Magnetic Connection Hypersurfaces in Relativistic Ideal Plasmas: a step towards a frame independent geometric definition of relativistic magnetic reconnection

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Motivation and main result

- The concept of magnetic reconnection has been transferred to relativistic regimes in the astrophysical and in the laser plasma contexts see e.g. M. Hoshino, *ApJ.*, **773**, 118 (2013), B. Cerutti, et al., *ApJ.*, **770**, (2013).
- In the MHD description magnetic field lines play a fundamental role by defining dynamically preserved "magnetic connections" between plasma elements W.A. Newcomb, Ann. Phys., 3, 347 (1958).
- The concept of magnetic connection needs to be generalized in the case of a relativistic MHD description where we require covariance under arbitrary Lorentz transformations.
- This is performed by defining 2-D *magnetic connection hypersurfaces* in the 4-D Minkowski space.

  It accounts for the *loss of simultaneity* in different frames between spatially separated events and for the *transformation between electric and magnetic fields* under a Lorentz boost.

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Main result and open problems

- Within this covariant relativistic framework , where the geometrical concept of field-hypersurfaces has taken the role played in three-dimensional space by field lines, the main open problem from a topological point of view is how to describe the breaking of magnetic connection hypersurfaces in the presence of non ideal effects
- In the presence of nonideal effects, as we will show, the condition that is required for the existence of the covariant magnetic hypersurfaces is in general violated
- If this violation is local in space and time we should provide a covariant description of magnetic reconnection by describing the dynamics of the local merging of these field hypersurfaces
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Ideal Ohm's law, connections and simultaneity

- Ideal 3-D Ohm's law $\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = 0, \Rightarrow \mathbf{E} \cdot \mathbf{B} = 0$, with $\mathbf{v}$ the 3-D plasma fluid velocity field and $\mathbf{E}$ and $\mathbf{B}$ the electric and the magnetic fields, is in not restricted to a nonrelativistic plasma regime or to a preferred reference frame. It can be written (unmodified) in the fully covariant form $\mathbf{F}_{\mu\nu}\mathbf{u}^{\nu} = 0$, where $\mathbf{F}_{\mu\nu}$ is the e.m. field tensor, $\mathbf{u}^{\mu}$ is a timelike 4-vector which we interpret as the relevant fluid velocity 4-vector field in the plasma.
- Using Faraday's equation we obtain in 3-D notation

$$d(d\mathbf{l} \times \mathbf{B})/dt = -(d\mathbf{l} \times \mathbf{B})\,(\nabla \cdot \mathbf{v}) - [(d\mathbf{l} \times \mathbf{B}) \times \nabla]\,\mathbf{v}.$$

Its interpretation, conservation of 3-D magnetic connections, cannot be directly transferred to a different reference frame, as a Lorentz boost will in general add a time component to the transformed vector field $d\mathbf{l}'$ so that it cannot be interpreted as the vector field tangent to a curve in 3-D (coordinate) space.

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Lichnerowicz-Anile representation in ideal MHD

- $\mathbf{F}_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma}\mathbf{b}^{\lambda}\mathbf{u}^{\sigma} + [\mathbf{u}_{\mu}\mathbf{e}_{\nu} - \mathbf{u}_{\nu}\mathbf{e}_{\mu}]$: $\mathbf{b}^{\mu}$ and $\mathbf{e}_{\mu}$ are the *4-vector magnetic and electric fields*, $\mathbf{u}^{\mu}\mathbf{e}_{\mu} = \mathbf{u}_{\mu}\mathbf{b}^{\mu} = 0, \quad \mathbf{u}_{\mu}\mathbf{u}^{\mu} = -1.$

  $\mathbf{e}_{\mu}$ and $\mathbf{b}^{\mu}$ are related to $\mathbf{E}$ and $\mathbf{B}$ by $\mathbf{b}^{\mu} = \gamma(\mathbf{B}+\mathbf{E}\times\mathbf{v}\,,\,\mathbf{B}\cdot\mathbf{v}\,)$, $\mathbf{e}_{\mu} = \gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}\,,\,-\mathbf{E}\cdot\mathbf{v})$, with $\mathbf{e}_{\mu}\mathbf{b}^{\mu} = \mathbf{E}\cdot\mathbf{B}$.

  The orthogonality conditions $\mathbf{u}^{\mu}\mathbf{e}_{\mu} = \mathbf{u}_{\mu}\mathbf{b}^{\mu} = 0$ make this representation unique.

- Dual $\mathbf{G}^{\mu\nu} \equiv \varepsilon^{\mu\nu\alpha\beta}\mathbf{F}_{\alpha\beta}/2$: $\mathbf{G}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\sigma}\mathbf{u}_{\lambda}\mathbf{e}_{\sigma} + [\mathbf{u}^{\mu}\mathbf{b}^{\nu} - \mathbf{u}^{\nu}\mathbf{b}^{\mu}]$,

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Frobenius condition and Covariant hypersurfaces

- From Maxwell's equations we have $\partial_\mu \mathbf{G}^{\mu\nu} = 0$, and thus[1] $\mathbf{u}^\mu \partial_\mu \mathbf{b}^\nu - \mathbf{b}^\mu \partial_\mu \mathbf{u}^\nu + \mathbf{b}^\nu \partial_\mu \mathbf{u}^\mu - \mathbf{u}^\nu \partial_\mu \mathbf{b}^\mu = 0,$
  It gives a Frobenius involution condition for the 4-vector fields $\mathbf{b}^\mu$ and $\mathbf{u}^\mu$ and it allows us[2] to construct in the 4-D space-time 2-D hypersurfaces generated by the vector fields $\mathbf{u}^\mu$ and $\mathbf{b}^\mu$.
- These hypersurfaces, which we call *connection hypersurfaces* because they will allow us to recast the connection theorem in a covariant form, are the 4-D counterpart of magnetic field lines in 3-D space when the ideal Ohm's law holds.

  They are not related to the magnetic surfaces defined in 3-D by the equation $\mathbf{B} \cdot \nabla \psi = 0$.

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[1] that can be rewritten as $\partial_\tau \mathbf{b}^\nu = \mathbf{u}^\nu \mathbf{b}^\alpha (\partial_\tau \mathbf{u}_\alpha) - \mathbf{b}^\nu \partial_\mu \mathbf{u}^\mu + \mathbf{b}^\mu \partial_\mu \mathbf{u}^\nu$.

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Magnetic gauge

- A gauge freedom is allowed in the definition of $\mathbf{b}^\mu$ in the LA representation if we relax the orthogonality condition $\mathbf{b}^\mu \mathbf{u}_\mu = 0$: $\mathbf{b}^\mu \to \mathbf{h}^\mu \equiv \mathbf{b}^\mu + g\,\mathbf{u}^\mu$, where g is a free scalar field.

- $\mathbf{G}^{\mu\nu}$ is unchanged if we insert $\mathbf{h}^\mu$ for $\mathbf{b}^\mu$ and the Frobenius condition holds independently of the gauge.

  The connection-hypersurfaces generated by $\mathbf{u}^\mu$ and $\mathbf{b}^\mu$ can also be seen as generated by $\mathbf{u}^\mu$ and $\mathbf{h}^\mu$.

- Taking in a given frame[3] the *magnetic gauge* $g = -\mathbf{v}\cdot\mathbf{B}$, we make the time component of $\mathbf{h}^\mu$ vanish and $\mathbf{h}||\mathbf{B}$ in that frame.

[3] The quantity $-\mathbf{v}\cdot\mathbf{B}$ is a Lorentz scalar. Its expression in a frame moving with respect to the chosen frame with velocity 4-vector V_μ is $-(V_\mu b^\mu)/(V_\nu u^\nu)$.

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Covariant connections and time resetting gauge

- We consider in a given frame a magnetic field line ℓ at a fixed time in 4-D space with tangent (spacelike) 4-vector field $d\mathbf{l}^{\mu}$.
  In this frame its time component $dl^{o} = 0$ and the condition $\mathbf{F}_{\mu\nu} d\mathbf{l}^{\nu} = 0$ implies[4] $d\mathbf{l} \times \mathbf{B} = d\mathbf{l} \times \mathbf{h} = 0$.
- The interpretation of the condition $\mathbf{F}_{\mu\nu} d\mathbf{l}^{\nu} = 0$ remains valid even if $dl^{o} \neq 0$ because of the "time gauge" freedom $d\mathbf{l}^{\mu} \to d\hat{\mathbf{l}}^{\mu} = = d\mathbf{l}^{\mu} + \mathbf{u}^{\mu} \, d\lambda$, with λ a scalar function, i.e. $d\hat{\mathbf{l}}^{\mu}$ remains in the hypersurface generated by $\mathbf{b}^{\mu}$ and $\mathbf{u}^{\mu}$ (or by $\mathbf{h}^{\mu}$ and $\mathbf{u}^{\mu}$).
- In a boosted frame the transformed vector field $d\mathbf{l}'^{\mu}$ will acquire a time component but will still lie on the boosted 2-D hypersurface generated by the boosted vector fields $\mathbf{b}'^{\mu}$ and $\mathbf{u}'^{\mu}$.
  Using the time gauge in reverse it is possible to set $dl'^{o} = 0$ without violating the condition in the boosted frame $\mathbf{F}'_{\mu\nu} d\mathbf{l}'^{\nu} = 0$ because of the ideal Ohm's law.

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Combining the magnetic and the time gauges

- After performing the time gauge, using the magnetic gauge we bring the boosted 4-vector field $\mathbf{b}'^{\mu}$ to the form $\mathbf{h}'^{\mu}_{||} = (0, \mathbf{B}'/\gamma)$. In the boosted frame $\mathbf{F}'_{\mu\nu} d\mathbf{l}'^{\nu} = 0$ implies $\mathbf{dl}' \times \mathbf{B}' = d\mathbf{l}' \times \mathbf{h}' = 0$.
- *This proves that it is possible to define magnetic connections in a covariant way, provided we refer to connection hypersurfaces instead of connection field lines and provided we properly "gauge" the 4-vector magnetic field $\mathbf{b}^{\mu}$ and the tangent (spacelike) 4-vector field $d\mathbf{l}^{\mu}$ within the connection hypersurface in order to compensate for the mixing between 3-D magnetic and electric fields under a Lorentz boost and for the loss of simultaneity in different frames.*
- Magnetic connections in 3-D space can then be recovered in any chosen reference frame by taking sections of these surfaces at a fixed (in that frame) time.

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- Magnetic connections in 3-D space can then be recovered in any chosen reference frame by taking sections of these surfaces at a fixed (in that frame) time.

Combining the magnetic and the time gauges

- After performing the time gauge, using the magnetic gauge we bring the boosted 4-vector field $\mathbf{b}'^{\mu}$ to the form $\mathbf{h}'^{\mu}_{||} = (0, \mathbf{B}'/\gamma)$. In the boosted frame $\mathbf{F}'_{\mu\nu} d\mathbf{l}'^{\nu} = 0$ implies $\mathbf{dl}' \times \mathbf{B}' = d\mathbf{l}' \times \mathbf{h}' = 0$.
- *This proves that it is possible to define magnetic connections in a covariant way, provided we refer to connection hypersurfaces instead of connection field lines and provided we properly "gauge" the 4-vector magnetic field $\mathbf{b}^{\mu}$ and the tangent (spacelike) 4-vector field $d\mathbf{l}^{\mu}$ within the connection hypersurface in order to compensate for the mixing between 3-D magnetic and electric fields under a Lorentz boost and for the loss of simultaneity in different frames.*
- Magnetic connections in 3-D space can then be recovered in any chosen reference frame by taking sections of these surfaces at a fixed (in that frame) time.

Relevant references

- F. Pegoraro, *EPL*, **99**, 35001 (2012)

  where no use of the LA representation is made

  E. D'Avignon, P.J. Morrison, F. Pegoraro, *Phys. Rev. D*, **91**, 084050 (2015)

  where a Hamiltonian point of view is used and the focus is on covariant dynamics, not on covariant topology.

  F. Pegoraro, *JPP*, **82**, 555820201, 1-11 (2016) doi:10.1017/S0022377816000325

  it gives an extended version of the approach presented here.

- A. Lichnerowicz, in *Relativistic Hydrodynamics and Magnetohydrodynamics*, (New York: Benjamin) 1967.

  M. Anile, in *Relativistic fluids and magneto-fluids*, Cambridge Monographs on Mathematical Physics, Cambridge 1989.

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