MAGNETIZED PARTICLE MOTION IN 3D FIELDS AND THE NATURE OF 3D EQUILIBRIA

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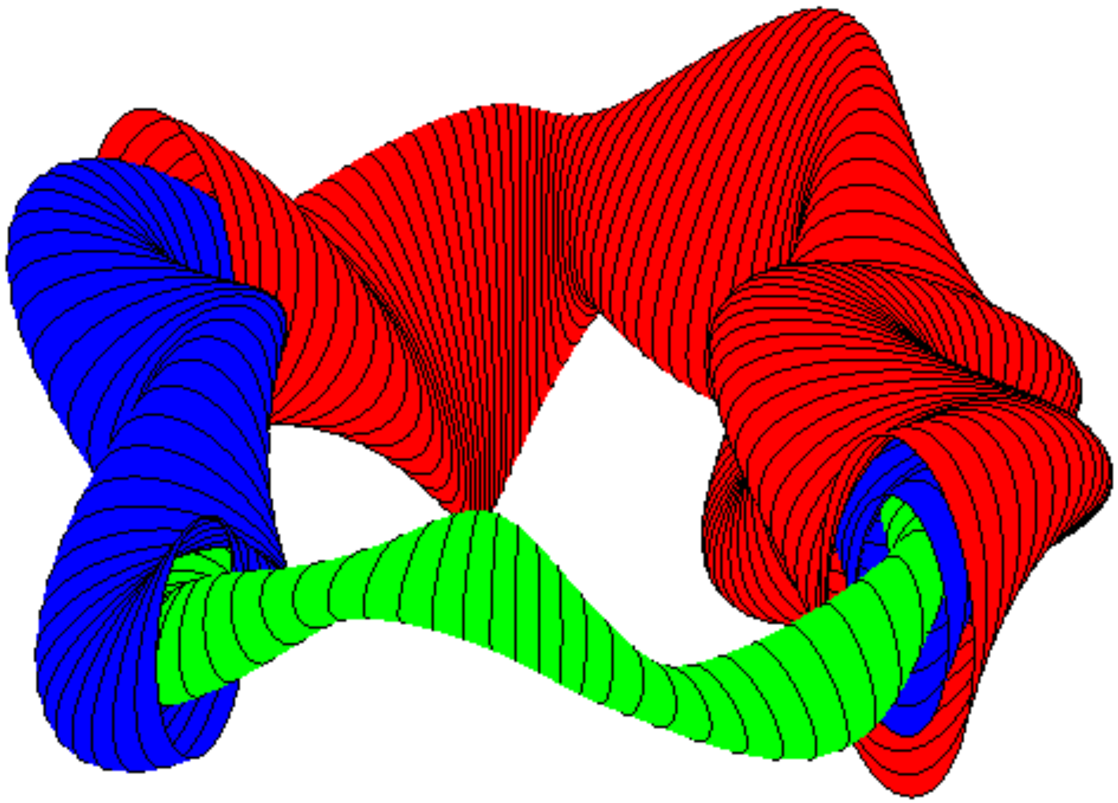
3D magnetic fields in stellarators

- Magnetic field $\mathbf{B}$ must be 3D (that is, without direction of symmetry) if we want
  - steady state
  - nested flux surfaces (surfaces || to $\mathbf{B}$)
  - $\mathbf{B}$ (mostly) generated by external currents



- Stellarators have inherent advantages
  - No current in the plasma $\Rightarrow$ no current drive, no current instabilities

Magnetized particle motion

- Assume steady state $\mathbf{E}$ and $\mathbf{B}$: $\mathbf{E} = -\nabla\phi \sim T/eL$
- Constant total energy

$$\mathcal{E} = \frac{v^2}{2} + \frac{Ze\phi}{m}$$

- Motion for $\rho_* = \rho/L << 1$
  - Magnetic moment (= adiabatic invariant) is constant

$$\mu = \frac{v_\perp^2}{2B} + O\left(\rho_* \frac{v_t^2}{B}\right)$$

  - Motion = fast parallel streaming + slow perpendicular drifts

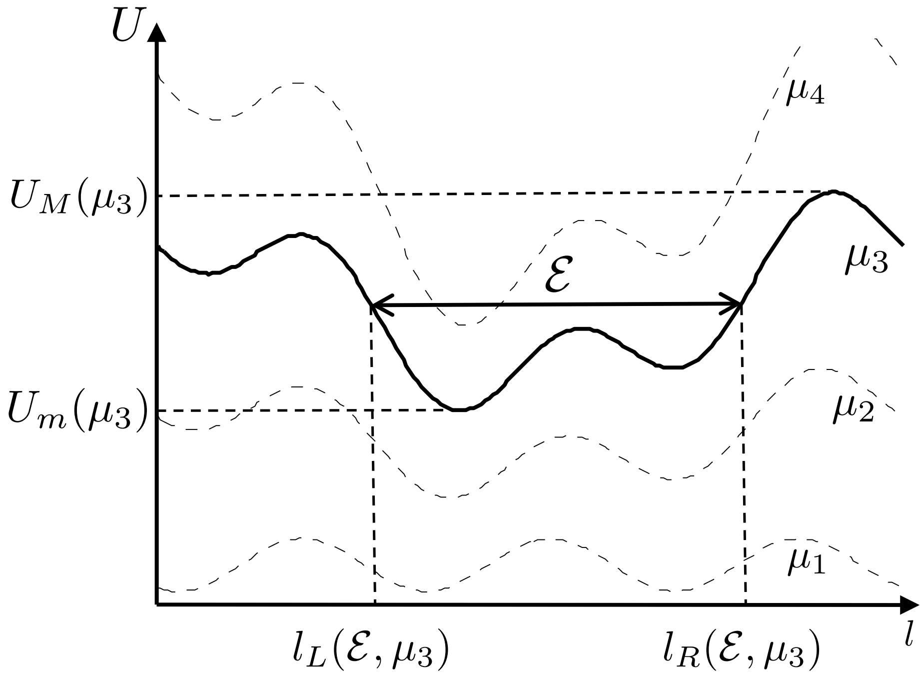
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \underbrace{v_\parallel \hat{\mathbf{b}}}_{\sim v_t} \underbrace{- \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}} + \frac{mc\mu}{ZeB}\hat{\mathbf{b}} \times \nabla B + \frac{mcv_\parallel^2}{ZeB}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})}_{=\mathbf{v}_d \sim \rho_* v_t}$$

Parallel motion

- To lowest order, particles move along magnetic field lines

$$\frac{\mathrm{d}l}{\mathrm{d}t} = v_{\|} = \pm\sqrt{2\left(\mathcal{E} - \mu B(l) - \frac{Ze\phi(l)}{m}\right)} = \pm\sqrt{2\left(\mathcal{E} - U(l)\right)}$$

- $\mathcal{E} > U_M(\mu) \Rightarrow v_{\|}$ does not change sign: passing particles
- $\mathcal{E} < U_M(\mu) \Rightarrow v_{\|}$ vanishes at bounce points: trapped particles

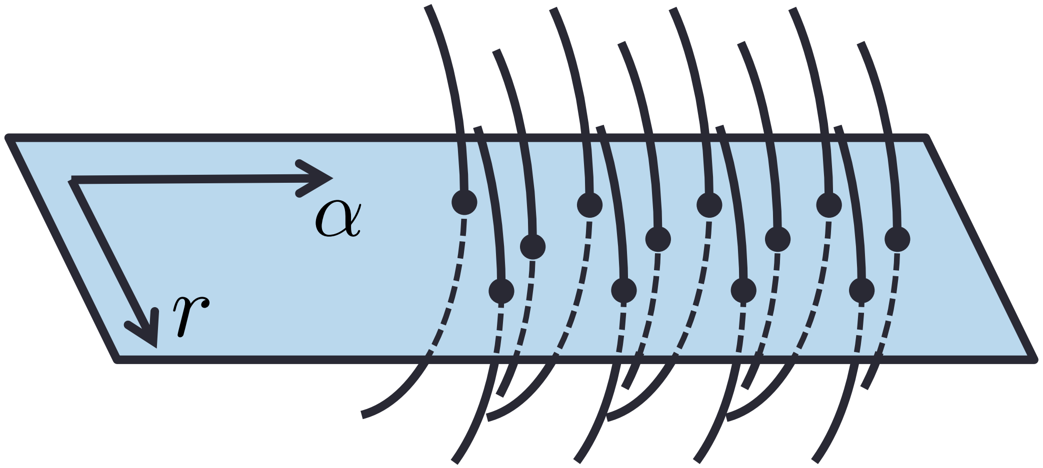


Perpendicular motion

- Ignore perpendicular motion for passing particles: perpendicular drifts give small correction to position
  - We’ll have to come back to this
- Trapped particles do not leave initial region unless we keep perpendicular drifts
- Use flux coordinates to describe perpendicular motion of trapped particles

Flux coordinates

- Simple picture: coordinates on plane cutting through $\mathbf{B}$ lines



- r and α constant along $\mathbf{B}$

$$\mathbf{B} \cdot \nabla r = 0 = \mathbf{B} \cdot \nabla \alpha \Rightarrow \mathbf{B} = \Psi(\mathbf{x}) \nabla r \times \nabla \alpha$$

- Magnetic flux across surface $\mathrm{d}r\,\mathrm{d}\alpha$ = $\Psi\,\mathrm{d}r\,\mathrm{d}\alpha$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \Psi(r, \alpha) \nabla r \times \nabla \alpha$$

Motion of trapped particles

- Perpendicular motion in r-direction

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \nabla r = \mathbf{v}_d \cdot \nabla r$$

- Since parallel motion is periodic with period $\tau_b = \oint \mathrm{d}l / v_{\|}$, decompose r into quasiperiodic and secular pieces

$$r(t) = \overline{r}(t) + \widetilde{r}(l(t)) \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} \simeq \frac{\mathrm{d}\overline{r}}{\mathrm{d}t} + v_{\|} \frac{\partial \widetilde{r}}{\partial l}$$

- Averaging over parallel trapped orbit

$$\boxed{\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = \frac{1}{\tau_b} \oint \mathbf{v}_d \cdot \nabla r \frac{\mathrm{d}l}{v_{\|}}}$$

- Integrating in l, $v_{\|} \frac{\partial \widetilde{r}}{\partial l} = \mathbf{v}_d \cdot \nabla r - \frac{\mathrm{d}\overline{r}}{\mathrm{d}t} \Rightarrow \widetilde{r} \sim \rho_* r \ll \boxed{r \simeq \overline{r}}$

Final equations

- Since α is equivalent to r

$$\frac{\mathrm{d}r}{\mathrm{d}t} \simeq \frac{1}{\tau_b} \oint \mathbf{v}_d \cdot \nabla r \, \frac{\mathrm{d}l}{v_\parallel} \sim \rho_* \frac{v_t}{L} r$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} \simeq \frac{1}{\tau_b} \oint \mathbf{v}_d \cdot \nabla \alpha \, \frac{\mathrm{d}l}{v_\parallel} \sim \rho_* \frac{v_t}{L} \alpha$$

- We can simplify these equations

Second adiabatic invariant

- Adiabatic invariant of periodic motion of trapped particles

$$J(r,\alpha,\mathcal{E},\mu)=\oint v_\parallel\,\mathrm{d}l=2\int_{l_L}^{l_R}\sqrt{2\left(\mathcal{E}-\mu B(l)-\frac{Ze\phi(l)}{m}\right)}\,\mathrm{d}l$$

- Equations based on second adiabatic invariant

$$\frac{\mathrm{d}r}{\mathrm{d}t}\simeq\frac{mc}{Ze\Psi\tau_b}\frac{\partial J}{\partial\alpha}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}\simeq-\frac{mc}{Ze\Psi\tau_b}\frac{\partial J}{\partial r}$$

- Particles move conserving their second adiabatic invariant

$$\frac{\mathrm{d}J}{\mathrm{d}t}=\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial J}{\partial r}+\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\partial J}{\partial\alpha}=0$$

Summary of particle motion

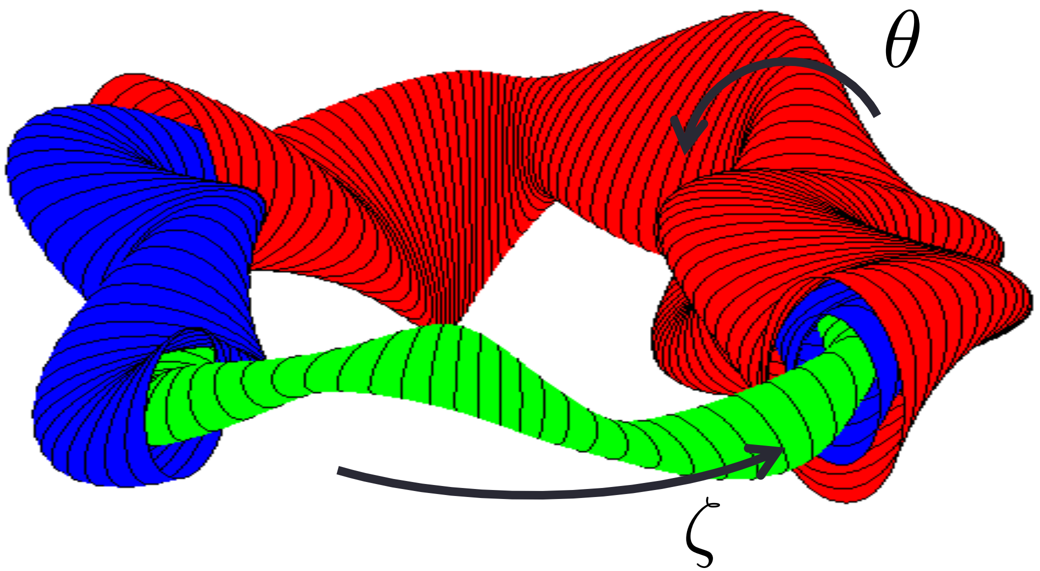
- Passing particles move mostly along $\mathbf{B}$ lines

- Trapped particles move along paths of constant J
  - In addition, particles have "oscillations" (small in ρ_*) around the paths of constant J

- To describe particle motion, we need maps of J
  - J can be easily constructed for given ϕ and $\mathbf{B}$

Particle motion in stellarators

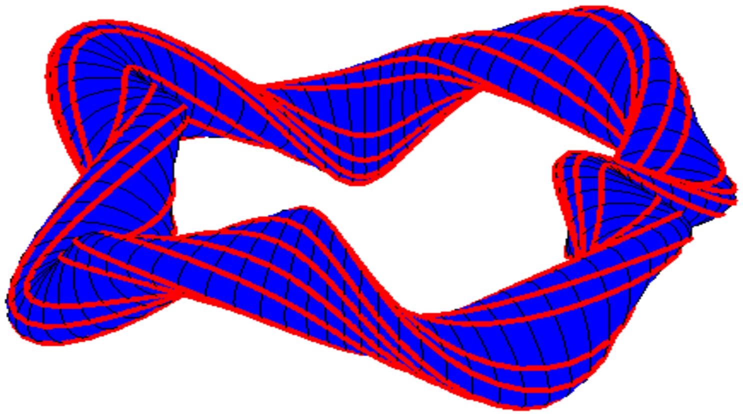
- Stellarator: confine to have nuclear fusion collisions
  - Need flux surfaces
  - A large number of Coulomb collisions will occur
- r labels flux surfaces and α lines within flux surfaces
  - Alternatively, poloidal and toroidal angles θ and ζ within flux surface



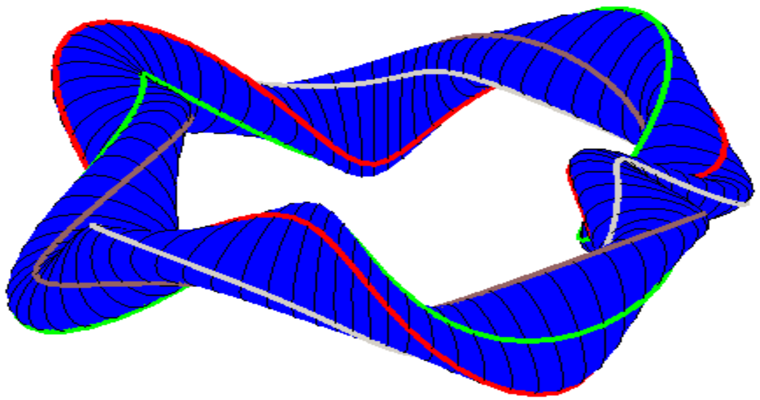
Passing particles

- Passing particles follow $\mathbf{B}$ lines $\Rightarrow$ two types of surfaces

- In most surfaces, pitch of $\mathbf{B}$ $\Rightarrow$ one $\mathbf{B}$ line defines whole flux surface = ergodic



- In a few surfaces, pitch of $\mathbf{B}$ $\Rightarrow$ $\mathbf{B}$ lines close on themselves = rational



Passing particles on ergodic surfaces

- To lowest order, passing particles stay on flux surfaces
- Can passing particles move across surfaces (in r-direction)? No on ergodic surfaces
  - Passing particles sample ergodic surfaces entirely
  - r-direction drift $\propto$ gradients & curvature parallel to flux surface

$$\mathbf{v}_d \cdot \nabla r = -\frac{mc}{ZeB}\left(\frac{Ze}{m}\nabla\phi + \mu\nabla B + v_\parallel^2 \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}\right) \cdot (\hat{\mathbf{b}} \times \nabla r)$$

  - Gradients & curvature “average” to zero due to periodicity in θ & ζ $\Rightarrow$ average r-direction drift = 0
- r can be described by quasiperiodic piece

$$v_\parallel \frac{\partial\tilde{r}}{\partial l} = \mathbf{v}_d \cdot \nabla r \Rightarrow \tilde{r} \sim \rho_* r \ll r$$

Passing particles on rational surfaces

- Particles do not sample rational surfaces entirely. Do they move across flux surfaces (in r-direction)? Not quite
  - Naively, equation for quasiperiodic piece is singular

$$v_{\parallel}\frac{\partial\widetilde{r}}{\partial l} = v_{\parallel}\boxed{\hat{\mathbf{b}}\cdot\nabla}\widetilde{r} = \mathbf{v}_d\cdot\nabla r \Rightarrow \boxed{\int_{\mathbf{B}\ \mathrm{lines}} \mathbf{v}_d\cdot\nabla r\,\frac{\mathrm{d}l}{v_{\parallel}} = 0}$$

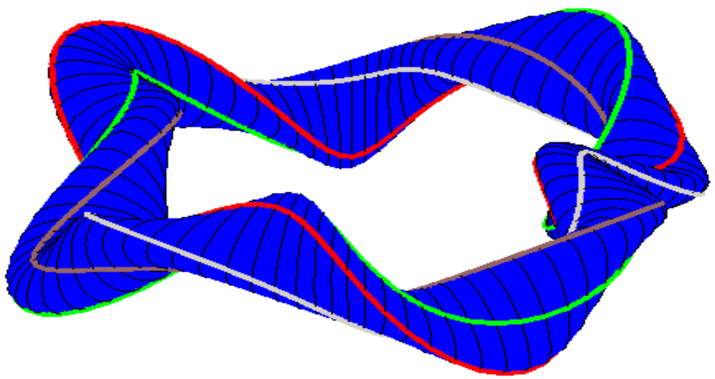
Singular

Not satisfied in general

  - Passing particles are still confined because rational surface is surrounded by ergodic flux surfaces

$$\left[v_{\parallel}\hat{\mathbf{b}} + \widetilde{r}\frac{\partial}{\partial r}\left(v_{\parallel}\hat{\mathbf{b}}\right)\right]\cdot\nabla\widetilde{r} = \mathbf{v}_d\cdot\nabla r$$

$$\Rightarrow \tilde{r} \sim \sqrt{\rho_*}r \ll r$$



Collisions

- Distribution function must be Maxwellian

$$f = f_M = \eta(\mathbf{x}) \left(\frac{m}{2\pi T(\mathbf{x})} \right)^{3/2} \exp\left(-\frac{m\mathcal{E}}{T(\mathbf{x})} \right)$$

- Passing particles stay at constant r to lowest order in $\rho_* \Rightarrow$ η and T must only depend on r
- Trapped particles follow paths of constant J, in general different from flux surfaces $\Rightarrow$ η and T must be constants to be compatible with passing particles
- To achieve confinement, η and T must depend on $r \Rightarrow J$ paths must coincide with flux surfaces

$$\boxed{J = J(r, \mathcal{E}, \mu) \Rightarrow \frac{\partial J}{\partial \alpha} = 0}$$ <u>Omnigeneity</u>

Omnigeneous stellarators

- Maxwellian with η and T only depending on r

$$n(\mathbf{x}) = \eta(r)\exp\left(-\frac{Ze\phi(\mathbf{x})}{T}\right)$$

$\Rightarrow$ Quasineutrality imposes that ϕ depend only on r

- Since ϕ does not depend on l, convenient to use variables v and $\lambda = 2\mu/v^2$

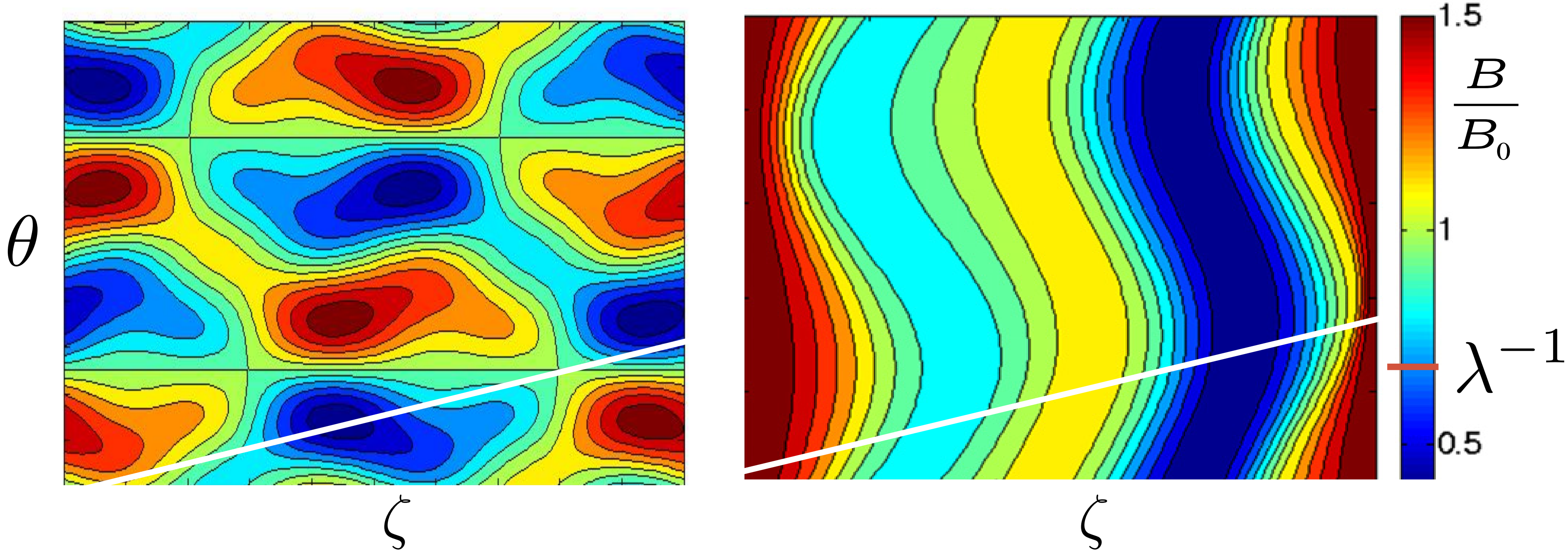
$$J = 2\int_{l_L}^{l_R}\sqrt{2\left(\mathcal{E} - \mu B(l) - \frac{Ze\phi}{m}\right)}\,\mathrm{d}l = \boxed{2v\int_{l_L}^{l_R}\sqrt{1-\lambda B(l)}\,\mathrm{d}l}$$

  - Bounce points $B(l_L) = 1/\lambda = B(l_R)$
  - Passing particles $\lambda < 1/B_M$, trapped particles $\lambda > 1/B_M$

"Constructing" omnigeneous fields

[Cary & Shasharina, PRL 97, PoP 97]

- Trapped particles need paths that do not close
  - Choose θ and ζ such that $\mathbf{B}$ lines are parallel straight lines



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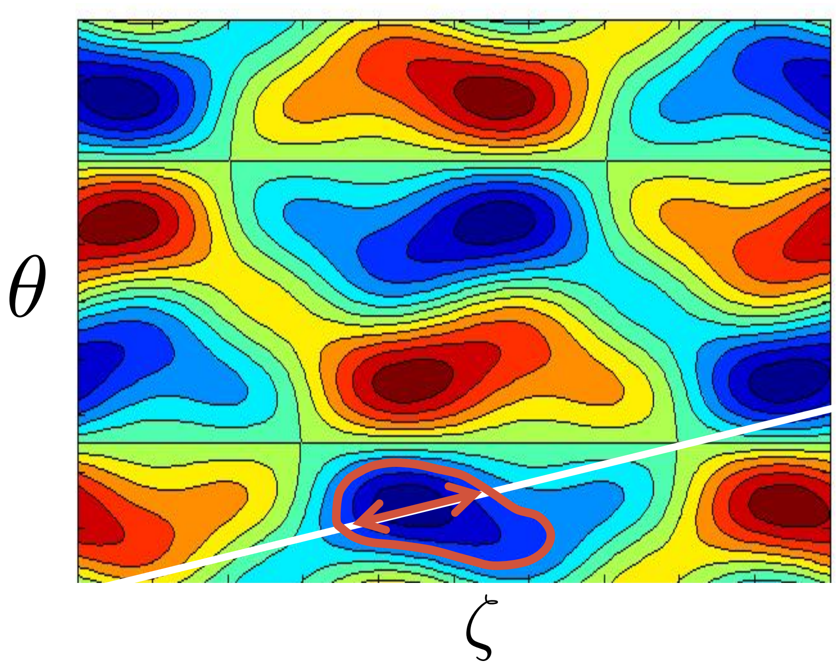

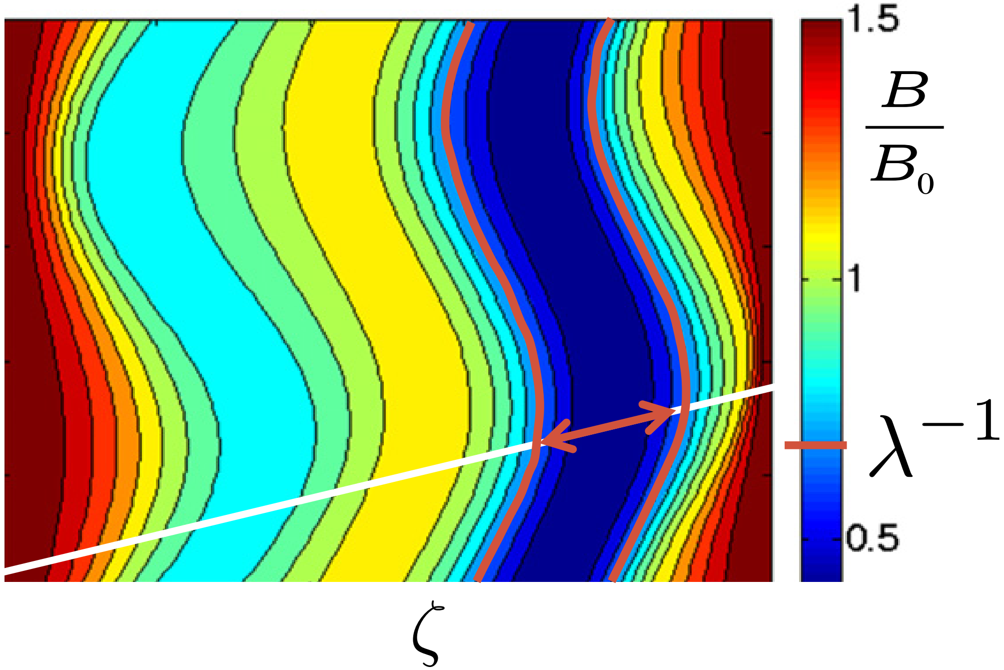

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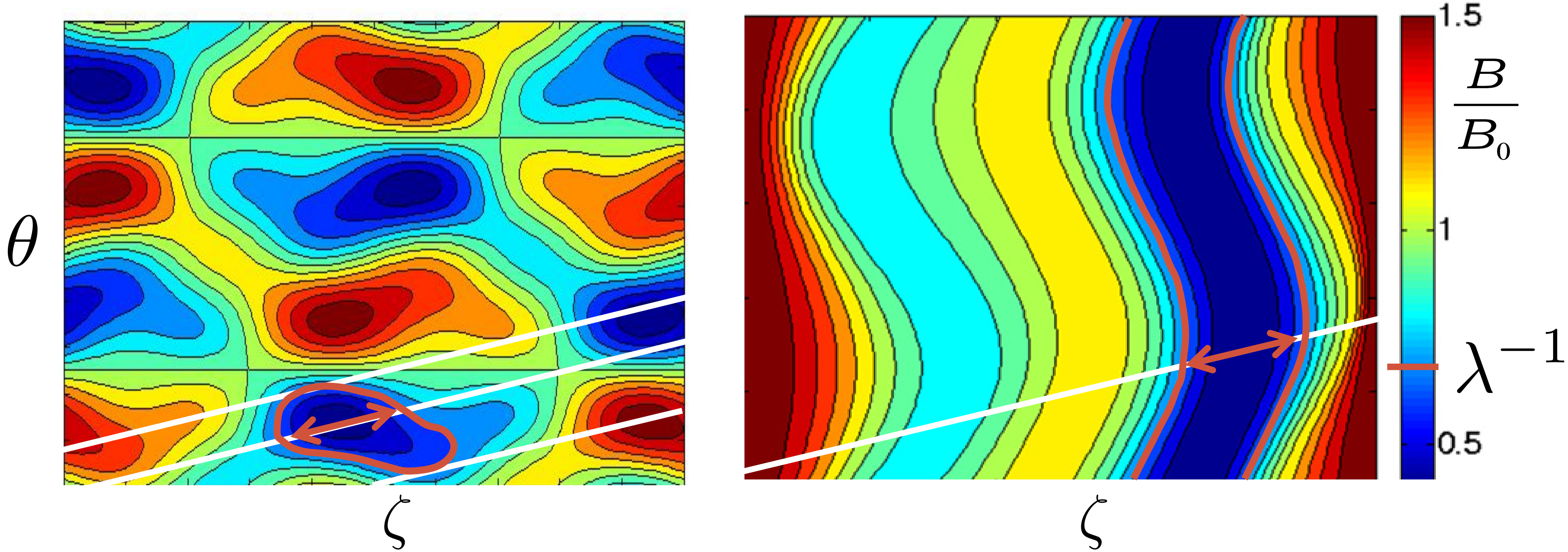

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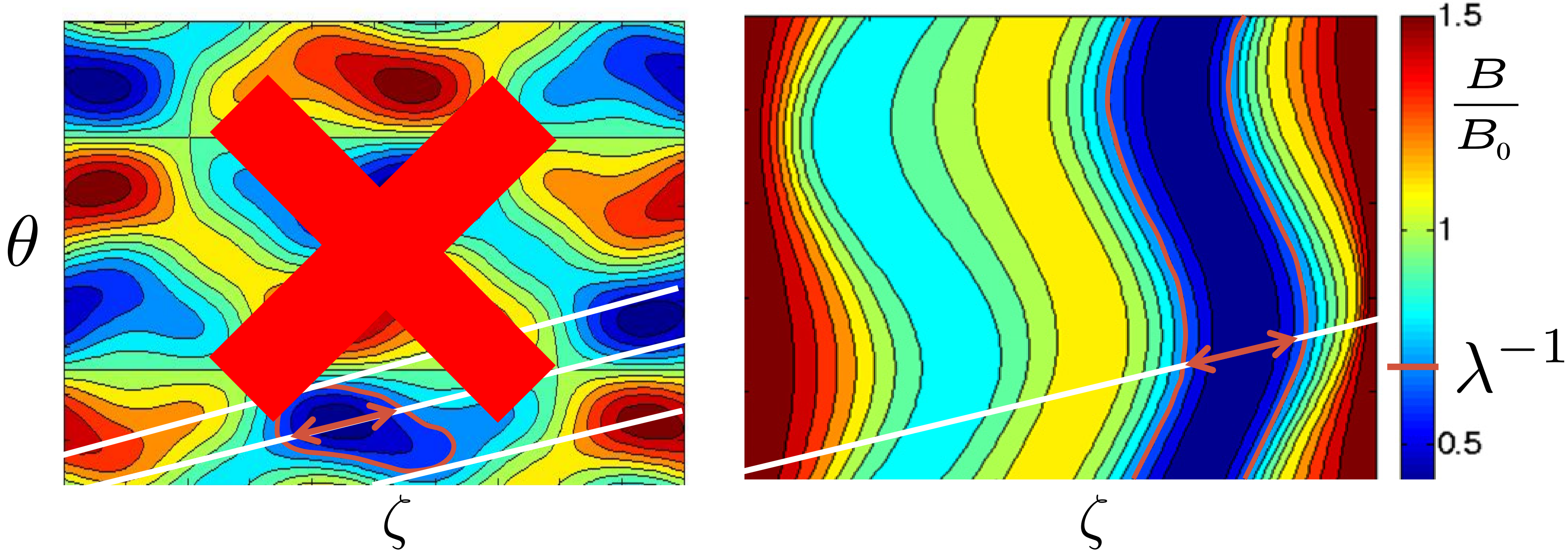


B-contours imply J cannot be constant on flux surface

Maxima B_M and minima B_m along $\mathbf{B}$ lines are the same

Keeping $\partial J/\partial\alpha = 0$ for all λ

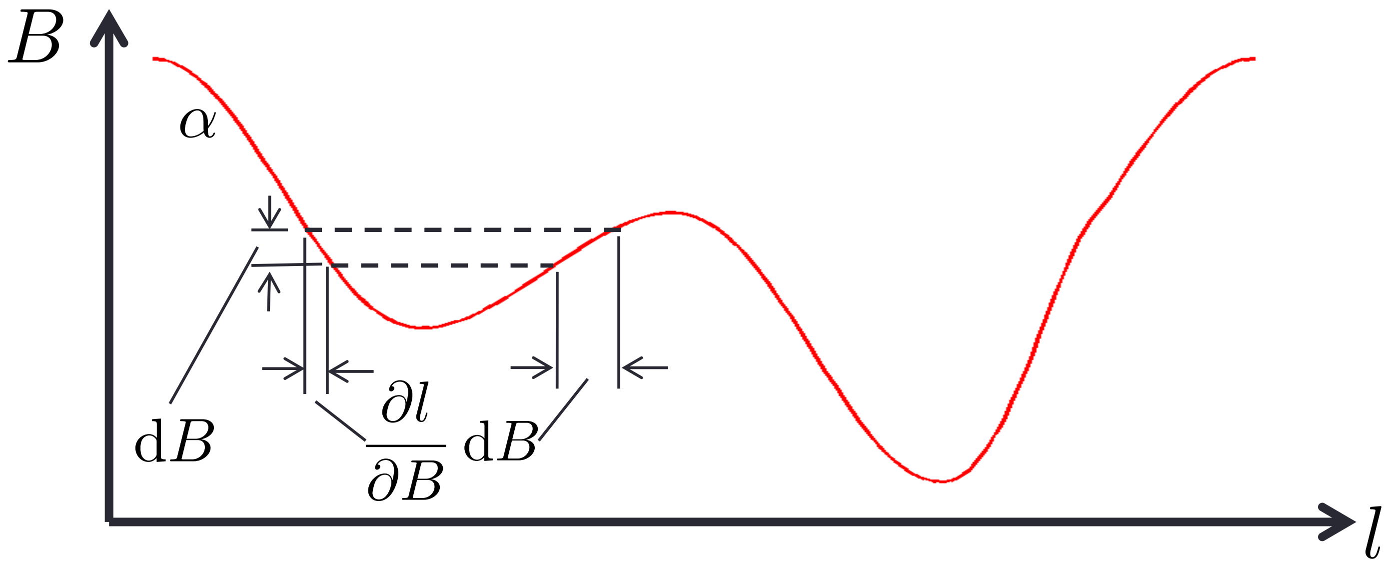
[Cary & Shasharina, PRL 97, PoP 97]

- Rewrite integral as integral over B

$$J = 2v\int_{l_L}^{l_R}\sqrt{1-\lambda B(l)}\,\mathrm{d}l = 2v\int_{B_m}^{1/\lambda}\sqrt{1-\lambda B}\,\frac{\partial l}{\partial B}\,\mathrm{d}B$$

$\Rightarrow$ $\partial l/\partial B$ independent of α

$\Rightarrow$ length along $\mathbf{B}$ line between two points with same B



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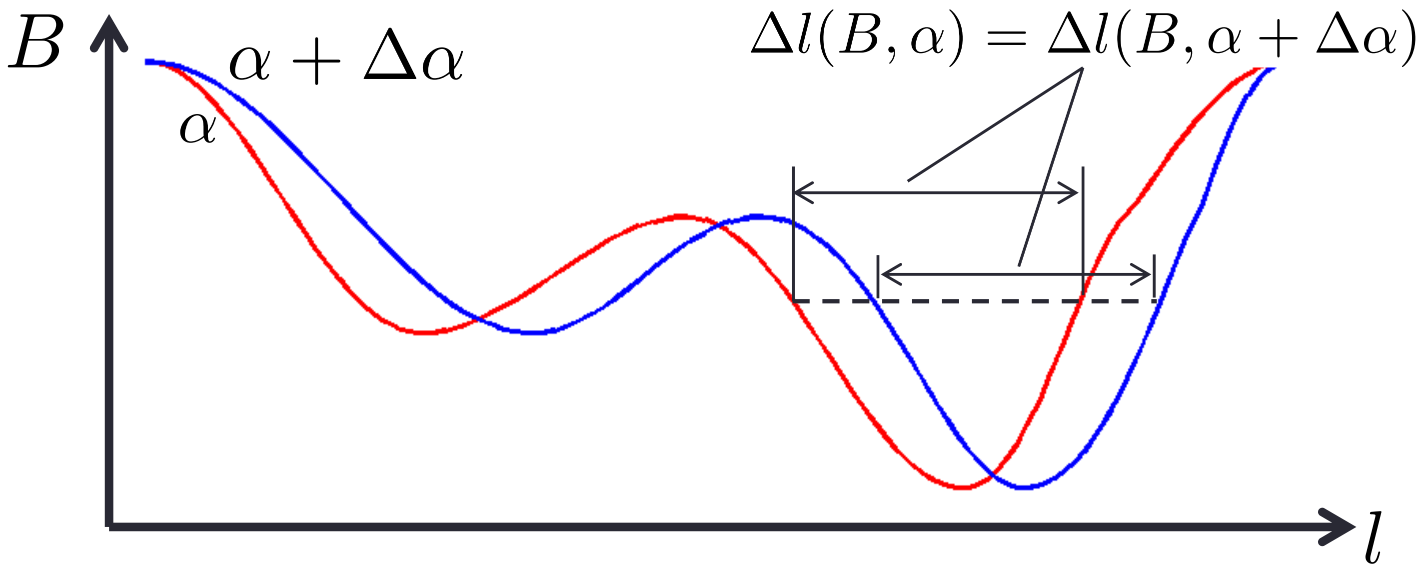
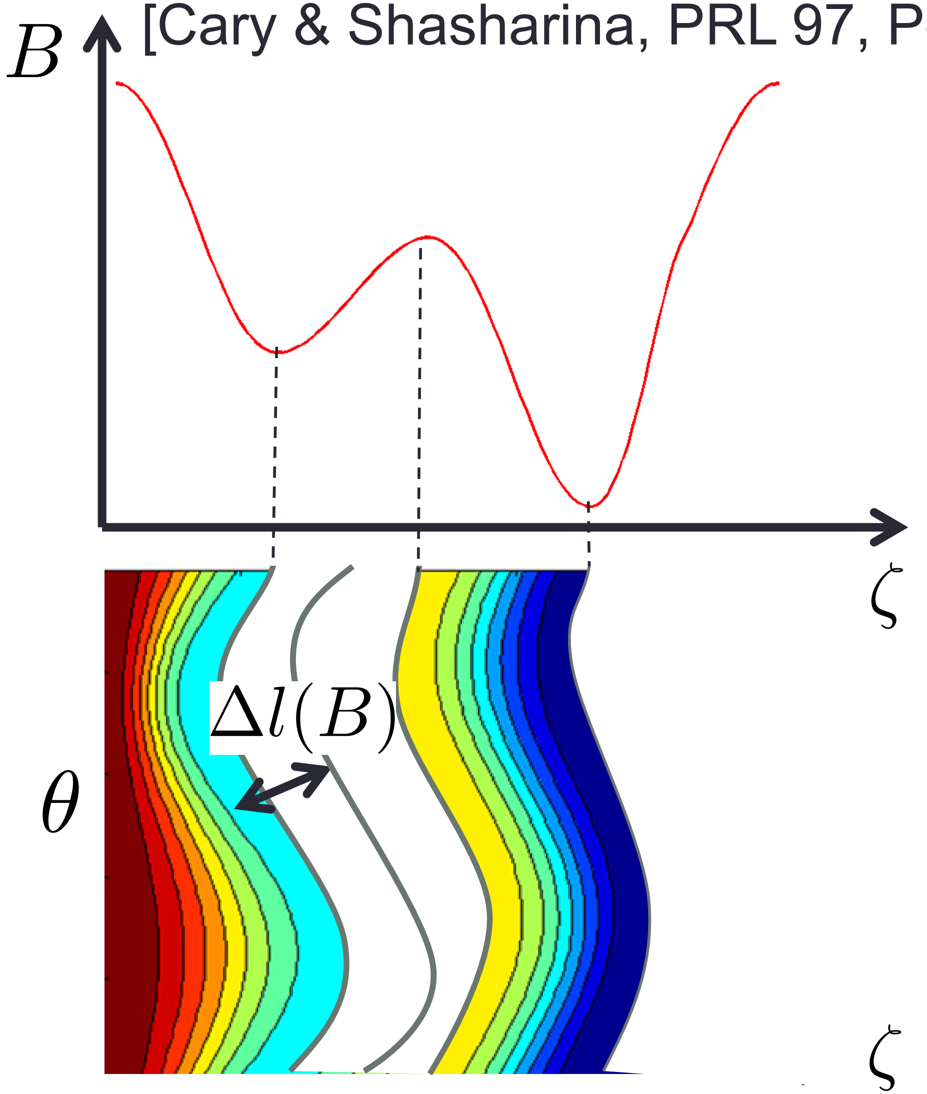


Construct omnigeneous B maps

[Cary & Shasharina, PRL 97, PoP 97]

- Choose a rough 1D B profile
  - Number of B_M and B_m
- Choose "half" of your contours
  - From B_M to B_m
- Choose $\Delta l\,(B)$
  - Boozer θ and ζ: $\mathbf{B}$ lines are straight lines, and $\Delta l \propto \Delta\theta, \Delta\zeta$
- Infinite $B\,(\theta,\,\zeta)$ that are omnigeneous!



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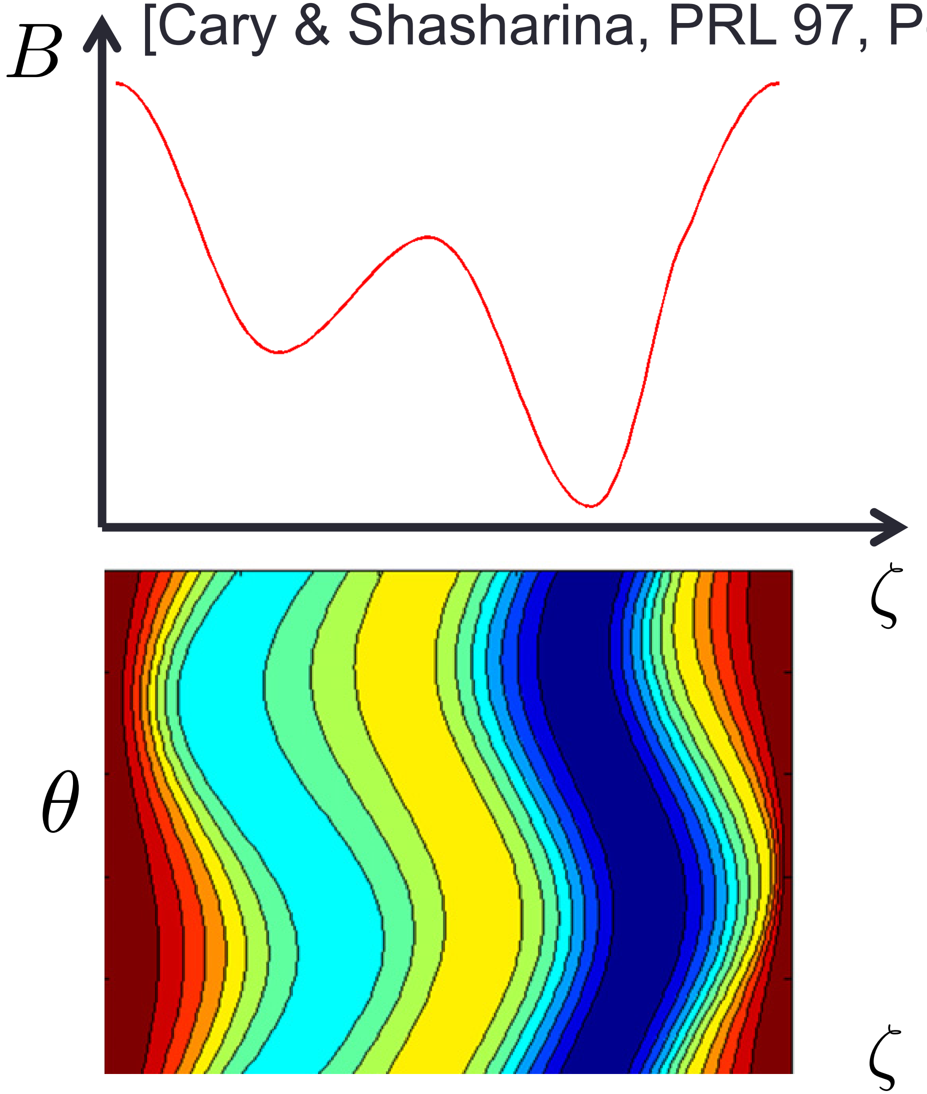


Omnigeneous stellarator equilibria

- Omnigeneous stellarators can reach Maxwellian equilibrium $\Rightarrow$ satisfy MHD force balance

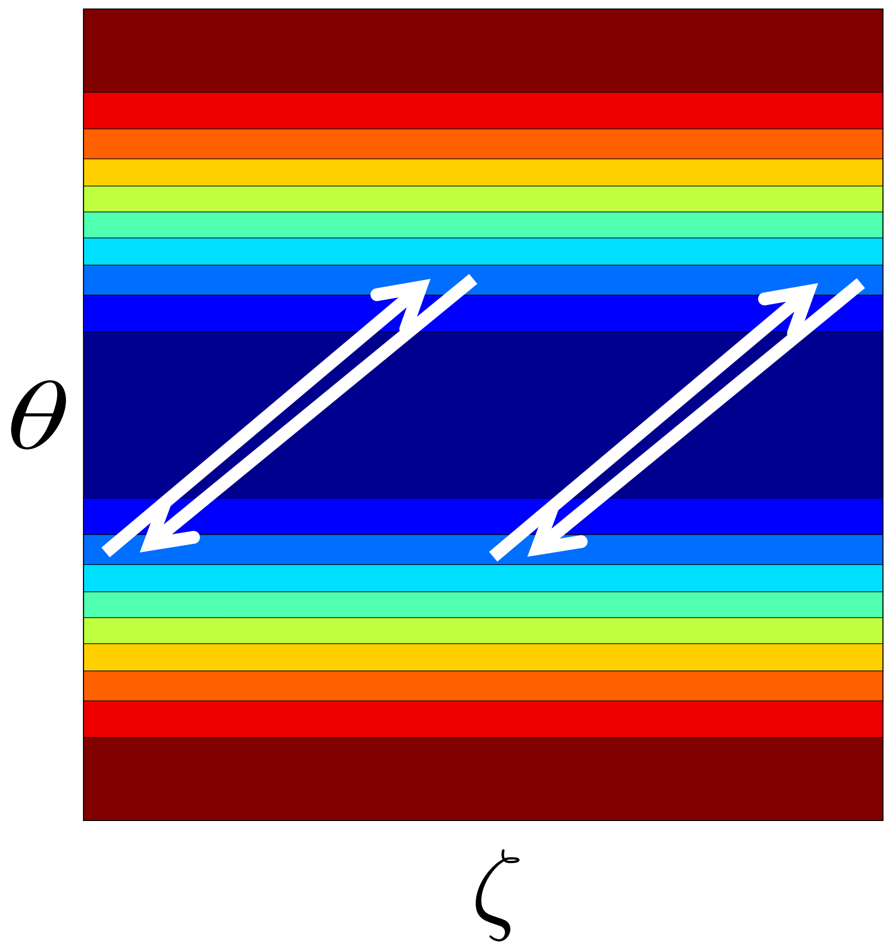
$$\frac{B^2}{4\pi}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} - \nabla_\perp\left(\frac{B^2}{8\pi}\right) = \nabla r\frac{\partial p}{\partial r}$$

- Assume known a flux surface r and $B\,(\theta,\,\zeta)$ on it
    - MHD equilibrium parallel to flux surface $\Rightarrow$ $\hat{\mathbf{b}}(\theta,\zeta)$
    - $\nabla\cdot\mathbf{B} = 0$ $\Rightarrow$ determines surface shape of surface $r - \Delta r$
    - MHD equilibrium perpendicular to flux surface $\Rightarrow$ $B\,(r - \Delta r,\,\theta,\,\zeta)$
- Integrate radially inwards to find equilibrium
- Regularity at the magnetic axis $\Rightarrow$ only the shape of flux surface r (and not $B\,(\theta,\,\zeta)$ on it) can be specified

Omnigeneous equilibria

[Garren & Boozer, PoP 91]

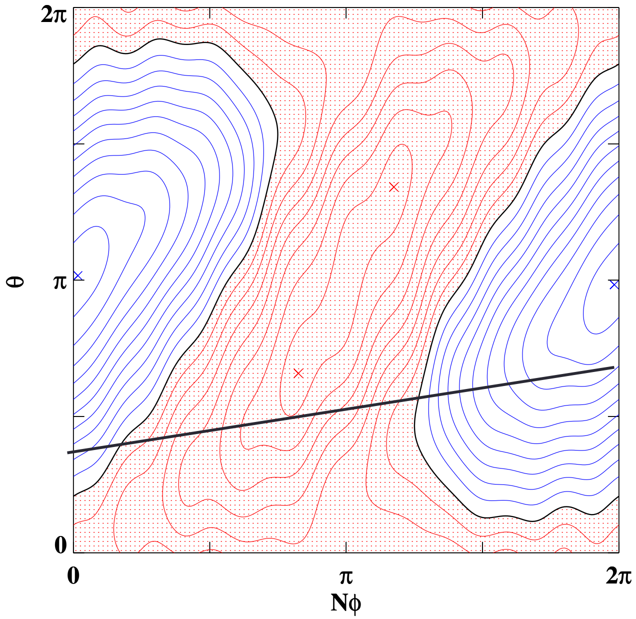
- Can choose one surface to be omnigeneous: choose 2D function $z\,(x,\,y)$ to obtain $B\,(\theta,\,\zeta)$
- In general, omnigeneity lost in other surfaces
- Is there an omnigeneous surface that propagates radially? For B with all derivatives continuous, only perfect axisymmetry: TOKAMAK



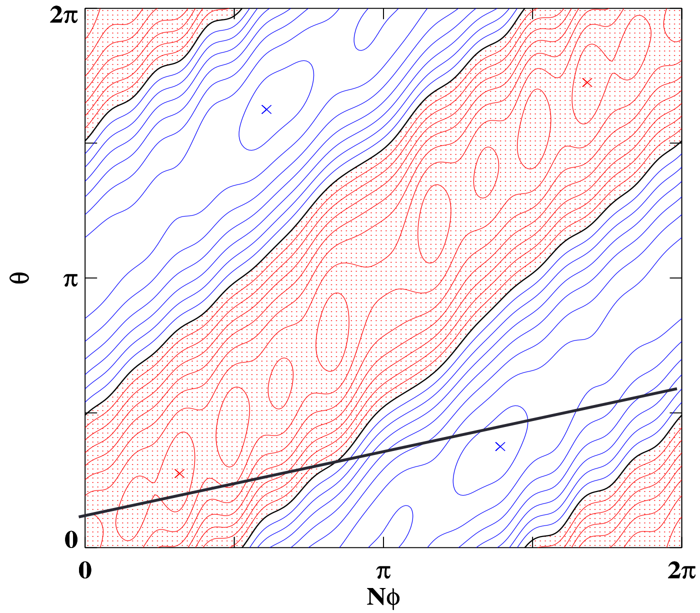
Any other omnigeneous equilibria?

[Beidler et al, NF 11]

- Stellarators can be optimized to be near-omnigenous
- Some examples
  - Boozer θ and ζ: $\mathbf{B}$ lines are straight lines, and $\Delta l \propto \Delta\theta, \Delta\zeta$



W7X, IPP-MPI, Germany



HSX, U. Wisconsin, USA

Some caveats

- Description of particle motion based on particles moving sufficiently far in between collisions

$$\nu \lesssim \rho_* \frac{v_t}{L}$$

- For "high" $\nu \gtrsim \rho_* v_t / L$, radial motion controlled by collisions

$$\frac{\Delta r}{r} \sim \frac{1}{\nu}\frac{\mathrm{d}r}{\mathrm{d}t} \sim \frac{mc}{Ze\nu\tau_b\Psi}\frac{\partial J}{\partial\alpha} \lesssim \frac{\rho_* v_t}{\nu L} \lesssim 1$$

- MHD equilibrium equations are not solvable in <u>rational</u> flux surfaces for <u>non-omnigeneous</u> stellarators

$$\nabla \cdot (J_{||}\hat{\mathbf{b}}) + \nabla \cdot \mathbf{J}_\perp = 0 \Rightarrow \underbrace{\boxed{\mathbf{B} \cdot \nabla}}_{\text{Singular}}\left(\frac{J_{||}}{B}\right) = -\nabla \cdot \left(\frac{c}{B}\hat{\mathbf{b}} \times \nabla p\right)$$

What else is there to do?

- If we relax continuity of derivatives, are there omnigeneous solutions other than the tokamak
  - This is independent of optimization programme: maybe the other omnigeneous solutions require unbuildable magnets!
- If perfect omnigeneity can ever be reached, how do we model low collisionality regions?
  - Inherently global problem with complicated particle orbits?
  - Some attempts of understanding the problem by expanding in either aspect ratio or “closeness” to omnigeneity
- How do we solve the problem of MHD equilibrium?
  - More of this in Stellarator sessions!
- Is this theory of interest elsewhere?