Electron-scale reduced fluid models with gyroviscous effects

T. PASSOT[1] † , P.L. SULEM[1] and E TASSI[2]

[1] Université Côte d'Azur, CNRS, Observatoire de la Côte d'Azur, Laboratoire J.L. Lagrange, Boulevard de l'Observatoire, CS 34229, 06304 Nice Cedex 4, France

[2] Aix Marseille Univ, Univ Toulon, CNRS, CPT, Marseille, France

(Received xx; revised xx; accepted xx)

Reduced fluid models for collisionless plasmas, including electron inertia and finite Larmor radius corrections, are derived for scales ranging from the ion to the electron gyroradii. Based either on pressure balance or on the incompressibility of the electron fluid, they respectively capture kinetic Alfvén waves (KAWs) or whistler waves (WWs), and can provide suitable tools for both reconnection and turbulence. Isothermal and Landau-fluid closures are considered. For small β_e, a perturbative computation of the gyroviscous force valid at the electron inertial length is performed at order $O(\beta_e)$, which requires second-order contributions in a scale expansion. Anisotropy of pressure fluctuations enter the model equations in this regime. Comparisons with kinetic theory are performed in the linear regime, with special attention paid to the influence of Landau damping. The spectrum of transverse magnetic fluctuations in either strong or weak turbulence energy cascades is also phenomenologically predicted for both types of waves. New regimes, possibly related to the steep spectra observed in space plasmas in the sub-electron range, are obtained for KAWs at scales smaller than the electron inertial length, characterized by magnetic energy spectra decaying faster than those of WWs.

1. Introduction

Exploring the dynamics of magnetized plasmas in the range of scales extending from the ion (ρ_i) to the electron (ρ_e) Larmor radii is of great importance in various contexts, including magnetic reconnection (Daughton *et al.* 2011; Treumann & Baumjohann 2013; Zweibel & Yamada 2017) and turbulence in space plasmas such as the solar wind (Sahraoui *et al.* 2010, 2013; Matteini *et al.* 2017) or the auroral regions (Chaston *et al.* 2008). At scales comparable to or smaller than the electron inertial length d_e, electron inertia cannot be neglected, while electron finite Larmor radius (FLR) corrections play a role at scales approaching ρ_e. Although a fully kinetic approach is a priori required to describe plasma dynamics at these scales, "reduced fluid models" can provide an interesting insight when considering relatively small fluctuations about a Maxwellian equilibrium state. Up to the assumptions needed to close the fluid hierarchy, such models can indeed be obtained using systematic asymptotic expansions in regimes where nonlinearities are small and characteristic length scales appropriately selected in order to permit a rigorous estimate of the non-gyrotropic components of the pressure tensor. At scales small compared to ρ_i, ions are mostly static. Their gyroviscous stress tensor is also negligible at these scales, which leads to drastic simplifications. Concerning the electron fluid, two regimes are classically distinguished. It can be viewed as incompressible when

† Email address for correspondence: passot@oca.eu

considering whistler waves (WWs), as usually assumed in electron magnetohydrodynamics (EMHD), except in the relativistic regimes of large Alfvén velocities (Kuvshinov *et al.* 1998). Differently, kinetic Alfvén waves (KAWs) are compressible at the sub-ion scales, pressure fluctuations being governed by the perpendicular pressure balance, a description valid as long as their frequency remains small enough, leading to the electron reduced magnetohydrodynamics (ERMHD) (see Schekochihin *et al.* (2009) for a review). The electron gyroviscous stress tensor is negligible at scales large compared to ρ_e but, except in the case of very large values of the ion to electron temperature ratio τ, this condition strongly constrains the validity range when scales are also supposed to be small compared to ρ_i. This point is exemplified in Tassi *et al.* (2016), where comparison with the kinetic theory shows that, in the linear regime, the accuracy of the fluid model for KAWs with τ and the electron beta parameter β_e of order unity, is limited to scales such that $k_\perp \rho_s \simeq 15$ (here, $\rho_s = (\frac{m_i}{2m_e})^{1/2}\rho_e$ is the sonic Larmor radius). At smaller scales, non-gyrotropic pressure contributions are to be retained. Their calculation is performed in Appendix A, using a perturbation expansion in terms of the scale separation, when the coupling to the non-gyrotropic part of the heat flux is not retained.

In the sub-ion range, the electron inertial length $d_e = (\frac{2m_e}{m_i\beta_e})^{1/2}\rho_s$ plays an important role as dispersive properties of KAWs and WWs display a qualitative change across this characteristic scale, thus potentially affecting turbulent cascades. In order to study scales smaller than d_e, but still larger than ρ_e, the parameter β_e should be taken small. When assuming $\beta_e = O(m_e/m_i)$ as in Zocco & Schekochihin (2011), electron inertia should be retained, but electron FLR corrections can be neglected, up to β_e-independent terms involved in gyroviscous cancellation. For values of β_e that, although small compared to unity, exceed m_e/m_i, the electron gyroviscous force becomes relevant, and it turns out that its computation to order β_e, at scales comparable to d_e, requires the expansion to be pushed to second order in the scale separation.

The resulting equations for the electrostatic and the parallel magnetic potentials must be supplemented by conditions concerning density and temperature fluctuations. As already mentioned, neglecting density fluctuations or prescribing perpendicular pressure balance, leads to discriminate between WWs and KAWs. Concerning temperature fluctuations, we have considered two different regimes. The first one assumes isothermal electrons, which leads to two-field models. Beyond its simplicity, such an assumption appears realistic for turbulent applications when not addressing questions related to plasma heating, and preferable to an adiabatic assumption (Tassi *et al.* 2016). Temperature indeed tends to be homogenized along the magnetic field lines which, in a turbulent regime, are expected to be stochastic (Schekochihin *et al.* 2009). A more elaborate model retains dynamical equations for the temperatures and involves a Landau-fluid (LF) closure to express the gyrotropic heat fluxes, in terms of lower order moments, in a way consistent with the linear kinetic theory (Hammett & Perkins 1990; Snyder *et al.* 1997). As discussed by Hesse *et al.* (2004), retaining electron heat fluxes can be an important issue in guide-field magnetic reconnection.

The models derived in this paper could be most useful to address questions such as the role of electron pressure anisotropy, Landau damping and FLR corrections in collisionless reconnection. Being three-dimensional, they are also well adapted to study turbulence dynamics at electron scales. The regime at scales smaller than d_e has been extensively studied mostly for WWs, in the framework of EMHD (Biskamp *et al.* 1996, 1999; Galtier & Bhattacharjee 2003; Galtier & Meyrand 2015; Lyutikov 2013) or using an incompressible bi-fluid model (Andrés *et al.* 2014; Andrés *et al.* 2016*a*,*b*). A purpose of the present paper is to address the KAWs dynamics at scales smaller than the electron inertial length, where a new regime of strong and weak turbulence is phenomenologically studied. Due to the compressibility of these waves, magnetic spectra steeper than in the case of WWs are obtained, a property that could be of interest to compare to the fast-decaying spectra observed in the terrestrial magnetosheath (Huang *et al.* 2014).

The paper is organized as follows. Sections 2, 3 and 4 provide a derivation of reduced models for KAWs and WWs, including the non-gyrotropic pressure force $\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e$, whose calculation is presented in Appendix A. Comparisons with previous estimates are made in Appendices B and C. In Section 5, closed systems of equations resulting from the assumption of isothermal electrons or from a Landau-fluid closure, are presented, both in the KAWs and WWs regimes. In Section 6, accuracy of these two closures is checked against kinetic theory in the linear regime. In Section 7, the isothermal models are used as a basis for a phenomenological theory of KAWs and WWs turbulent cascades, at scales either large or small compared with the electron inertial length. The influence of the ion to electron temperature ratio τ on the small-scale KAWs spectral exponent is in particular discussed. Section 8 is the conclusion.

2. Reduced form of the Faraday equation

We consider the Faraday equation for the magnetic field $\boldsymbol{B}$

$$\partial_t \boldsymbol{B} = -c\boldsymbol{\nabla}\times\boldsymbol{E}, \tag{2.1}$$

where the electric field $\boldsymbol{E}$ is given by the generalized Ohm's law

$$\boldsymbol{E} = -\frac{1}{c}\boldsymbol{u}_e\times\boldsymbol{B} - \frac{1}{en}\boldsymbol{\nabla}\cdot\boldsymbol{P}_e - \frac{m_e}{e}\frac{D^{(e)}}{Dt}\boldsymbol{u}_e. \tag{2.2}$$

Here c is the speed of light, $D^{(e)}/Dt = \partial_t + \boldsymbol{u}_e\cdot\boldsymbol{\nabla}$ holds for the material derivative associated to the electron velocity field $\boldsymbol{u}_e$, and the electron pressure tensor $\boldsymbol{P}_e$ is given by

$$\boldsymbol{P}_e = p_{\perp e}\boldsymbol{I} + (p_{\| e} - p_{\perp e})\boldsymbol{\tau} + \boldsymbol{\Pi}_e, \tag{2.3}$$

where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\tau} = \widehat{\boldsymbol{b}}\otimes\widehat{\boldsymbol{b}}$ (with $\widehat{\boldsymbol{b}} = \boldsymbol{B}/|\boldsymbol{B}|$), $p_{\| e}$ and $p_{\perp e}$ are the gyrotropic parallel and perpendicular electron pressures, and $\boldsymbol{\Pi}_e$ refers to the FLR contribution to the electron pressure tensor. As usual, n holds for the number density of the electrons whose mass and charge are denoted by m_e and e respectively.

The equation for the parallel magnetic fluctuations B_z thus reads

$$\frac{D^{(e)}}{Dt}B_z + (\boldsymbol{\nabla}\cdot\boldsymbol{u}_e)B_z - (\boldsymbol{B}\cdot\boldsymbol{\nabla})u_{ze} - \widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(\frac{c}{ne}\boldsymbol{\nabla}\cdot\boldsymbol{P}_e) - \frac{m_e c}{e}\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\frac{D^{(e)}}{Dt}\boldsymbol{u}_e = 0, \tag{2.4}$$

where $\widehat{\boldsymbol{z}}$ denotes the unit vector along the ambient magnetic field. The divergence of the electron velocity is given by the continuity equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_e = -\frac{1}{n}\frac{D^{(e)}}{Dt}n. \tag{2.5}$$

To describe the evolution of the transverse magnetic fluctuations, it is convenient to consider the parallel component $A_\|$ of the magnetic potential $\boldsymbol{A}$, defined by $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. One classically gets

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\partial_t\boldsymbol{A}, \tag{2.6}$$

where φ is the electrostatic potential. By projection on the parallel direction defined by the unit vector $\widehat{\boldsymbol{b}}$,

$$\partial_t A_\| + c\nabla_\| \varphi = -cE_\|, \tag{2.7}$$

where the subscript $\|$ refers to the component along the local magnetic field. In particular, the parallel gradient is given by $\nabla_\| = \widehat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} = \partial_z - [A_\|, \cdot]$. Here the bracket of two scalar functions f and g is defined as $[f, g] = \widehat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g)$.

In addition, when concentrating on sub-ion scales where the electron velocity strongly exceeds that of the ions, the Ampère equation reduces to

$$\boldsymbol{u}_e = \frac{c}{4\pi en} \varDelta \boldsymbol{A}. \tag{2.8}$$

We consider in the following relatively weak fluctuations about an equilibrium state characterized by a number density n_0, isotropic ion and electron temperatures T_{i0} and T_{e0}, and a guide field of magnitude B_0 taken in the z-direction. In order to deal with dimensionless quantities, we rescale time by the inverse ion gyrofrequency $\varOmega_i = eB_0/(m_i c)$, velocities by the sound speed $c_s = (T_{e0}/m_i)^{1/2}$, space coordinates by the sonic Larmor radius $\rho_s = c_s/\varOmega_i$, density by the equilibrium density n_0, magnetic field by the equilibrium field B_0, electric potential by T_{e0}/e, parallel magnetic potential by $B_0\rho_s$, ion pressures by $n_0 T_{i0}$, electron pressures by $n_0 T_{e0}$, parallel and perpendicular electron heat fluxes by $c_s n_0 T_{e0}$, and fourth-rank moments by $n_0 T_{e0}^2/m_e$. For convenience, we keep the same notation for the rescaled fields. We also define the non-dimensional parameters $\tau = T_{i0}/T_{e0}$, $\delta = \sqrt{m_e/m_i}$ and $\beta_e = 8\pi n_0 T_{e0}/B_0^2$.

We consider both KAWs and WWs for which we perform asymptotic reductions by introducing the small parameters ε and μ such that $u_{\perp e} = O(\varepsilon)$ and $\boldsymbol{\nabla}_\perp = O(1/\mu)$. Concentrating on scales around d_e leads to assume $\delta^2/(\beta_e\mu^2) = 1$. At these scales, the condition (in dimensional units) $k_\perp \rho_e \ll 1$ reduces to $\beta_e \ll 1$. Based on their dispersion relations, we are led to prescribe that for both kinds of waves, $\partial_t = O(\varepsilon/\mu)$, $\varphi = O(\mu\varepsilon)$, $B_z = O(\beta_e \mu\varepsilon)$. Differently, we have $\nabla_\| = O(\delta\varepsilon/\mu) = O(\beta^{1/2}\varepsilon)$, $A_\| = O(\delta\mu\varepsilon) = O(\beta^{-1/2}\delta^2\varepsilon)$, and $u_{\|e} = O(\mu\varepsilon/\delta) = O(\beta^{-1/2}\varepsilon)$ for the KAWs, while for the WWs, $\nabla_\| = O(\beta\varepsilon)$, $A_\| = O(\beta\mu^2\varepsilon) = O(\delta^2\varepsilon)$, and $u_{\|e} = O(\varepsilon)$. Furthermore, $B_z/|\boldsymbol{B}_\perp| = O(\beta^{1/2})$ for the KAWs, but is of order unity for the WWs. Similarly, the density, pressure and temperature fluctuations are $O(\mu\varepsilon)$ for the KAWs, while the density fluctuations are negligible for the WWs. For both types of waves, the gyrotropic heat fluxes $q_{\|e}$ and $q_{\perp e}$ scale like the parallel velocity $u_{\|e}$ and the fourth-rank cumulants like the temperature fluctuations.

Under the above assumptions, a drift expansion of the transverse electron velocity gives (in terms of the rescaled variables)

$$\boldsymbol{u}_{\perp e} = \widehat{\boldsymbol{z}} \times (\boldsymbol{\nabla}_\perp(\varphi - p_{\perp e}) - \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varPi}_e) \tag{2.9}$$

where, to leading order (see Appendix A), $\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varPi}_e = (\delta^2/2)\boldsymbol{\nabla}\omega_{ze}$, in terms of the parallel electron vorticity ω_{ze}. We write

$$\boldsymbol{u}_{\perp e} = \widehat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \varphi^* \tag{2.10}$$

with

$$\varphi^* = \varphi - p_{\perp e} - \frac{\delta^2}{2}\omega_{ze}. \tag{2.11}$$

In Eq. (2.11) and hereafter, the various fields refer to fluctuations, except for the pressure in primitive equations.

Furthermore, using (2.8),

$$u_{\parallel e} = \frac{2}{\beta_e}\Delta_\perp A_\parallel \tag{2.12}$$

$$\omega_{ze} = \Delta_\perp \varphi^* \tag{2.13}$$

$$\varphi^* = \frac{2}{\beta_e} B_z \tag{2.14}$$

$$(1+\frac{\delta^2}{2}\Delta_\perp)\varphi^* = \varphi - p_{\perp e}. \tag{2.15}$$

It follows that

$$\frac{D^{(e)}}{Dt} = \frac{d}{dt} - [p_{\perp e} + \frac{\delta^2}{\beta_e}\Delta_\perp B_z, \cdot], \tag{2.16}$$

where $\frac{d}{dt} = \partial_t + [\varphi, \cdot]$. Equations (2.7) rewrites

$$\frac{d}{dt}\left(1 - \frac{2\delta^2}{\beta_e}\Delta_\perp\right) A_\parallel + \partial_z\varphi - \nabla_\parallel p_{\parallel e} + \delta^2[p_{\perp e} + \frac{\delta^2}{2}\omega_{ze}, u_{\parallel e}] - \widehat{\boldsymbol{b}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e) = 0, \tag{2.17}$$

where the FLR contributions $\widehat{\boldsymbol{b}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e) = \widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(1)} + \widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(2)}$ are given by Eqs. (A 7), (A 8), (A 10) and (A 18). Equation (2.17) rewrites

$$\begin{aligned}
&\frac{d}{dt}\left(1 - \frac{2\delta^2}{\beta_e}\Delta_\perp + \frac{2\delta^4}{\beta_e}\Delta_\perp^2\right) A_\parallel + \partial_z\varphi - \nabla_\parallel p_{\parallel e} - [p_{\perp e}, \frac{2\delta^4}{\beta_e}\Delta_\perp^2 A_\parallel] \\
&\quad +\delta^2[B_z, u_{\parallel e} + q_{\perp e}] - \delta^2\nabla_\parallel \Delta_\perp\varphi^* - \delta^2 \sum_{i=x,y}[\partial_i\varphi^*, (1 - \frac{2\delta^2}{\beta_e}\Delta_\perp)\partial_i A_\parallel] \\
&\quad +\delta^4\frac{d}{dt}\Delta_\perp q_{\perp e} - \delta^4[p_{\perp e}, \Delta_\perp q_{\perp e}] - \delta^4[\Delta_\perp\varphi^*, q_{\perp e}] = 0,
\end{aligned} \tag{2.18}$$

where the gyroviscous cancellation eliminates the $\delta^2[p_{\perp e} + \frac{\delta^2}{2}\omega_{ze}, u_{\parallel e}]$ term.

We now turn to Eq. (2.4) where, assuming an isotropic equilibrium state, we write

$$\boldsymbol{\nabla}\cdot\boldsymbol{P}_e = \nabla_\perp p_{\perp e} + \nabla_\parallel p_{\parallel e}\widehat{\boldsymbol{b}} + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e. \tag{2.19}$$

At the order of the present expansion, we have

$$\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\frac{D^{(e)}}{Dt}\boldsymbol{u}_e = \frac{D^{(e)}}{Dt}\omega_z = \frac{2}{\beta_e}\frac{D^{(e)}}{Dt}\Delta_\perp B_z \tag{2.20}$$

and

$$\begin{aligned}
\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e) &= \frac{2\delta^2}{\beta_e}[p_{\perp e} - B_z, \Delta_\perp B_z] + \delta^2\sum_{i=x,y}[\partial_i(p_{\perp e} - B_z), \partial_i\varphi^*] \\
&\quad -\frac{\delta^2}{2}\Delta_\perp\boldsymbol{\nabla}\cdot\boldsymbol{u}_e - \frac{\delta^2}{2}\nabla_\parallel\Delta_\perp u_{\parallel e} - \delta^2\nabla_\parallel\Delta_\perp q_{\perp e} + \delta^2\sum_{i=x,y}[\partial_i A_\parallel, \partial_i q_{\perp e}].
\end{aligned} \tag{2.21}$$

Using Eq. (A 21), we get

$$\frac{d}{dt}\Big((1-\frac{2\delta^2}{\beta_e}\Delta_\perp)B_z\Big)-[p_{\perp e}+\frac{\delta^2}{\beta_e}\Delta_\perp B_z, B_z+n]+(1+\frac{\delta^2}{2}\Delta_\perp)(\boldsymbol{\nabla}\cdot\boldsymbol{u}_e)$$
$$-\nabla_\|\Big(1-\frac{\delta^2}{2}\Delta_\perp\Big)u_{\|e}+\delta^2[B_z,\Delta_\perp\varphi^*]-\delta^2\sum_{i=x,y}[\partial_i p_{\perp e},\partial_i\varphi^*]$$
$$+\delta^2\nabla_\|\Delta_\perp q_{\perp e}-\delta^2\sum_{i=x,y}[\partial_i A_\|,\partial_i q_{\perp e}]=0 \tag{2.22}$$

or, when using Eq. (2.5),

$$\frac{d}{dt}\Big((1-\frac{2\delta^2}{\beta_e}\Delta_\perp)B_z-(1+\frac{\delta^2}{2}\Delta_\perp)n\Big)-[p_{\perp e}+\frac{3\delta^2}{\beta_e}\Delta_\perp B_z, B_z]-\frac{\delta^2}{2}[\Delta_\perp\varphi,n]$$
$$+\frac{\delta^2}{2}\Delta_\perp[p_{\perp e}+\frac{\delta^2}{\beta_e}\Delta_\perp B_z,n]-\frac{2}{\beta_e}\nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\Delta_\perp A_\|+\delta^2\sum_{i=x,y}[\partial_i n,\partial_i\varphi]$$
$$-\delta^2[\partial_i p_{\perp e},\partial_i\varphi^*]+\delta^2\nabla_\|\Delta_\perp q_{\perp e}-\delta^2\sum_{i=x,y}[\partial_i A_\|,\partial_i q_{\perp e}]=0. \tag{2.23}$$

3. Pressure equations

The equations for the electron gyrotropic pressures read

$$\frac{D^{(e)}}{Dt}p_{\|e}+p_{\|e}\boldsymbol{\nabla}\cdot\boldsymbol{u}_e+2p_{\|e}\nabla_\|\boldsymbol{u}_e\cdot\widehat{\boldsymbol{b}}+\boldsymbol{\nabla}\cdot\boldsymbol{q}_e:\boldsymbol{\tau}+\Big[(\boldsymbol{\Pi}_e\cdot\boldsymbol{\nabla}\boldsymbol{u}_e)^S:\boldsymbol{\tau}-\boldsymbol{\Pi}_e:\frac{d\boldsymbol{\tau}}{dt}\Big]=0 \tag{3.1}$$
$$\frac{D^{(e)}}{Dt}p_{\perp e}+2p_{\perp e}\boldsymbol{\nabla}\cdot\boldsymbol{u}_e-p_{\perp e}\nabla_\|\boldsymbol{u}_e\cdot\widehat{\boldsymbol{b}}+\frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{q}_e:\boldsymbol{n}+\frac{1}{2}\Big[(\boldsymbol{\Pi}_e\cdot\boldsymbol{\nabla}\boldsymbol{u}_e)^S:\boldsymbol{n}+\boldsymbol{\Pi}_e:\frac{d\boldsymbol{\tau}}{dt}\Big]=0, \tag{3.2}$$

where $\mathbf{n}=\mathbf{I}-\boldsymbol{\tau}$ and the electron heat flux tensor $\boldsymbol{q}_e$ can be written $\boldsymbol{q}_e=\boldsymbol{S}_e+\boldsymbol{\sigma}_e$. Here the tensor $\boldsymbol{\sigma}_e$ obeys the conditions $\boldsymbol{\sigma}_e:\boldsymbol{n}=0$ and $\boldsymbol{\sigma}_e:\boldsymbol{\tau}=0$. The elements of the tensor $\boldsymbol{S}_r$ are classically expressed (see e.g. Goswami *et al.* (2005)) in terms of the components of two vectors $\boldsymbol{S}_r^\|$ and $\boldsymbol{S}_r^\perp$ defined by $\boldsymbol{S}_e^\|=\boldsymbol{q}_e:\boldsymbol{\tau}$ and $\boldsymbol{S}_e^\perp=(1/2)\ \boldsymbol{q}_e:\boldsymbol{n}$ that measure the directional fluxes of the parallel and perpendicular heats respectively. The usual perpendicular and parallel gyrotropic heat fluxes are given by $q_{\perp e}=\boldsymbol{S}_e^\perp\cdot\boldsymbol{b}$ and $q_{\|e}=\boldsymbol{S}_e^\|\cdot\boldsymbol{b}$, and thus correspond to the fluxes along the magnetic field. We write $\boldsymbol{S}_r^\perp=q_{\perp e}\boldsymbol{b}+\boldsymbol{S}_{\perp e}^\perp$ and $\boldsymbol{S}_e^\|=q_{\|e}\boldsymbol{b}+\boldsymbol{S}_{\perp e}^\|$.

In the present asymptotics, the contribution to the pressure equations of the tensor $\boldsymbol{\sigma}_e$, whose expression is given in Ramos (2005*a*) and also in Sulem & Passot (2015), is negligible. To leading order, we are thus led to write

$$(\nabla\cdot\mathbf{q}_e):\boldsymbol{\tau}=\nabla_\|q_{\|e}+\nabla\cdot\mathbf{S}_{\perp e}^\| \tag{3.3}$$
$$\frac{1}{2}(\nabla\cdot\mathbf{q}_e):\mathbf{n}=\nabla_\|q_{\perp e}+\nabla\cdot\mathbf{S}_{\perp e}^\perp. \tag{3.4}$$

Here, the non-gyrotropic heat fluxes contributions, described by the vectors $\mathbf{S}_{\perp e}^\|$ and $\mathbf{S}_{\perp e}^\perp$, are obtained by changing sign in the equations Eqs. (3.6) and (3.7) of Sulem & Passot

(2015) for the ion heat flux vectors. We get, at the order of the asymptotics,

$$\mathbf{S}^{\parallel}_{\perp e} = -\frac{1}{B}\widehat{\mathbf{b}} \times (p_{\perp e}\boldsymbol{\nabla} T_{\parallel e} + 2\delta^2 q_{\perp e}\widehat{\boldsymbol{b}}\times\boldsymbol{\omega}_e + \boldsymbol{\nabla}\widetilde{r}_{\parallel\perp e}) \tag{3.5}$$

$$\mathbf{S}^{\perp}_{\perp e} = -\frac{2}{B}\widehat{\mathbf{b}} \times (p_{\perp_e}\boldsymbol{\nabla} T_{\perp e} + \boldsymbol{\nabla}\widetilde{r}_{\perp\perp e}), \tag{3.6}$$

where the $\widetilde{r}$'s refer to the gyrotropic components of the fourth-rank cumulants.

Among the terms associated with the work of the non-gyrotropic pressure force, that involving the time derivative scales like $O(\varepsilon^3)$ and is thus negligible. In the other terms, only the linear part of $\boldsymbol{\Pi}_e$ is possibly relevant at the prescribed order, but it is easily checked that the corresponding contribution in fact vanishes.

Using the electron continuity equation to eliminate the velocity divergence, we are led to write the equations for the temperature fluctuations

$$\frac{d}{dt}T_{\parallel e} - [B_z, T_{\parallel e} + \widetilde{r}_{\parallel\perp e}] + \nabla_\parallel(2u_{\parallel e} + q_{\parallel e}) - \frac{\delta^2}{\beta_e}[\varDelta_\perp B_z, T_{\parallel e}] + 2\delta^2[q_{\perp e}, u_{\parallel e}] = 0 \tag{3.7}$$

$$\frac{d}{dt}(T_{\perp e} - n) - 2[B_z, T_{\perp e} + \widetilde{r}_{\perp\perp e}] + \nabla_\parallel(q_{\perp e} - u_{\parallel e}) - \frac{\delta^2}{\beta_e}[\varDelta_\perp B_z, T_{\perp e} - n] = 0, \tag{3.8}$$

where $u_{\parallel e}$ and B_z are given by Eqs. (2.12) and (2.14) respectively.

4. KAWs or WWs, depending on the compressibility level

At this stage, it is important to note that, beyond the estimate of the gyrotropic electron pressure fluctuations which are prescribed by the closure assumption for the fluid hierarchy, the system given by Eqs. (2.18) and (2.23) involves three unknown quantities $A_\parallel$, φ and n. An additional relation is obtained when specifying the type of waves the system is describing. KAWs and WWs are indeed characterized by different levels of compressibility. The relative importance of these two types of waves in collisionless reconnection, depending on the strength of the guide field, is discussed by Treumann & Baumjohann (2013).

In the case of KAWs, the characteristic frequencies are low, and the plasma is thus in pressure balance in the transverse direction. We have

$$(\boldsymbol{\nabla}\cdot(\tau\boldsymbol{P}_i + \boldsymbol{P}_e))_\perp - \frac{2}{\beta_e}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\widehat{\boldsymbol{z}} = 0, \tag{4.1}$$

where the index i refers to the ions. Applying the transverse divergence, we get

$$\varDelta_\perp(\tau p_{\perp i} + p_{\perp e}) + \boldsymbol{\nabla}_\perp\cdot(\boldsymbol{\nabla}\cdot(\boldsymbol{\Pi}_i + \boldsymbol{\Pi}_e)) + \frac{2}{\beta_e}\varDelta_\perp B_z = 0. \tag{4.2}$$

As shown in Tassi *et al.* (2016), at the sub-ion scales, $\boldsymbol{\Pi}_i$ is negligible and the ions are isothermal (making in the present units electron pressure and density fluctuations equal). To leading order in ε, Eq. (4.2) rewrites

$$\tau n + p_{\perp e} + \frac{\delta^2}{2}\varDelta\varphi^* + \frac{2}{\beta_e}B_z = 0. \tag{4.3}$$

Using Eqs. (2.14) and (2.15), it is then easily checked that

$$n = -\frac{1}{\tau}\varphi. \tag{4.4}$$

It results that

$$n = -\frac{1}{1+\tau}\left(T_{\perp e} + (1+\frac{\delta^2}{2}\varDelta_\perp)\varphi^*\right) \tag{4.5}$$

$$\varphi = \frac{\tau}{1+\tau}\left(T_{\perp e} + (1+\frac{\delta^2}{2}\varDelta_\perp)\varphi^*\right). \tag{4.6}$$

A different behavior of the density holds in the case of the WWs. The linear kinetic theory (see e.g. Gary & Smith (2009)) shows that the electron compressibility of WWs are significantly smaller than those of the KAWs, at least when β_e is sufficiently small and for sufficiently large perpendicular wavenumbers. This suggests to assume $\boldsymbol{\nabla}\cdot\boldsymbol{u}_e$ smaller than ε^2, and thus to neglect the density fluctuations. This regime formally corresponds to taking the limit $\tau \to \infty$ in Eqs. (4.5)-(4.6). Note however than in auroral zones where the Alfvén velocity can exceed several thousands of km/s, quasi-neutrality no longer holds at large enough wavenumber, making electron density fluctuations relevant for WW dynamics (Kuvshinov *et al.* 1998).

In the next Section, two different closures of the fluid hierarchy will be examined, namely the isothermal case, a regime often considered in turbulence studies, and a Landau fluid closure.

5. Closure assumptions

5.1. *General form of the four-field model*

Two asymptotics which are similar but not identical can be distinguished. When considering scales comparable to d_e, we should take β_e as the expansion parameter. We refer in the following to this regime as the small β_e regime. Differently, when considering β_e of order unity, a gradient expansion is performed in terms of $\delta^2\varDelta_\perp$ (hereafter, large-scale regime). In fact both approaches can be captured simultaneously, but this leads to retain subdominant terms of order $\beta_e\delta^2\varDelta_\perp = O(\beta_e^2)$ in the former regime and of order $\delta^4\varDelta_\perp^2/\beta_e$ in the latter.

In the KAW regime, using Eqs. (4.5)-(4.6), Eqs. (2.18) and (2.23) rewrite,

$$\begin{aligned}
&\partial_t\left((1-\frac{2\delta^2}{\beta_e}\varDelta_\perp + \frac{2\delta^4}{\beta_e}\varDelta_\perp^2)A_\parallel + \delta^4\varDelta_\perp q_{\perp e}\right) + \nabla_\parallel(T_{\perp e}-T_{\parallel e}) + \nabla_\parallel(1-\frac{\delta^2}{2}\varDelta_\perp)\varphi^* \\
&+[\varphi^*, \delta^2\varDelta_\perp A_\parallel + \frac{2\delta^4}{\beta_e}\varDelta_\perp^2 A_\parallel] - \frac{\tau}{1+\tau}[T_{\perp e} + (1+\frac{\delta^2}{2}\varDelta_\perp)\varphi^*, \frac{2\delta^2}{\beta_e}\varDelta_\perp A_\parallel] \\
&-\delta^2\sum_{i=x,y}[\partial_i\varphi^*, (1-\frac{2\delta^2}{\beta_e}\varDelta_\perp)\partial_i A_\parallel] + [\varphi^*, \frac{\beta_e}{2}\delta^2 q_{\perp e} + \delta^4\varDelta_\perp q_{\perp e}] - \delta^4[\varDelta_\perp\varphi^*, q_{\perp e}] = 0,
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
&\partial_t\Big(\frac{\beta_e}{2}(1-\frac{2\delta^2}{\beta_e}\varDelta_\perp)\varphi^* + \frac{1}{1+\tau}(1+\delta^2\varDelta_\perp)\varphi^* + \frac{1}{1+\tau}(1+\frac{\delta^2}{2}\varDelta_\perp)T_{\perp e}\Big) \\
&-\frac{2}{\beta_e}\nabla_\parallel(1-\frac{\delta^2}{2}\varDelta_\perp)\varDelta_\perp A_\parallel + \delta^2\nabla_\parallel\varDelta_\perp q_{\perp e} + \frac{2\tau+1}{2(\tau+1)}[\delta^2\varDelta_\perp\varphi^*, T_{\perp e}] + \delta^2\sum_{i=x,y}[\partial_i\varphi^*, \partial_i T_{\perp e}] \\
&-[\varphi^*, (\frac{\tau}{1+\tau} - \frac{\beta_e}{2})\delta^2\varDelta_\perp\varphi^* - \frac{1}{1+\tau}\frac{\delta^2}{2}\varDelta_\perp T_{\perp e}] - \delta^2\sum_{i=x,y}[\partial_i A_\parallel, \partial_i q_{\perp e}] = 0.
\end{aligned} \tag{5.2}$$

Note that retaining $O(\beta_e)$ contributions make temperature anisotropy relevant in Eqs.

(5.1) and (5.2), while neglecting these terms and using φ instead of φ^* eliminates $T_{\perp e}$ from these equations.

Equations (3.7)-(3.8) become

$$\partial_t T_{\| e} + \nabla_\| (\frac{4}{\beta_e}\Delta_\perp A_\| + q_{\| e}) + \frac{\tau}{1+\tau}[T_{\perp e} + (1+\frac{\delta^2}{2}\Delta_\perp)\varphi^*, T_{\| e}]$$
$$-\frac{\beta_e}{2}[\varphi^*, T_{\| e} + \widetilde{r}_{\|\perp e}] - [\frac{\delta^2}{2}\Delta_\perp\varphi^*, T_{\| e}] + \frac{4\delta^2}{\beta_e}[q_{\perp e}, \Delta_\perp A_\|] = 0 \quad (5.3)$$
$$\partial_t\left(\frac{2+\tau}{1+\tau}T_{\perp e} + \frac{1}{1+\tau}(1+\frac{\delta^2}{2}\Delta_\perp)\varphi^*\right) + \nabla_\|(q_{\perp e} - \frac{2}{\beta_e}\Delta_\perp A_\|)$$
$$-\frac{\tau}{1+\tau}[T_{\perp e}, (1+\frac{\delta^2}{2}\Delta_\perp)\varphi^*] - \beta_e[\varphi^*, T_{\perp e} + \widetilde{r}_{\perp\perp e}] - \frac{\delta^2}{2}[\Delta_\perp\varphi^*, \frac{2+\tau}{1+\tau}T_{\perp e} + \frac{1}{1+\tau}\varphi^*] = 0. \quad (5.4)$$

In the limit $\delta = 0$, Eqs. (5.1)-(5.4) reduce to Eqs. (3.72)-(3.78) of Tassi *et al.* (2016). In the adiabatic limit (where $q_{\| e}$, $q_{\perp e}$, $\widetilde{r}_{\|\perp e}$ and $\widetilde{r}_{\perp\perp e}$ are taken equal to zero), this system possesses a conserved energy, a property easily established when noting that the brackets involving gradients are eliminated within the integrals by using equalities of the type

$$2\int \Delta_\perp A_\| \sum_{i=x,y}[\partial_i\varphi^*, \partial_i A_\|]d\boldsymbol{x} = \int (\Delta_\perp^2 A_\|[\varphi^*, A_\|] - \Delta_\perp A_\|[\Delta_\perp\varphi^*, A_\|])d\boldsymbol{x} \quad (5.5)$$
$$\int \Delta_\perp A_\| \sum_{i=x,y}[\partial_i\varphi^*, \partial_i\Delta_\perp A]d\boldsymbol{x} = \int \Delta_\perp^2 A[\varphi^*, \Delta_\perp A_\|]. \quad (5.6)$$

These identities are obtained by expanding $\Delta_\perp[\varphi^*, f]$ within the equality $\int \Delta_\perp A_\| \Delta_\perp[\varphi^*, f]d\boldsymbol{x} = \int \Delta_\perp^2 A_\|[\varphi^*, f]d\boldsymbol{x}$, where f holds for $A_\|$ or $\Delta_\perp A_\|$, and using the identity $\int f[g,h]d\boldsymbol{x} = \int h[f,g]d\boldsymbol{x}$. The energy reads

$$\mathcal{E}_{KAW} = \frac{1}{2}\int \frac{2}{\beta_e}\Big(|\boldsymbol{\nabla} A_\||^2 + \frac{2\delta^2}{\beta_e}(\Delta_\perp A_\|)^2 + \frac{2\delta^4}{\beta_e}|\boldsymbol{\nabla}\Delta_\perp A_\||^2\Big) + \Big(\frac{1}{\tau+1} + \frac{\beta_e}{2}\Big)\varphi^{*2}$$
$$+\frac{\tau}{1+\tau}\delta^2|\boldsymbol{\nabla}\varphi^*|^2 + \frac{T_{\| e}^2}{2} + \frac{2+\tau}{1+\tau}T_{\perp e}^2 + \frac{2}{1+\tau}T_{\perp e}(1+\frac{\delta^2}{2}\Delta_\perp)\varphi^* d\boldsymbol{x}. \quad (5.7)$$

Writing that, to leading order, $\varphi^* = -(T_{\perp e} + (1+\tau)n)$, it can be shown that the thermodynamic terms in Eq. (5.7) coincide with those of Eq. (3.39) of Tassi, Sulem & Passot (2016), once the various fields are transformed from the gyrofluid to the particle formulation.

The system of equations describing WWs dynamics is conveniently obtained by taking the limit $\tau \to \infty$.

5.2. *The isothermal case*

Isothermal electrons is a good approximation when one does not focus on dissipative effects, as long as $k_\| \ll k_\perp$ and $k_\perp \rho_e \ll 1$, as discussed in Schekochihin *et al.* (2009).

5.2.1. *KAWs reduced model*

Taking temperature fluctuations and heat fluxes equal to zero, one gets, in the small β_e regime,

$$\partial_t \left(1-\frac{2\delta^2}{\beta_e}\Delta_\perp+\frac{2\delta^4}{\beta_e}\Delta_\perp^2\right)A_\| + \nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\varphi^* + [\varphi^*, \delta^2\Delta_\perp A_\| + \frac{2\delta^4}{\beta_e}\Delta_\perp^2 A_\|]$$
$$-\frac{\tau}{\tau+1}[(1+\frac{\delta^2}{2}\Delta_\perp)\varphi^*, \frac{2\delta^2}{\beta_e}\Delta_\perp A_\|] - \delta^2\sum_{i=x,y}[\partial_i\varphi^*, (1-\frac{2\delta^2}{\beta_e}\Delta_\perp)\partial_i A_\|] = 0 \qquad (5.8)$$
$$\partial_t\left(\frac{\beta_e}{2}(1-\frac{2\delta^2}{\beta_e}\Delta_\perp)+\frac{1}{\tau+1}(1+\delta^2\Delta_\perp)\right)\varphi^* - \frac{2}{\beta_e}\nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\Delta_\perp A_\|$$
$$-\frac{\tau}{1+\tau}[\varphi^*, \delta^2\Delta_\perp\varphi^*] = 0, \qquad (5.9)$$

while at finite β_e and scales large compared to ρ_e, the system reduces to

$$\partial_t\left(1-\frac{2\delta^2}{\beta_e}\Delta_\perp\right)A_\| + \nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\varphi^* + [\varphi^*, \delta^2\Delta_\perp A_\|]$$
$$-\frac{\tau}{\tau+1}[\varphi^*, \frac{2\delta^2}{\beta_e}\Delta_\perp A_\|] - \delta^2\sum_{i=x,y}[\partial_i\varphi^*, \partial_i A_\|] = 0 \qquad (5.10)$$
$$\partial_t\left(\frac{\beta_e}{2}(1-\frac{2\delta^2}{\beta_e}\Delta_\perp)+\frac{1}{\tau+1}(1+\delta^2\Delta_\perp)\right)\varphi^* - \frac{2}{\beta_e}\nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\Delta_\perp A_\|$$
$$-(\frac{\tau}{1+\tau}-\frac{\beta_e}{2})[\varphi^*, \delta^2\Delta_\perp\varphi^*] = 0. \qquad (5.11)$$

In the small β_e regime, the energy, which reduces to

$$\mathcal{E}_{KAW} = \frac{1}{2}\int\Big(\frac{1}{\tau+1}+\frac{\beta_e}{2}\Big)\varphi^{*2}+\frac{\tau}{1+\tau}\delta^2|\boldsymbol{\nabla}\varphi^*|^2+\frac{2}{\beta_e}\Big(|\boldsymbol{\nabla}A_\||^2+\frac{2\delta^2}{\beta_e}(\Delta_\perp A_\|)^2+\frac{2\delta^4}{\beta_e}|\boldsymbol{\nabla}\Delta_\perp A|^2\Big)d\boldsymbol{x}, \qquad (5.12)$$

is conserved during the time evolution. In the case of finite β_e, the term proportional to δ^4 is absent in the definition of the energy.

5.2.2. *WWs reduced model*

Proceeding as above, we get in the case of small β_e

$$\partial_t \left(1-\frac{2\delta^2}{\beta_e}\Delta_\perp+\frac{2\delta^4}{\beta_e}\Delta_\perp^2\right)A_\| + \nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\varphi^* + [\varphi^*, \delta^2\Delta_\perp A_\| + \frac{2\delta^4}{\beta_e}\Delta_\perp^2 A_\|]$$
$$-[(1+\frac{\delta^2}{2}\Delta_\perp)\varphi^*, \frac{2\delta^2}{\beta_e}\Delta_\perp A_\|] - \delta^2\sum_{i=x,y}[\partial_i\varphi^*, (1-\frac{2\delta^2}{\beta_e}\Delta_\perp)\partial_i A_\|] = 0 \qquad (5.13)$$
$$\partial_t\left(1-\frac{2\delta^2}{\beta_e}\Delta_\perp\right)\varphi^* - \frac{4}{\beta_e^2}\nabla_\|(1-\frac{\delta^2}{2}\Delta_\perp)\Delta_\perp A_\| + \Big(1-\frac{2}{\beta_e}\Big)[\varphi^*, \delta^2\Delta_\perp\varphi^*] = 0. \qquad (5.14)$$

The corresponding energy reads

$$\mathcal{E}_{WW} = \frac{1}{2}\int \varphi^{*2} + \frac{2\delta^2}{\beta_e}|\boldsymbol{\nabla}\varphi^*|^2 + \frac{4}{\beta_e^2}\Big(|\boldsymbol{\nabla}A_\||^2 + \frac{2\delta^2}{\beta_e}(\Delta_\perp A_\|)^2 + \frac{2\delta^4}{\beta_e}|\boldsymbol{\nabla}\Delta_\perp A|^2\Big)d\boldsymbol{x}. \qquad (5.15)$$

In the finite β_e regime, the only change in the system of equations and in the corresponding energy consists in the absence of the δ^4 contributions.

Note that in the range $\rho_i \ll k_\perp \ll d_e$, where both electron inertia and FLR corrections are negligible, the equations for KAWs and for WWs formally identify, although φ^* refers in fact to different quantities, being proportional to the electron density in the former case and to the parallel magnetic field in the latter (Boldyrev *et al.* 2013). The resulting model is usually referred to as electron reduced magnetohydrodynamics (ERMHD) (see e.g. Schekochihin *et al.* (2009)). When electron inertia is retained, but not the FLR corrections, the equations for the WWs appear as an extension to the anisotropic three-dimensional regime of the 2.5D equations given in Biskamp *et al.* (1996, 1999).

5.3. *Landau-fluid closure*

In Eqs (5.1)-(5.4) for KAWs, which are valid both in the small β_e and the large-scale regimes (or in the equations for WWs which, as previously mentioned, are obtained by taking the limit $\tau \to \infty$), the heat fluxes and fourth-rank cumulants are still to be specified. A simple closure, aimed at capturing Landau damping, in a way consistent with linear kinetic theory, is provided by directly expressing the heat fluxes in terms of lower order fluctuations, as done in Eqs. (3.17)-(3.19) of Sulem & Passot (2015), which here reduce to

$$q_{\parallel e} = -2\alpha H T_{\parallel e} \tag{5.16}$$

$$q_{\perp e} = -\alpha H T_{\perp e} - \alpha\delta^2 H \omega_{ze} \tag{5.17}$$

where, at the considered scales, the vorticity term arising in Eq. (3.18) of the above reference has, in fact, to be also retained. Here, $\alpha = (\frac{2}{\pi})^{1/2}\delta^{-1}$ and the operator H denotes the negative Hilbert transform along the magnetic field lines (see Sulem & Passot (2015) for a discussion on its modelization). The fourth order cumulants are given by Eqs. (3.21) and (3.27) of Sulem & Passot (2015), which here take the form

$$\widetilde{r}_{\parallel\perp e} = -T_{\perp e} - \alpha\delta^2 H \omega_{ze} \tag{5.18}$$

$$\widetilde{r}_{\perp\perp e} = 0. \tag{5.19}$$

In contrast with the small-scale model discussed in Tassi *et al.* (2016), the Reduced Landau fluid (RLF) model discussed here includes an explicit closure of the fluid hierarchy and retains electron inertia together with FLR corrections.

Remark: The present RLF model can be extended to larger scales by relating φ and n, as in Zocco & Schekochihin (2011), by means of the more general relation provided by the gyrokinetic Poisson equation (Krommes 2002)

$$n = -\left(\frac{1 - \mathit{\Gamma}_0(\tau k_\perp^2)}{\tau}\right)\varphi. \tag{5.20}$$

Here $\mathit{\Gamma}_0(x) = e^{-x} I_0(x)$, where I_0 denotes the modified Bessel function of order 0. In this approach, the ion response is taken into account and the agreement with kinetic theory improved at values of $k_\perp$ close to unity, as discussed in Section 6. The model then appears as a Landau fluid version of the kinetic model of Zocco & Schekochihin (2011), where FLR corrections have been supplemented.

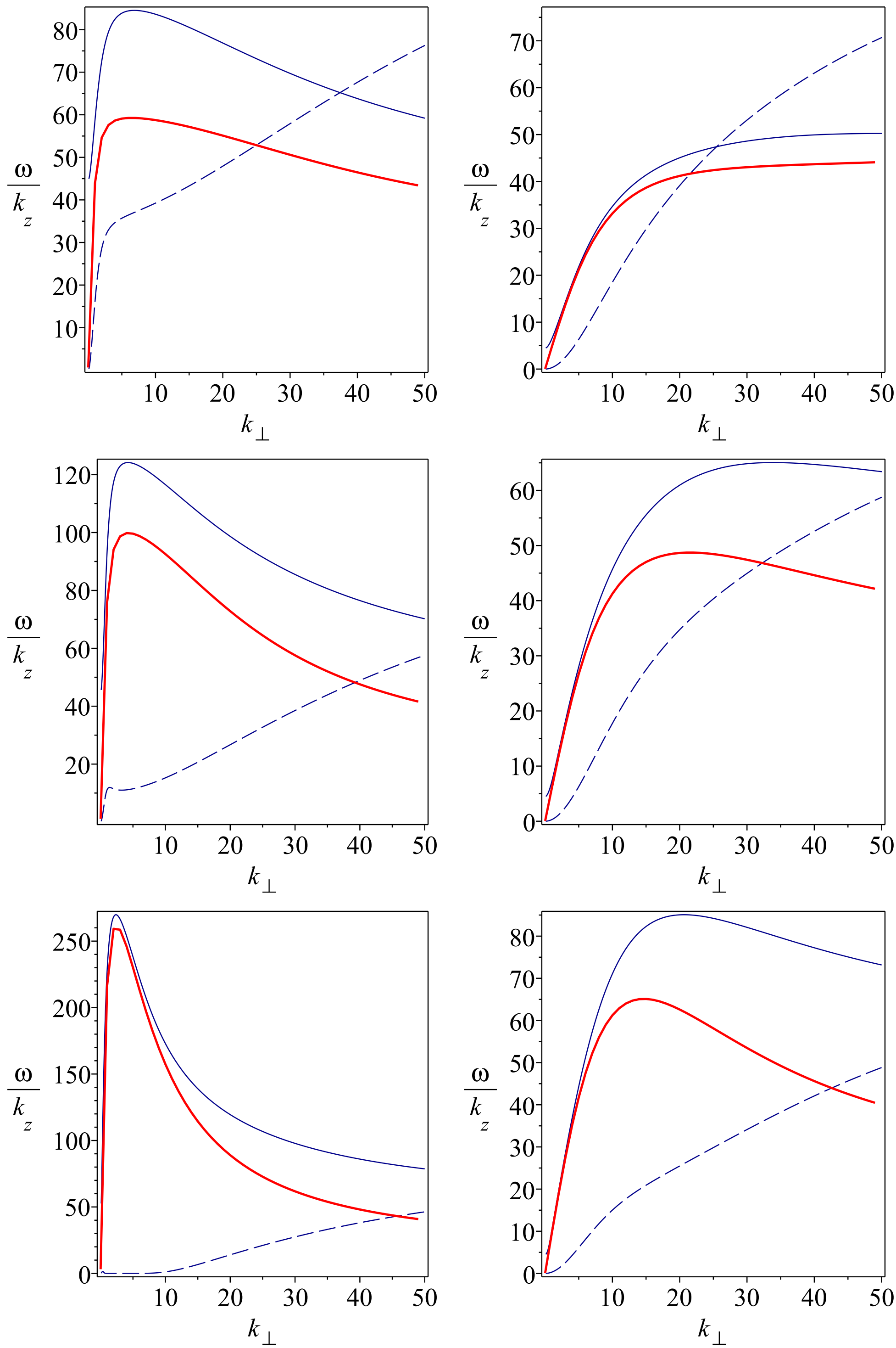

Figure 1. $\Re(\omega)/k_z$ (solid line) and $-\Im(\omega)/k_z$ (dashed line) from kinetic theory (in dark blue) and ω/k_z from the two-field model (thick red line) for KAWs at $\beta = 0.001$ (left) for $\tau = 1$ (top), $\tau = 5$ (middle) and $\tau = 50$ (bottom), and at $\beta = 0.1$ (right) for $\tau = 0.2$ (top), $\tau = 1$ (middle) and $\tau = 5$ (bottom).

6. Linear theory

6.1. *Isothermal regime*

Linearizing Eqs. (5.8)-(5.9) and (5.13)-(5.14), which correspond to the small β_e regime, we obtain the dispersion relation for the KAWs

$$\frac{\omega}{k_z} = \sqrt{\frac{2}{\beta_e}} \frac{(1+\frac{\delta^2 k_\perp^2}{2})k_\perp}{\left(1+\frac{2\delta^2 k_\perp^2}{\beta_e}+\frac{2\delta^4 k_\perp^4}{\beta_e}\right)^{1/2}\left(\frac{\beta_e}{2}(1+\frac{2\delta^2 k_\perp^2}{\beta_e})+\frac{1}{1+\tau}(1-\delta^2 k_\perp^2)\right)^{1/2}}, \tag{6.1}$$

and for the WWs

$$\frac{\omega}{k_z} = \frac{2}{\beta_e} \frac{(1+\frac{\delta^2 k_\perp^2}{2})k_\perp}{\left(1+\frac{2\delta^2 k_\perp^2}{\beta_e}+\frac{2\delta^4 k_\perp^4}{\beta_e}\right)^{1/2}\left(1+\frac{2\delta^2 k_\perp^2}{\beta_e}\right)^{1/2}}, \tag{6.2}$$

respectively, where ω, k_z and $k_\perp$ denote the frequency and the parallel and transverse wavenumbers. For both types of waves, the eigenvector is associated to the relation

$$\widehat{\varphi}^* = \frac{\omega}{k_z}\left(\frac{1+\frac{2\delta^2 k_\perp^2}{\beta_e}+\frac{2\delta^4 k_\perp^4}{\beta_e}}{1+\frac{\delta^2}{2}k_\perp^2}\right)\widehat{A_\parallel}, \tag{6.3}$$

where the hat symbol refers to Fourier modes. Note that, when $(1+\tau)\beta_e \gg 1$, the dispersion relation of KAWs and WWs coincide. In that case, when neglecting FLR corrections, the frequency of the waves saturates at a value $\omega_s = \dfrac{k_z}{k_\perp}\dfrac{1}{\delta^2}$, which, at large enough propagation angle $\theta_{\boldsymbol{kB}}$, identifies (in dimensional units) with $\Omega_e \cos\theta_{\boldsymbol{kB}}$. On the other hand, when $(1+\tau)\beta_e \leqslant 1$, KAWs obey, in the high frequency domain (and still neglecting FLR effects), $\omega/k_z \simeq (1+\tau)^{1/2}/\delta$.

It is of interest to compare the predictions of the models with those of the kinetic theory obtained using the WHAMP software (Rönnmark 1982). The first question concerns the domain of existence of the two types of waves. For the KAWs, a propagation angle $\theta_{\boldsymbol{kB}}$ close to 90° is required in order to prevent the occurrence of cyclotron resonance within the considered range of wavenumbers. Note that this condition is relaxed when $\beta_i = \tau\beta_e$ is large (typically a few units) (Sahraoui *et al.* 2012; Passot *et al.* 2012). For these large values of β_i, and at large angles of propagation, there is only one electromagnetic mode at scales smaller than the ion Larmor radius. It is a continuation of the shear Alfvén branch, and is named Alfvén-whistler mode because its frequency can greatly exceed the ion gyrofrequency (Sahraoui *et al.* 2012). For smaller β_i, KAWs and WWs are clearly distinct modes which appear to be the continuation of shear Alfvén and fast waves respectively. They furthermore exist in different domains of $\theta_{\boldsymbol{kB}}$ and β_e, whistler modes requiring a smaller angle of propagation (see also Boldyrev *et al.* (2013)). If one assumes that the frequency ω_W of WWs obeys $\omega_W > k_\perp v_{th\,i} > \Omega_i$, inserting the approximate (dimensional) formula $\omega_W \simeq \frac{\Omega_i}{\beta_e} k_z k_\perp \rho_s^2$, (valid in the range $kd_e < 1$), one finds that, for $k_\perp\rho_s \simeq 1$, one should have $k_z\rho_i \gtrsim \beta_i$, while for $k_\perp\rho_s = O(1/\mu) = \beta^{1/2}/\delta$, the condition is $k_z/k_\perp \gtrsim (\tau\beta_e)^{1/2}\delta$. On the other hand, taking for the KAWs, $\omega_K \simeq \Omega_i\beta_e^{-1/2}k_z k_\perp\rho_s^2$, the condition $\omega_K < \Omega_i$ taken for $k_\perp\rho_s \simeq \beta^{1/2}/\delta$ gives the condition $k_z/k_\perp \lesssim \delta^2\beta_e^{-1/2}$. In the following, we choose $\theta_{\boldsymbol{kB}} = 82°$ for the WWs and $\theta_{\boldsymbol{kB}} = 89.99°$ for the KAWs.

Figure 1 displays the ratios $\Re(\omega)/k_z$ (in dark-blue solid lines) and $-\Im(\omega)/k_z$ (in dark-blue dashed lines) obtained from the linear kinetic theory for KAWs at $\beta_e = 0.001$ for $\tau = 1$, $\tau = 5$ and $\tau = 50$ (left) and at $\beta_e = 0.1$ for $\tau = 0.2$, $\tau = 1$ and $\tau = 5$ (right). Superimposed in thick red solid lines are the corresponding ratios ω/k_z of the isothermal dispersion relation given by Eq. (6.1). At small β_e, the agreement between kinetic theory and the isothermal model is better for large values of τ, in part due to a smaller damping rate. A much faster decrease of the ratio $\Re(\omega)/k_z$ is observed for $\tau = 50$ and the behavior is indeed close to that of WWs (see Fig. 2, right), as predicted by inspection of the dispersion relations. For smaller values of τ and/or larger values of β_e, the damping becomes quite strong when $k_\perp$ reaches a few units.

Turning to the WWs, it is of interest to first briefly discuss their properties when varying angles and β_e. As mentioned previously, for small enough β_i, WWs can be found as a continuation to small scales of the fast mode. At a sufficiently small angle of propagation, e.g. 60°, they do not encounter any resonance, even at β_i of order unity, but as the angle and/or β_i increases, branches of Bernstein modes cross the whistler branch, that nevertheless remains continuous throughout the considered range of wavenumbers, if $\theta_{\boldsymbol{kB}}$ and β_i remain below certain thresholds. This is illustrated in Fig. 2 (left), which displays the frequencies (solid) and damping rates (dashed) at $\beta_e = 0.01$, $\tau = 8$ and $\theta_{\boldsymbol{kB}} = 82°$ for an Alfvén wave (red), a fast wave ending in the first Bernstein mode (blue) and in the second one (brown), and for the whistler wave (green) continuing to small scale (as a red dotted line in the right panel). Choosing $\theta_{\boldsymbol{kB}} = 82°$ and $\beta_e = 0.01$, we display in the right panel of Fig. 2, $\Re(\omega)/k_z$ (solid lines) and $-\Im(\omega)/k_z$ (dashed lines) for WWs at $\tau = 0.2$ (dark blue), $\tau = 1$ (brown), $\tau = 5$ (green) and $\tau = 8$ (red dots), superimposed with the prediction of linear theory (Eq. (6.2)) (thick black line). As τ increases, the kinetic results converge to the same curve which is very close to the prediction of the model, as long as dissipation remains small (i.e. for $k_\perp \leqslant 10$). A very similar behavior is observed for a 75° propagation angle.

Left panel of Fig. 3 displays the electron density fluctuation of the eigenmode, obtained from the kinetic theory for KAWs (red line) at $\theta_{\boldsymbol{kB}} = 89.99°$ and WWs (dark blue line) at $\theta_{\boldsymbol{kB}} = 82°$ for $\beta = 0.01$ and $\tau = 5$. This result, which remains true at larger values of β_e, confirms the assumption that was used to distinguish the two waves, namely that the WWs are associated to almost incompressible motions.

Finally, the right panel of Fig. 3 displays the case of a smaller β_e for KAWs, a situation where the behavior at small $k_\perp$ is now quite sensitive to the ion response. We display $\Re(\omega)/k_z$ for $\beta_e = 10^{-4}$ and $\tau = 1$ for the kinetic theory (dark blue solid line) and different models. The prediction of Eq. (6.1), which is displayed with a thick red solid line, shows a strong disagreement for $k_\perp < 5$. The agreement with kinetic theory is however much better at these large scales when using the dispersion relation of the model of Zocco & Schekochihin (2011) taken in the isothermal limit (black dashed line). Note that the different curvature of the dispersion relation, observed at small $k_\perp$ when β_e crosses m_e/m_i, is associated to the well-known transition from KAWs to inertial Alfvén waves. Interestingly, when modified by using Eq. (5.20) (green dots), our model reproduces for ω/k_z both the qualitative behavior at large scales and the decrease at small scales, in agreement with kinetic theory, although with a different absolute level due to the existence of a strong Landau damping.

6.2. *Reduced Landau-fluid model*

We shall here restrict the discussion to the impact of Landau damping on the dispersion relation for KAWs. Equations (5.1)-(5.4) supplemented by Eqs. (5.16)-(5.19), are linearized, and the dispersion relation computed using MAPLE software. Figure 4 displays $\Re(\omega)/k_z$ (solid lines) and $-\Im(\omega)/k_z$ (dashed lines) for KAWs at $\tau = 1$, in both cases $\beta_e = 1$ (left) and $\beta_e = 0.01$ (right), for different reduced models, with the kinetic theory prediction superimposed in dark blue. The green and thick red lines correspond to the isothermal and RLF models respectively. A clear extension of the spectral validity range

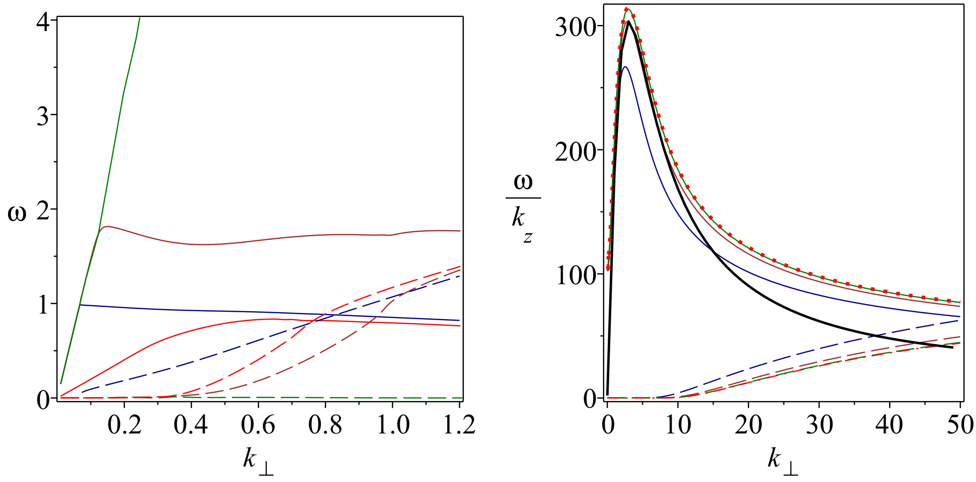

FIGURE 2. Left: frequencies (solid) and damping rates (dashed) at $\beta_e = 0.01$, $\tau = 8$ and $\theta_{\boldsymbol{kB}} = 82°$ for an Alfvén wave (red), a fast wave ending in the first Bernstein mode (blue) and in the second one (brown) and for the whistler wave (green) continuing to small scale (see right in red dotted line). Right: $\Re(\omega)/k_z$ (solid lines) and $-\Im(\omega)/k_z$ (dashed lines) from kinetic theory for whistlers at $\beta_e = 0.01$ for $\tau = 0.2$ (dark blue), $\tau = 1$ (brown), $\tau = 5$ (green) and $\tau = 8$ (red dotted) and ratio ω/k_z from the two-field model (thick black line).

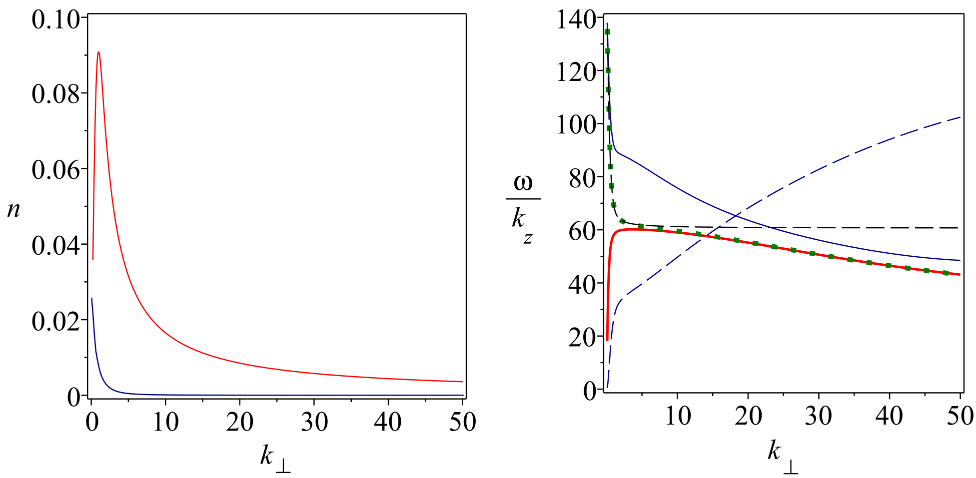

FIGURE 3. Left: electron density fluctuations for whistlers at $\theta_{\boldsymbol{kB}} = 82°$ (dark blue) and KAWs at $\theta_{\boldsymbol{kB}} = 89.99°$ (red) for $\beta_e = 0.01$ and $\tau = 5$. Right: $\Re(\omega)/k_z$ for a KAW at $\beta_e = 0.0001$ and $\tau = 1$, from kinetic theory (in dark blue) and frequency from various two-field models: from Eq. (6.1) (thick red line), from the model of Zocco & Schekochihin (2011) taken in the isothermal limit (dashed black line), and from an extension of the current model using the improved relation between n and φ given by Eq. (5.20) (green dotted line). The dark blue dashed line displays $-\Im(\omega)/k_z$.

is obtained when Landau damping is retained. The damping rate is however globally weaker than predicted by the kinetic theory, except at the largest scales. Noticeably for $\beta_e = 1$, the comparison with kinetic theory for the real part is satisfactory up to $k = 25$ (very close to the inverse electron Larmor radius). In the case where electron inertia and FLRs are not included (violet line), a regime where no saturation is expected,

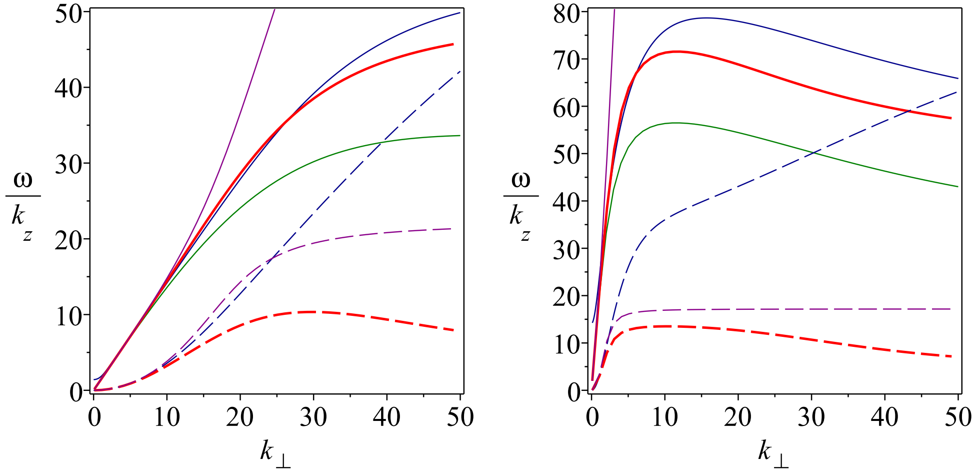

FIGURE 4. $\Re(\omega)/k_z$ (solid lines) and $-\Im(\omega)/k_z$ (dashed lines) for KAWs at $\tau = 1$ and $\beta_e = 1$ (left) or $\beta_e = 0.01$ (right) from kinetic theory (dark blue), the RLF model (thick red lines), the isothermal two-field model (green) and the RLF model with $\delta = 0$ (violet).

the disagreement starts around $k = 10$. This latter model, which can be viewed as an extension of the model of Boldyrev *et al.* (2013) including Landau damping, was originally derived in Tassi *et al.* (2016) (where the heat fluxes were estimated directly from the linear kinetic theory, instead of using the above approximate closure). For $\beta_e = 0.01$, globally similar graphs are obtained except that, saturation occurring at smaller wavenumbers, the domain where the kinetic theory is accurately reproduced is more limited.

7. Turbulent regimes

In this section, we concentrate on the isothermal models, considered in the spectral ranges $\mathcal{D}_e^< = \{1/\rho_s \ll k_\perp \ll 1/d_e\}$ and $\mathcal{D}_e^> = \{1/d_e \ll k_\perp \ll 1/\rho_e\}$ separately. While in the former domain, where $\delta^2 k_\perp^2/\beta_e \ll 1$, electron inertia is negligible, this effect is dominant in the latter, where $\delta^2 k_\perp^2/\beta_e \gg 1$. In both ranges, FLR effects are subdominant corrections that we will thus neglect. The energy becomes

$$\mathcal{E}_{KAW} \propto \frac{1}{2}\int \left(\frac{1}{1+\tau}\varphi^{*2} + \frac{2}{\beta_e}|\boldsymbol{\nabla}_\perp A_\|\|^2 + \frac{4\delta^2}{\beta_e^2}(\Delta_\perp A_\|)^2 \right) d\boldsymbol{x}, \tag{7.1}$$

for the KAWs, and

$$\mathcal{E}_{WW} \propto \frac{1}{2}\int \left(\varphi^{*2} + \frac{2\delta^2}{\beta_e}|\boldsymbol{\nabla}_\perp \varphi^*|^2 + \frac{4}{\beta_e^2}|\boldsymbol{\nabla}_\perp A_\|\|^2 + \frac{8\delta^2}{\beta_e^3}(\Delta_\perp A_\|)^2 \right) d\boldsymbol{x}, \tag{7.2}$$

for the WWs. In more physically explicit terms, we can write

$$\mathcal{E}_{KAW} \propto \frac{1}{2}\int \left(\frac{2}{\beta_e}|\boldsymbol{B}_\perp|^2 + (\tau+1)n^2 + \delta^2 u_{\|e}^2 \right) d\boldsymbol{x} \tag{7.3}$$

and

$$\mathcal{E}_{WW} \propto \frac{1}{2}\int \left(\frac{2}{\beta_e}|\boldsymbol{B}|^2 + \delta^2|\boldsymbol{u}_e|^2 \right) d\boldsymbol{x}. \tag{7.4}$$

Note that these energies are dominated by the transverse magnetic energy at the largest scales and by the parallel kinetic energy at the smallest ones. In a turbulent regime, the energy is expected to cascade to small transverse scales, resulting in an inertial range characterized by a scale-independent energy flux ϵ. The aim of this section is to phenomenologically evaluate the scaling properties of the transverse magnetic energy spectrum $E_{B_\perp}(k_\perp)$ in such a cascade, in both regimes of strong and weak turbulence, assumed to exist and to result from three-wave interactions.

The energy flux ϵ can be estimated as the energy carried by the modes of wavenumber $k_\perp$ (or equivalently, when assuming transverse isotropy, by $k_\perp$ times the one-dimensional spectral density), divided by the characteristic transfer time τ_{tr} at this wavenumber. The regimes of strong and weak turbulence differ in the estimate of this time that, in both cases, can nevertheless be written $\tau_{tr} = \tau_{NL}^2/\tau_L$, where τ_{NL} refers to the characteristic time of the nonlinear interactions at the corresponding scale, and $\tau_L \sim 1/\omega$ to the inverse frequency of the considered wave at this scale.

In the strong-turbulence regime, where a so-called critical balance holds, the rate of the nonlinear interaction at a given scale is comparable to the frequency of the linear wave at this scale ($\tau_{NL} \sim \tau_L$), leading to identify τ_{tr} with τ_{NL}. In this approach, τ_{NL} is defined as the shortest of the characteristic times associated with the various nonlinear couplings, which turns out to be ge given by $\tau_{NL}^{-1} \sim [\widehat{\varphi}^*, \cdot]$ associated with the transverse strain. Note that the same estimate of the transfer time holds in the absence of waves. We are thus led to write $1/\tau_{NL} \sim k_\perp^2 \widehat{\varphi}^*$. An additional element used to relate the magnitude of the various fields is provided by the condition that the considered solution is an eigenmode of the linear problem, thus obeying Eq. (6.3). In both spectral ranges defined above, one can write the phase velocity in the asymptotic form $\omega/k_\| \sim k_\perp^\alpha$. In $\mathcal{D}_e^<$, one thus has $\widehat{\varphi}^* \sim (\omega/k_\|)\widehat{A}_\|$, leading to $\tau_{NL}^{-1} \sim k_\perp^{1+\alpha}\widehat{B}_\perp$, while in $\mathcal{D}_e^>$, $\widehat{\varphi}^* \sim (\omega/k_\|)k_\perp^2\widehat{A}_\|$, leading to $\tau_{NL}^{-1} \sim k_\perp^{3+\alpha}\widehat{B}_\perp$.

In the strong turbulent regime, it follows that in $\mathcal{D}_e^<$, where $\epsilon \sim k_\perp^{1+\alpha}\widehat{B}_\perp^3$, one gets $E_{B_\perp}(k) \sim \epsilon^{2/3}k_\perp^{-(5+2\alpha)/3}$, while in $\mathcal{D}_e^>$, where $\epsilon \sim k_\perp^{5+\alpha}\widehat{B}_\perp^3$, one has $E_{B_\perp}(k) \sim \epsilon^{2/3}k_\perp^{-(13+2\alpha)/3}$.

In the weak turbulent regime, one easily obtains that in $\mathcal{D}_e^<$, $\tau_{tr}^{-1} \sim k_\perp^{(2+\alpha)}B_\perp^2$, while in $\mathcal{D}_e^>$, $\tau_{tr} \sim k_\perp^{(6+\alpha)}B_\perp^2$. It follows that in $\mathcal{D}_e^<$, $E_{B_\perp}(k) \sim \epsilon^{1/2}k_\perp^{-(2+\alpha/2)}$, while in $\mathcal{D}_e^>$, $E_{B_\perp}(k) \sim \epsilon^{1/2}k_\perp^{-(5+\alpha/2)}$. Let us now consider more specifically the cases of KAWs and WWs.

In $\mathcal{D}_e^<$, the dispersion relation is the same for KAWs and WWs, with $\alpha = 1$. As a consequence, for both types of waves, $E_{B_\perp}(k_\perp) \sim \epsilon^{2/3}k_\perp^{-7/3}$ in the strong turbulent regime and $E_{B_\perp}(k_\perp) \sim \epsilon^{1/2}k_\perp^{-5/2}$ in the weak turbulent regime, a result obtained as the finite flux solution of the weak-turbulence spectral equations in Galtier & Bhattacharjee (2003).

In $\mathcal{D}_e^>$, different cases are to be distinguished. For the KAWs, when the ion-electron temperature ratio τ is moderate, $\alpha = 0$. One thus gets $E_{B_\perp}(k_\perp) \sim \epsilon^{2/3}k_\perp^{-13/3}$ in the strong turbulent regime and $E_{B_\perp}(k_\perp) \sim \epsilon^{1/2}k_\perp^{-5}$ in the weak turbulent regime. Note that a $-13/3$ exponent for the magnetic field spectra has also been reported in the different context of a generalized helicity cascade in incompressible so-called extended MHD (Krishan & Mahajan 2004; Abdelhamid *et al.* 2016). Differently, for the WWs, but also for the KAWs when the ion-electron temperature ratio is large ($\tau \gg 1$), $\alpha = -1$. It follows that in this case $E_{B_\perp}(k_\perp) \sim \epsilon^{2/3}k_\perp^{-11/3}$ in the strong turbulence regime and $E_{B_\perp}(k_\perp) \sim \epsilon^{1/2}k_\perp^{-9/2}$ in the weak turbulence regime.

Numerical simulations of incompressible bi-fluid equations in 2.5D have indeed shown a transition from a $-7/3$ spectrum to a steeper slope (close to $-11/3$) at scales smaller than

d_e (Andrés *et al.* 2014). Three-dimensional particle-in-cell simulations of WW turbulence also clearly indicate the development of two sub-ion spectral ranges, but the exponents are steeper than those predicted by the above phenomenology, and non-universal (Chang *et al.* 2011; Gary *et al.* 2012). Kinetic effects, such as electron Landau damping could be at the origin of this discrepancy. This issue could be addressed using the RLF model discussed in Section 5.3), both numerically and using a phenomenological approach as in Passot & Sulem (2015).

It is of interest to consider the effect of a finite value of β_e on the energy spectrum. Considering the case of KAWs, Fig. 3 (right) shows that taking $\beta_e = 10^{-4}$ causes $\omega/k_\|$ to decrease with $k_\perp$ (red solid and green dotted lines), while it tends to a constant when $\beta_e = 0$ (black dashed line). As the spectrum can be written in the form $k_\perp^{-13/3}(\omega/k_\|)^{-2/3}$, this suggests that a finite β_e will tend to make the spectrum shallower.

8. Conclusion

Reduced fluid models have been derived for the sub-ion scale dynamics of collisionless plasmas, retaining electron inertia and leading-order electron FLR corrections. Neglecting the ion dynamics, we are led to discriminate between two limiting cases: a low-frequency regime involving perpendicular pressure balance and a high-frequency one, where the electron fluid is essentially incompressible. The two resulting models capture KAWs and WWs respectively, in the context of isothermal or Landau-fluid closures. They extend the validity range of previously existing models. At small β_e and scales large compared to d_e, they both reduce to ERMHD (Schekochihin *et al.* 2009; Boldyrev *et al.* 2013) where the same equations govern different fields depending on the kind of waves. Furthermore, at scales comparable to d_e, in 2.5D dimensions (i.e. three-dimensional velocities and magnetic fields but two-dimensional space coordinates) and for β_e small enough for electron FLR corrections to be negligible (i.e. $\beta_e = O(m_e/m_i)$), the system for WWs reproduces EMHD (see e.g. Eqs. (10)-(11) of Biskamp *et al.* (1999)). Differently, in the same β_e-range, the equations for KAWs with the Landau-fluid closure appear as a fluid reduction of those of Zocco & Schekochihin (2011) (see Loureiro *et al.* (2013) for their numerical simulations in the case of fast collisionless reconnection). While the latter model is based on a drift kinetic description of the electrons, the present RLF model includes first-order FLR corrections. Such terms, together with the contribution of parallel magnetic fluctuations, which are both $O(\beta_e)$, induce a sensitivity of the system to pressure anisotropy, an effect potentially important in a reconnection context (Lee *et al.* 2016).

Computation of the electron gyroviscous force $\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e$ is performed within an asymptotic expansion based on temporal and spatial scale separation between the electron gyroradius and the considered quasi-transverse scales, a procedure which, at scales comparable to d_e, results in an expansion in terms of β_e. The computation involves a recursive process involving both the non-gyrotropic pressure and heat flux tensors, two iterations being needed in order to obtain the leading order in β_e. Because of the algebraic complexity in the general framework (see e.g. Ramos (2005*a*)), we resorted in the present paper to assume a gyrotropic heat flux in the expression of the gyro-viscous force. Relations with previous FLR estimates and implementation of the gyroviscous cancellation is discussed in Appendices B and C. Note that including the gyroviscous force in a generalized Ohm's law may in particular be useful for enriching hybrid models in the context of collisionless reconnection.

Both isothermal and Landau-fluid closures are considered for KAWs and WWs. The linear regime is examined in comparison with the fully kinetic theory to evaluate the validity of these closures. In the isothermal case, a qualitative agreement is found, which turns out to be more accurate when Landau damping is weak. For the propagation angles exceeding 80° that we have considered, WWs are more weakly damped than KAWs, at least at scales larger than ρ_e. As expected, for the values of $\beta_e \lesssim 0.1$ we have considered, the range of angles where KAWs can propagate without resonance turns out to be restricted to quasi-perpendicular angles. The Landau-fluid closure improves the model accuracy. In particular, when β_e is pushed to values of order unity, the range of wavenumbers that are accurately described is significantly enlarged.

Observations in the solar wind and the magnetosheath provide evidence of power law magnetic energy spectra at scales smaller than d_e, possibly associated with turbulent cascades. Assuming that KAWs and WWs are still present at such scales, we explored the spectrum of the transverse magnetic fluctuations in both strong and weak turbulence energy cascades. As a first step, a phenomenological approach where FLR corrections are neglected is presented. In addition to the well-known WWs magnetic spectra both above and below d_e^{-1}, as well as the $-7/3$ sub-ion KAWs spectrum, a new regime of KAW turbulence is obtained at scales smaller than d_e, characterized by a $-13/3$ exponent for strong turbulence and -5 in the weak regime. These exponents are to be compared with satellite observational data that display slopes steeper than the WWs $-11/3$ spectrum. Note that other physical effects such as Landau damping (Passot & Sulem 2015; Sulem *et al.* 2016)) and intermittency corrections associated with coherent structures, such as current sheets (Boldyrev *et al.* 2013), can also lead to steeper spectra. It should be stressed that both phenomenological arguments and numerical simulations predict a transition at d_e, while observational spectra in the terrestrial magnetosheath have been reported (Huang *et al.* 2014) where the transition takes place at ρ_e. This could suggest that additional effects could play a role, including other kind of waves or the presence of structures.

As shown in Section 5, our four-field model, as well as its two-field isothermal reductions, conserve energy. To the best of our knowledge, this is the first example of reduced fluid model for inertial reconnection conserving energy and accounting for electron FLR effects. A question we intend to address in the future concerns the Hamiltonian structure of this model. Previous results concerning the inclusion of (ion) FLR effects in Hamiltonian reduced fluid models were presented in Morrison *et al.* (1984); Hazeltine *et al.* (1987); Dagnelund & Pavlenko (2005); Izacard *et al.* (2011). It could in particular be of interest to investigate whether the idea of the gyromap transformation, adopted in Morrison *et al.* (1984); Hazeltine *et al.* (1987); Izacard *et al.* (2011), together with the Hamiltonian structure of the model in the absence of FLR contributions, can help in building a Hamiltonian model including electron FLR effects. Further developments should also include numerical simulations to test the predictions for the turbulent energy spectra, and in particular to evaluate the role of coherent structures and analyze the difference between two and three-dimensional geometries. Existence and stability of Alfvén or electron vortices (Mikhalovskii *et al.* 1987; Schep *et al.* 1994) are also open questions. Other open issues concern the relative importance of KAWs and WWs in space plasmas. Such questions cannot be addressed using gyrokinetic simulations or gyrofluid models which, involving a perpendicular pressure balance, concentrate on low-frequency waves. Furthermore, the models for the WWs and the KAWs considered in this paper differentiate at the level of the determination of the magnitude of the density fluctuations. When the latter are neglected, the system can be viewed as an extension of EMHD for WWs to smaller scales, while when pressure balance is prescribed, it describes KAWs dynamics. It would be of great interest to derive a reduced fluid model able to simultaneously capture the two types of waves. In addition to turbulence applications discussed in this paper, an important issue that the present models can address concerns collisionless magnetic reconnection, in particular in three dimensions where fully kinetic simulations require huge computational resources.

Appendix A. FLR pressure tensor

As derived in Schekochihin *et al.* (2010), an exact equation for the electron pressure tensor $\boldsymbol{P}_e$ reads, when neglecting collisions,

$$P_{e,ij} = p_{\perp,e}\delta_{ij} + (p_{\perp e} - p_{\parallel e})\widehat{b}_i\widehat{b}_j + \delta^2 \frac{M_{ijkl}}{4B}\Big[\frac{D^{(e)}}{Dt}P_{e,kl} + \partial_m Q_{e,mkl} + (\delta_{mn}P_{e,kl} + \delta_{kn}P_{e,ml} + \delta_{ln}P_{e,mk})\,\partial_m u_{e,n}\Big]. \tag{A 1}$$

In this equation, $\boldsymbol{Q}_e$ denotes the heat flux tensor and

$$M_{ijkl} = \left(\delta_{ik} + 3\widehat{b}_i\widehat{b}_k\right)\epsilon_{jln}\widehat{b}_n + \epsilon_{iln}\widehat{b}_n\left(\delta_{jk} + 3\widehat{b}_j\widehat{b}_k\right). \tag{A 2}$$

Equation (A 1) can be solved recursively, using as small parameters $\delta^2\omega$ and δ^2k^2, corresponding in the dimensional variables to the conditions that the considered scales k^{-1} be large and the frequencies ω small compared to the electron Larmor radius and cyclotron frequency respectively. For β_e of order unity, this also prescribes that scales must be large compared to d_e. In this regime, the appropriate scaling is that given by Eqs. (3.13) of Tassi *et al.* (2016). Differently, for small values of β_e, scales comparable to d_e can be considered, the appropriate scaling being that given in Section 2. Keeping the leading order corrections in β_e nevertheless requires in this case to also retain the second order relatively to the scale expansion, as shown below. In the following, we shall concentrate on the small β_e regime, but the resulting gyroviscous force remains valid at $\beta_e = O(1)$ (recalling that, in the present units, k must remain small compared to $(\frac{m_i}{2m_e})^{1/2} \approx 30$), although it then includes subdominant contributions.

The iteration mentioned above involves in fact the coupling with another equation for the heat flux tensor (not written here). Solving the coupled system proves to be quite involved and falls outside the scope of the present paper. A linear version can be found in Goswami *et al.* (2005). Here, for the sake of simplicity, we resorted to only retain the contribution of the gyrotropic part of $\boldsymbol{Q}_e$ in the evaluation of the gyroviscous stress.

A.1. *First order contributions*

At zeroth order, the pressure tensor is simply given by its gyrotropic expression $\boldsymbol{P}_e^G = p_{\perp e}\boldsymbol{I} + (p_{\perp e} - p_{\parallel e})\boldsymbol{\tau}$. At first order, replacing $\boldsymbol{P}_e$ by $\boldsymbol{P}_e^G$ in the rhs of (A 1), we easily get for the gyroviscous tensor

$$\boldsymbol{\Pi}_e^{(1)} = -\frac{\delta^2}{4B}\Big[\widehat{\boldsymbol{b}}\times\boldsymbol{W}\cdot(\boldsymbol{I} + 3\,\boldsymbol{\tau}) - (\boldsymbol{I} + 3\,\boldsymbol{\tau})\cdot\boldsymbol{W}\times\widehat{\boldsymbol{b}}\Big] - \frac{\delta^2}{B}\Big[\widehat{\boldsymbol{b}}\otimes(\boldsymbol{w}\times\widehat{\boldsymbol{b}}) + (\boldsymbol{w}\times\widehat{\boldsymbol{b}})\otimes\widehat{\boldsymbol{b}}\Big].$$

Here,

$$\boldsymbol{W} = \Big[p_{\perp e}\boldsymbol{\nabla}\boldsymbol{u}_e + \boldsymbol{\nabla}(q_{\perp e}\,\widehat{\boldsymbol{b}})\Big]^S \tag{A 3}$$

and

$$\boldsymbol{w} = (p_{\perp e} - p_{\parallel e})\Big(\frac{d\widehat{\boldsymbol{b}}}{dt} + \nabla_\parallel\boldsymbol{u}_e\Big) + (3q_{\perp e} - q_{\parallel e})\nabla_\parallel\widehat{\boldsymbol{b}}, \tag{A 4}$$

where, for a given tensor $\boldsymbol{T}$, the notation $\boldsymbol{T}^S$ denotes the sum of the tensor with the ones obtained by circular permutation of the indices. Assuming an equilibrium state with isotropic temperatures, the term w is of order $O(\varepsilon^3)$ in the present asymptotics and can thus be neglected. We thus write $\boldsymbol{\Pi}_e^{(1)} = \boldsymbol{\Pi}_e^{(1,u)} + \boldsymbol{\Pi}_e^{(1,q)}$ where

$$\boldsymbol{\Pi}_e^{(1,u)} = -\frac{\delta^2}{4B}\Big[\widehat{\boldsymbol{b}}\times\left[p_{\perp e}\boldsymbol{\nabla}\boldsymbol{u}_e\right]^S\boldsymbol{\cdot}(\boldsymbol{I}+3\,\boldsymbol{\tau}) - (\boldsymbol{I}+3\,\boldsymbol{\tau})\boldsymbol{\cdot}\left[p_{\perp e}\boldsymbol{\nabla}\boldsymbol{u}_e\right]^S\times\widehat{\boldsymbol{b}}\Big], \tag{A 5}$$

and where $\boldsymbol{\Pi}_e^{(1,q)}$ and all the quantities derived from it, can obtained from the formula involving $\boldsymbol{\Pi}_e^{(1,u)}$ after the replacements $p_{\perp e}$ by 1 and $\boldsymbol{u}_e$ by $q_{\perp e}\widehat{\boldsymbol{b}}$. Equation (A 5) identifies with Eq. (C1) of Ramos (2005*b*), up to the minus sign due to the electron charge.

A.1.1. *Velocity contributions*

Carrying out calculations similar to those performed in Ramos (2005*b*), we obtain, without neglecting any possibly subdominant contribution,

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}_e^{(1,u)} = &\overbrace{-\delta^2\left[\boldsymbol{\nabla}\times\left(\frac{p_{\perp e}}{B}\widehat{\boldsymbol{b}}\right)\boldsymbol{\cdot}\boldsymbol{\nabla}\right]\boldsymbol{u}_e}^{①} + \overbrace{\frac{\delta^2}{2}\boldsymbol{\nabla}\left(\frac{p_{\perp e}\omega_{ze}}{B}\right)}^{②}\\ &\overbrace{+\delta^2\boldsymbol{\nabla}\times\left(\frac{p_{\perp e}}{B}\nabla_\parallel\boldsymbol{u}_e\right)}^{③}\overbrace{+\frac{\delta^2}{2}\boldsymbol{\nabla}\times\left((\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}_e)\widehat{\boldsymbol{b}}\right)}^{④}\overbrace{-\frac{3\delta^2}{2}\boldsymbol{\nabla}\times\left(\widehat{\boldsymbol{b}}\boldsymbol{\cdot}(\nabla_\parallel\boldsymbol{u}_e)\,\widehat{\boldsymbol{b}}\right)}^{⑤}\\ &\overbrace{-3\delta^2B\nabla_\parallel\left(\frac{p_{\perp e}}{B^2}\widehat{\boldsymbol{b}}\times\nabla_\parallel\boldsymbol{u}_e\right)}^{⑥}\overbrace{-\frac{3\delta^2}{2}B\nabla_\parallel\left(\frac{p_{\perp e}}{B^2}\omega_{ze}\widehat{\boldsymbol{b}}\right)}^{⑦}\overbrace{+\delta^2B\nabla_\parallel\left(\frac{p_{\perp e}}{B^2}\boldsymbol{\omega}_e\right)}^{⑧},\end{aligned} \tag{A 6}$$

where $\boldsymbol{\omega}_e = \boldsymbol{\nabla}\times\boldsymbol{u}_e$.

Let us first calculate $\widehat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\Pi_e^{(1,u)}$, which enters Eq. (2.17) where all the terms scale as $\delta\varepsilon^2$ for the KAWs or $\beta_e^{1/2}\delta\varepsilon^2$ for the WWs (when the scaling parameter μ is replaced by $\delta/\beta_e^{1/2}$). Thus, in Eq. (A 6), only contributions of these orders, possibly up to factors of order $O(\beta_e)$ (which is assumed to be only "moderately" small) are to be kept. It follows that the contributions of the terms ④ - ⑥, which scale like ε^3 (since in particular $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}_e$ is of second order in ε for the KAWs and even smaller for the WWs), are negligible. The relevant contributions are: from term ①, $\delta^2[p_{\perp e}-B_z, u_{\parallel e}]$; from term ②, $(\delta^2/2)\nabla_\parallel\omega_{ze}$; from term ③, $\delta^2\widehat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\nabla_\parallel\boldsymbol{u}_{\perp e}$; from term ⑦, $-(3\delta^2/2)\nabla_\parallel\omega_{ze}$; from term ⑧, $\delta^2\nabla_\parallel\omega_{ze}$. We thus obtain

$$\widehat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}_e^{(1,u)} = \delta^2\Big\{[p_{\perp e}-B_z, u_{\parallel e}] + \widehat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\nabla_\parallel\boldsymbol{u}_{\perp e}\Big\}. \tag{A 7}$$

Note that, in terms of φ^*, we can write

$$\widehat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\nabla_\parallel\boldsymbol{u}_{\perp e} = \nabla_\parallel\Delta_\perp\varphi^* + \sum_{i=x,y}[\partial_i\varphi^*, \partial_i A_\parallel]. \tag{A 8}$$

Furthermore, the term $\delta^2[p_{\perp e}-B_z, u_{\parallel e}]$, which participates to the so-called gyroviscous cancellation is $O(\delta\varepsilon^2)$ for the KAWs, thus larger than all the other contributions, which are $O(\beta_e\delta\varepsilon^2)$. Differently, for the WWs, all the terms are $O(\beta^{3/2}\delta\varepsilon^2)$.

It is also easy to obtain the transverse component of $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}_e^{(1,u)}$, which reads

$$\begin{aligned}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}_e^{(1,u)})_\perp = \ &\delta^2\Big\{[p_{\perp e}-B_z, \boldsymbol{u}_{\perp e}] + \frac{1}{2}\boldsymbol{\nabla}_\perp\left(\frac{p_{\perp e}}{B}\omega_{ze}\right)\\ &+\frac{1}{2}\boldsymbol{\nabla}_\perp\times\left[\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}_e - \nabla_\parallel u_{\parallel e}\right)\widehat{\boldsymbol{z}}\right] + \nabla_\parallel\boldsymbol{\nabla}\times\left(u_{\parallel e}\widehat{\boldsymbol{z}}\right)\Big\},\end{aligned} \tag{A 9}$$

where an expression involving a bracket with a scalar as one argument and a vector as the other one stands for the vector whose components are obtained as the brackets of the scalar with each components of the vector.

The dominant contribution to $(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(1,u)})_\perp$, which enters Eq. (2.9), originates from the second term in the rhs and writes $(\delta^2/2)\boldsymbol{\nabla}\omega_{ze}$. It is $O(\beta_e\varepsilon)$ for both types of waves. For the KAWs, the other terms are $O(\beta_e^{1/2}\delta\varepsilon^2)$, or $O(\beta_e^{3/2}\delta\varepsilon^2)$ for the term involving B_z, while for the WWs, they are all $O(\beta^{3/2}\delta\varepsilon^2)$.

A.1.2. *Heat flux contributions*

Let us now consider the contributions involving $\boldsymbol{\Pi}_e^{(1,q)}$. It is easy to find that

$$\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(1,q)} = -\delta^2[B_z, q_{\perp e}] \tag{A 10}$$

and

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(1,q)})_\perp = \delta^2\left\{-\frac{1}{2}\boldsymbol{\nabla}_\perp\left(\frac{\beta_e}{2}q_{\perp e}u_{\| e} + \widehat{\boldsymbol{z}}\cdot(\widehat{\boldsymbol{b}}_\perp\times\boldsymbol{\nabla}_\perp q_{\perp e})\right) + \nabla_\|\boldsymbol{\nabla}\times(q_{\perp e}\widehat{\boldsymbol{z}})\right\}, \tag{A 11}$$

where we have used that, to the required order, $\widehat{\boldsymbol{z}}\times(\boldsymbol{\nabla}_\perp\times\widehat{\boldsymbol{b}}_\perp) = -\Delta_\perp A_\| = -\dfrac{\beta_e}{2}u_{\| e}$.

A.2. *Second order contributions*

At second order, since terms of order $O(\varepsilon^3)$ or smaller are discarded, we can write

$$\Pi_{e,ij}^{(2)} = \frac{\delta^2}{4}\mathsf{M}_{ijkl}\Big[\frac{D^{(e)}\Pi_{e,kl}^{(1)}}{Dt} + \left(\delta_{kn}\Pi_{e,ml}^{(1)} + \delta_{ln}\Pi_{e,mk}^{(1)}\right)\partial_m u_{e,n}\Big]. \tag{A 12}$$

In this formula, it is also sufficient to use for $\boldsymbol{\Pi}^{(1)}$ (keeping only terms of order $O(\varepsilon)$)

$$\Pi_{e,ij}^{(1,1)} = -\frac{\delta^2}{4}\Big[\epsilon_{i3q}\left(\partial_q {u^*}_{e,j} + \partial_j {u^*}_{e,q}\right) + 3\epsilon_{i3q}\partial_q {u^*}_{\| e}\delta_{j3}\Big]^S, \tag{A 13}$$

where $\boldsymbol{u}_e^* = \boldsymbol{u}_e + q_{\perp e}\widehat{\boldsymbol{z}}$, while $\boldsymbol{\mathsf{M}}$ reduces to $\mathsf{M}_{ijkl} = ((\delta_{ik} + 3\delta_{i3}\delta_{k3})\epsilon_{jl3})^S$.

We shall now calculate the second order contribution for the perpendicular components of the gyroviscous force $(\nabla_j\Pi_{e,ij}^{(2)})_\perp$ by considering all terms separately.

In the second term in the rhs of Eq. (A 12), we find by inspection that $m\neq 3$, $j\neq 3$ (the terms involving parallel derivatives would be too small), $l\neq 3$ (as implied by the symmetries of M_{ijkl}) and $k\neq 3$ (because $k=3$ only contributes to the parallel gyroviscous force). It is then easy to see that $\Pi_{e,i\neq 3,j\neq 3}^{(1,1)}$ only involves perpendicular components of the velocity. As a result, the order of magnitude of the considered term is $\delta^4\varepsilon^2/\mu^3 = \beta_e^{3/2}\delta\varepsilon^2$ for both types of waves, which is too small to be retained. Similar arguments apply to the third and the first terms which also contribute only at most at order $\beta_e^{3/2}\delta\varepsilon^2$. We thus conclude that $(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(2)})_\perp = 0$.

Let us then turn to the parallel component $\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^{(2)}$ which reduces to $\partial_j\Pi_{e,3j}^{(2)}$. We have $\mathsf{M}_{3jkl} = 4\delta_{k3}\epsilon_{jl3}$ and $\Pi_{e,3j}^{(1,1)} = \delta^2\epsilon_{jq3}\partial_q u_{\| e}^* = \delta^2\boldsymbol{\nabla}\times(u_{\| e}^*\widehat{\boldsymbol{z}})$. Up to terms of order $O(\epsilon^3)$, the first term of the rhs of Eq. (A 12) writes

$$R_1 = \delta^4\epsilon_{jl3}\partial_j\frac{D^{(e)}}{Dt}(\epsilon_{lq3}\partial_q u_{\| e}^*) = -\delta^4\left(\frac{D^{(e)}}{Dt}\Delta_\perp(u_{\| e} + q_{\perp e}) + \partial_j\boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp\partial_j(u_{\| e} + q_{\perp e})\right), \tag{A 14}$$

the second one writes

$$\begin{aligned}R_2 &= -\frac{\delta^4}{4}\epsilon_{jl3}\partial_j\left\{\epsilon_{m3q}\left(\partial_q u^*_{e,l} + \partial_l u^*_{e,q}\right)\partial_m u_{\parallel e} + \epsilon_{l3q}\left(\partial_q u^*_{e,m} + \partial_m u^*_{e,q}\right)\partial_m u_{\parallel e}\right\}\\ &= -\frac{\delta^4}{4}\left([\omega_{ze}, u_{\parallel e}] + 2\epsilon_{jl3}[u_{e,l}, \partial_j u_{\parallel e}] + \Delta_\perp \boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp u_{\parallel e} + 2\partial_j \boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp\partial_j u_{\parallel e}\right),\end{aligned} \tag{A 15}$$

and the third one

$$R_3 = \delta^4\left([\omega_{ze}, u^*_{\parallel e}] + \epsilon_{jl3}[u_{e,l}, \partial_j u^*_{e\parallel}]\right). \tag{A 16}$$

We thus have

$$\begin{aligned}\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(2)}_e = &-\delta^4\frac{D^{(e)}}{Dt}\Delta_\perp u_{\parallel e} - \frac{3\delta^4}{2}\partial_j\boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp\partial_j u_{\parallel e} - \frac{\delta^4}{4}\Delta_\perp\boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp u_{\parallel e}\\ &+\frac{3\delta^4}{4}[\omega_{ze}, u_{\parallel e}] + \frac{\delta^4}{2}\epsilon_{jl3}[u_{e,l}, \partial_j u_{\parallel e}]\\ &-\delta^4\frac{D^{(e)}}{Dt}\Delta_\perp q_{\perp e} - \delta^4\partial_j\boldsymbol{u}_{e\perp}\cdot\boldsymbol{\nabla}_\perp\partial_j q_{\perp e} + \delta^4[\omega_{ze}, q_{\perp e}]\\ &+\delta^4\epsilon_{jl3}[u_{e,l}, \partial_j q_{\perp e}]\end{aligned} \tag{A 17}$$

or, when expressing $\boldsymbol{u}_{\perp e}$ in terms of φ^*,

$$\begin{aligned}\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(2)}_e = &-\delta^4\frac{D^{(e)}}{Dt}\Delta_\perp u_{\parallel e} + \frac{\delta^4}{2}[\Delta_\perp\varphi^*, u_{\parallel e}] - \delta^4\sum_{j=x,y}[\partial_j\varphi^*, \partial_j u_{\parallel e}]\\ &-\delta^4\frac{D^{(e)}}{Dt}\Delta_\perp q_{\perp e} + \delta^4[\Delta_\perp\varphi^*, q_{\perp e}].\end{aligned} \tag{A 18}$$

Since all the terms at this order in the expansion of Eq. (A 1) are $O(\epsilon^2)$, the higher orders cannot contribute to the gyroviscous force at order $O(\epsilon^2)$.

A.3. *Curl of the gyroviscous force*

We shall now turn to the expressions entering Eq. (2.4). Since only first order terms contribute to the perpendicular gyroviscous force, one gets from Eqs. (A 9) and (A 11)

$$\begin{aligned}\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e) = \delta^2\Big\{&[p_{\perp e} - B_z, \omega_{ze}] + \sum_{i=x,y}[\partial_i(p_{\perp e} - B_z), \partial_i\varphi^*]\\ &+\frac{1}{2}\Delta_\perp\left(\nabla_\parallel u_{\parallel e} - \boldsymbol{\nabla}\cdot\boldsymbol{u}_e\right) - \nabla_\parallel\Delta_\perp u_{\parallel e} + \sum_{i=x,y}[\partial_i A_\parallel, \partial_i u_{\parallel e}]\\ &-\nabla_\parallel\Delta_\perp q_{\perp e} + \sum_{i=x,y}[\partial_i A_\parallel, \partial_i q_{\perp e}]\Big\}\end{aligned} \tag{A 19}$$

which, after some algebra, rewrites

$$\begin{aligned}\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e) = \delta^2\Big\{&[p_{\perp e} - B_z, \Delta_\perp\varphi^*] + \sum_{i=x,y}[\partial_i(p_{\perp e} - B_z), \partial_i\varphi^*]\\ &-\frac{1}{2}\Delta_\perp\boldsymbol{\nabla}\cdot\boldsymbol{u}_e - \frac{1}{2}\nabla_\parallel\Delta_\perp u_{\parallel e} - \nabla_\parallel\Delta_\perp q_{\perp e} + \sum_{i=x,y}[\partial_i A_\parallel, \partial_i q_{\perp e}]\Big\}.\end{aligned} \tag{A 20}$$

For the KAWs, all the terms in this equation (except that involving B_z) are $O(\beta_e\epsilon^2)$, while those of Eq. (2.4) are $O(\epsilon^2)$. In this formula, the term $\boldsymbol{\nabla}\cdot\boldsymbol{u}_e$ can be replaced by $-\dfrac{D^{(e)}}{Dt}n$. Differently, for the WWs, $\boldsymbol{\nabla}\cdot\boldsymbol{u}_e$ is negligible and the other terms are all $O(\beta^2\varepsilon^2)$, while those of Eq. (2.4) are $O(\beta_e\epsilon^2)$.

Finally, the term $\widehat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}n\times(\boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}}_e))$ simply writes at order $O(\epsilon^2)$

$$\widehat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}n\times(\boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}}_e))=\frac{\delta^2}{2}[n,\varDelta_\perp\varphi^*], \tag{A 21}$$

which is $O(\beta\varepsilon^2)$ for the KAWs where $n=(1/\tau)\varphi$ and negligible for the WWs.

Appendix B. Relation with classical FLR formulations

It is of interest to relate the FLR pressure tensor formulas derived using a systematic asymptotics in Appendix A, to the classical formulations given in Hazeltine & Meiss (1985) and Hsu *et al.* (1986).

Neglecting the heat flux contribution and using the identity

$$[p_{\perp e},\boldsymbol{u}_{\perp e}]=(\boldsymbol{\nabla}_\perp p_{\perp e}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{\nabla}_\perp\varphi^*-\varDelta_\perp\varphi^*\boldsymbol{\nabla}_\perp p_{\perp e}, \tag{B 1}$$

it is easily checked that, up to the sign coming from the electron charge, Eqs. (A 7) and (A 9) (which provide the leading-order contribution) identify with Eq. (4.133) of Hazeltine & Meiss (1985) once we perform the assumptions made in this paper, namely isothermal electrons ($p_{\perp e}=n$), no longitudinal magnetic fluctuations ($B_z=0$) nor longitudinal variations ($\nabla_\|=0$), negligible compressibility of the transverse electron flow ($\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}_{\perp e}=0$).

We now turn to Eq. (24) of Hsu *et al.* (1986) which, in comparison with that of Hazeltine & Meiss (1985), takes into account parallel gradients, parallel flow and compressibility. Defining the electron diamagnetic drift $\boldsymbol{V}_{De}=-\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}p_{\perp e}$, and using Eq. (B 1), we note that, at order $O(\epsilon^2/\mu)$,

$$\frac{D^{(e)}}{Dt}\boldsymbol{V}_{De}=\varDelta_\perp\varphi^*\boldsymbol{\nabla}_\perp p_{\perp e}-(\boldsymbol{\nabla}_\perp p_{\perp e}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{\nabla}_\perp\varphi^*-(\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp)\,\frac{D^{(e)}}{Dt}p_{\perp e}. \tag{B 2}$$

Under the assumption of isothermal electrons, the last term is evaluated using the continuity equation. Putting together Eqs. (A 7), (A 9), (B 1) and neglecting B_z, we obtain to leading order

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}}_e=\ &\delta^2\Big\{(\boldsymbol{\nabla}_\perp p_{\perp e}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{\nabla}_\perp\varphi^*-\varDelta_\perp\varphi^*\boldsymbol{\nabla}_\perp p_{\perp e}+\frac{1}{2}\boldsymbol{\nabla}_\perp\left(p_{\perp e}\varDelta_\perp\varphi^*\right)\\&+\nabla_\|\varDelta_\perp\varphi^*\widehat{\boldsymbol{z}}+\widehat{\boldsymbol{z}}\cdot\left(\boldsymbol{\nabla}_\perp p_{\perp e}\times\boldsymbol{\nabla}_\perp u_{\|e}\right)\widehat{\boldsymbol{z}}-\frac{1}{2}\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}_e\right)\\&+\frac{1}{2}\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\left(\nabla_\| u_{\|e}\right)-\nabla_\|\left(\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp u_{\|e}\right)+\sum_{i=x,y}[\partial_i\varphi^*,\partial_i A_\|]\widehat{\boldsymbol{z}}\Big\}.\end{aligned} \tag{B 3}$$

If the terms of the last line are neglected, this expression rewrites, using Eq. (B 2),

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}}_e=\ &\delta^2\Big\{-\frac{D^{(e)}}{Dt}\boldsymbol{V}_{De}+\frac{1}{2}\boldsymbol{\nabla}_\perp\left(p_{\perp e}\varDelta_\perp\varphi^*\right)\\&+\nabla_\|\varDelta_\perp\varphi^*\widehat{\boldsymbol{z}}-\widehat{\boldsymbol{z}}\cdot\left(\boldsymbol{\nabla}_\perp u_{\|e}\times\boldsymbol{\nabla}_\perp p_{\perp e}\right)\widehat{\boldsymbol{z}}+\frac{1}{2}\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}_e\right)\Big\},\end{aligned} \tag{B 4}$$

which identifies with Eq. (24) of Hsu *et al.* (1986) (which refers to ions), once we make the appropriate sign change for the electron charge (remembering that the diamagnetic drift is here defined with the opposite sign). We note however that, with the present scalings, additional terms involving the parallel derivative and the parallel flow contribute to the same order in the formulation given by Eq. (B 3).

Appendix C. Comment on gyroviscous cancellation

It is of interest to comment on the gyroviscous cancellations at work in Eq. (2.23). In Fitzpatrick & Porcelli (2004) the term proportional to $\frac{d}{dt}\Delta_\perp B_z$ disappears, as shown in Fitzpatrick & Porcelli (2007). This results from the use of the gyroviscous force given in Hazeltine & Meiss (1985), which turns out to be incomplete as flow compressibility is neglected in the latter reference, while it is taken into account in Fitzpatrick & Porcelli (2004). To be more specific , let us evaluate $\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e)$ when the only term that contributes to the gyroviscous force is the term $-\frac{D^{(e)}}{Dt}\boldsymbol{V}_{De}$. Using Eqs. (B 1) and (B 2), we have

$$\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\left(\frac{D^{(e)}}{Dt}\boldsymbol{V}_{De}\right)=\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times([p_{\perp e},\boldsymbol{u}_e])-\Delta_\perp(\boldsymbol{\nabla}\cdot\boldsymbol{u}_e) \tag{C 1}$$

In the framework of Fitzpatrick & Porcelli (2004), using their equations (19) and (23) (which assumes cold ions and perpendicular pressure balance), we write

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_e=-\frac{dp_{\perp e}}{dt}=\frac{2}{\beta_e}\frac{d}{dt}B_z. \tag{C 2}$$

After simple algebra, using the fact that, at this order, $\varphi^*\propto B_z$ and that $p_{\perp e}=-B_z$, we find that

$$\widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\left(\frac{D^{(e)}}{Dt}\boldsymbol{V}_{De}\right)=-\frac{2}{\beta_e}\frac{D^{(e)}}{Dt}\Delta_\perp B_z. \tag{C 3}$$

Using the latter equation together with Eq. (2.20), we indeed find a cancellation of the last two terms of (2.4). This cancellation is however not complete when using a more general form of the gyroviscous force.

REFERENCES

Abdelhamid, H. M., Lingam, M. & Mahajan, S. M. 2016 Extended MHD turbulence and its applications to the solar wind. *Astrophys. J.* **829**, 87.

Andrés, N., Galtier, S. & Sahraoui, F. 2016*a* Exact scaling laws for helical three-dimensional two-fluid turbulent plasmas. *Phys. Rev. E* **94**, 063206.

Andrés, N., Gonzalez, C., Martin, L., Dmitruk, P. & Gómez, D. 2014 Two-fluid turbulence including electron inertia. *Phys. Plasmas* **21**, 122305.

Andrés, N., Mininni, P.D., Dmitruk, P. & Gómez, D.O. 2016*b* von Kármán-Howarth equation for three-dimensional two-fluid plasmas. *Phys. Rev. E* **93**, 063202.

Biskamp, D., Schwarz, E. & Drake, J. F. 1996 Two-dimensional electron magnetohydrodynamic turbulence. *Phys. Rev. Lett.* **76**, 1264–1267.

Biskamp, D., Schwarz, E., Zeiler, A., Celani, A. & Drake, J. F. 1999 Electron magnetohydrodynamic turbulence. *Phys. Plasmas* **6**, 751–758.

Boldyrev, S., Horaites, K., Xia, Q. & Perez, J.C. 2013 Toward a theory of astrophysical plasma turbulence at subproton scales. *Astrophys. J.* **777**, 41.

Chang, O., Gary, S. P. & Wang, J. 2011 Whistler turbulence forward cascade: Three-dimensional particle-in-cell simulations. *Geophys. Res. Lett.* **38**, L22102.

Chaston, C. C., Salem, C., Bonnell, J. W., Carlson, C. W., Ergun, R. E., Strangeway, R. J. & McFadden, J. P. 2008 The turbulent Alfvénic aurora. *Phys. Rev. Lett.* **100**, 175003.

Dagnelund, D. & Pavlenko, V. P. 2005 Hamiltonian description and stability of vortex flows in interchange mode turbulence. *Phys. Scripta* **71**, 293–297.

Daughton, W., Roytershteyn, V., Karimabadi, H., Yin, L., Albright, B. J., Bergen, B. & Bowers, K. J. 2011 Role of electron physics in the development of turbulent magnetic reconnection in collisionless plasmas. *Nature Phys.* **7**, 539–542.

Fitzpatrick, R. & Porcelli, F. 2004 Collisionless magnetic reconnection with arbitrary guide field. *Phys. Plasmas* **11**, 4713–4718.

Fitzpatrick, R. & Porcelli, F. 2007 Erratum: Collisionless magnetic reconnection with arbitrary guide-field [Phys. Plasmas 11, 4713 (2004)]. *Phys. Plasmas* **14**, 049902.

Galtier, S. & Bhattacharjee, A. 2003 Anisotropic weak whistler wave turbulence in electron magnetohydrodynamics. *Phys. Plasmas* **10**, 3065–3075.

Galtier, S. & Meyrand, R. 2015 Entanglement of helicity and energy in kinetic Alfvén wave/whistler turbulence. *J. Plasma Phys.* **81**, 325810106.

Gary, S. P., Chang, O. & Wang, J. 2012 Forward cascade of whistler turbulence: three-dimensional particle-in-cell simulations. *Astrophys. J.* **755**, 142.

Gary, S. P. & Smith, C. W 2009 Short-wavelength turbulence in the solar wind: linear theory of whistler and kinetic Alfvén fluctuations. *J. Geophys. Res.* **114**, A12105.

Goswami, P., Passot, T. & Sulem, P. L. 2005 A Landau fluid model for warm collisionless plasmas. *Phys. Plasmas* **12**, 102109.

Hammett, G. W. & Perkins, F. W. 1990 Fluid moment models for Landau damping with application to the ion-temperature gradient instability. *Phys. Rev. Lett* **64**, 3019–3022.

Hazeltine, R. D., Hsu, C. T. & Morrison, P. J. 1987 Hamiltonian four-field model for nonlinear tokamak dynamics. *Phys. Fluids* **30**, 3204–3211.

Hazeltine, R. D. & Meiss, J. D. 1985 Shear-Alfvén dynamics of toroidally confined plasmas. *Phys. Rep.* **121**, 1–164.

Hesse, M., Kuznetsova, M. & Birn, J. 2004 The role of electron heat flux in guide-field magnetic reconnection. *Phys. Plasmas* **11**, 5387–5397.

Hsu, C. T., Hazeltine, R. D. & Morrison, P. J. 1986 A generalized reduced fluid model with finite iongyroradius effects. *Phys. Fluids* **29**, 1480–1487.

Huang, S. Y., Sahraoui, F., Deng, X. H., He, J. S., Yuan, Z. G., Zhou, M., Pang, Y. & Fu, H. S. 2014 Kinetic turbulence in the terrestrial magnetosheath: Cluster observations. *Astrophys. J. Lett.* **789**, L28.

Izacard, O., Chandre, C., Tassi, E. & Ciraolo, G. 2011 Gyromap for a two-dimensional Hamiltonian fluid model derived from Braginskii's closure for magnetized plasmas. *Phys. Plasmas* **18**, 062105.

Krishan, V. & Mahajan, S. M. 2004 Magnetic fluctuations and Hall magnetohydrodynamic turbulence in the solar wind. *J. Geophys. Res.* **109**, A11105.

Krommes, J. A. 2002 Fundamental statistical descriptions of plasma turbulence in magnetic fields. *Phys. Reports* **360**, 1–352.

Kuvshinov, B. N., Westerhof, E., Schep, T. J. & Berning, M. 1998 Electron magnetohydrodynamics of magnetized, inhomogeneous plasmas. *Phys. Lett. A* **241**, 287–292.

Lee, A., Daughton, W., Karimabadi, H. & Egedal, J. 2016 Hybrid simulations of magnetic reconnection with kinetic ions and fluid electron pressure anisotropy. *Phys. Plasmas* **23**, 032114.

Loureiro, N. F., Schekochihin, A. A. & Zocco, A. 2013 Fast collisionless reconnection and electron heating in strongly magnetized plasmas. *Phys. Rev. Lett.* **111**, 025002.

Lyutikov, M. 2013 Electron magnetohydrodynamics: Dynamics and turbulence. *Phys. Rev. E* **88**, 053103.

Matteini, L., Alexandrova, O., Chen, C. H. K. & Lacombe, C. 2017 Electric and magnetic spectra from MHD to electron scales in the magnetosheath. *Month. Not. Roy. Astron. Soc.* **466**, 945–951.

Mikhalovskii, A. B., Lakhin, V. P., Aburdzhaniya, G. D., Mikhailovskaya, L. A., Onishchenko, O. G. & Smolyakov, A. I. 1987 On the theory of Alfvén vortices. *Plasma Phys. Contr. Fusion* **29**, 1–25.

Morrison, P. J., Caldas, I. L. & Tasso, H. 1984 Hamiltonian formulation of two-dimensional gyroviscous MHD. *Z. Naturforschung A* **39**, 1023.

Passot, T. & Sulem, P. L. 2015 A model for the non-universal power-law of the solar wind sub-ion scale magnetic spectrum. *Astrophys. J. Lett.* **812**, L37.

Passot, T., Sulem, P. L. & Hunana, P. 2012 Extending magnetohydrodynamics to the slow dynamics of collisionless plasmas. *Phys. Plasmas* **19**, 082113.

RAMOS, J. J. 2005*a* Fluid formalism for collisionless magnetized plasmas. *Phys. Plasmas* **12**, 052102.

RAMOS, J. J. 2005*b* General expression of the gyroviscous force. *Phys. Plasmas* **12**, 112301.

RÖNNMARK, K. 1982 Waves in homogeneous, anisotropic multicomponent plasmas (WHAMP). *Tech. Rep.* 179. Kiruna Geophysical Institute.

SAHRAOUI, F., BELMONT, G. & GOLDSTEIN, M. L. 2012 New insight into short-wavelength solar wind fluctuations from Vlasov theory. *Astrophys. J.* **748**, 100.

SAHRAOUI, F., GOLDSTEIN, M. L., BELMONT, G., CANU, P. & REZEAU, L. 2010 Three dimensional anisotropic k spectra of turbulence at subproton scales in the solar wind. *Phys. Rev. Lett.* **105**, 131101.

SAHRAOUI, F., HUANG, S. Y., BELMONT, G., GOLDSTEIN, M. L., RÉTINO, A., ROBERT, P. & DE PATOUL, J. 2013 Scaling of the electron dissipation range of solar wind turbulence. *Astrophys. J.* **777**, 15.

SCHEKOCHIHIN, A. A., COWLEY, S. C., DORLAND, W., HAMMETT, G. W., HOWES, G. G., QUATAERT, E. & TATSUNO, T. 2009 Astrophysical gyrokinetics: kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl.* **182**, 310–377.

SCHEKOCHIHIN, A. A., COWLEY, S. C., RINCON, F. & ROSIN, M. S. 2010 Magnetofluid dynamics of magnetized cosmic plasma: firehose and gyrothermal instabilities. *Mon. Not. R. Astron. Soc.* **405**, 291–300.

SCHEP, J. T., PEGORARO, F. & KUVSHINOV, B. N. 1994 Generalized two-fluid theory of nonlinear magnetic structures. *Phys. Plasmas* pp. 2843–2852.

SNYDER, P. B., HAMMETT, G. W. & DORLAND, W. 1997 Landau fluid models of collisionless magnetohydrodynamics. *Phys. Plasmas* **4**, 3974–3985.

SULEM, P. L & PASSOT, T. 2015 Landau fluid closures with nonlinear large-scale finite Larmor radius corrections for collisionless plasmas. *J. Plasma Phys.* **81**, 325810103.

SULEM, P. L., PASSOT, T., LAVEDER, D. & BORGOGNO, D. 2016 Influence of the nonlinearity parameter on the solar wind sub-ion magnetic energy spectrum: FLR-Landau fluid simulations. *Astrophys. J.* **818**, 66.

TASSI, E., SULEM, P. L & PASSOT, T. 2016 Reduced models accounting for parallel magnetic perturbations: gyrofluid and finite Larmor radius-Landau fluid approaches. *J. Plasma Phys.* **82**, 705820601.

TREUMANN, R. A. & BAUMJOHANN, W. 2013 Collisionless reconnection in space plasmas. *Front. Phys.* **1**, 31.

ZOCCO, A. & SCHEKOCHIHIN, A. 2011 Reduced fluid-kinetic equations for low-frequency dynamics, magnetic reconnection, and electron heating in low-beta plasmas. *Phys. Plasmas* **18**, 102309.

ZWEIBEL, E. G. & YAMADA, M. 2017 Perspectives on magnetic reconnection. *Proc. R. Soc. A* **472**, 20160479.