sim \frac{\delta_{in}}{\lambda} \sim \frac{1}{\Delta' \lambda} \sim k_s \lambda. \quad (C36) $$

The amplitudes $\delta b_x$ and $\delta b_y$ of the tearing perturbation at the onset of nonlinearity
can be worked out by observing that the typical angular distorsion of a field line due to
the perturbation is

$$ wk_s \sim \frac{\delta b_x}{\nu A_y} \sim \frac{w^2 k_s}{\nu} \sim (k_s \lambda)^3, \quad \frac{\delta b_y}{\nu A_y} \sim \frac{w}{\lambda} \sim k_s \lambda. \quad (C38) $$

Note that the second of these relations implies $\delta b_y \sim b_0(x \sim w)$, i.e., the perturbed field
is locally (at $x \sim w$) as large as the equilibrium field.

Let us confirm that (C36) was a good estimate for the onset of nonlinearity, i.e., that,
once it is achieved, the characteristic rate of the nonlinear evolution of the tearing per-
turbation becomes comparable to its linear growth rate (C30). The nonlinear evolution
rate can be estimated as $k_s \delta u_y$, where $\delta u_y$ is outflow velocity from the tearing region.
When $\text{Pm} \lesssim 1$, this is obviously Alfvénic, $\delta u_y \sim \delta b_y$. When $\text{Pm} \gg 1$, the situation is
more subtle as the viscous relaxation of the flows is in fact faster than their Alfvénic
evolution (as we are about to see). Then the outflow velocity must be determined from
the force balance between viscous and magnetic stresses: using (C38),

$$ \nu \frac{w^2}{w^2} \delta u_y \sim k_s \delta b_y^2 \quad \Rightarrow \quad \delta u_y \sim \frac{k_s w^2 \delta b_y}{\nu} \sim \frac{k_s w^3 v_{A_y}}{\lambda \nu} \sim (k_s \lambda)^4 \frac{S_{\lambda}}{\text{Pm}} \sim \frac{1}{\sqrt{\text{Pm}}}. \quad (C39) $$

Combining the small- and large-Pm cases, we get

$$ \delta u_y \sim \frac{\delta b_y}{\sqrt{1 + \text{Pm}}} \quad \Rightarrow \quad k_s \delta u_y \sim \frac{k_s^2 \lambda v_{A_y}}{\sqrt{1 + \text{Pm}}} \sim \gamma. \quad (C40) $$
In the last expression, we used (C 38) for the perturbation amplitude and then (C 30) to ascertain that the nonlinear and linear rates are indeed the same.

Once these nonlinear effects come in, the tearing perturbation becomes subject to ideal-MHD evolution (for $Pm \gg 1$, also viscosity). This leads to collapse of the X-points separating the islands of the tearing perturbation into current sheets (Waelbroeck 1993; Jemella et al. 2003, 2004)—with the time scale for this process being the same as that for the Coppi mode’s growth (Loureiro et al. 2005) (which, as we have just seen, is the same as the ideal-MHD timescale for a perturbation that is gone nonlinear). If we now assume that, as a result, the islands circularise while preserving their area in the perpendicular plane, we find the saturated island size to be

$$w_{sat} \sim (w k_x^{-1})^{1/2} \sim \lambda.$$  
(C 41)

Thus, at the end of the tearing mode’s evolution, the associated perturbation finally breaks its scale separation with the equilibrium.

C.3. Sweet–Parker Sheet

Let me flesh out what was meant by the X-point collapse at the end of appendix C.2. The idea is that, once the nonlinearity takes hold and Alfvénic (or visco-Alfvénic) outflows from the reconnection region develop, the reconnecting site will suck plasma in, carrying the magnetic field with it, leading to formation of an extended sheet, which is a singularity from the ideal-MHD viewpoint, but resolved, of course, by resistivity and acting as a funnel both for magnetic flux and plasma (figure 19). After the collapse has occurred and a sheet has been established, the magnetic field just outside the resistive layer (the “upstream field”) is now the full equilibrium field, which has been brought in by the incoming flow $\delta u_x$ of plasma. In terms of the discussion in appendix C.2, this means $\delta b_y \sim v_{A y}$ [and so the islands at the ends of the sheet are large: $w \sim \lambda$, cf. (C 41)].
C.3.1. *Sweet–Parker Reconnection*

The flux brought in by this flow must be destroyed by resistivity (reconnected and turned into \( b_x \)), so\(^\text{54}\)

\[
u_x v_A^y \sim \eta j_z \sim \eta \frac{v_A^y}{\delta} \quad \Rightarrow \quad \delta \sim \frac{\eta}{u_x} \sim \frac{l}{S_l} v_A^y, \quad S_l = \frac{v_A^y l}{\eta},
\]

where \( \delta \) is the resistive layer’s width and \( u_x \) the inflow velocity. We have, in line with the prevailing convention (and physics) of the reconnection theory, introduced a Lundquist number based on the sheet length \( l \). In the context of a sheet formed between two islands of a tearing perturbation, \( l \sim k^{-1} \).

Since the sheet has to process matter as well as flux and since matter must be conserved, we may balance the inflow (\( u_x \)) and the outflow (\( u_y \)):

\[
u_x l \sim u_y \delta \quad \Rightarrow \quad u_x \sim \frac{\delta}{l} u_y \quad \Rightarrow \quad \delta \sim \frac{l}{\sqrt{S_l}} \left( \frac{v_A^y}{u_y} \right)^{1/2},
\]

where the third equation is the result of combining the second with (C 42).

Finally, the outflow velocity is inevitably Alfvénic in the absence of viscosity: this follows by balancing Reynolds and Maxwell stresses (inertial and tension) in the momentum equation (in either \( y \) or \( x \) direction; note that \( b_x \sim v_A^y \delta / l \)). Physically, this is just saying that the tension in the “parabolic”-shaped freshly reconnected magnetic field line will accelerate plasma and propel it out of the sheet. In the presence of viscosity, i.e., when \( Pm \gg 1 \), we must balance the magnetic stress with the viscous one, exactly like we did in (C 39), but with a narrower channel and a greater upstream field:

\[\frac{\nu}{\delta^2} u_y \sim \frac{v_A^y}{l} \quad \Rightarrow \quad \frac{u_y}{v_A^y} \sim \frac{v_A^y \delta^2}{\nu} \sim \frac{1}{\sqrt{Pm}}.\]

To get the last expression, we used \( \delta \) from (C 43). Just as I have done everywhere else, let us combine the low- and high-Pm cases [cf. (C 40)]:

\[u_y \sim \frac{v_A^y}{\sqrt{1 + Pm}} \quad \Rightarrow \quad \frac{\delta}{l} \sim \frac{(1 + Pm)^{1/4}}{\sqrt{S_l}} \quad \equiv \quad \frac{1}{\sqrt{S_l}}, \quad \tilde{S}_l = \frac{u_y l}{\eta},\]

where \( \tilde{S}_l \), the Lundquist number based on the outflow velocity is an obviously useful shorthand.\(^\text{55}\) Other relevant quantities can now be calculated, e.g., the effective rate at which flux is being reconnected:

\[rac{\partial \Psi}{\partial t} \sim u_x v_A^y \sim \frac{u_y v_A^y}{\sqrt{S_l}} \sim \frac{v_A^2}{(1 + Pm)^{1/4} \sqrt{S_l}}.
\]

The argument that I have just presented is one of the enduring classics of the genre and is due to *Sweet* (1958) and *Parker* (1957) (hereafter SP; the large-Pm extension was done by *Park* et al. 1984). While the argument is qualitative, it does work, in the sense both that one can construct unique solutions of the SP kind, in a manner pleasing to rigorous theoreticians (*Uzdensky* & *Kulsrud* 2000), and that SP reconnection has been

\(^{54}\)Formally, this is just a statement of balance between the advective and resistive terms in the induction equation.

\(^{55}\)Note that replacing in this argument \( l \to k^{-1} \), \( u_y \to \delta u_y \), \( v_A^y \to \delta b_y \sim v_A^y w / \lambda \sim v_A^y k \lambda \) gives us back the scalings associated with the tearing mode at the onset of nonlinearity (appendix C.2), with \( \delta \sim \delta_n \). This is, of course, inevitable as both theories are based on the same balances in the reconnection region, except the tearing before \( X \)-point collapse has a smaller upstream field \( \delta b_y \).
Figure 20. Plasmoid instability in current sheets with, from top to bottom, $S_\xi = 10^4, 10^5, 10^6, 10^7, 10^8$. The domain shown is 0.12 of the full length of the sheet. This plot is adapted from Samtaney et al. (2009), who confirmed the scalings (C 47) numerically.

measured and confirmed experimentally (Ji et al. 1998, 1999) (figure 21 shows an SP sheet measured in the MRX experiment at Princeton, where this was done).

C.3.2. Plasmoid Instability

However, an SP sheet is a sheet like any other and so, like for any sheet, one can work out a tearing instability for it. The results of appendix C.1.3 can be ported directly to this situation, by identifying $\lambda = \delta$ (and $l = \xi$). This gives instantly

$$\gamma \sim \frac{u_y}{\xi} \tilde{S}_\xi^{1/4}, \quad k_* \xi \sim \tilde{S}_\xi^{3/8}, \quad \frac{\delta_{in}}{\delta} \sim \tilde{S}_\xi^{-1/8}. \quad (C 47)$$

This is the so-called plasmoid instability (Tajima & Shibata 1997; Loureiro et al. 2007, 2013a; Bhattacharjee et al. 2009; Comisso & Grasso 2016; see figure 20). The realisation that SP sheets must be unstable can in fact be traced back to Bulanov et al. (1978, 1979), with the first numerical demonstration achieved by Biskamp (1986) (see also Biskamp 1982, Steinolfson & van Hoven 1984, Matthaeus & Lamkin 1985 and Lee & Fu 1986). However, this knowledge did not seem to have impacted the field as much as it should have done until the appearance of the analytical paper by Loureiro et al. (2007) and the rise of the plasmoid-chain simulation industry in 2D (Lapenta 2008; Daughton et al. 2009; Cassak et al. 2009; Huang & Bhattacharjee 2010, 2012, 2013; Huang et al. 2017; Bárta et al. 2011; Loureiro et al. 2012; Shen et al. 2013; Tenerani et al. 2015b), followed, more recently, by its more turbulent counterpart in 3D (Oishi et al. 2015; Huang & Bhattacharjee 2016; Beresnyak 2017; Kowal et al. 2017). Perhaps this was because plasmoids had to wait for their moment in the sun until they could be properly resolved numerically and that required relatively large simulations. Indeed, for an SP sheet to start spawning plasmoids, a sizeable Lundquist number is needed: asking for $\delta_{in}/\delta$ to be a reasonably small number, say, at least $1/3$, (C 47) gives us

$$\tilde{S}_\xi \gtrsim \tilde{S}_\xi^{(\text{plasmoid})} \sim 10^4. \quad (C 48)$$

Arguably the most important observation about (C 47) is, however, that the plasmoid instability of an SP sheet is massively supercritical: at large enough $\tilde{S}_\xi$, it is nowhere near marginal stability and so the question really is whether we should expect SP sheets ever to be formed in natural circumstances. This brings us to our next topic.

C.4. Formation and Disruption of Sheets

Let us put SP sheets aside and talk more generally about MHD sheets of the kind that we envisioned as the background equilibrium for tearing. The naturally occurring tearing-unstable ideal-MHD solutions are in fact not static equilibria: they arise, basically, because of the dynamical tendency in MHD for $X$-points to collapse into sheets (which I invoked at the transition between appendices C.2 and C.3), illustrated in figure 21.
Figure 21. Formation of a sheet from an X-point in the MRX experiment at Princeton (plot taken from Yamada et al. 1997).

An elementary example is the classic Chapman & Kendall (1963) collapsing solution of MHD equations:

\[ \Phi_0 = \Gamma(t)xy, \quad \Psi_0 = \frac{\nu A y}{2} \left[ \frac{x^2}{\lambda(t)} - \frac{y^2}{\xi(t)} \right]. \]  \hspace{1cm} (C 49)

Here \( \Gamma(t) \) can be specified arbitrarily and then \( \lambda(t) \) and \( \xi(t) \) follow upon direct substitution of the ansatz (C 49) into the RMHD equations (C 2–C 3) (with \( \eta = 0 \)). The original Chapman & Kendall (1963) version of this was the exponential collapse

\[ \Gamma(t) = \Gamma_0 = \text{const}, \quad \lambda(t) = \lambda_0 e^{-2\Gamma_0 t}, \quad \xi(t) = \xi_0 e^{2\Gamma_0 t}. \]  \hspace{1cm} (C 50)

A later, perhaps more physically relevant example, due to Uzdensky & Loureiro (2016), is obtained by fixing the outflow velocity at the end of the sheet to be a constant parameter: \( u_y = \partial \Phi_0 / \partial x = u_0 y / \xi \) and so

\[ \Gamma(t) = \frac{u_0}{\xi(t)}, \quad \lambda(t) = \frac{\lambda_0 \xi_0}{\xi_0 + 2u_0 t}, \quad \xi(t) = \xi_0 + 2u_0 t. \]  \hspace{1cm} (C 51)

In this, or any other conceivable model of sheet formation, the aspect ratio increases with time as the sheet’s width \( \lambda \) decreases and/or its length \( \xi \) increases.

The traditional thinking about sheets in MHD held that an ideal collapsing solution such as (C 49) (or an explosively collapsing one obtained by Syrovatskii 1971 for compressible MHD) would culminate in a steady-state current sheet, which, from the ideal-MHD point of view, would be a singularity, but resolved in resistive MHD by Ohmic diffusion, leading to an SP sheet. One could then discuss magnetic reconnection in such a sheet (appendix C.3.1). However, as we saw in appendix C.3.2, an examination of the stability of this object to tearing perturbations shows that it is massively unstable and will break up into a multitude of islands (“plasmoids”). Uzdensky & Loureiro (2016) and Pucci & Velli (2014) argued that it would never form anyway as tearing perturbations growing against the background of a collapsing ideal-MHD solution will disrupt it before it reaches its steady-state, resistive SP limit.

The detailed demonstration of this result involves realising that not only does the instantaneous aspect ratio of a forming sheet decide what types of tearing perturbations are allowed (single-island FKR modes or multi-island fastest-growing, “Coppi” modes), but that in principle this can change as the sheet evolves, that many different modes can
coexist and that these perturbations will grow on different time scales not only linearly but also nonlinearly (the FKR modes having to go through the secular Rutherford 1973 regime, the Coppi ones not). A careful analysis of all this can be found in the paper by Uzdensky & Loureiro (2016), but for our purposes here, it suffices to say that if the fastest-growing linear mode (C30) fits into the sheet, it will also be the one that first reaches the nonlinear regime and disrupts the formation of the sheet—with the moment of disruption defined as the moment when the width of the islands associated with the perturbation becomes comparable to the width $\lambda$ of the sheet.

Let us focus on the last point a little more closely. At the onset of the nonlinear regime of the tearing mode the width of the islands is given by (C36). Since $w \ll \lambda$, islands this size are, in fact, short of what is needed to disrupt the sheet. Uzdensky & Loureiro (2016) argue that the collapse of the inter-island $X$-points, which I already discussed at the end of appendix C.2, will produce saturated islands of size $\lambda$ [see (C41)], just right to be properly disruptive. This is a key ingredient for the discussion of “reconnecting turbulence” in section 7.2.

C.4.1. “Ideal Tearing”

So what kind of sheets can form before disruption occurs? This is equivalent to asking what aspect ratio a sheet can reach before the growth rate of the tearing mode triggered in the sheet becomes larger than the rate at which the sheet is collapsing in the course of its ideal-MHD evolution. The former rate is given by (C30) and the latter is $\Gamma \sim v_{\rm A \parallel} / \xi$, as is illustrated by the Uzdensky–Loureiro solution (C51). Then

$$\gamma \gtrsim \Gamma \iff \frac{\xi}{\lambda} \gtrsim S^{1/2} \lambda (1 + Pm)^{1/4} \iff \frac{\xi}{\lambda} \gtrsim S^{1/3} \xi (1 + Pm)^{1/6}. \quad \text{(C52)}$$

The last expression contains the Lundquist number referred to the length $\xi$ rather than the width $\lambda$ of the sheet, as it customarily done in magnetic-reconnection theory (cf. appendix C.3). Note that the assumption that it is the fastest-growing Coppi mode (C30) that should be used in this estimate is confirmed a posteriori by checking that the mode does fit into the sheet [cf. (C35)]:

$$k_y \xi \sim S^{1/4} \lambda (1 + Pm)^{3/8} \sim S^{1/6} \xi (1 + Pm)^{1/3} \gg 1. \quad \text{(C53)}$$

The scaling (C52) of the aspect ratio of the sheet with $S_\xi$ was put forward by Pucci & Velli (2014) as the maximum possible attainable one before the sheet is destroyed by what they termed “ideal tearing”, i.e., tearing modes that grow on the same time scale as the ideal-MHD sheet evolves [the extension of this result to $Pm \gg 1$ is due to Tenerani et al. 2015a; it has also been generalised by Pucci et al. 2018 to the case of $\Delta'$ having different scaling with $k_y$ than (C10)]. The conclusion that the sheet is indeed destroyed depends on the $X$-point-collapse argument described above, because the tearing modes by themselves do not produce islands as wide as the sheet.

The argument in section 7.1 is essentially the application of the criterion (C52) to the aligned structures of which Boldyrev’s MHD turbulent cascade consists.

Since the aspect ratio of the sheet described by (C45) is smaller than that of the SP sheet ($S^{1/3}_\xi$, rather than $S^{1/2}_\xi$), Pucci & Velli (2014) argued that global SP sheets can

---

56Assuming an Alfvénic outflow. This is OK even when $Pm \gg 1$ as long as the sheet is macroscopic, i.e., viscosity is unimportant at scale $\lambda$. If instead we are considering a microscopic “equilibrium”, like the secondary $X$-points between the islands of a tearing perturbation (appendix C.2), one should use $\Gamma \sim u_y / \xi$, where $u_y$ is the visco-Alfvénic outflow: see (C45). The condition (C52) then becomes $\xi/\lambda \gtrsim S^{1/2} \lambda = S^{1/2} \xi (1 + Pm)^{-1/4}$. 

Figure 22. This is a plot from Huang et al. (2017) illustrating the evolution of tearing perturbations of an evolving sheet in a 2D MHD simulation with $S_\xi \sim 10^6$ and $Pm \ll 1$. Their $(x,y,z)$ are our $(y,z,x)$, their $L$ is our $\xi$ (sheet length), their $a$ is our $\lambda$ (sheet width), their $\tau_A$ is our $\Gamma^{-1} \sim \xi/v_A$ (characteristic time of the sheet evolution), their $\delta$ is our $\delta_{in}$ (width of the tearing inner layer). The colour in the upper halves of their plots shows out-of-page current (colour bar “$J_y$”) and in the lower halves the outflow velocity along the sheet (colour bar “$v_x$”). The solid magenta lines are separatrices demarcating two “global” coalescing islands that they set up to form the sheet. The four snapshots are (a) at the moment the tearing mode goes nonlinear ($w \sim \delta_{in}$; see appendix C.2), (b) a little later, showing formation of secondary sheets (and so collapse of inter-island X-points), (c) later on, with secondary instability of these sheets manifesting itself as more plasmoids appear (cf. appendix C.4.2), and (d) after saturation, which for them is the period of stochastic but statistically steady and fast (with rate independent of $S_\xi$) reconnection and which obviously also corresponds to islands reaching the width of the sheet and starting to form a stochastic chain, moving and coalescing (see Uzdensky et al. 2010). Note that all of this evolution happens within one Alfvén time, although the initial-growth stage does need a few Alfvén times to get going.

never form. A recent extensive numerical study by Huang et al. (2017) of the instability of forming current sheets has indeed confirmed explicitly that the plasmoid-instability scalings (C 47) derived for an SP sheet only survive up to to a certain critical value $S_{\xi,\text{c}}^{(\text{ideal})} \sim 10^5 - 10^6$ [which obviously has to be bigger than the critical Lundquist number (C 48) for the plasmoid instability itself], with the “ideal-tearing” scalings (C 52) and
(C 53) taking over at \( S_{\xi} \gtrsim S_{\xi,c}^{(\text{ideal})} \).

Figure 22, taken from their paper, is an excellent illustration of the evolution of tearing perturbations and plasmoid chains.

### C.4.2. Recursive Tearing

It is not a difficult leap to realise that if a collapsing “global” MHD sheet-like configuration (which, the way it was introduced at the beginning of appendix C.4, was manifestly an \( X \)-point configuration) is unstable to tearing, the secondary \( X \)-points generated by the tearing can also be unstable to (secondary) tearing and thus might not “complete” the collapse into “proper” SP sheets that was posited for them above. This happens if the secondary tearing has a shorter growth time than the primary one, which, as we are about to see, is always the case. This conjures up an image of recursive tearings proceeding \( ad \ infinitum \) or, rather, until the inter-island sheets become short enough to be stable [see (C 48)]. At that point, they all collapse properly into reconnecting mini-

SP-sheets and one is left with a multiscale population of islands, which now have time to circularise and finally break up the “mother sheet” (and/or interact with each other).

For the purposes of our discussion in the main text (section 7.2), the issue is whether we should be concerned that the outcome of this break up is not just a number of flux ropes of one size (C 41), but a whole multiscale distribution of them.

Let us work on the assumption that the secondary tearing of an inter-island \( X \)-point works in the same way as the primary tearing described in appendix C.1.3 (see figure 23), except the width of the “equilibrium” is now the island width (C 36) of the primary tearing mode and the length of the secondary sheet is the wavelength \( k^{-1} \) of the primary mode [see (C 30)]—we already saw in appendix C.2 that fields associated with this new “equilibrium” are locally at least of the same size as the original equilibrium field. We now assign our old equilibrium parameters to the \( i \)-th level of tearing and the perturbation’s parameters (worked out in appendix C.2) to the \((i + 1)\)-st:

\[
\begin{align*}
  v_i &\equiv v_{Ay}, & v_{i+1} &\equiv \delta b_y, & \lambda_i &\equiv \lambda, & \lambda_{i+1} &\equiv w, & \xi_i &\equiv \xi, & \xi_{i+1} &\equiv k^{-1}_x, \\
\end{align*}
\]

with \( i = 0 \) corresponding to the mother sheet. Then

\[
\begin{align*}
  \gamma_i &\sim \frac{u_i}{\lambda_i} \tilde{S}_i^{-1/2}, & \frac{v_{i+1}}{v_i} &\sim \frac{\lambda_{i+1}}{\lambda_i} &\sim \frac{\lambda_i}{\xi_{i+1}} &\sim \tilde{S}_i^{-1/4},
\end{align*}
\]

They also find that \( S_{\xi,c}^{(\text{ideal})} \) gets smaller when larger initial background noise is present in the system and that the onset of tearing instability (and, therefore, of fast reconnection) is generally facilitated by such noise (the same is true for the plasmoid instability of SP sheets: see Loureiro et al. 2009). Their paper is written in a way that might give one the impression that they disagree profoundly with both Uzdensky & Loureiro (2016) and Pucci & Velli (2014): the main point of disagreement is their observation that the disruption of the sheet happens when \( \gamma \) is equal a few times \( \Gamma \), rather than \( \gamma/\Gamma \approx \frac{1}{1} \) [see (C 52)], and that exactly how many times \( \Gamma \) it must be depends on the initial noise level. In the context of the turbulence-disruption arguments advanced in section 7, this may be a useful practical caveat pointing to the value of \( \lambda_D \) [see (7.2)] possibly being an overestimate by a factor of a few. However, nearly all my arguments in this paper are order-unity-inaccurate “twiddle” theory, so I am not too bothered by this complication (although anyone attempting a quantitative numerical study should be). In any event, the fact that the disruption of the sheet is helped by more noise is surely a good thing for the validity of \( \gamma/\Gamma \sim \frac{1}{1} \) as the disruption criterion in a turbulent environment, where there is noise aplenty. Another (related) complication that matters quantitatively but probably not qualitatively is the possible presence of logarithmic corrections and other subtleties in the tearing-instability scalings for time-dependent sheets (Comisso et al. 2016, 2017).
Figure 23. A plot, adapted from Tenerani et al. (2015b), of the $b_x = -i k_y \psi(x)$ profiles (cf. figure 18) for nested tearing modes: primary (black), secondary (red) and tertiary (black). They extracted these from a direct numerical simulation of a recursively tearing sheet. This is a remarkably clean example of the similarity of tearing at ever smaller scales.

where [cf. (C 40)]

$$u_i = \frac{v_i}{\sqrt{1 + P_m}}, \quad \tilde{S}_i = \frac{S_i}{\sqrt{1 + P_m}}, \quad S_i = \frac{v_i \lambda_i}{\eta}.$$ (C 56)

Using the second relation in (C 55),

$$\frac{\tilde{S}_{i+1}}{\tilde{S}_i} = \frac{v_{i+1} \lambda_{i+1}}{v_i \lambda_i} \sim \tilde{S}_i^{-1/2} \Rightarrow \tilde{S}_{i+1} \sim \tilde{S}_i^{1/2} \Rightarrow \tilde{S}_i \sim \tilde{S}_0^{(1/2)^i}.$$ (C 57)

If there is some critical Lundquist number $\tilde{S}_c$ required for tearing modes to be unstable, (C 57) allows one to work out the maximum number of times that the recursive tearing will be iterated before $X$-points can collapse unimpeded into proper, stable, reconnecting current sheets:

$$i_{\text{max}} \sim \ln \frac{\ln \tilde{S}_0}{\ln \tilde{S}_c}. \quad \text{(C 58)}$$

It is obvious that in practice this will not be a large number at all. However, this detail does not matter for our purposes and, in any event, there is no harm in thinking in wildly asymptotic terms, so let us press on.

The second relation in (C 55) tells us that the amplitude of the $i$-th perturbation is proportional to its transverse scale:

$$\frac{v_i}{\lambda_i} \sim \text{const} \sim \frac{v_0}{\lambda_0}. \quad \text{(C 59)}$$

When translated into a spectral slope, this gives $k^{3-\perp}$, i.e., while islands at all scales below $\lambda_0$ are produced, they do not contain much energy. If this is true, we should be allowed to dismiss recursive tearing as a side show in the context of the disruption-range turbulence described in section 7.

The result (C 59) follows from the relation between the island width and the amplitude of the tearing perturbation at the onset of nonlinearity [see (C 38)]. In order for this to be useable, it must be the case that the $(i + 1)$-st tearing starts right at the onset of nonlinearity and outperforms the collapse of the $i$-th tearing perturbation’s $X$-point.\(^{58}\)

\(^{58}\)This is the main difference between my “naïve” recursive model and those of Shibata &
This appears to be easy: in view of (C 40) and (C 55),
\[
\frac{\gamma_{i+1}}{u_{i+1}/\xi_{i+1}} \sim \frac{\gamma_{i+1}}{\gamma_i} \sim \left(\frac{v_{i+1}}{v_i}\right)^{1/2} \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{-3/2} \sim \frac{\lambda_i}{\lambda_{i+1}} \sim \tilde{S}_{i+1}^{1/4} \gg 1. \tag{C 60}
\]
Let us check also that the fastest-growing mode always fits into its sheet: using (C 55) and (C 57),
\[
\frac{\xi_{i+1}}{\xi_i} \sim \frac{\lambda_i}{\lambda_{i-1}} \frac{\lambda_{i-1}}{\xi_i} \sim \tilde{S}_{i+1}^{1/4} \tilde{S}_{i-1}^{-1/2} \sim \tilde{S}_{i}^{3/4} \ll 1. \tag{C 61}
\]
While this all looks good, let me hedge by acknowledging that it may be a bit of a bold leap to assume that the local “equilibrium” set up by the \(i\)-th tearing perturbation, which features flows as well as fields, will be tearing unstable in exactly the same way as a very simple equilibrium that I reviewed above. It appears, however, that the flows are only expected to be seriously stabilising if \(u_{i+1}/\xi_{i+1} \sim \gamma_{i+1}\) (Bulanov et al. 1978, 1979; Biskamp 1986), so perhaps we are safe.

Finally, for completeness, let me give explicit expressions for everything. Using (C 57) and the recursion relations stated or derived above, we get
\[
\gamma_i \sim \frac{u_0}{\lambda_0} \tilde{S}_0^{-(1/2)^{i+1}} \rightarrow \frac{u_0}{\lambda_0}, \tag{C 62}
\]
\[
\frac{v_i}{v_0} \sim \frac{\lambda_i}{\lambda_0} \sim \tilde{S}_0^{-[1-(1/2)^i]/2} \rightarrow \tilde{S}_0^{-1/2}, \tag{C 63}
\]
\[
\frac{\xi_i}{\lambda_0} \sim \tilde{S}_0^{-[1-3(1/2)^i]/2} \rightarrow \tilde{S}_0^{-1/2}. \tag{C 64}
\]
The limits are all for \(i \to \infty\). Note that the relationship between \(\xi_0\) and \(\lambda_0\) does not satisfy (C 64) because the first sheet in the sequence was not itself produced by tearing and \(\xi_0\) could have been anything. Taking this first sheet to satisfy the “ideal-tearing”

Tanuma (2001) and Tenerani et al. (2015b, 2016). They assume effectively that, before any secondary tearing occurs, the X-point collapse proceeds at least far enough for the reconnecting field to be assumed the same at all levels of tearing: \(v_i \sim v_0\). Secondly, they assume that the tearing at each level only just outperforms the X-point collapse, or, equivalently, the outflows: \(\gamma_i \sim u_i/\xi_i\), whence \(\lambda_i/\xi_i \sim \tilde{S}_i^{-1/2}\) [this is the “ideal-tearing” threshold (C 52), except, for \(\text{Pm} \gg 1\), the Alfvénic outflow is tempered by viscosity, as secondary-sheet dynamics, as well as tearing, happen at scales where viscosity matters]. Finally, Shibata & Tanuma (2001) assume that the length of the sheet at the \(i\)-the level \(\xi_i\) is the wave length \(k_i^{-1}\) of the tearing mode at the \((i-1)\)-st level (which I also do), viz., \(\xi_i \sim \lambda_{i-1} \tilde{S}_{i-1}^{1/4}\), but let the width \(\lambda_i\) be determined from the rest (one only needs three equations for the three unknowns \(v_i, \xi_i, \lambda_i\)). The result is \(\tilde{S}_i \sim \tilde{S}_0^{(5/6)^i}\), etc. In contrast, Tenerani et al. (2015b, 2016) assume (based on their simulations) that the width of the sheet at the \(i\)-th level \(\lambda_i\) is the island width \(w\) [equivalently, \(\delta_i\); see (C 36)] of the tearing mode at the \((i-1)\)-st level (which, again, I also do), viz., \(\lambda_i \sim \lambda_{i-1} \tilde{S}_{i-1}^{1/4}\), but let \(\xi_i\) be determined from the rest. The result is \(\tilde{S}_i \sim \tilde{S}_0^{(3/4)^i}\), etc. I do not see why the local X-point “equilibria” produced in the nonlinear stage of the primary tearing should stay stable until X-point collapse makes \(v_i \sim v_0\) [which, according to (C 60), it will do slower than the notional secondary tearing perturbation would grow], but determining definitely whether they do so clearly requires a careful quantitative theory of the secondary tearing. Note finally that my model can be viewed as a version of the earlier model by Cassak & Drake (2009), who posit that the width \(\lambda_{i+1}\) of the secondary sheet is the SP width \(\delta\) given by (C 45) with \(l \rightarrow \xi_{i+1}\) but with reduced upstream field \(v_Ay \rightarrow v_{i+1} \sim v_i/\lambda_{i+1}/\lambda_i\). But that is nothing but \(\delta_i\) (and, therefore, \(w\)) for a tearing mode at the onset of nonlinearity (see appendix C.3.1). The simulations of Cassak & Drake (2009) appear to support the notion that secondary tearing gets going in these circumstances, but such simulations are always somewhat in the eye of the beholder—thus, Tenerani et al. (2015b, 2016) claim that their simulations support their picture.
criterion (C 52), we find that $\xi_\infty/\xi_0 \sim S_0^{-1}$, i.e., the total number of islands generated at all levels of recursive tearing scales as $S_0$.

All of this happens very quickly (on the $\gamma_0^{-1}$ time scale), then the islands circularise, with only the largest ones being energetically of any consequence [see (C 59)], and the sheet breaks up. Were it to persist for a long time, everything would change in the course of the subsequent dynamics of its plasmoid (island) population: plasmoid shapes, their number (they will travel along the sheet, coalesce, and eventually get ejected from the sheet), field amplitudes in them (reconnection will continue via elementary inter-plasmoid current sheets that are short enough not to be unstable). Such stochastic plasmoid chains have been studied numerically by many people (see references in appendix C.3.2). The statistical steady state of such a chain is, I believe, correctly described by the theoretical model of Uzdensky et al. (2010) (see also Shibata & Tanuma 2001). In a turbulent system envisioned in section 7, there is no reason for the aligned structure hosting all this to survive for much longer than the disruption time $\sim \gamma_0^{-1}$—although one could imagine that during some transient period comparable to that time, it might support the kind of stochastic plasmoid chain that was studied by the authors cited above. If this does happen, the interactions between the energetically dominant level-0 plasmoids and their nonlinear progeny should still be well described by the reconnecting cascade proposed by Mallet et al. (2017b) and discussed in section 7.2.

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