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Electron Sub-Larmor Turbulence

Alex Schekochihin (Oxford)

w i t h

Steve Cowley (UKAEA/Oxford),

Bill Dorland (U of Maryland),

Tomo Tatsuno (UEC Tokyo),

Gabriel Plunk (IPP, Greifswald)

ApJS 182, 310 (2009), section 7.12

Electron Gyrokinetics @ Sub-Larmor Scales

$k_\perp \rho_e \gg 1$ electron Larmor rings are >> spatial scale of e-m fluctuations

$\omega \ll \Omega_e$ but electron Larmor period << time scale of e-m fluctuations

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this is simultaneously possible if $k_\parallel \ll k_\perp$, because $\omega \sim k_\parallel v_{\rm the}$

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$$f_e = F_0 + \varphi(t,\mathbf{r})F_0 + h(t,\mathbf{R},v_\perp,v_\|) \leftarrow$$

distribution of rings

F_0: equilibrium Maxwellian (yes, I know…)

$\varphi(t,\mathbf{r})F_0$: Boltzmann response $\varphi = e\phi/T_e$

$\mathbf{R}$: gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

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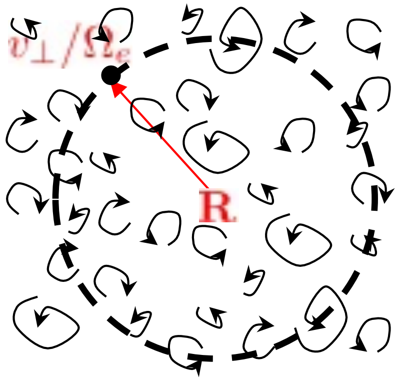
energy injection (from larger scales)

$$\boxed{\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi}$$

- $v_\parallel \nabla_\parallel h$: parallel particle streaming (more of it later!)
- $\langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h$: gyroaveraged **E**x**B** drift velocitty $\mathbf{u}_\perp = \dfrac{\rho_e v_{\rm the}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi$
- $-\dfrac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0$: gyroaveraged wave-ring interaction $-\left\langle \dfrac{d\varepsilon}{dt} \dfrac{\partial f_e}{\partial \varepsilon} \right\rangle_\mathbf{R}$
- $C[h]$: collisions
- χ: energy injection (from larger scales)



$$\langle \varphi \rangle_\mathbf{R} = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \, \varphi\left(\mathbf{R} + \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}\right)$$

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$$f_e = F_0 + \varphi(t,\mathbf{r})F_0 + h(t,\mathbf{R},v_\perp,v_\|) \leftarrow$$

distribution of rings

equilibrium Maxwellian (yes, I know…)

Boltzmann response $\varphi = e\phi/T_e$

gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

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$$\boxed{\frac{\partial h}{\partial t} + v_\| \nabla_\| h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi}$$

parallel particle streaming (more of it later!)

gyroaveraged **E**x**B** drift velocitty

$$\mathbf{u}_\perp = \frac{\rho_e v_{\rm the}}{2}\hat{\mathbf{b}} \times \nabla_\perp \varphi$$

gyroaveraged wave-ring interaction

$$-\left\langle \frac{d\varepsilon}{dt}\frac{\partial f_e}{\partial \varepsilon}\right\rangle_{\mathbf{R}}$$

collisions

$$\langle \varphi \rangle_{\mathbf{R}} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta\, \varphi\left(\mathbf{R} + \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}\right) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right)\varphi_{\mathbf{k}}$$

Gyroaveraging is a Bessel operator, so, at $k_\perp \rho_e \gg 1$, $\langle \varphi \rangle_{\mathbf{R}} = \hat{J}_0 \varphi \sim \dfrac{\varphi}{\sqrt{k_\perp \rho_e}}$

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$$f_e = \underset{\substack{\uparrow\\ \text{equilibrium}\\ \text{Maxwellian}\\ \text{(yes, I know...)}}}{F_0} + \underset{\substack{\uparrow\\ \text{Boltzmann}\\ \text{response}\\ \varphi = e\phi/T_e}}{\varphi(t,\mathbf{r})F_0} + \underset{\substack{\uparrow\\ \text{gyrocentre}}}{h(t,\mathbf{R},v_\perp,v_\parallel)} \leftarrow \text{distribution of rings}$$

$$\mathbf{R} = \mathbf{r} - \frac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$$

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

To calculate φ, use quasineutrality:

$$\frac{\delta n_e}{n_e} = \varphi + \frac{1}{n_e}\int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \frac{\delta n_i}{n_i}$$

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gyrocentre $\mathbf{R} = \mathbf{r} - \dfrac{\mathbf{v}_\perp \times \hat{\mathbf{b}}}{\Omega_e}$

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Ions have Boltzmann response
because everything else averages
out over their (huge!) Larmor orbits

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$$f_e = F_0 + \varphi(t,\mathbf{r})F_0 + h(t,\mathbf{R},v_\perp,v_\parallel) \leftarrow$$ distribution of rings

equilibrium Maxwellian (yes, I know…) — F_0

Boltzmann response $\varphi = e\phi/T_e$

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$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e}\int d^3\mathbf{v}\, J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1+T_e/T_i}$$

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Closed system

$$\boxed{\varphi(\mathbf{r}) = \frac{\alpha}{n_e} \int d^3\mathbf{v}\, \langle h \rangle_{\mathbf{r}} = \alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_e} \int d^3\mathbf{v}\, J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_{\mathbf{k}}}$$

$$\alpha = -\frac{1}{1 + T_e/T_i}$$

Magnetic Fluctuations @ Sub-Larmor Scales

Our equations are electrostatic. Is this a good approximation?

Closed system

$$\boxed{\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|} h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi}$$

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Parallel Ampere's law: $\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} j_\parallel = \frac{4\pi e}{c} \int d^3\mathbf{v}\, v_\parallel \langle h \rangle_\mathbf{r}$

$$\boxed{\frac{\delta \mathbf{B}_{\perp\mathbf{k}}}{B_0} = -\frac{\hat{\mathbf{b}} \times i\mathbf{k}_\perp A_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp \rho_e} \hat{\mathbf{b}} \times i\mathbf{k}_\perp \frac{1}{n_e} \int d^3\mathbf{v}\, \frac{v_\parallel}{v_{the}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k} \ll \varphi}$$

small factor!

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Perpendicular Ampere's law:

$$\nabla_\perp^2 \delta B_\parallel = -\frac{4\pi}{c}\hat{\mathbf{b}}\cdot(\nabla_\perp\times\mathbf{j}_\perp) = \frac{4\pi e}{c}\hat{\mathbf{b}}\cdot\left(\nabla_\perp\times\int d^3\mathbf{v}\,\langle\mathbf{v}_\perp h\rangle_\mathbf{r}\right)$$

$$\boxed{\frac{\delta B_{\parallel\mathbf{k}}}{B_0} = \frac{\beta_e}{k_\perp\rho_e}\frac{1}{n_e}\int d^3\mathbf{v}\,\frac{v_\perp}{v_{\mathrm{the}}} J_1\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k} \ll \varphi}$$

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small factor!

<u>Key point</u>: magnetic spectra are slaved to the spectra of density and of φ :

$$\frac{\delta B}{B_0} \sim \frac{\beta_e}{k_\perp \rho_e} \varphi$$

Plan: Theory $\Rightarrow$ Observables

1. Solve this system for h and φ:

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi$$

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…and get spectra $E_\varphi(k_\perp) \propto k_\perp^{-\mu}, \quad E_h(k_\perp) \propto k_\perp^{-\nu}$

2. Infer density spectra: $E_n(k_\perp) \propto k_\perp^{-\mu}$ because $\frac{\delta n_e}{n_e} = \frac{\varphi}{\alpha} = -\left(1 + \frac{T_e}{T_i}\right) \varphi$

magnetic-field spectra: $E_B(k_\perp) \propto k_\perp^{-\mu-2}$ because $\frac{\delta B}{B} \sim \frac{\beta_e}{k_\perp \rho_e} \varphi$

electric-field spectra: $E_E(k_\perp) \propto k_\perp^{-\mu+2}$ because $\mathbf{E}_\perp = -\nabla_\perp \phi \propto k_\perp \varphi$

Free Energy

1. Solve this system for h and φ :

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

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Rather than "solving," we can resort to Kolmogorov-ology: scalings will be set assuming constant flux of some conserved quantity, viz., free energy:

$$\frac{d}{dt}\left[\underbrace{\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h^2}{2F_0} + \int d^3\mathbf{r}\, \frac{\varphi^2}{2\alpha}}_{\text{free energy}}\right] = \underbrace{\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{h\chi}{F_0}}_{\text{injection } \equiv \varepsilon} + \underbrace{\frac{1}{n_e}\iint d^3\mathbf{v} d^3\mathbf{R}\, \frac{hC[h]}{F_0}}_{\text{collisional dissipation (negative definite!)}}$$

free energy

injection

$\equiv \varepsilon$

collisional dissipation

(negative definite!)

[AAS et al. 2008, PPCF 50, 24024]

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NB: free energy has to get to small scales in velocity space, to dissipate.

Free Energy

In general, the free energy in δf kinetics is

$$\boxed{\begin{aligned}\mathcal{F} &= -\sum_s T_s \delta S = -\sum_s T_s\, \delta\left[\iint d^3\mathbf{v}d^3\mathbf{r}\, f_s \ln f_s\right] = \sum_s \iint d^3\mathbf{v}d^3\mathbf{r}\, \frac{T_s \delta f_s^2}{2F_{0s}} \\ &= n_e T_e \left[\frac{1}{n_e}\iint d^3\mathbf{v}d^3\mathbf{R}\,\frac{h^2}{2F_0} + \int d^3\mathbf{r}\,\frac{\varphi^2}{2\alpha}\right] \quad \text{in our case}\end{aligned}}$$

This has a long history:

Kruskal & Oberman 1958
Bernstein 1958
Fowler 1963, 68
Krommes & Hu 1994
Krommes 1999
Sugama et al. 1996
Hallatschek 2004
Howes et al. 2006
Schekochihin et al. 2007-09
Scott 2010
Banon, Jenko et al. 2011-14
Plunk et al 2012
Abel et al. 2013
Kunz et al. 2015…

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So our conserved quantity is (minus) entropy!

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\frac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \frac{h}{F_0}$ at each scale $k_\perp^{-1}$

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cascade time

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cascade time

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi$$

$$\mathbf{u}_\perp = \frac{\rho_e v_{\mathrm{the}}}{2} \hat{\mathbf{b}} \times \nabla_\perp \varphi \sim \rho_e^2 \Omega_e k_\perp \varphi$$

$$\langle \varphi \rangle_\mathbf{R} = \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{R}} J_0\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \varphi_\mathbf{k} \sim \hat{J}_0 \varphi \sim \frac{\varphi}{\sqrt{k_\perp \rho_e}}$$

Cascade time: $\boxed{\tau^{-1} \sim k_\perp \langle u_\perp \rangle_\mathbf{R} \sim \Omega_e (k_\perp \rho_e)^2 \hat{J}_0 \varphi \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi}$

[AAS et al. 2008, PPCF 50, 24024]

Constant-Flux Cascade

Constant flux of free energy: $\boxed{\dfrac{\hat{h}^2}{\tau} \sim \varepsilon,}$ $\hat{h} \equiv \dfrac{h}{F_0}$ at each scale $k_\perp^{-1}$

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NB: $\tau^{-1} \ll \Omega_e$ provided $\varphi \ll \dfrac{1}{(k_\perp \rho_e)^{3/2}}$ (we'll check this later)

[AAS et al. 2008, PPCF 50, 24024]

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[AAS et al. 2008, PPCF 50, 24024]

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Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

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NB: $\tau^{-1} \ll \Omega_e$ provided $\varphi \ll \frac{1}{(k_\perp\rho_e)^{3/2}}$ (we'll check this later)

[AAS et al. 2008, PPCF 50, 24024]

Gyroaveraged Response

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

…and we now need a relationship between φ and $\hat{h}$:

$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_{\mathbf{r}} = \alpha\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\,J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right)h_{\mathbf{k}}(v_\perp)$$

[AAS et al. 2008, PPCF 50, 24024]

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we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp\rho_e}$

$$\approx \left(\frac{2\Omega_e}{\pi k_\perp v_\perp}\right)^{1/2}\cos\left(\frac{k_\perp v_\perp}{\Omega_e} - \frac{\pi}{4}\right)$$

oscillatory integral, sign changes with period

$$\frac{\Delta v_\perp}{v_{the}} = \frac{2\pi}{k_\perp\rho_e}$$

[AAS et al. 2008, PPCF 50, 24024]

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$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

from J_0 (gyroaverging)

integral accumulates as a random walk,

$$N \sim \frac{v_{the}}{\delta v_\perp} \sim k_\perp\rho_e$$

$$\approx \left(\frac{2\Omega_e}{\pi k_\perp v_\perp}\right)^{1/2}\cos\left(\frac{k_\perp v_\perp}{\Omega_e} - \frac{\pi}{4}\right)$$

oscillatory integral, sign changes with period

$$\frac{\Delta v_\perp}{v_{the}} = \frac{2\pi}{k_\perp\rho_e}$$

[AAS et al. 2008, PPCF 50, 24024]

Nonlinear Phase Mixing

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$

we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{\rm the}} \sim \frac{1}{k_\perp\rho_e}$

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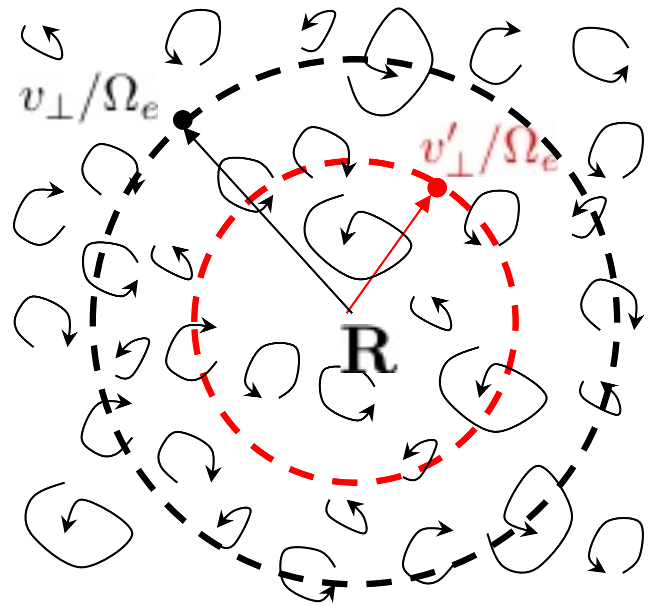
$$\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}$$

$$\frac{\partial h}{\partial t} + v_\parallel\nabla_\parallel h + \langle\mathbf{u}_\perp\rangle_{\mathbf{R}}\cdot\nabla_\perp h = -\frac{\partial\langle\varphi\rangle_{\mathbf{R}}}{\partial t}F_0 + C[h] + \chi$$



Two values of gyroveraged $\mathbf{E}\times\mathbf{B}$ velocity $\langle\mathbf{u}_\perp\rangle_{\mathbf{R}}(v_\perp)$ and $\langle\mathbf{u}_\perp\rangle_{\mathbf{R}}(v'_\perp)$ come from spatially decorrelated fluctuations if

$$\left|\frac{v_\perp}{\Omega_e} - \frac{v'_\perp}{\Omega_e}\right| \gtrsim \frac{1}{k_\perp} \quad\Rightarrow\quad \boxed{\frac{\delta v_\perp}{v_{\rm the}} \sim \frac{1}{k_\perp\rho_e}}$$

coherence scale in velocity space, q.e.d.

[AAS et al. 2008, PPCF 50, 24024]

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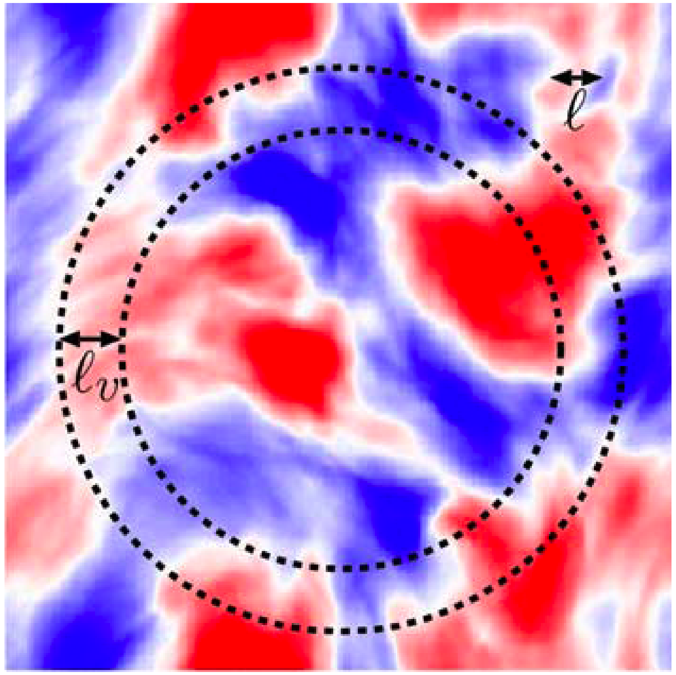

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coherence scale in velocity space, q.e.d.

[Tatsuno et al. 2009, PRL 103, 015003]

Entropy Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp \rho_e)^{-3/2}}$

we'll show this decorrelates on the scale $\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp \rho_e}$

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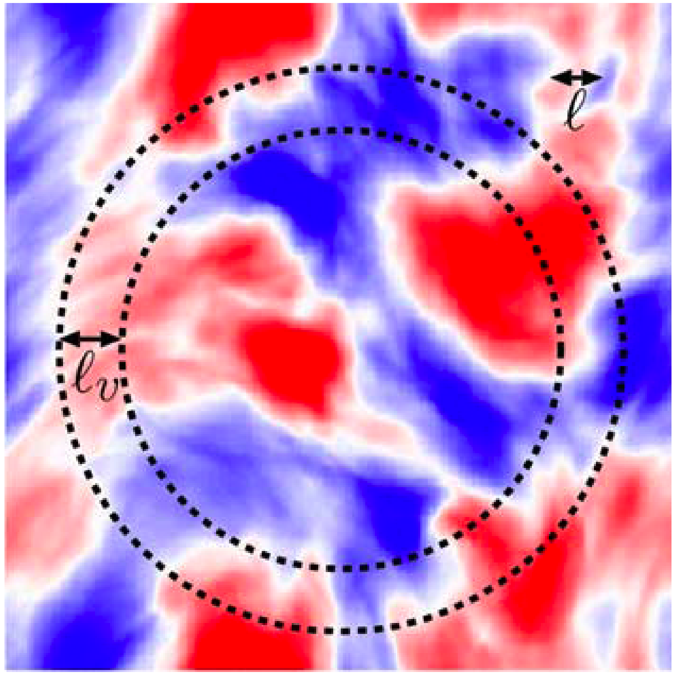
$$\varphi(\mathbf{r}) = \frac{\alpha}{n_e}\int d^3\mathbf{v}\,\langle h\rangle_\mathbf{r} = \alpha\sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{n_e}\int d^3\mathbf{v}\, J_0\!\left(\frac{k_\perp v_\perp}{\Omega_e}\right) h_\mathbf{k}(v_\perp)$$

$$\varphi \sim \frac{1}{\sqrt{k_\perp \rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp \rho_e}$$



Thus, we have a phase-space cascade ("entropy cascade"), simultaneous in position and velocity.

$$\boxed{\frac{\delta v_\perp}{v_{the}} \sim \frac{1}{k_\perp \rho_e}}$$ coherence scale in velocity space.

[Tatsuno et al. 2009, PRL 103, 015003]

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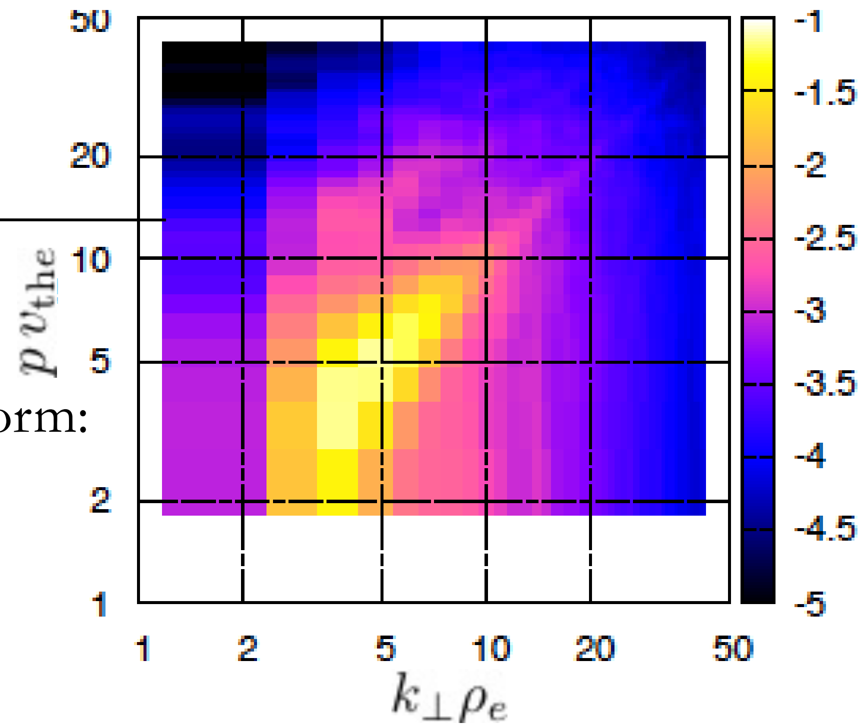
Thus, we have a phase-space cascade ("entropy cascade"), simultaneous in position and velocity.

Spectral representation in terms of Hankel transform:

$$\tilde{h}_\mathbf{k}(p) = 2\pi\int dv_\perp v_\perp J_0(pv_\perp)h_\mathbf{k}(v_\perp)$$

Phase-space spectrum: $E_h(k_\perp, p) = p\,\overline{\left|\tilde{h}_\mathbf{k}(p)\right|^2}$

[Plunk et al. 2010, JFM, 664, 407]



$\boxed{p\,v_{\rm the} \sim k_\perp\rho_e}$ coherence scale in velocity space.

[Tatsuno et al. 2009, PRL 103, 015003]

Entropy Cascade

Constant flux of free energy: $\boxed{\hat{h}^2\varphi \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-3/2}}$ $\Rightarrow$ $\hat{h}^3 \sim \frac{\varepsilon}{\Omega_e}(k_\perp\rho_e)^{-1/2}$

$$\boxed{\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}}$$

Entropy Cascade

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$$\boxed{\hat{h} \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-1/6}} \Rightarrow E_h \propto k_\perp^{-4/3}$$

$$\boxed{\varphi \sim \frac{1}{\sqrt{k_\perp\rho_e}}\frac{\hat{h}}{\sqrt{N}} \sim \frac{\hat{h}}{k_\perp\rho_e}}$$

$$\boxed{\varphi \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-7/6}} \Rightarrow E_\varphi \propto k_\perp^{-10/3}$$

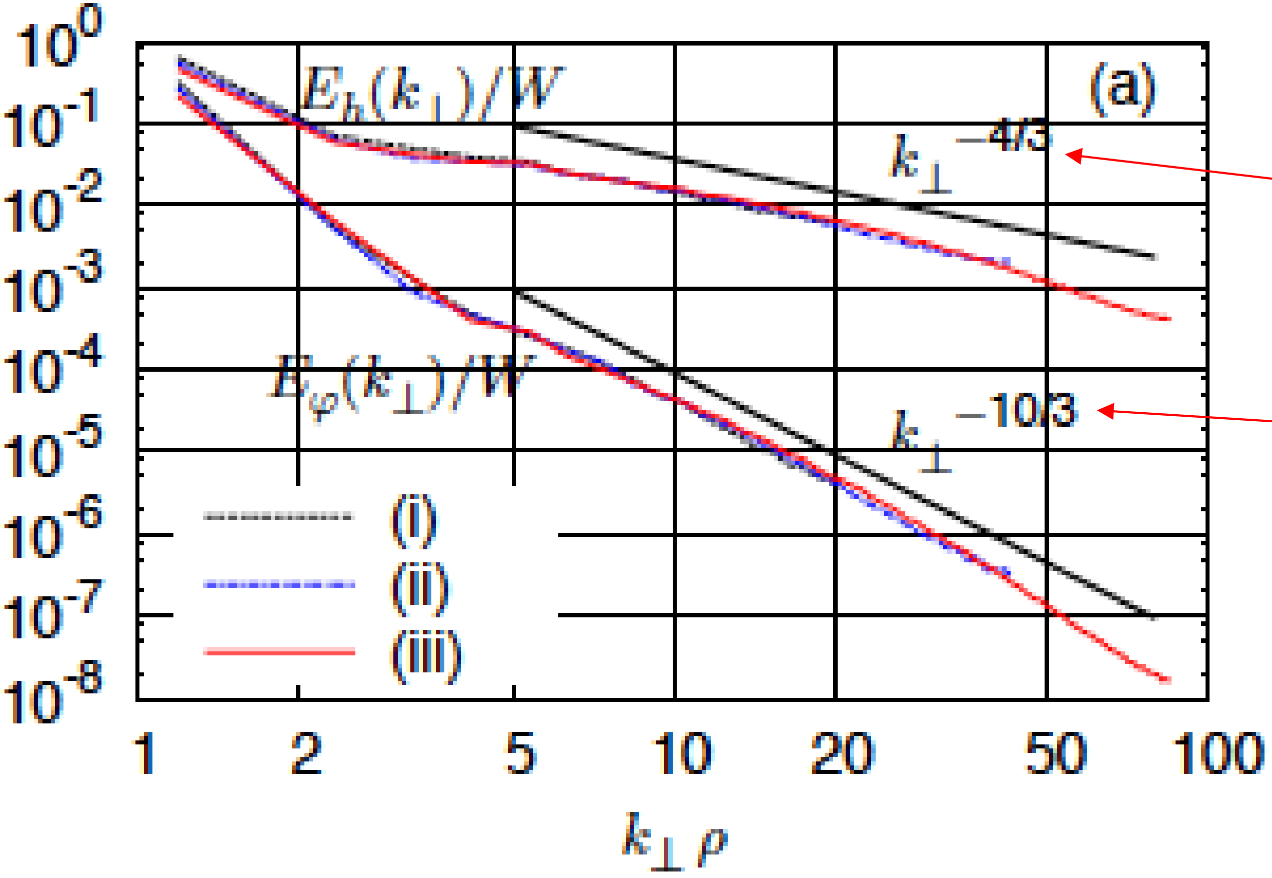
$$E \sim k_\perp\phi \Rightarrow E_E \propto k_\perp^{-4/3}$$

$$\boxed{\frac{\delta B}{B_0} \sim \frac{\beta_e}{k_\perp\rho_e}\varphi \sim \beta_e\left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_\perp\rho_e)^{-13/6}}$$

$$\Rightarrow E_B \propto k_\perp^{-16/3}$$

Theory vs. Simulations

GK SIMULATIONS by T. Tatsuno (2D, electrostatic, decaying):



[Tatsuno et al. 2009, PRL 103, 015003]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

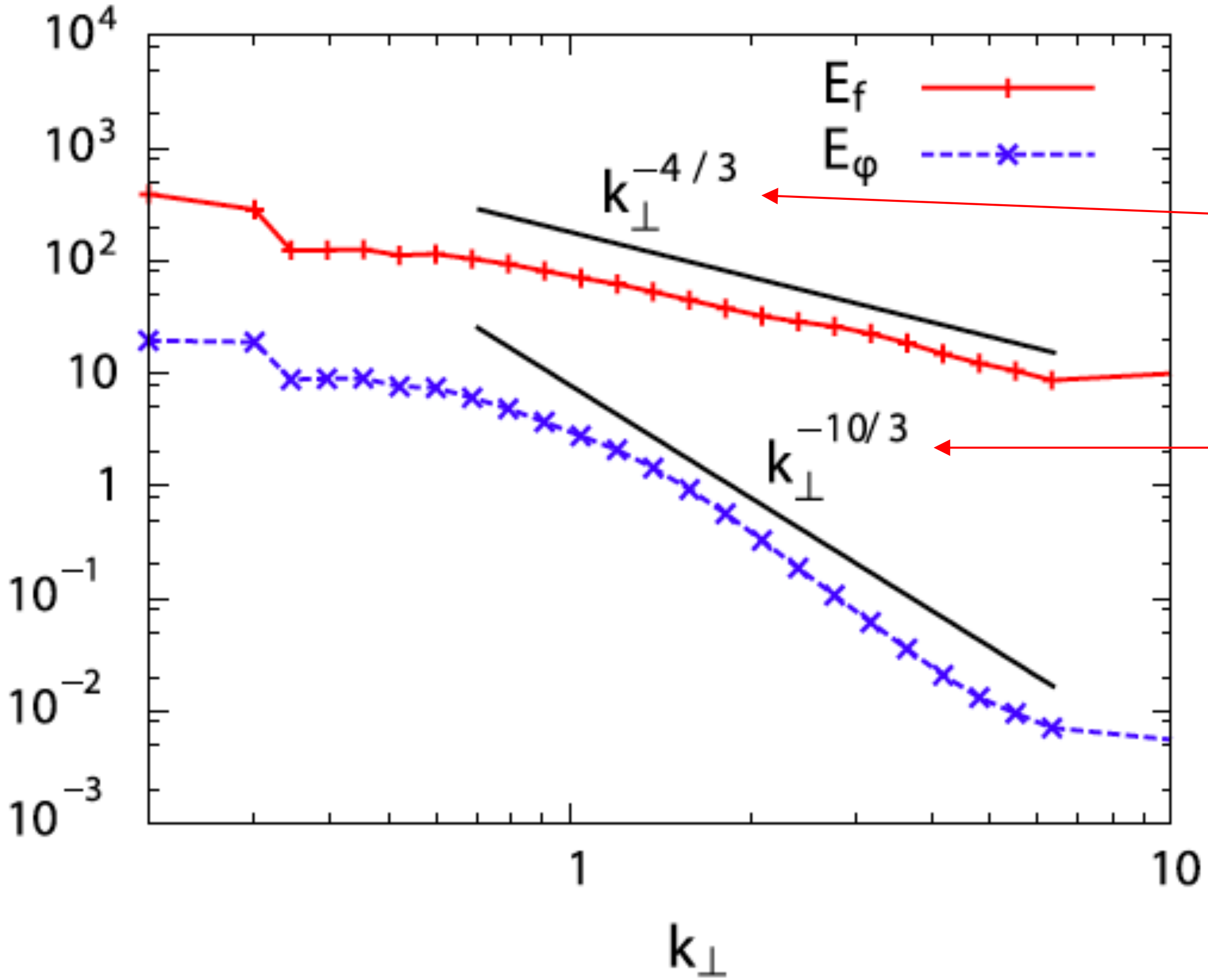
$$E_\varphi \propto k_\perp^{-10/3}$$

$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

Theory vs. Simulations

GK SIMULATIONS (3D electrostatic, ITG):



[Banon Navarro et al. 2011, PRL 106, 055001]

THEORY:

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$E_E \propto k_\perp^{-4/3}$

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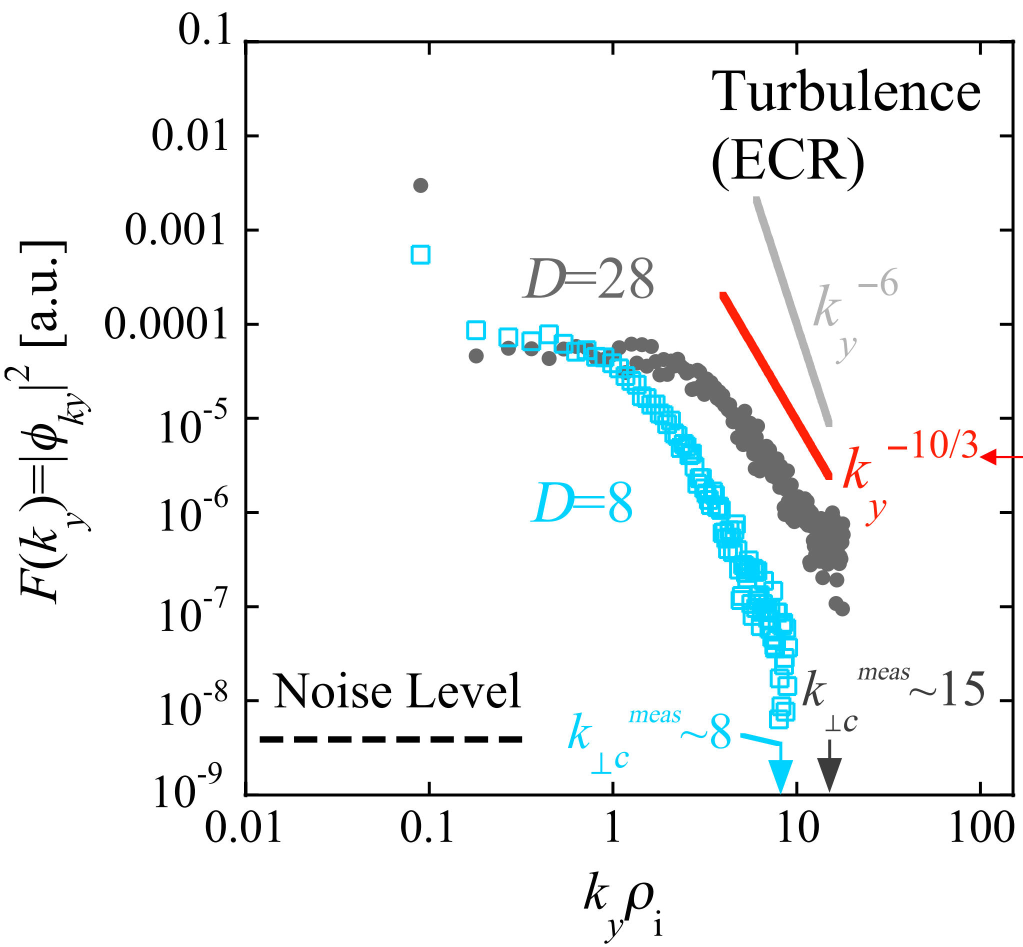
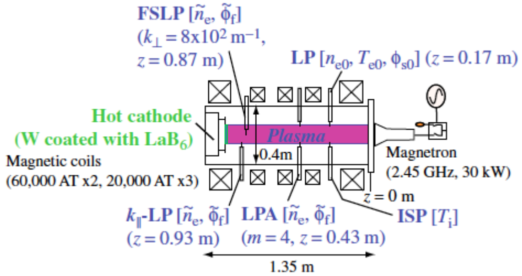
This was done for ion entropy cascade, but in the electrostatic limit,
the theory and results are exactly the same [AAS et al. 2008, PPCF 50, 24024]

Theory vs. Experiment!

LABORATORY EXPERIMENT:



[Kawamori (2013), PRL 110, 195001]



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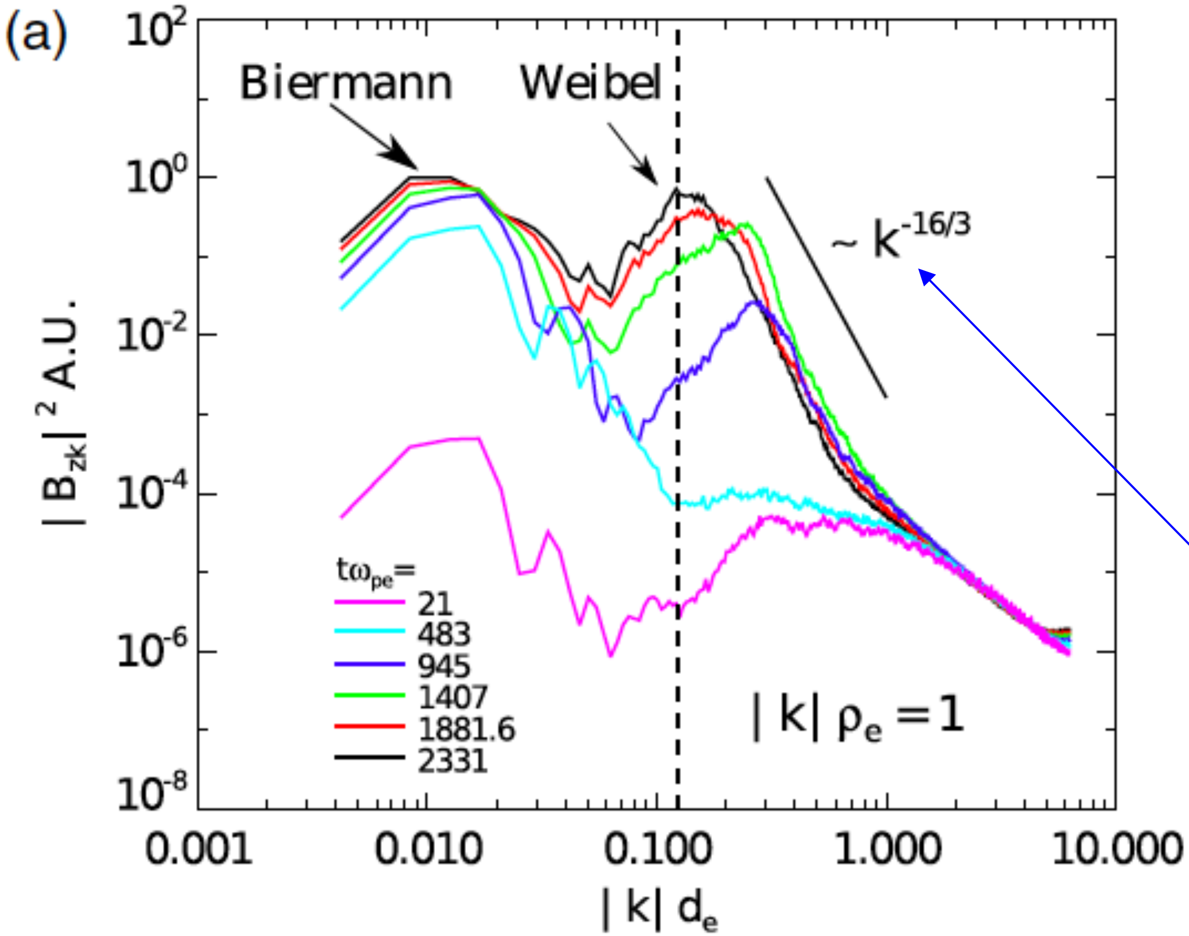
This was done for ion entropy cascade, but in the electrostatic limit,
the theory and results are exactly the same [AAS et al. 2008, PPCF 50, 24024]

Theory vs. Simulations

PIC SIMULATIONS (3D, self-generated m. field):



[Schoeffler et al. (2014), PRL 112, 175001]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

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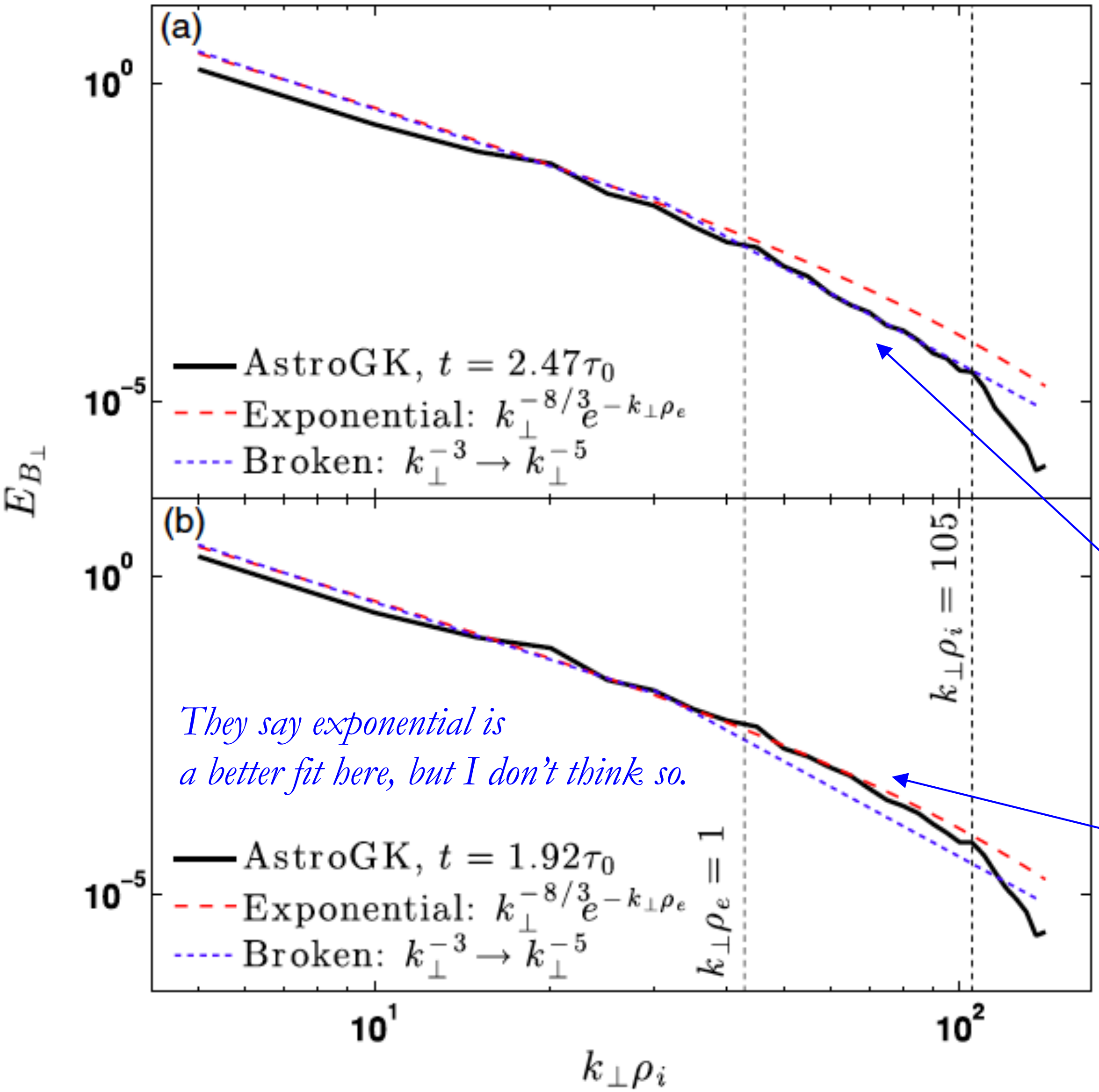
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

Theory vs. Simulations

GK SIMULATIONS by J. TenBarge (3D, forced):



[TenBarge et al. (2013), ApJ 774, 139]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

$$E_\varphi \propto k_\perp^{-10/3}$$

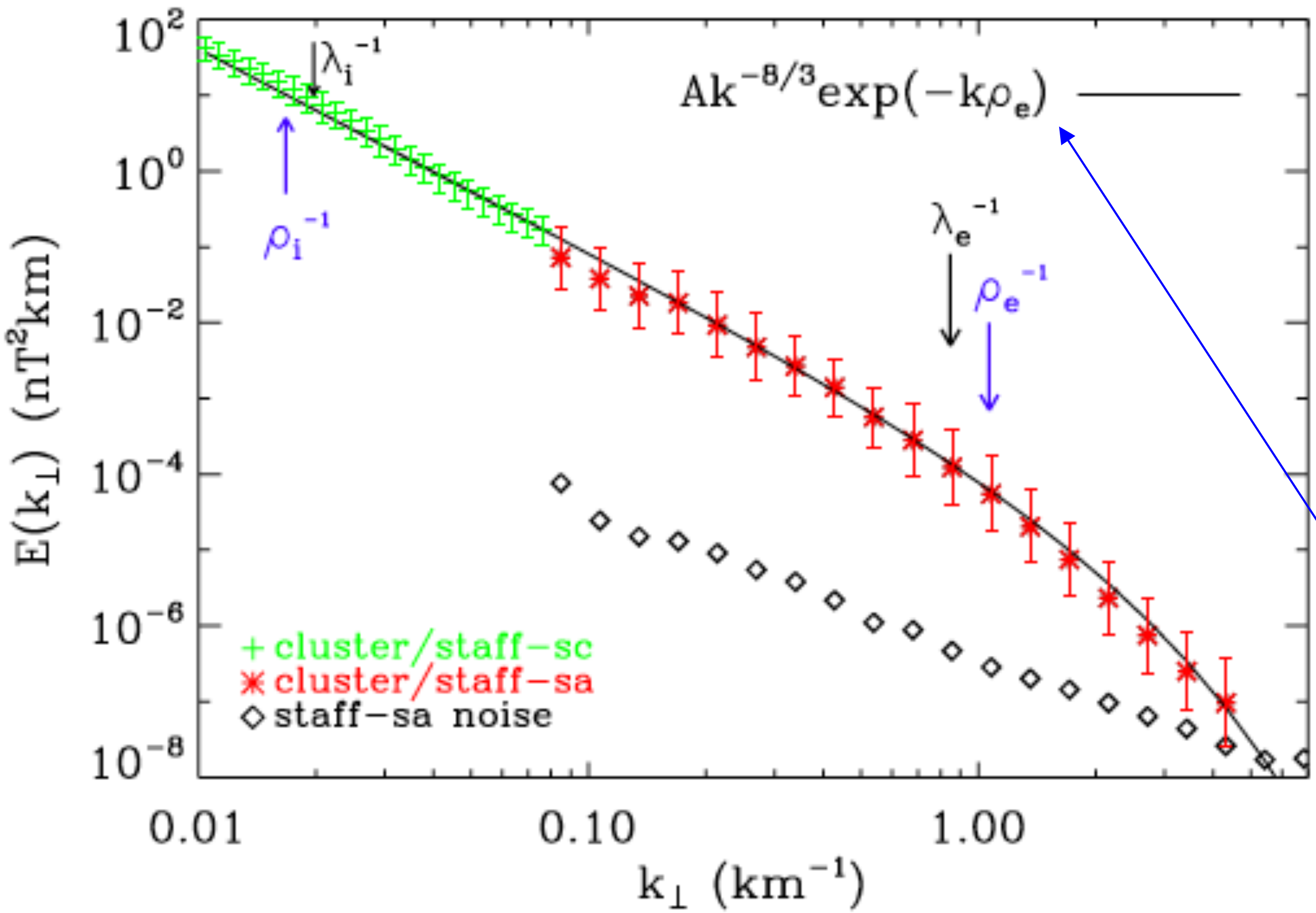
$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

$$-\frac{16}{3} \approx -5.3$$

Theory vs. Observations

SOLAR WIND OBSERVATIONS (Cluster):



[Alexandrova et al. (2012), ApJ 760, 121]

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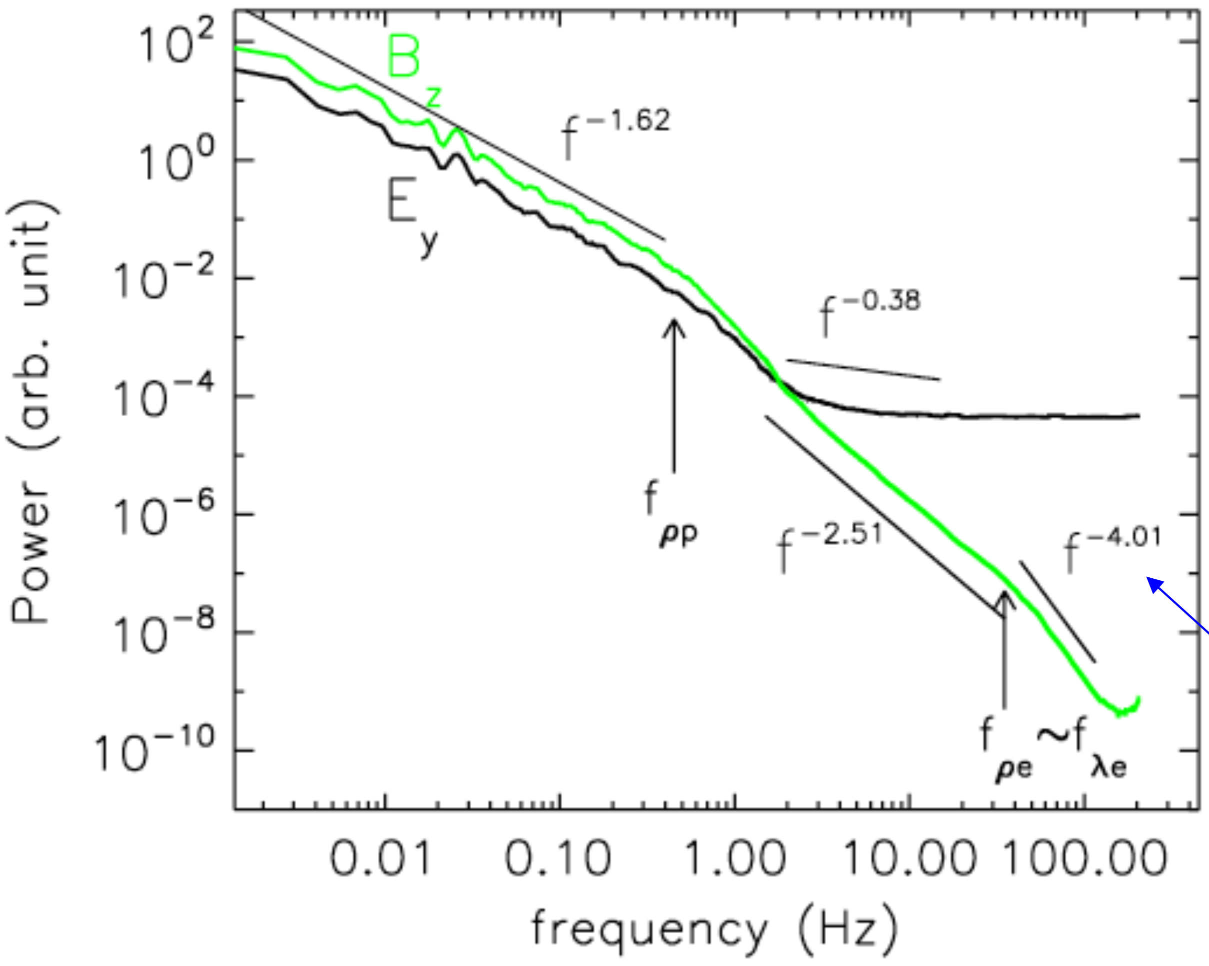

[Sahraoui et al. (2009), PRL 102, 231102]

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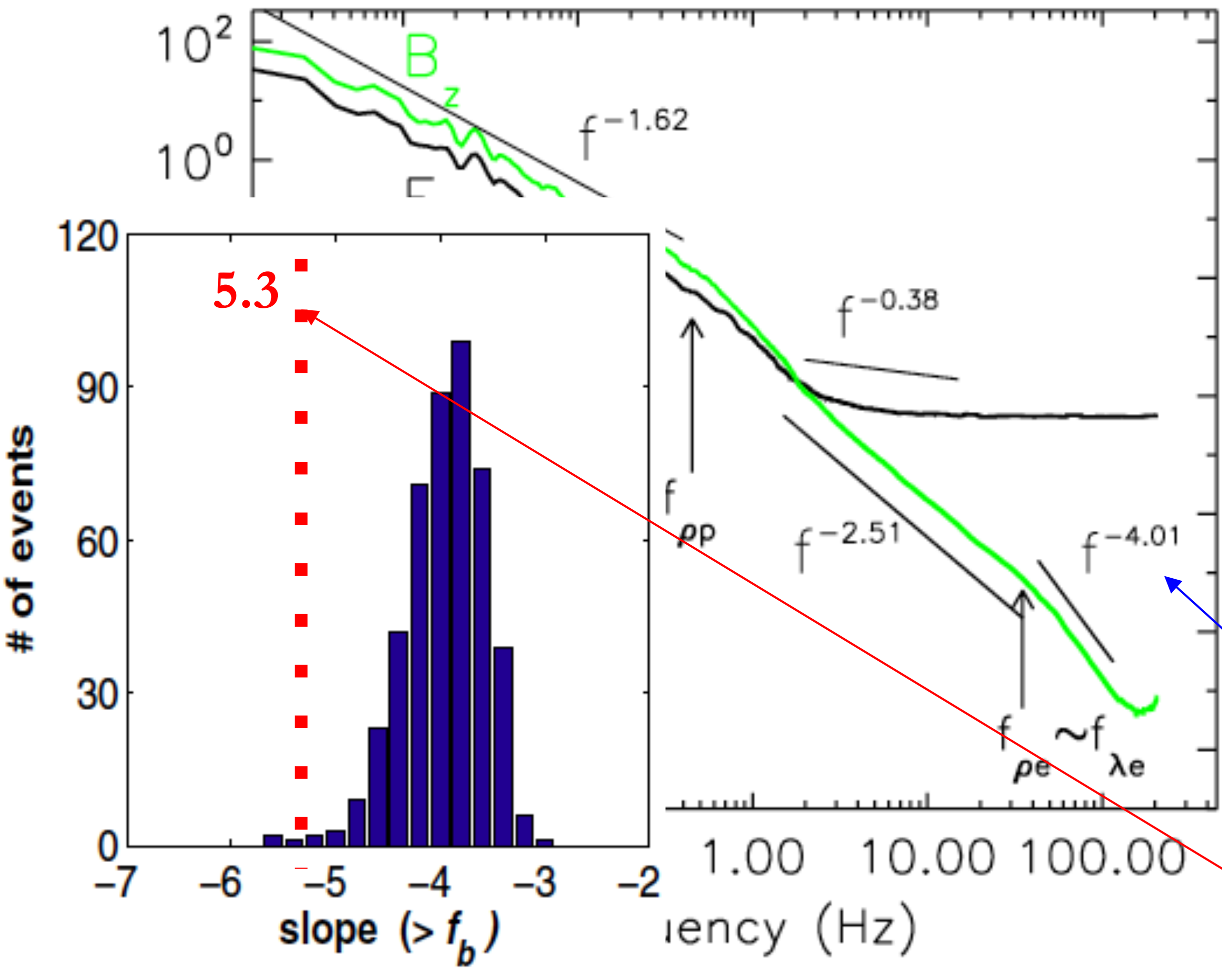

[Sahraoui et al. (2013), ApJ 777, 15]

THEORY:

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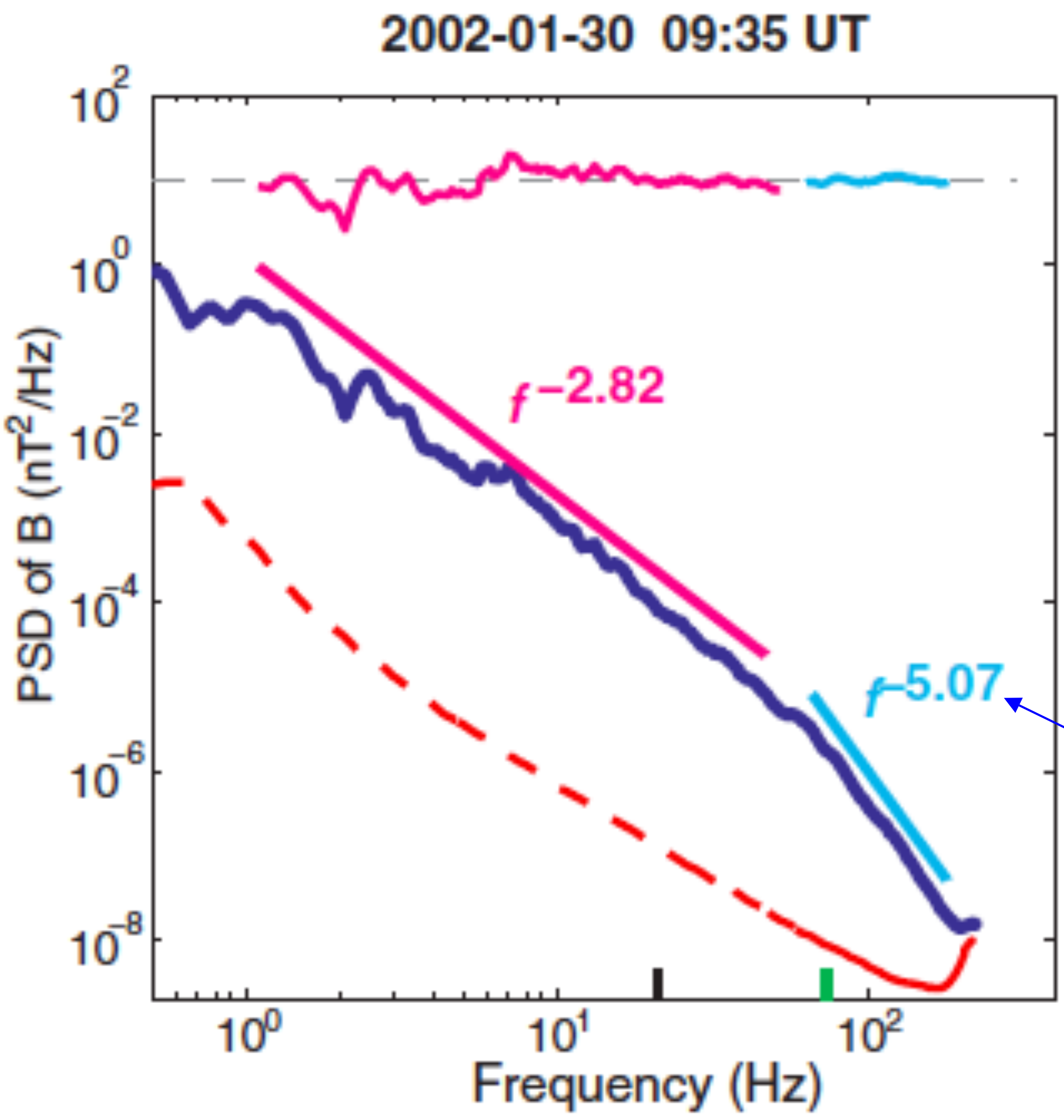
Theory vs. Observations

MAGNETOSHEATH OBSERVATIONS (Cluster):



[Huang, Sahraoui et al. (2014), ApJ 789, L28]

THEORY:

$$E_h \propto k_\perp^{-4/3}$$

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$$E_E \propto k_\perp^{-4/3}$$

$$E_B \propto k_\perp^{-16/3}$$

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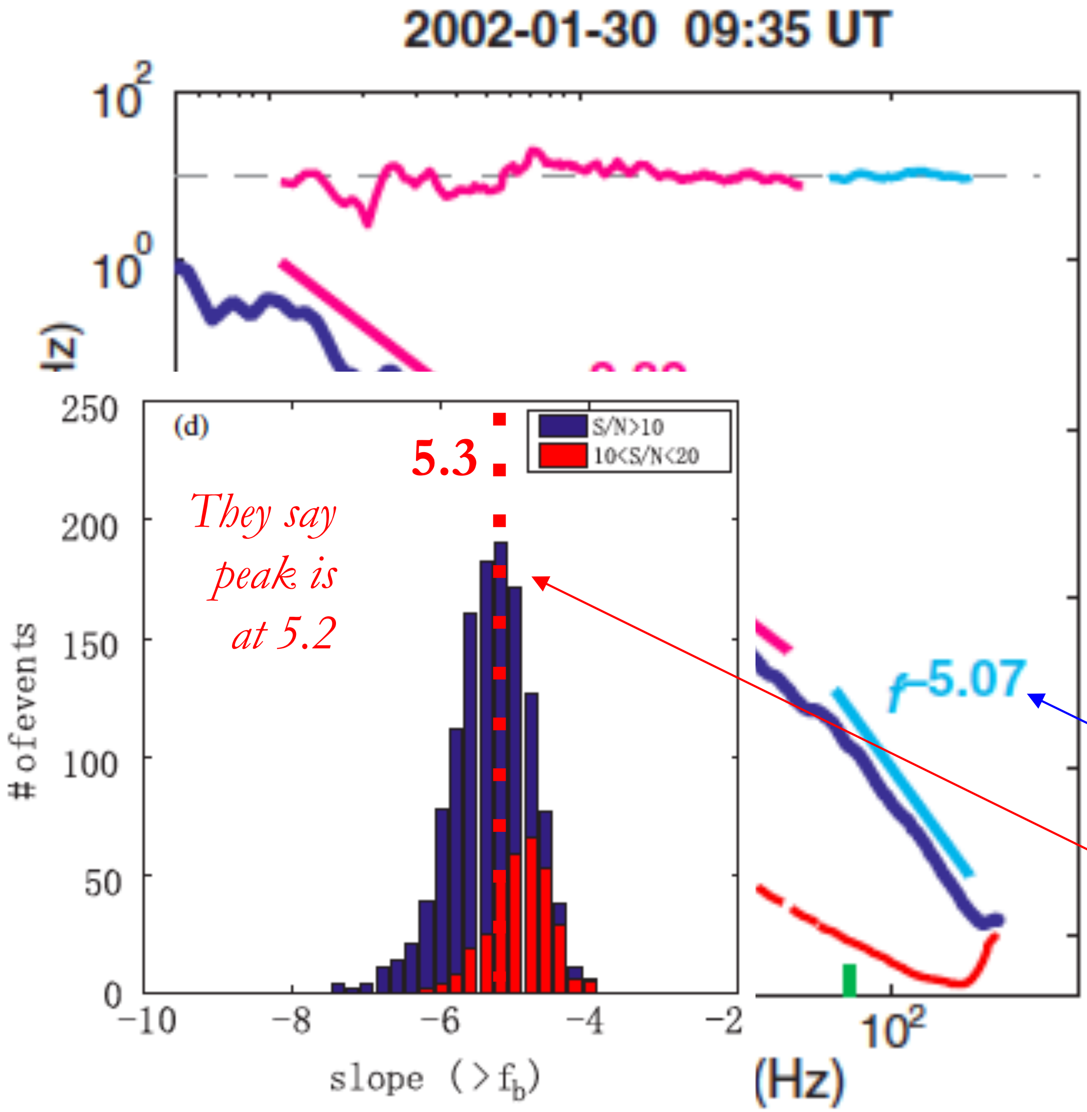
Theory vs. Observations

MAGNETOSHEATH OBSERVATIONS (Cluster):



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$$-\frac{16}{3} \approx -5.3$$

[Huang, Sahraoui et al. (2014), ApJ 789, L28]

"Kolmogorov" Scale

Where does the electron entropy cascade cut off?

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_\mathbf{R} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_\mathbf{R}}{\partial t} F_0 + C[h] + \chi$$

nonlinear advection

collisional dissipation

$$\tau^{-1} \sim k_\perp \langle u_\perp \rangle_\mathbf{R} \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi$$

$$\sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$$

$$C \sim \nu_e v_{\mathrm{the}}^2 \frac{\partial^2}{\partial v_\perp^2} \sim \nu_e \left(\frac{\delta v_\perp}{v_{\mathrm{the}}}\right)^{-2}$$

$$\sim \nu_e (k_\perp \rho_e)^2$$

because

$$\varphi \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{-7/6}$$

[AAS et al. 2008, PPCF 50, 24024]

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nonlinear advection

collisional dissipation

$$\tau^{-1} \sim k_\perp \langle u_\perp \rangle_\mathbf{R} \sim \Omega_e (k_\perp \rho_e)^{3/2} \varphi \qquad C \sim \nu_e v_{the}^2 \frac{\partial^2}{\partial v_\perp^2} \sim \nu_e \left(\frac{\delta v_\perp}{v_{the}}\right)^{-2}$$

$$\boxed{\Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3} \sim \nu_e (k_\perp \rho_e)^2}$$

$$\Downarrow$$

<u>Collisional cutoff:</u> $\displaystyle \frac{1}{k_{\perp c} \rho_e} \sim \frac{\delta v_{\perp c}}{v_{the}} \sim \left(\nu_e \tau_{\rho_e}\right)^{3/5} \equiv \mathrm{Do}^{-3/5}$

NB: spatial and velocity resolution are linked!

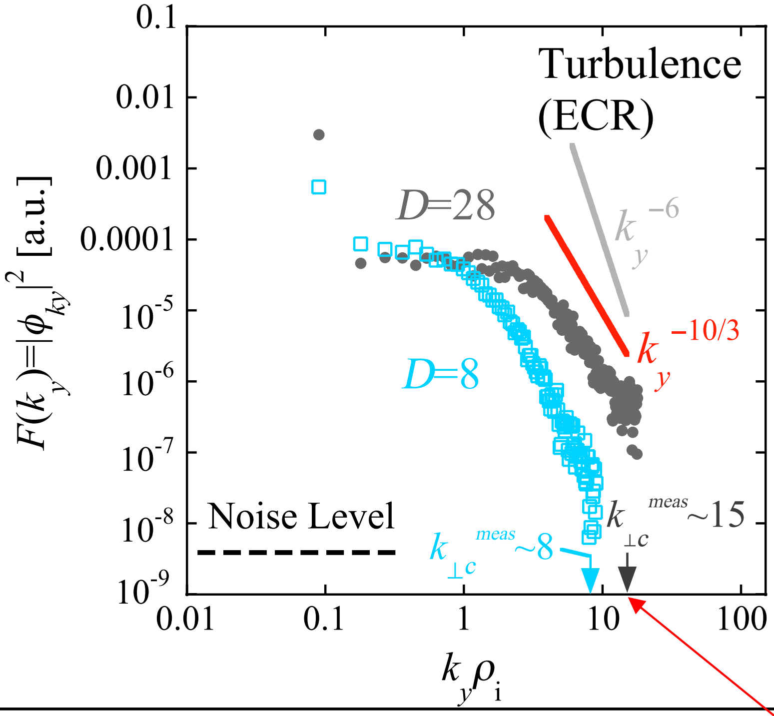
nonlinear time at Larmor scale

“Dorland number”

$$\tau_{\rho_e}^{-1} \sim \Omega_e \varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e} \frac{\delta B_{\rho_e}}{B_0}$$

[AAS et al. 2008, PPCF 50, 24024]

“Kolmogorov” Scale



This appears to have been checked in a laboratory experiment (for ions)

[Kawamori (2013), PRL 110, 195001]

Collisional cutoff:

$$\boxed{\frac{1}{k_{\perp c}\rho_e} \sim \frac{\delta v_{\perp c}}{v_{\text{the}}} \sim (\nu_e \tau_{\rho_e})^{3/5} \equiv \text{Do}^{-3/5}}$$

NB: spatial and velocity resolution are linked!

nonlinear time at Larmor scale

“Dorland number”

$$\tau_{\rho_e}^{-1} \sim \Omega_e \varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}$$

[AAS et al. 2008, PPCF 50, 24024]

Validity of Low-Frequency Limit

$$\tau^{-1} \sim \tau_{\rho_e}^{-1}(k_\perp \rho_e)^{1/3} \ll \Omega_e \quad \Leftrightarrow \quad k_\perp \rho_e \ll (\Omega_e \tau_{\rho_e})^3 \sim \varphi_{\rho_e}^{-3} \sim \left(\frac{1}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}\right)^{-3}$$

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Thus, the entropy cascade stays within low-frequency limit if $\varphi_{\rho_e} \ll \mathrm{Do}^{-1/5}$, or

$$\boxed{\varphi_{\rho_e} \ll \left(\frac{\nu_e}{\Omega_e}\right)^{1/6}}$$

can't be too difficult!

Otherwise all sorts of high-frequency physics will kick in…

Collisional cutoff:

$$\boxed{\frac{1}{k_{\perp c}\rho_e} \sim \frac{\delta v_{\perp c}}{v_{the}} \sim (\nu_e \tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5}}$$

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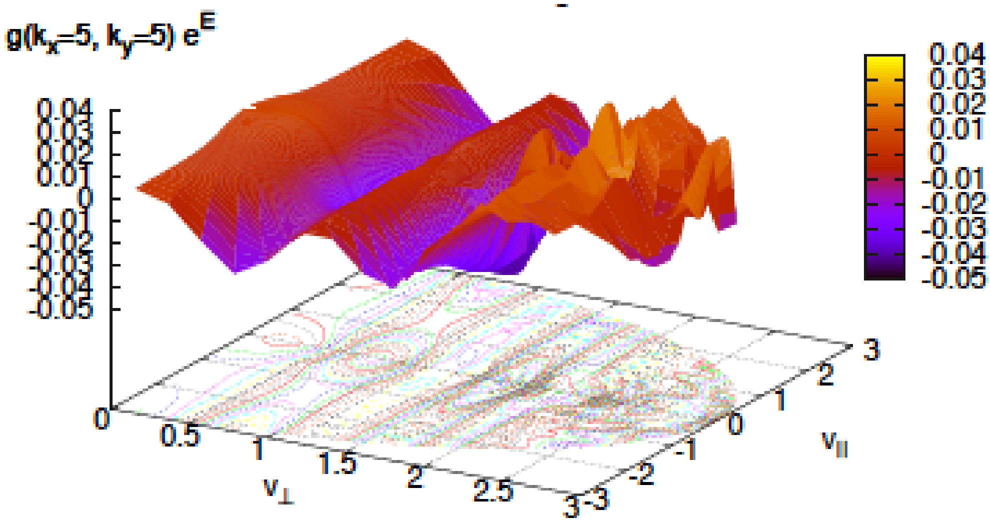
[AAS et al. 2008, PPCF 50, 24024]

Linear (||) vs. Nonlinear (⊥) Phase Mixing

Quick treatment:



NONLINEAR (perpendicular):

$$\boxed{\frac{\delta v_{\perp c}}{v_{\rm the}} \sim (\nu_e \tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5} \ll 1}$$

Since cascade is nonlinear, mixing occurs in one turnover time (fast)

Collisional cutoff: $\frac{1}{k_{\perp c}\rho_e} \sim \frac{\delta v_{\perp c}}{v_{\rm the}} \sim (\nu_e \tau_{\rho_e})^{3/5} \equiv \mathrm{Do}^{-3/5}$

NB: spatial and velocity resolution are linked!

nonlinear time at Larmor scale

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$$\tau_{\rho_e}^{-1} \sim \Omega_e \varphi_{\rho_e} \sim \frac{\Omega_e}{\beta_e}\frac{\delta B_{\rho_e}}{B_0}$$

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Linear (∥) vs. Nonlinear (⊥) Phase Mixing

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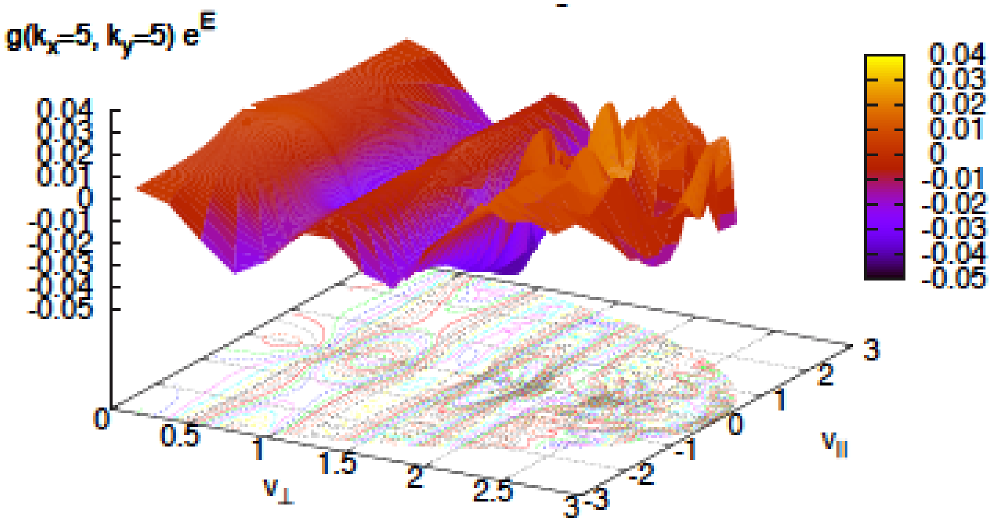

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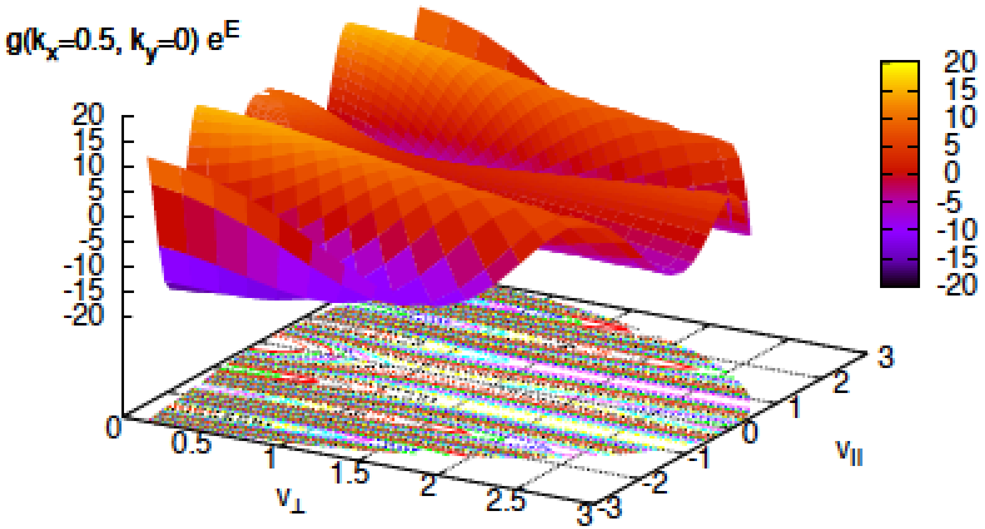


LINEAR (parallel):

"ballistic response"

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h = \dots \quad \Rightarrow \quad h \propto e^{-ik_\parallel v_\parallel t}$$

$$\boxed{\frac{\delta v_\parallel}{v_{the}} \sim \frac{1}{k_\parallel v_{the} t} \sim 1}$$

after one turnover time

if *"critical balance"* holds,
so linear phase mixing is slow

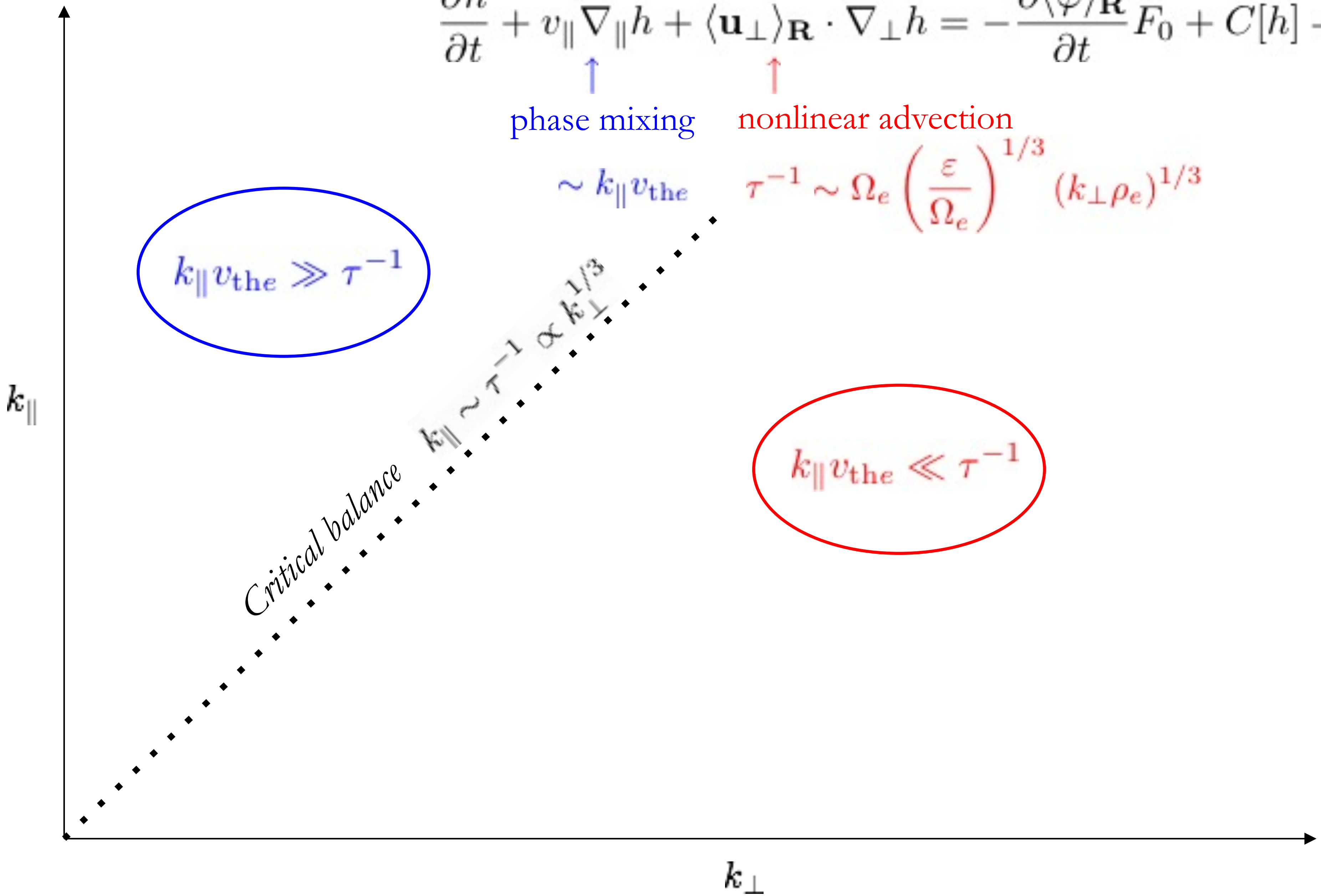
[AAS et al. 2008, PPCF 50, 24024]

Linear Phase Mixing and Critical Balance

$$\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|}h + \langle \mathbf{u}_{\perp}\rangle_{\mathbf{R}}\cdot\nabla_{\perp}h = -\frac{\partial\langle\varphi\rangle_{\mathbf{R}}}{\partial t}F_0 + C[h] + \chi$$

phase mixing $\sim k_{\|}v_{\mathrm{the}}$

nonlinear advection $\tau^{-1} \sim \Omega_e\left(\frac{\varepsilon}{\Omega_e}\right)^{1/3}(k_{\perp}\rho_e)^{1/3}$



[analogous to AAS et al. 2016, JPP 82, 905820212]

Linear Phase Mixing and Critical Balance

$$\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|}h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

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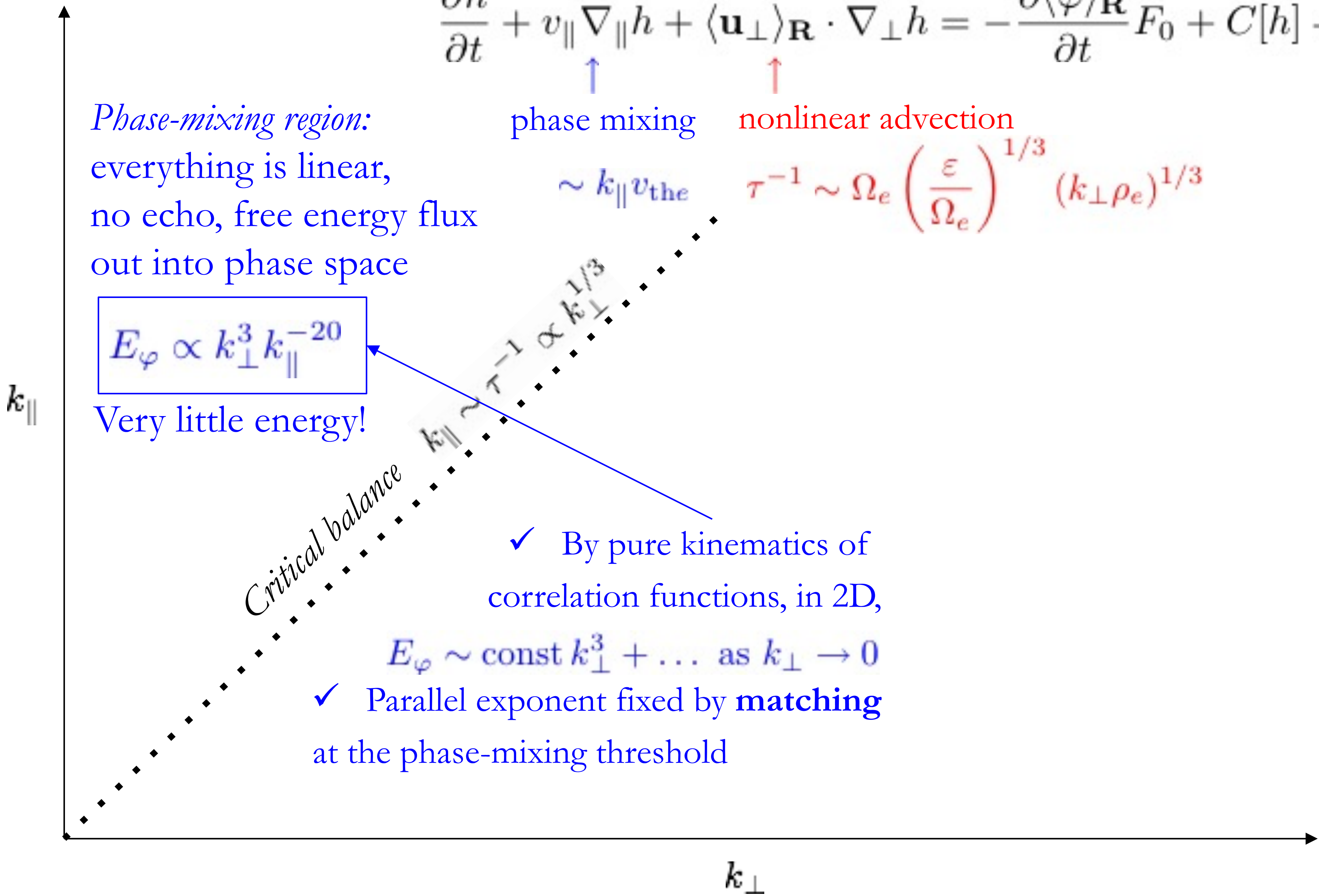

Phase-mixing region:
everything is linear,
no echo, free energy flux
out into phase space

$$\boxed{E_\varphi \propto k_\perp^3 k_{\|}^{-20}}$$

Very little energy!

✓ By pure kinematics of correlation functions, in 2D,

$$E_\varphi \sim \mathrm{const}\, k_\perp^3 + \ldots \;\text{ as } k_\perp \to 0$$

✓ Parallel exponent fixed by **matching** at the phase-mixing threshold

[analogous to AAS et al. 2016, JPP 82, 905820212]

Linear Phase Mixing and Critical Balance

$$\frac{\partial h}{\partial t} + v_{\|}\nabla_{\|}h + \langle \mathbf{u}_{\perp}\rangle_{\mathbf{R}} \cdot \nabla_{\perp}h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t}F_0 + C[h] + \chi$$

phase mixing

$\sim k_{\|}v_{\mathrm{th}e}$

nonlinear advection

$$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_{\perp}\rho_e)^{1/3}$$

Phase-mixing region:
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$$\boxed{E_{\varphi} \propto k_{\perp}^{3} k_{\|}^{-20}}$$

Very little energy!

Advection-dominated region:
fully nonlinear, perfect echo,
free energy flux to phase space vanishes

$$\boxed{E_{\varphi} \propto k_{\|}^{0} k_{\perp}^{-11/3}}$$

<u>Most energy along critical balance curve</u>

Critical balance $k_{\|} \sim \tau^{-1} \propto k_{\perp}^{1/3}$

✓ Perpendicular exponent
fixed by **constant flux in 3D**
✓ Parallel exponent is
white noise: loss of correlation
at long distances

$k_{\|}$

$k_{\perp}$

[analogous to AAS et al. 2016, JPP 82, 905820212]

2D Spectra

$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

phase mixing

$\sim k_\parallel v_{\mathrm{the}}$

nonlinear advection

$$\tau^{-1} \sim \Omega_e \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{1/3}$$

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$k_\parallel$

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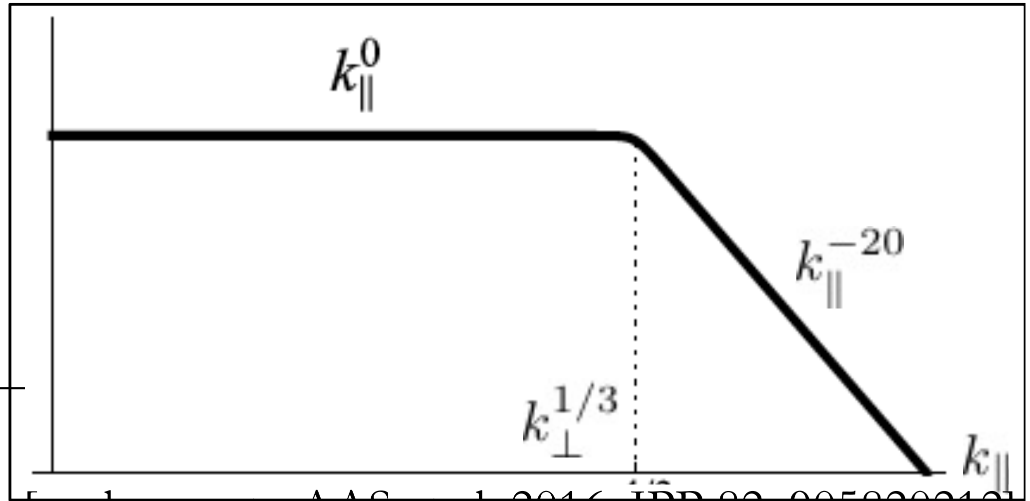

[analogous to AAS et al. 2016, JPP 82, 905820212]

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$$\frac{\partial h}{\partial t} + v_\parallel \nabla_\parallel h + \langle \mathbf{u}_\perp \rangle_{\mathbf{R}} \cdot \nabla_\perp h = -\frac{\partial \langle \varphi \rangle_{\mathbf{R}}}{\partial t} F_0 + C[h] + \chi$$

phase mixing $\sim k_\parallel v_{\mathrm{th}e}$

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Phase-mixing region:
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Very little energy!

$k_\parallel \sim \tau^{-1} \propto k_\perp^{1/3}$

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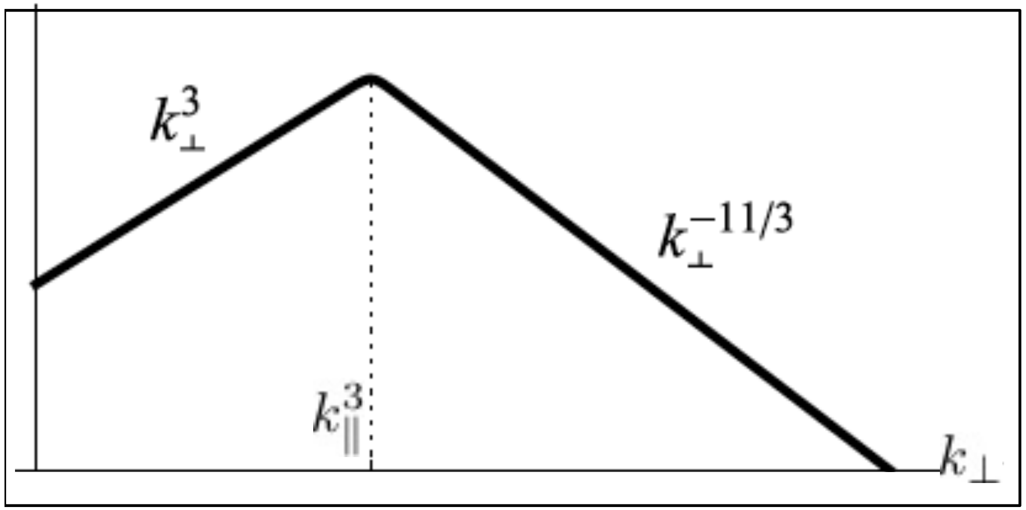
Most energy along critical balance curve

$k_\parallel$

$k_\perp$

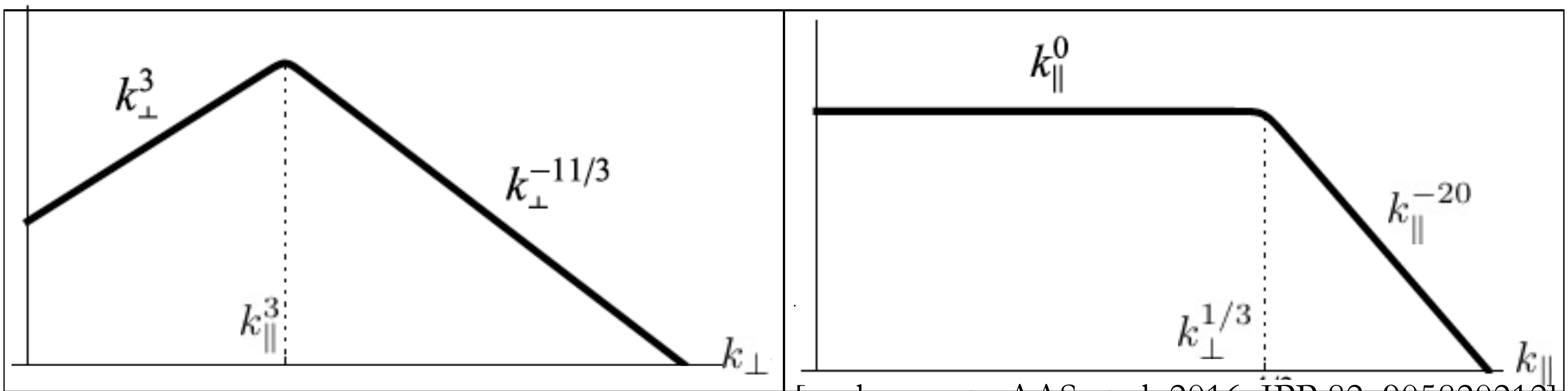


[analogous to AAS et al. 2016, JPP 82, 905820212]

2D Spectra

These are "2D spectra" of φ.

➢ Magnetic-field spectra are $E_B(k_\parallel, k_\perp) \propto \dfrac{E_\varphi(k_\parallel, k_\perp)}{k_\perp^2}$



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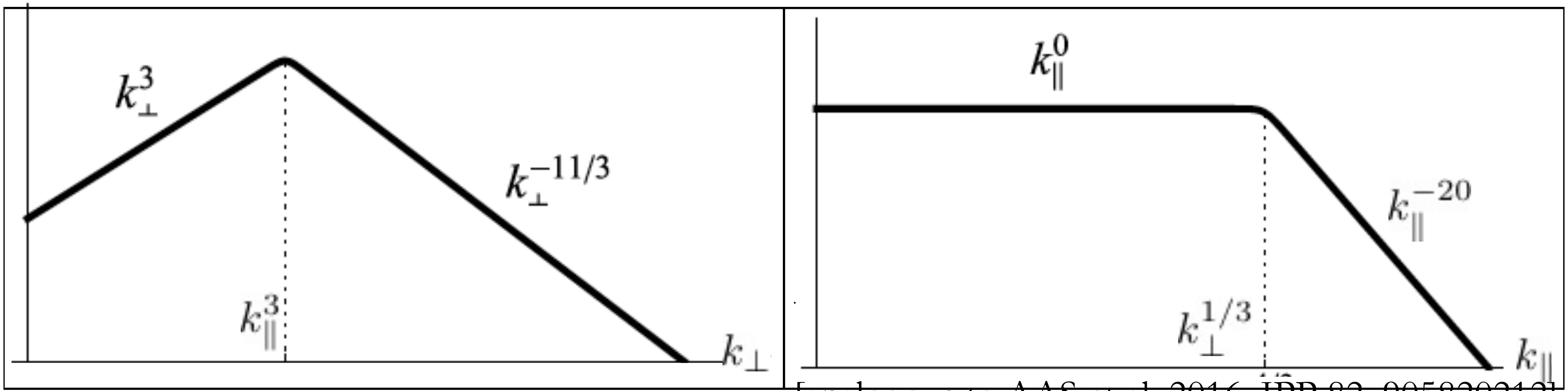
These are "2D spectra" of φ.

- Magnetic-field spectra are $E_B(k_\parallel, k_\perp) \propto \dfrac{E_\varphi(k_\parallel, k_\perp)}{k_\perp^2}$
- To get "1D spectra," integrate over wavenumber ranges bounded by critical balance:

$$E_\varphi^{(1\mathrm{D})}(k_\perp) \propto \int_0^{k_\perp^{1/3}} dk_\parallel k_\parallel^0 k_\perp^{-11/3} \sim k_\perp^{-10/3}, \quad E_\varphi^{(1\mathrm{D})}(k_\parallel) \propto \int_{k_\parallel^3}^{\infty} dk_\perp k_\parallel^0 k_\perp^{-11/3} \sim k_\parallel^{-8}$$

(same as derived above)

very steep!
NB: this is also the
frequency spectrum



[analogous to AAS et al. 2016, JPP 82, 905820212]

Phase-Space Spectra

These are "2D spectra" of φ.

- Magnetic-field spectra are $E_B(k_\parallel, k_\perp) \propto \dfrac{E_\varphi(k_\parallel, k_\perp)}{k_\perp^2}$
- To get "1D spectra," integrate over wavenumber ranges bounded by critical balance:

$$E_\varphi^{(1D)}(k_\perp) \propto \int_0^{k_\perp^{1/3}} dk_\parallel k_\parallel^0 k_\perp^{-11/3} \sim k_\perp^{-10/3}, \quad E_\varphi^{(1D)}(k_\parallel) \propto \int_{k_\parallel^3}^{\infty} dk_\perp k_\parallel^0 k_\perp^{-11/3} \sim k_\parallel^{-8}$$

- This all the tip of a larger iceberg – **PHASE-SPACE TURBULENCE:**

Hermite spectrum: $\boxed{E_h(m, k_\parallel) \propto m^{-19/2}}$

$m \sim (\delta v_\parallel / v_{\mathrm{the}})^{-2}$

Spectrum of parallel phase-mixing: super-steep, so **Landau damping is heavily reduced!**

Hankel spectrum: $\boxed{E_h(p) \propto p^{-4/3}}$

$p \sim (\delta v_\perp / v_{\mathrm{the}})^{-1}$

Spectrum of perpendicular phase-mixing (entropy cascade) [Plunk et al. 2010, JFM, 664, 407]

Cf. linear case: $E_h \propto m^{-1/2}$ [Kanekar et al. 2014, JPP 81, 305810104]

Details: another talk… or (exercise) derive this yourself by analogy with this paper

[analogous to AAS et al. 2016, JPP 82, 905820212]

Conclusions

- Turbulence associated with the kinetic species at sub-Larmor scales can be understood in terms of entropy cascade, intimately associated with nonlinear perpendicular phase mixing (small-scale spatial structure imprints itself on the velocity space due to Larmor gyration of particles).

- Spectra at electron sub-Larmor scales:

  density $E_n \propto k_\perp^{-10/3}$, electric field $E_E \propto k_\perp^{-4/3}$, magnetic field $E_B \propto k_\perp^{-16/3}$

  These appear to have numerical, experimental and perhaps observational support.

- Parallel phase-mixing is a subdominant effect (but this has **not** been checked!)

- Phase-space dynamics, statistics, scalings, etc. remain largely unexplored.
  THIS IS THE NEW FRONTIER (imho): both for theoreticians & for observers.

"Turbulent Dissipation Challenge"
what it should be about:
cascade via phase space or position space?

THOR?
velocity-space
structure!

PPCF 50, 24024 (2008)
ApJS 182, 310 (2009), sec. 7.12
PRL 103, 015003 (2009)
JPP 82, 905820212 (2016)