Dissipation-range kinetic turbulence in pressure-anisotropic astrophysical plasmas

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(Received ?; revised ?; accepted ?. - To be entered by editorial office)

In this paper, we continue our investigation of the gyrokinetic turbulent phase-space cascade of free energy in a multi-species, magnetized, pressure-anisotropic plasma, with a focus on the sub-Larmor "dissipation" range. After a brief recapitulation of Kunz *et al.* (2015), the study of which was the long-wavelength "inertial" range of gyrokinetic turbulence ("kinetic reduced magnetohydrodynamics"; KRMHD), we present the linear gyrokinetic theory of pressure-anisotropic "astrophysical" (i.e., slab) plasmas. Special attention is paid to the effect of electron pressure anisotropy on kinetic Alfvén waves. This is followed by a derivation and discussion of a very general gyrokinetic free-energy conservation law, which captures both the KRMHD free-energy conservation at long wavelengths and dual cascades of kinetic Alfvén waves and entropy at short wavelengths. Our results have implications for how pressure anisotropy affects the differential heating of ions and electrons as well as the ratio of parallel versus perpendicular phase mixing.

PACS codes:

1. Introduction

In a previous paper (Kunz *et al.* 2015, hereafter Paper I), we presented a theoretical framework for low-frequency electromagnetic (drift-)kinetic turbulence valid at scales larger than the particles' Larmor radii in a collisionless, multi-species plasma. The result generalised reduced magnetohydrodynamics (RMHD; Kadomtsev & Pogutse 1974; Strauss 1976, 1977; Zank & Matthaeus 1992) and kinetic RMHD (Schekochihin *et al.* 2009) to the case where the mean distribution function of the plasma is pressure-anisotropic and different ion species are allowed to drift with respect to each other – a situation routinely encountered in the solar wind (e.g. Hundhausen *et al.* 1967; Feldman *et al.* 1973; Marsch *et al.* 1982*a*,*b*; Marsch 2006) and presumably ubiquitous in hot dilute astrophysical plasma such as the intracluster medium of galaxy clusters (e.g. Schekochihin *et al.* 2005; Schekochihin & Cowley 2006). This framework was obtained via two

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routes: one starting from Kulsrud's formulation of kinetic MHD (Kulsrud 1964, 1983) and one starting from applying the nonlinear gyrokinetic ordering (e.g. Frieman & Chen 1982; Howes *et al.* 2006) to the Vlasov-Maxwell set of equations. The latter approach also enables a study of fluctuations at and below the ion Larmor scale, the subject of this Paper.

Before embarking on any quantitative analysis or even qualitative discussion of what the gyrokinetic framework entails, it seems most prudent to catalogue here the main theoretical achievements and implications of Paper I. First, we showed that the main physical feature of low-frequency long-wavelength plasma turbulence survives the generalisation to non-Maxwellian particle distribution functions: Alfvénic and compressive fluctuations are energetically decoupled, with the latter passively advected by the former. The Alfvénic cascade is fluid, satisfying RMHD equations (with the Alfvén speed modified by pressure anisotropy and interspecies drifts), whereas the compressive cascade is kinetic and subject to collisionless damping. For a bi-Maxwellian plasma, the kinetic cascade splits into three independent collisionless cascades. Secondly, the organising principle of this long-wavelength turbulence was elucidated in the form of a conservation law for the appropriately generalised kinetic free energy. Using this alongside linear theory, we showed that non-Maxwellian features in the distribution function reduce the rate of collisionless damping and the efficacy of magnetic stresses, and that these changes influence the partitioning of free energy amongst the various cascade channels. As the firehose or mirror instability thresholds are approached, the dynamics of the plasma are modified so as to reduce the energetic cost of bending magnetic-field lines or of compressing/rarefying them.

In this paper, the second in a series, we concentrate on the sub-Larmor-scale "dissipation" range. We investigate the linear properties of kinetic Alfvén waves and their nonlinear phase-space cascade in a plasma whose particle distribution functions exhibit pressure anisotropy and interspecies drifts. We find that the stability conditions imposed on the kinetic Alfvén waves by the anisotropy of the distribution functions are distinct from those experienced by Alfvén waves and compressive fluctuations at long wavelengths. We further show that, similar to the dual Alfvénic-kinetic cascade of free energy in the inertial range (Paper I), there are two sub-ion-Larmor-scale kinetic cascades: one of kinetic Alfvén waves, which is governed by a set of fluid-like electron reduced magnetohydrodynamic (ERMHD) equations, and a passive cascade of ion entropy fluctuations both in configuration and velocity space. While these cascades have been considered already for a single-ion-species isotropic plasma (Schekochihin *et al.* 2009), here we focus on whether and to what extent those results carry over to the more general case. Special attention is paid to the transition from the inertial range across the ion-Larmor scale to the dissipation range, and the effect of pressure anisotropy on the spectral location of this transition and on the amount of Landau-damped energy that ultimately makes its way to collisional scales.

All sections and equations in Paper I are referenced using the prefix "I–"; e.g. (I–C1) refers to equation (C1) of Paper I and §I–4 refers to section 4 of Paper I.

2. Prerequisites

2.1. *Basic equations and notation*

Although it would be wise for any reader of this paper to have at least surveyed its immediate predecessor, for completeness, we provide here the basic equations derived in Paper I from which this paper's results follow, as well as the notation introduced in Paper

I by which this paper's results may be understood.† This recapitulation starts with the Vlasov-Landau equation,

$$\dot{f}_s \doteq \frac{\partial f_s}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} f_s + \frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\boldsymbol{\cdot}\frac{\partial f_s}{\partial \boldsymbol{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{\rm coll}, \tag{2.1}$$

governing the space-time evolution of the particle distribution function of species s, $f_s = f_s(t, \boldsymbol{v}, \boldsymbol{r})$, where $\boldsymbol{v}$ is the velocity-space variable and $\boldsymbol{r}$ is the real-space variable. The charge and mass of species s are denoted q_s and m_s, respectively; c is the speed of light. The electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are expressed in terms of scalar and vector potentials:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad \text{and} \quad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \boldsymbol{\nabla}\times\boldsymbol{A}, \tag{2.2}$$

where $B_0\hat{\boldsymbol{z}}$ is the guide magnetic field, taken to lie along the z axis, and $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{A} = 0$ (the Coulomb gauge). These fields satisfy the plasma quasineutrality constraint,

$$0 = \sum_s q_s n_s = \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, f_s, \tag{2.3}$$

and the pre-Maxwell version of Ampère's law,

$$-\nabla^2\boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j} = \frac{4\pi}{c}\sum_s q_s n_s \boldsymbol{u}_s = \frac{4\pi}{c}\sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v} f_s, \tag{2.4}$$

where n_s and $\boldsymbol{u}_s$ are the number density and mean velocity of species s and $\boldsymbol{j}$ is the current density.

The term on the right-hand side of (2.1) represents the effect of collisions on the distribution function; in this paper, collisions are assumed to be sub-dominant and thus its specific form will not be required (precisely what 'sub-dominant' means will be stated below). The assumption of weak collisionality gives the pressure tensor

$$\mathsf{P}_s \doteq \int \mathrm{d}^3\boldsymbol{v}\, m_s(\boldsymbol{v}-\boldsymbol{u}_s)(\boldsymbol{v}-\boldsymbol{u}_s) f_s \tag{2.5}$$

the freedom to be anisotropic, even in the mean (zeroth-order) background. An example of such a pressure tensor is that describing a gyrotropic plasma (see §2.3),

$$\mathsf{P}_s = p_{\perp s}\big(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) + p_{\parallel s}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}, \tag{2.6}$$

where I is the unit dyadic, $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ is the unit vector in the direction of the magnetic field, the subscript $\perp$ ($\parallel$) denotes the component perpendicular (parallel) to $\hat{\boldsymbol{b}}$, and

$$p_{\parallel s} \doteq n_s T_{\parallel s} = \int \mathrm{d}^3\boldsymbol{v}\, m_s (v_\parallel - u_{\parallel s})^2 f_s, \tag{2.7}$$

$$p_{\perp s} \doteq n_s T_{\perp s} = \int \mathrm{d}^3\boldsymbol{v}\, \frac{m_s}{2}|\boldsymbol{v}_\perp - \boldsymbol{u}_{\perp s}|^2 f_s \tag{2.8}$$

are the parallel and perpendicular pressures, respectively, of species s. An oft-employed distribution function that exhibits such pressure anisotropy is the bi-Maxwellian

$$f_{bi\text{-}M,s}(v_\parallel, v_\perp) \doteq \frac{n_s}{\sqrt{\pi}v_{th\parallel s}}\exp\left[-\frac{(v_\parallel - u_{\parallel s})^2}{v^2_{th\parallel s}}\right]\frac{1}{\pi v^2_{th\perp s}}\exp\left[-\frac{|\boldsymbol{v}_\perp - \boldsymbol{u}_{\perp s}|^2}{v^2_{th\perp s}}\right], \tag{2.9}$$

† A glossary of frequently used symbols can be found in Appendix E of Paper I.

where

$$v_{th\|s} \doteq \sqrt{\frac{2T_{\|s}}{m_s}} \quad \text{and} \quad v_{th\perp s} \doteq \sqrt{\frac{2T_{\perp s}}{m_s}} \tag{2.10}$$

are the parallel and perpendicular thermal speeds of species s. Pressure anisotropy is caused in a weakly collisional plasma by adiabatic invariance; conservation of the magnetic moment $\mu_s \doteq m_s|\boldsymbol{v}_\perp - \boldsymbol{u}_{\perp s}|^2/2B$ implies that a slow change in magnetic-field strength must be accompanied by a proportional change in the perpendicular temperature of species s. While such velocity-space anisotropy is generically exhibited by the gyrokinetic fluctuations regardless of whether the mean distribution function is proved (or assumed) to be isotropic and Maxwellian, in what follows we allow for the possibility of a background pressure anisotropy. In doing so, we relax two of the common assumptions of standard gyrokinetics – that the mean distribution functions of all species are Maxwellian and that there is only one ionic species.

2.2. *Gyrokinetic ordering*

Our aim is to reduce (2.1)–(2.4) so that they describe only those fields whose fluctuating parts are small compared to the mean field, are spatially anisotropic with respect to it, have frequencies ω small compared to the Larmor frequency $\Omega_s \doteq q_s B_0/m_s c$, and have parallel length scales $k_\|^{-1}$ large compared to the Larmor radius $\rho_s \doteq v_{th\perp s}/\Omega_s$. While such specifications may appear to be quite restrictive, modern theories (e.g. Goldreich & Sridhar 1995) and numerical simulations (e.g. Shebalin *et al.* 1983; Oughton *et al.* 1994; Cho & Vishniac 2000; Maron & Goldreich 2001) of magnetized turbulence provide a strong foundation for expecting such anisotropic low-frequency fluctuations to comprise much of the turbulent cascade. Such spatial anisotropy is also now routinely measured in the solar wind (e.g. Bieber *et al.* 1996; Horbury *et al.* 2008; Podesta 2009; Wicks *et al.* 2010; Chen *et al.* 2011) and suggested by observations of turbulent density fluctuations in the interstellar medium (e.g. Armstrong *et al.* 1990; Rickett *et al.* 2002).

The reduction is carried out in detail in appendix C of Paper I; here we describe its primary ingredients and principal consequences. The fields are split into their mean parts (denoted with a subscript '0') and fluctuating parts (denoted with δ), the former characterized by spatial homogeneity. The latter are taken to satisfy the asymptotic ordering

$$\frac{\delta f_{1s}}{f_{0s}} \sim \frac{\delta \boldsymbol{B}}{B_0} \sim \frac{\delta \boldsymbol{E}}{(v_{th\|s}/c)B_0} \sim \frac{k_\|}{k_\perp} \sim \frac{\omega}{\Omega_s} \doteq \epsilon \ll 1, \quad k_\perp \rho_s \sim 1, \tag{2.11}$$

where we have expanded the distribution function in powers of ϵ:

$$f_s = f_{0s} + \delta f_s = f_{0s} + \delta f_{1s} + \delta f_{2s} + \dots. \tag{2.12}$$

Note that the fluctuations are permitted to have (perpendicular) scales on the order of the Larmor radius. We further assume that the collision frequency $\nu_s \ll \epsilon^2 \Omega_s$, thus allowing non-Maxwellian f_{0s} (cf. §A2.2 of Howes *et al.* 2006).

The gyrokinetic ordering guarantees that (to lowest order) all species drift perpendicularly to the magnetic field with identical velocities, $\boldsymbol{u}_{\perp s} = \boldsymbol{u}_\perp = c\boldsymbol{E}\times\boldsymbol{B}/B^2$. It then follows that the mean drift of any species relative to the centre-of-mass velocity $\boldsymbol{u} \doteq \sum_s m_s n_s \boldsymbol{u}_s / \sum_s m_s n_s$ must be in the parallel direction, *viz.*, $\boldsymbol{u}_s = \boldsymbol{u} + u'_{\|s}\hat{\boldsymbol{b}}$, with

$$u'_{\|s} = \frac{1}{n_s}\int \mathrm{d}^3\boldsymbol{v}\,(v_\| - u_\|) f_s. \tag{2.13}$$

Our collisionless ordering permits parallel interspecies drifts (denoted by $u'_{\|0s}$) in the background state, and we formally take $u'_{\|0s} \sim v_{th\|s}$ for all species s. We further assume that the Alfvén speed,

$$v_A \doteq \frac{B_0}{\sqrt{4\pi\rho_0}}, \tag{2.14}$$

is of order the thermal speed, where ρ_0 is the mean mass density of the plasma. This implies that the parallel and perpendicular plasma beta parameters,

$$\beta_{\|s} \doteq \frac{8\pi p_{\|0s}}{B_0^2} \quad \text{and} \quad \beta_{\perp s} \doteq \frac{8\pi p_{\perp 0s}}{B_0^2} \tag{2.15}$$

respectively, are considered to be of order unity in the gyrokinetic expansion. The other dimensionless parameters in the system – namely, the electron-ion mass ratio m_e/m_i, the charge ratio $Z = q_i/|q_e| = q_i/e$, the parallel and perpendicular temperature ratios

$$\tau_{\|s} \doteq \frac{T_{\|0s}}{T_{\|0e}} \quad \text{and} \quad \tau_{\perp s} \doteq \frac{T_{\perp 0s}}{T_{\perp 0e}}, \tag{2.16}$$

and the temperature anisotropy $T_{\perp 0s}/T_{\|0s}$ of species s – are all considered to be of order unity as well. Subsidiary expansions with respect to these parameters can (and will) be made after the gyrokinetic expansion is performed.

Since we have $\omega \sim k_\| v_{ths} \sim k_\| v_A$, fast magnetosonic fluctuations are ordered out of our equations. Such fast-wave fluctuations are rarely seen in the solar wind (Howes *et al.* 2012). Observations of turbulence in the solar wind confirm that it is primarily Alfvénic (e.g. Belcher & Davis 1971) and that its compressive component is substantially pressure-balanced (Burlaga *et al.* 1990; Roberts 1990; Marsch & Tu 1993; Bavassano *et al.* 2004). A more serious limitation of our analysis is perhaps the exclusion of cyclotron resonances, which have been traditionally considered necessary to explain the strong perpendicular heating observed in the solar wind (Leamon *et al.* 1998; Isenberg 2001; Kasper *et al.* 2013). The same is true of Larmor-scale fluctuations whose amplitudes are large enough to break adiabatic invariance and thus drive chaotic gyromotion and stochastic particle heating (Chandran *et al.* 2010, 2013). That being said, the gyrokinetic framework does capture a lot of physics within both the inertial and dissipative ranges of kinetic turbulence, and so it is a sensible step to incorporate realistic background distribution functions into the gyrokinetic description of weakly collisional astrophysical plasmas. It is with that goal in mind that we commence with a presentation of the gyrokinetic theory.

2.3. *Gyrokinetic reduction*

2.3.1. *Gyrotropy of the background distribution function*

Under the ordering (2.11), the largest term in the Vlasov-Landau equation (2.1) corresponds to Larmor motion of the mean distribution about the uniform guide field:

$$-\Omega_s \hat{\boldsymbol{z}} \boldsymbol{\cdot} \left(\boldsymbol{v} \times \frac{\partial f_{0s}}{\partial \boldsymbol{v}} \right) = 0. \tag{2.17}$$

This directional bias allows us to set up a local Cartesian coordinate system and decompose the particle velocity in terms of the parallel velocity $v_\|$, the perpendicular velocity $v_\perp$, and the gyrophase angle ϑ,

$$\boldsymbol{v} = v_\| \hat{\boldsymbol{z}} + v_\perp (\cos\vartheta\, \hat{\boldsymbol{x}} + \sin\vartheta\, \hat{\boldsymbol{y}}). \tag{2.18}$$

Equation (2.17) then takes on the simple form

$$-\Omega_s \frac{\partial f_{0s}}{\partial \vartheta} = 0, \tag{2.19}$$

which states that the mean distribution function is gyrotropic (independent of the gyrophase):

$$f_{0s} = f_{0s}(v_\parallel, v_\perp, t). \tag{2.20}$$

All velocity-space derivatives of f_{0s} that enter (2.1) are thus with respect to $v_\parallel$ and $v_\perp$, *viz.*

$$\frac{q_s}{m_s}\frac{\partial f_{0s}}{\partial \boldsymbol{v}} = -(v_\parallel - u'_{\parallel 0s})\hat{\boldsymbol{b}}\,\frac{q_s f^{\parallel}_{0s}}{T_{\parallel 0s}} - \boldsymbol{v}_\perp\,\frac{q_s f^{\perp}_{0s}}{T_{\perp 0s}}, \tag{2.21}$$

where

$$f^{\parallel}_{0s} \doteq -v^2_{th\parallel s}\,\frac{\partial f_{0s}}{\partial (v_\parallel - u'_{\parallel 0s})^2} \qquad \text{and} \qquad f^{\perp}_{0s} \doteq -v^2_{th\perp s}\,\frac{\partial f_{0s}}{\partial v^2_\perp} \tag{2.22}$$

are dimensionless derivatives of a species' mean distribution function with respect to the square of the parallel velocity (peculiar to the species drift velocity) and the perpendicular velocity, respectively. Their weighted difference,

$$\mathfrak{D}f_{0s} \doteq \frac{T_{\perp 0s}}{T_{\parallel 0s}} f^{\parallel}_{0s} - f^{\perp}_{0s}, \tag{2.23}$$

measures the velocity-space anisotropy of the mean distribution function; for a bi-Maxwellian distribution, $f^{\parallel}_{0s} = f^{\perp}_{0s} = f_{0s}$ and

$$\mathfrak{D}f_{0s} = \left(\frac{T_{\perp 0s}}{T_{\parallel 0s}} - 1\right) f_{0s} \doteq \Delta_s f_{0s}, \tag{2.24}$$

where Δ_s, defined *in situ*, is the temperature (or, equivalently, pressure) anisotropy of the mean distribution function of species s.

2.3.2. *Boltzmann response*

At $\mathcal{O}(\epsilon\Omega_s f_{0s})$, we learn from (2.1) that the first-order distribution function δf_{1s} may be split into two parts. The first of these is the so-called adiabatic (or 'Boltzmann') response,

$$\delta f_{1s,Boltz} = -\frac{q_s\varphi'_s}{T_{\parallel 0s}} f^{\parallel}_{0s} + \frac{q_s}{T_{\perp 0s}}\left(\varphi - \frac{v_\parallel A_\parallel}{c}\right)\mathfrak{D}f_{0s}, \tag{2.25}$$

where

$$\varphi'_s \doteq \varphi - u'_{\parallel 0s}A_\parallel/c \tag{2.26}$$

is the fluctuating electrostatic potential in the frame of the parallel-drifting species s. This part of δf_{1s} represents the (leading-order) evolution of f_{0s} under the influence of the perturbed electromagnetic fields. Indeed, if we introduce the total particle energy in the parallel-drifting frame,

$$\overline{\varepsilon}_s = \frac{1}{2}m_s\big|\boldsymbol{v} - u'_{\parallel 0s}\hat{\boldsymbol{z}}\big|^2 + q_s\varphi'_s \tag{2.27}$$

and the (gyrophase-dependent part of the) first adiabatic invariant,

$$\overline{\mu}_s = \frac{m_s v^2_\perp}{2B_0} + \frac{q_s}{B_0}\left(\varphi - \frac{v_\parallel A_\parallel}{c}\right), \tag{2.28}$$

both written out to first order in the fluctuation amplitudes (e.g. Kruskal 1958; Hastie *et al.* 1967; Taylor 1967; Catto *et al.* 1981; Parra 2013), it is straightforward to show (see §I–C.4) that the sum of the mean distribution function and the Boltzmann response is simply

$$f_{0s}(v_\parallel, v_\perp) + \delta f_{1s,Boltz} = f_{0s}(\overline{\varepsilon}_s, \overline{\mu}_s) + \mathcal{O}(\epsilon^2). \tag{2.29}$$

In other words, the Boltzmann response does not change the form of the mean distribution function if the latter is written as a function of sufficiently precisely conserved particle invariants.

2.3.3. *Gyrokinetic response*

The second part of δf_{1s}, which we denote by h_s, represents the response of charged rings to the fluctuating fields, and is thus referred to as the *gyrokinetic response*. It satisfies

$$\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp h_s - \Omega_s \frac{\partial h_s}{\partial \vartheta}\bigg|_{\boldsymbol{r}} = -\Omega_s \frac{\partial h_s}{\partial \vartheta}\bigg|_{\boldsymbol{R}_s} = 0, \tag{2.30}$$

where we have transformed the ϑ derivative taken at constant position $\boldsymbol{r}$ to one taken at constant guiding centre

$$\boldsymbol{R}_s = \boldsymbol{r} + \frac{\boldsymbol{v} \times \hat{\boldsymbol{z}}}{\Omega_s}. \tag{2.31}$$

Thus, h_s is independent of the gyrophase angle at constant guiding centre $\boldsymbol{R}_s$ (but not at constant position $\boldsymbol{r}$):

$$h_s = h_s(t, \boldsymbol{R}_s, v_\parallel, v_\perp). \tag{2.32}$$

2.3.4. *Gyrokinetic equation*

At $\mathcal{O}(\epsilon^2 \Omega_s f_{0s})$, we find from (2.1) that the gyrokinetic response evolves via the *gyrokinetic equation*

$$\begin{aligned}\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\} = {} & \frac{q_s f_{0s}^{\parallel}}{T_{\parallel 0s}}\left(\frac{\partial}{\partial t} + u'_{\parallel 0s}\frac{\partial}{\partial z}\right)\langle\chi\rangle_{\boldsymbol{R}_s} \\ & - \frac{q_s\,\mathfrak{D}f_{0s}}{T_{\perp 0s}}\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z}\right)\langle\chi\rangle_{\boldsymbol{R}_s},\end{aligned} \tag{2.33}$$

where

$$\chi \doteq \varphi - \frac{v_\parallel A_\parallel}{c} - \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \tag{2.34}$$

is the gyrokinetic potential and

$$\langle\chi(t,\boldsymbol{r},\boldsymbol{v})\rangle_{\boldsymbol{R}_s} \doteq \frac{1}{2\pi}\oint \mathrm{d}\vartheta\, \chi\left(t, \boldsymbol{R}_s - \frac{\boldsymbol{v}\times\hat{\boldsymbol{z}}}{\Omega_s}, \boldsymbol{v}\right) \tag{2.35}$$

denotes the ring average of χ at fixed guiding centre $\boldsymbol{R}_s$. The Poisson bracket

$$\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\} \doteq \hat{\boldsymbol{z}} \boldsymbol{\cdot} \left(\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial \boldsymbol{R}_s} \times \frac{\partial h_s}{\partial \boldsymbol{R}_s}\right) \tag{2.36}$$

represents the nonlinear interaction between the gyrocentre rings and the electromagnetic fields.

The gyrokinetic equation (2.33) can also be written in the following, perhaps more physically illuminating form,

$$\frac{\partial h_s}{\partial t} + \langle\dot{\boldsymbol{R}}_s\rangle_{\boldsymbol{R}_s} \boldsymbol{\cdot} \frac{\partial h_s}{\partial \boldsymbol{R}_s} = -\langle\dot{\bar{\varepsilon}}_s\rangle_{\boldsymbol{R}_s}\frac{\partial f_{0s}}{\partial \bar{\varepsilon}_s} - \langle\dot{\bar{\mu}}_s\rangle_{\boldsymbol{R}_s}\frac{\partial f_{0s}}{\partial \bar{\mu}_s}, \tag{2.37}$$

where

$$\langle\dot{\boldsymbol{R}}_s\rangle_{\boldsymbol{R}_s} = v_\parallel \hat{\boldsymbol{z}} - \frac{c}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial \boldsymbol{R}_s} \times \hat{\boldsymbol{z}} \tag{2.38}$$

is the ring velocity,

$$\langle \dot{\overline{\varepsilon}}_s \rangle_{\boldsymbol{R}_s} = q_s \left(\frac{\partial}{\partial t} + u'_{\parallel 0s} \frac{\partial}{\partial z} \right) \langle \chi \rangle_{\boldsymbol{R}_s} \tag{2.39}$$

is the ring-averaged rate of change of the particle energy (equation (2.27)), and

$$\langle \dot{\overline{\mu}}_s \rangle_{\boldsymbol{R}_s} = \frac{q_s}{B_0} \left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z} \right) \langle \chi \rangle_{\boldsymbol{R}_s} \tag{2.40}$$

is the ring-averaged rate of change of the (gyrophase-dependent part of the) first adiabatic invariant (equation (2.28)). The right-hand side of (2.37) represents the effect of collisionless work done on the rings by the fields (the wave-ring interaction). Written in this way, (2.33) is simply the ring-averaged Vlasov equation, $\langle \dot{f}_s(t, \boldsymbol{r}, \overline{\varepsilon}_s, \overline{\mu}_s) \rangle_{\boldsymbol{R}_s} = 0$ to lowest order in ϵ.

It is a manifestly good idea in much of what follows to absorb the final term of (2.33) (and, likewise, of (2.37)), into h_s by writing the latter in terms of the velocity-space coordinates $(v_\parallel, \mu_s)$, where

$$\mu_s \doteq \overline{\mu}_s - \frac{q_s}{B_0} \langle \chi \rangle_{\boldsymbol{R}_s} \tag{2.41}$$

is the full adiabatic invariant; to wit, $\langle \dot{\mu}_s \rangle_{\boldsymbol{R}_s} \sim \mathcal{O}(\epsilon^2 \omega T_{\perp 0s} / B_0)$. Note that, at long wavelengths satisfying $k_\perp \rho_s \ll 1$,

$$\mu_s \simeq \frac{m_s |\boldsymbol{v} - \boldsymbol{u}_\perp - \boldsymbol{v} \cdot \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}|^2}{2B} \doteq \frac{m_s w_\perp^2}{2B}, \tag{2.42}$$

which is simply the magnetic moment of a particle in a magnetic field of strength B drifting across said field at the $\boldsymbol{E} \times \boldsymbol{B}$ velocity. Then, introducing

$$\widetilde{h}_s(v_\parallel, \mu_s) \doteq h_s(v_\parallel, v_\perp) + \frac{q_s \, \mathfrak{D} f_{0s}}{T_{\perp 0s}} \langle \chi \rangle_{\boldsymbol{R}_s}, \tag{2.43}$$

the gyrokinetic equation reads

$$\frac{\partial \widetilde{h}_s}{\partial t} + \langle \dot{\boldsymbol{R}}_s \rangle_{\boldsymbol{R}_s} \cdot \frac{\partial \widetilde{h}_s}{\partial \boldsymbol{R}_s} = \frac{q_s f_{0s}^{\parallel}}{T_{\parallel 0s}} \left(\frac{\partial}{\partial t} + u'_{\parallel 0s} \frac{\partial}{\partial z} \right) \langle \chi \rangle_{\boldsymbol{R}_s}. \tag{2.44}$$

This form of the gyrokinetic equation is particularly well suited for deriving the gyrokinetic invariants (§4). Its right-hand side represents the collisionless work done on the rings by the fields in a frame comoving with the parallel drift velocity of species s. It will also prove useful in what follows to modify the energy variable $\overline{\varepsilon}_s$ to obtain

$$\varepsilon_s \doteq \overline{\varepsilon}_s - q_s \langle \varphi'_s \rangle_{\boldsymbol{R}_s}, \tag{2.45}$$

which is the kinetic energy of the particle as measured in the frame moving with the $u'_{\parallel 0s}$ and $\boldsymbol{E} \times \boldsymbol{B}$ drifts; indeed,

$$\varepsilon_s \simeq \frac{1}{2} m_s (v_\parallel - u'_{\parallel 0s})^2 + \frac{1}{2} m_s w_\perp^2 \tag{2.46}$$

at long wavelengths. If the mean distribution function is expressed in terms of these new velocity-space variables, *viz.* $f_s = \widetilde{f}_{0s}(\varepsilon_s, \mu_s) + \delta \widetilde{f}_s$, then the perturbed distribution function $\delta \widetilde{f}_s$ becomes

$$\delta \widetilde{f}_s(\varepsilon_s, \mu_s) = \widetilde{h}_s(v_\parallel, \mu_s) - \frac{q_s f_{0s}^{\parallel}}{T_{\parallel 0s}} \langle \varphi'_s \rangle_{\boldsymbol{R}_s}. \tag{2.47}$$

This particular form of the perturbed distribution function is quite useful; it is the

$k_\perp \rho_s \sim 1$ generalisation of the perturbed distribution function that prominently features in the generalised free energy of KRMHD (§I–5.1), and thus is anticipated to appear in the generalised free energy of the gyrokinetic theory. The latter is derived in Section 4.

2.3.5. *Field equations*

The equations governing the electromagnetic potentials are most easily obtained by substituting the decomposition

$$f_s = f_{0s}(v_\parallel, v_\perp) + \delta f_{1s,Boltz} + h_s(t, \boldsymbol{R}_s, v_\parallel, v_\perp) + \delta f_{2s} + \dots \tag{2.48}$$

into the leading-order expansions of the quasineutrality constraint (2.3) and Ampère's law (equation (2.4)). The result is (see Paper I, §C.3)

$$0 = \sum_s q_s \left[\int \mathrm{d}^3\boldsymbol{v}\, \langle h_s \rangle_{\boldsymbol{r}} - \frac{q_s n_{0s}}{T_{\perp 0s}} \left(C^\perp_{0s}\varphi - C^\perp_{1s} \frac{u'_{\parallel 0s} A_\parallel}{c} \right) \right], \tag{2.49}$$

$$\nabla^2_\perp A_\parallel = -\frac{4\pi}{c} \sum_s q_s \left[\int \mathrm{d}^3\boldsymbol{v}\, v_\parallel \langle h_s \rangle_{\boldsymbol{r}} - \frac{q_s n_{0s} v_{th\parallel s}}{2T_{\perp 0s}} \left(C^\perp_{1s}\varphi \frac{2u'_{\parallel 0s}}{v_{th\parallel s}} + \widetilde{\Delta}_s \frac{v_{th\parallel s} A_\parallel}{c} \right) \right], \tag{2.50}$$

$$\nabla^2_\perp \delta B_\parallel = -\frac{4\pi}{c} \hat{\boldsymbol{z}} \boldsymbol{\cdot} \left[\boldsymbol{\nabla}_\perp \times \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, \langle \boldsymbol{v} h_s \rangle_{\boldsymbol{r}} \right], \tag{2.51}$$

where

$$\widetilde{\Delta}_s \doteq \frac{T_{\perp 0s}}{T_{\parallel 0s}} - C^\perp_{2s} \left(1 + \frac{2u'^2_{\parallel 0s}}{v^2_{th\parallel s}} \right) \tag{2.52}$$

is the temperature anisotropy of species s augmented by the parallel ram pressure from background parallel drifts, $C^\perp_{\ell s}$ are parallel moments of the perpendicular-differentiated mean distribution function (all of which equate to unity for a drifting bi-Maxwellian distribution; see Appendix B), and

$$\langle h_s(t, \boldsymbol{R}_s, v_\parallel, v_\perp) \rangle_{\boldsymbol{r}} \doteq \frac{1}{2\pi} \oint \mathrm{d}\vartheta\, h_s \left(t, \boldsymbol{r} + \frac{\boldsymbol{v} \times \hat{\boldsymbol{z}}}{\Omega_s}, v_\parallel, v_\perp \right) \tag{2.53}$$

denotes the gyro-average of h_s at fixed $\boldsymbol{r}$. Together with the gyrokinetic equation (2.33), the field equations (2.49)–(2.51) constitute a closed system that describes the evolution of a gyrokinetic plasma with non-Maxwellian f_{0s} and parallel interspecies drifts.

This completes our abbreviated review of the material derived in Paper I on the gyrokinetic framework for homogeneous, non-Maxwellian plasmas. Everything henceforth is new.

3. Linear gyrokinetic theory

3.1. *From rings to gyrocentres*

The most straightforward way of making contact with the results of Paper I, while facilitating the extension of the theoretical framework into the dissipation range, is via the linear theory. This is obtained most easily by shifting the description of the plasma from one composed of extended rings of charge that move in a vacuum to one of a gas of point-particle-like gyrocentres moving in a polarisable medium. This transformation is enacted by working with the *gyrocentre distribution function*

$$g_s \doteq \widetilde{h}_s - \frac{q_s}{T_{\parallel 0s}} \left\langle \varphi'_s - \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \right\rangle_{\boldsymbol{R}_s} f^{\parallel}_{0s} \tag{3.1a}$$

$$= \langle \delta f_{1s} \rangle_{\boldsymbol{R}_s} + \frac{q}{T_{0s}} \left\langle \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \right\rangle_{\boldsymbol{R}_s} f^{\perp}_{0s}. \tag{3.1b}$$

This new function not only helps simplify the algebra involved in deriving the linear theory, but also makes a good deal of physical sense. In the electrostatic limit, the use of g_s (which, in this limit, equals $\langle \delta f_{1s} \rangle_{\boldsymbol{R}_s}$) aids in the interpretation of polarisation effects within gyrokinetics (Krommes 2012), places the gyrokinetic equation in a numerically convenient characteristic form (Lee 1983), and arises naturally from the Hamiltonian formulation of gyrokinetics (Dubin *et al.* 1983; Brizard & Hahm 2007). In the electromagnetic case, introducing g_s takes advantage of the fact that the Alfvénic fluctuations have a gyrokinetic response that is largely cancelled at long wavelengths by the Boltzmann response (Paper I, §C.4).

Using (3.1) to replace $\widetilde{h}_s$ in the gyrokinetic equation (2.44), we find that g_s evolves according to

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} + \frac{c}{B_0} \{ \langle \chi \rangle_{\boldsymbol{R}_s}, g_s \} = - \frac{q_s f^{\parallel}_{0s}}{T_{\parallel 0s}} \left(v_\parallel - u'_{\parallel 0s} \right) \left\langle \frac{1}{B_0} \{ A_\parallel, \varphi - \langle \varphi \rangle_{\boldsymbol{R}_s} \} \right.$$
$$\left. + \frac{1}{c} \frac{\partial A_\parallel}{\partial t} + \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} \varphi - \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} \left\langle \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \right\rangle_{\boldsymbol{R}_s} \right\rangle_{\boldsymbol{R}_s}, \tag{3.2}$$

where

$$\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} = \frac{\partial}{\partial z} + \frac{\delta \boldsymbol{B}_\perp}{B_0} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp = \frac{\partial}{\partial z} - \frac{1}{B_0} \{ A_\parallel, \dots \} \tag{3.3}$$

is the spatial derivative along the perturbed magnetic field. We have used compact notation in writing out the nonlinear terms: $\langle \{ A_\parallel, \varphi - \langle \varphi \rangle_{\boldsymbol{R}_s} \} \rangle_{\boldsymbol{R}_s} = \langle \{ A_\parallel(\boldsymbol{r}), \varphi(\boldsymbol{r}) \} \rangle_{\boldsymbol{R}_s} - \{ \langle A_\parallel \rangle_{\boldsymbol{R}_s}, \langle \varphi \rangle_{\boldsymbol{R}_s} \}$, where the first Poisson bracket involves derivatives with respect to $\boldsymbol{r}$ and the second with respect to $\boldsymbol{R}_s$. We now proceed to develop the linear theory.

3.2. *Linear gyrokinetic equation*

We begin by linearizing the gyrokinetic equation (3.2) in the fluctuations amplitudes:

$$\frac{\partial g_s}{\partial t} + v_\parallel \frac{\partial g_s}{\partial z} = - \frac{q_s f^{\parallel}_{0s}}{T_{\parallel 0s}} \left(v_\parallel - u'_{\parallel 0s} \right) \left\langle \frac{1}{c} \frac{\partial A_\parallel}{\partial t} + \frac{\partial \varphi}{\partial z} - \frac{\partial}{\partial z} \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \right\rangle_{\boldsymbol{R}_s}. \tag{3.4}$$

Decomposing the perturbed distribution function g_s and the fluctuating electromagnetic potentials φ and $\boldsymbol{A}$ into plane-wave solutions,

$$g_s(t, \boldsymbol{R}_s, v_\parallel, v_\perp) = \sum_{\boldsymbol{k}} g_{s\boldsymbol{k}}(v_\parallel, v_\perp) \, \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{R}_s)},$$
$$\varphi(t, \boldsymbol{r}) = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}} \, \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{r})}, \quad \boldsymbol{A}(t, \boldsymbol{r}) = \sum_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}} \, \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{r})},$$

and substituting these expressions into (3.4), we find that the Fourier coefficient

$$g_{s\boldsymbol{k}} = - \left[\mathrm{J}_0(a_s) \frac{q_s}{T_{\parallel 0s}} \left(\varphi_{\boldsymbol{k}} - \frac{\omega A_{\parallel \boldsymbol{k}}}{k_\parallel c} \right) + \frac{2 v_\perp^2}{v_{th\parallel s}^2} \frac{\mathrm{J}_1(a_s)}{a_s} \frac{\delta B_{\parallel \boldsymbol{k}}}{B_0} \right] \frac{v_\parallel - u'_{\parallel 0s}}{v_\parallel - \omega / k_\parallel} f^{\parallel}_{0s}, \tag{3.5}$$

where $\mathrm{J}_0(a_s)$ and $\mathrm{J}_1(a_s)$ are, respectively, the zeroth- and first-order Bessel functions of $a_s \doteq k_\perp v_\perp/\Omega_s$ (cf. equation I–B1).

3.3. *Gyrokinetic field equations*

Next, we insert (3.5) into the field equations (2.49)–(2.51). This procedure involves computing several $v_\|$-, $v_\perp$-, and Bessel-function–weighted Landau-like integrals over the mean distribution function. These integrals (denoted $\Gamma^\|_{\ell m}$ and $\Gamma^\perp_{\ell m}$ for integral ℓ and m) are defined and evaluated to leading order in $\alpha_s \doteq (k_\perp\rho_s)^2/2$ in Appendix B. Using these definitions, the quasineutrality constraint (2.49) and the parallel (2.50) and perpendicular (2.51) components of Ampère's law may be written, respectively, as

$$\begin{aligned}
&\sum_s \frac{q_s^2 n_{0s}\varphi_{\boldsymbol{k}}}{T_{\perp 0s}}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{00}(\xi_s,\alpha_s) + C^\perp_{0s} - \Gamma^\perp_{00}(\alpha_s)\right] \\
&\quad - \sum_s \frac{q_s^2 n_{0s} u'_{\|0s} A_{\|\boldsymbol{k}}}{cT_{\perp 0s}}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{01}(\xi_s,\alpha_s) + C^\perp_{1s} - \Gamma^\perp_{01}(\alpha_s)\right] \\
&\quad + \sum_s q_s n_{0s}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{10}(\xi_s,\alpha_s) - \Gamma^\perp_{10}(\alpha_s)\right]\frac{\delta B_{\|\boldsymbol{k}}}{B_0} = 0,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\sum_s \frac{q_s^2 n_{0s} u'_{\|0s}\varphi_{\boldsymbol{k}}}{T_{\perp 0s}}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{01}(\xi_s,\alpha_s) + C^\perp_{1s} - \Gamma^\perp_{01}(\alpha_s)\right] + \left\{\frac{c^2k_\perp^2}{4\pi} + \sum_s \frac{q_s^2 n_{0s}}{m_s}\right. \\
&\quad \left. - \sum_s \frac{q_s^2 n_{0s}}{m_s}\frac{T_{\|0s}}{T_{\perp 0s}}\left(1 + \frac{2u'^2_{\|0s}}{v^2_{th\|s}}\right)\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{02}(\xi_s,\alpha_s) + C^\perp_{2s} - \Gamma^\perp_{02}(\alpha_s)\right]\right\}\frac{A_{\|\boldsymbol{k}}}{c} \\
&\quad + \sum_s q_s n_{0s} u'_{\|0s}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{11}(\xi_s,\alpha_s) - \Gamma^\perp_{11}(\alpha_s)\right]\frac{\delta B_{\|\boldsymbol{k}}}{B_0} = 0,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sum_s \beta_{\perp s}\frac{q_s\varphi_{\boldsymbol{k}}}{T_{\perp 0s}}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{10}(\xi_s,\alpha_s) - \Gamma^\perp_{10}(\alpha_s)\right] \\
&\quad - \sum_s \beta_{\perp s}\frac{q_s u'_{\|0s} A_{\|\boldsymbol{k}}}{cT_{\perp 0s}}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{11}(\xi_s,\alpha_s) - \Gamma^\perp_{11}(\alpha_s)\right] \\
&\quad + \left\{\sum_s \beta_{\perp s}\left[\frac{T_{\perp 0s}}{T_{\|0s}}\Gamma^\|_{20}(\xi_s,\alpha_s) - \Gamma^\perp_{20}(\alpha_s)\right] - 2\right\}\frac{\delta B_{\|\boldsymbol{k}}}{B_0} = 0,
\end{aligned} \tag{3.8}$$

where $\xi_s \doteq (\omega - k_\| u'_{\|0s})/k_\| v_{th\|s}$ is the dimensionless phase velocity of the fluctuations in the parallel-drifting frame. The former two equations – quasineutrality (3.6) and the parallel component of Ampère's law (3.7) – can be combined by eliminating $\delta B_{\|\boldsymbol{k}}/B_0$ to yield what amounts to a statement of vorticity conservation:

$$\begin{aligned}
&\frac{\omega}{k_\|}\sum_s \frac{q_s^2 n_{0s}\varphi_{\boldsymbol{k}}}{T_{\perp 0s}}\left\{C^\perp_{0s} - \Gamma^\perp_{00}(\alpha_s) - \frac{k_\| u'_{\|0s}}{\omega}\left[C^\perp_{1s} - \Gamma^\perp_{01}(\alpha_s)\right]\right\} \\
&\quad - \frac{\omega}{k_\|}\sum_s \frac{q_s^2 n_{0s} u'_{\|0s} A_{\|\boldsymbol{k}}}{cT_{\perp 0s}}\left\{C^\perp_{1s} - \Gamma^\perp_{01}(\alpha_s) - \frac{k_\| u'_{\|0s}}{\omega}\left[C^\perp_{2s} - \Gamma^\perp_{02}(\alpha_s)\right]\right\} \\
&\quad = \frac{A_{\|\boldsymbol{k}}}{c}\sum_s \frac{q_s^2 n_{0s}}{m_s}\left\{\frac{2\alpha_s}{\beta_\perp} + 1 - \Gamma_{00}(\alpha_s) - \frac{T_{\|0s}}{T_{\perp 0s}}\left[C^\perp_{2s} - \Gamma^\perp_{02}(\alpha_s)\right]\right\}.
\end{aligned} \tag{3.9}$$

Note that the lowest-order terms in this equation are first order in $(k_\perp \rho_s)^2$.

3.4. *Gyrokinetic dispersion relation for arbitrary f_{0s}*

The linear dispersion relation for pressure-anisotropic multispecies slab gyrokinetics is obtained by combining (3.6)–(3.8) and demanding non-trivial solutions. We have found its general form to be neither physically illuminating nor particularly useful for our purposes. In lieu of numerically computing its general solutions across an expansive parameter space, we opt instead to examine a number of illustrative asymptotic limits for which analytical solutions may be obtained. These limits are treated in the remainder of this section.

3.5. *Long-wavelength limit for arbitrary f_{0s}: Linear KRMHD*

We first examine the long-wavelength ($k_\perp \rho_s \ll 1$) limit of (3.6)–(3.8), which is obtained by taking the leading-order expressions for the $\Gamma_{\ell m}(\xi_s, \alpha_s)$ factors given in (B 4) and (B 5) and by dropping the $k_\perp^2 c^2/4\pi$ term in (3.7). At this order, the parallel Ampère's law is redundant with the quasineutrality constraint. The remaining field equations (*viz.* quasineutrality and the perpendicular Ampère's law) may be written in the following compact form:

$$\begin{bmatrix} \sum_s c_s^2 C_{0s}^{\parallel} \dfrac{2}{\beta_{\parallel s}} & \sum_s c_s \Delta_{1s} \\ \\ \sum_s c_s \Delta_{1s} & \sum_s \beta_{\perp s} \Delta_{2s} - 1 \end{bmatrix} \begin{bmatrix} \dfrac{4\pi e n_{0e}}{B_0^2} \left(\varphi_{\boldsymbol{k}} - \dfrac{\omega A_{\parallel \boldsymbol{k}}}{k_\parallel c} \right) \\ \\ \dfrac{\delta B_{\parallel \boldsymbol{k}}}{B_0} \end{bmatrix} = 0. \tag{3.10}$$

There are two types of solutions to (3.10). The first is straightforwardly obtained by setting the determinant of the above 2×2 matrix to zero, yielding the dispersion relation

$$\left(\sum_s c_s^2 C_{0s}^{\parallel} \frac{2}{\beta_{\parallel s}} \right) \left(\sum_s \beta_{\perp s} \Delta_{2s} - 1 \right) = \left(\sum_s c_s \Delta_{1s} \right)^2 .$$

This equation is identical to the KRMHD dispersion relation for the compressive fluctuations (cf. I–B9).† The other type of solution is obtained by stipulating $\delta B_{\parallel \boldsymbol{k}} = 0$ and thus requiring $\varphi_{\boldsymbol{k}} = \omega A_{\parallel \boldsymbol{k}}/k_\parallel c$. Writing the potentials in terms of the perpendicular velocity and magnetic-field fluctuations, *viz.*

$$\boldsymbol{u}_\perp = \frac{c}{B_0} \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \varphi(\boldsymbol{r}), \tag{3.11a}$$

$$\frac{\delta \boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = -\frac{v_A}{B_0} \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp A_\parallel(\boldsymbol{r}), \tag{3.11b}$$

this gives $\boldsymbol{u}_{\perp \boldsymbol{k}} = -(\omega/k_\parallel)(\delta \boldsymbol{B}_{\perp \boldsymbol{k}}/B_0)$, which we recognize as the eigenvector describing the Alfvénic fluctuations. To obtain the corresponding eigenvalues, we use $\varphi_{\boldsymbol{k}} = \omega A_{\parallel \boldsymbol{k}}/k_\parallel c$ in (3.9) and examine the leading-order terms. After some straightforward algebra, we find

$$\frac{c^2 k_\perp^2}{4\pi} \left[\frac{\omega^2}{k_\parallel^2 v_A^2} - 1 - \sum_s \frac{\beta_{\parallel s}}{2} \left(\Delta_s - \frac{2 u_{\parallel 0s}'^2}{v_{th\parallel s}^2} \right) \right] \frac{A_{\parallel \boldsymbol{k}}}{c} = 0.$$

† There is a typo in (I–B8): the minus sign there should be a plus sign. This error does not affect any of the subsequent formulae or analysis in that paper.

Setting the term in brackets to zero leads to the Alfvén-wave eigenvalues (cf. I–3.1),

$$\omega = \pm k_\parallel v_A \left[1 + \sum_s \frac{\beta_{\parallel s}}{2} \left(\Delta_s - \frac{2u'^2_{\parallel 0s}}{v^2_{th\parallel s}} \right) \right]^{1/2} \equiv \pm k_\parallel v_{A*}, \tag{3.12}$$

where we have defined the effective Alfvén speed v_{A*}. For $p_{\perp 0} - p_{\parallel 0} - \sum_s m_s n_{0s} u'^2_{\parallel 0s} < 0$, the speed at which deformations in the magnetic field are propagated is effectively reduced by the excess parallel pressure, which undermines the restoring force exerted by the tension of the magnetic-field lines. When

$$1 + \sum_s \frac{\beta_{\parallel s}}{2} \left(\Delta_s - \frac{2u'^2_{\parallel 0s}}{v^2_{th\parallel s}} \right) < 0, \tag{3.13}$$

the effective Alfvén speed becomes imaginary and the *firehose instability* results.

Thus, at long wavelengths, the Alfvén- and slow-wave branches are decoupled and the linear gyrokinetic theory correctly reduces to the linear theory of KRMHD (Paper I).

3.6. *Gyrokinetic dispersion relation for an electron-ion bi-Maxwellian plasma*

As the ion gyroscale is approached, $k_\perp \rho_i \sim 1$, the Alfvén waves are no longer decoupled from the compressive fluctuations and therefore can be collisionlessly damped. The fraction of the Alfvén-wave energy that remains in the turbulent cascade is channeled to yet smaller scales, where the Alfvén-wave cascade transitions into a cascade of dispersive *kinetic Alfvén waves*. This cascade proceeds further to electron Larmor scales, $k_\perp \rho_e \sim 1$, at which point the kinetic Alfvén waves Landau damp on the electrons. In this section, the linear theory of Maxwellian collisionless slab gyrokinetics that forms the basis of these statements (Howes *et al.* 2006; Schekochihin *et al.* 2009) is extended to a bi-Maxwellian plasma of single-species ions and electrons.†

Before proceeding with the derivation, we note that, for $f_{0s} = f_{bi\text{-}M,s}(v_\parallel, v_\perp)$, the integrals over the perpendicular velocity space in the $\Gamma_{\ell m}(\xi_s, \alpha_s)$ coefficients (B 4) and (B 5) may be expressed in terms of the zeroth-order (I_0) and first-order (I_1) modified Bessel functions:

$$\Gamma_0(\alpha_s) \doteq \int_0^\infty \frac{\mathrm{d}v_\perp^2}{v^2_{th\perp s}} \left[\mathrm{J}_0(a_s)\right]^2 \mathrm{e}^{-v_\perp^2/v^2_{th\perp s}} = \mathrm{I}_0(\alpha_s)\, \mathrm{e}^{-\alpha_s}, \tag{3.14a}$$

$$\Gamma_1(\alpha_s) \doteq \int_0^\infty \frac{\mathrm{d}v_\perp^2}{v^2_{th\perp s}} \frac{v_\perp^2}{v^2_{th\perp s}} \frac{2\mathrm{J}_1(a_s)\mathrm{J}_1(a_s)}{a_s} \mathrm{e}^{-v_\perp^2/v^2_{th\perp s}} = \left[\mathrm{I}_0(\alpha_s) - \mathrm{I}_1(\alpha_s)\right] \mathrm{e}^{-\alpha_s}, \tag{3.14b}$$

$$\Gamma_2(\alpha_s) \doteq \int_0^\infty \frac{\mathrm{d}v_\perp^2}{v^2_{th\perp s}} \left[\frac{2v_\perp^2}{v^2_{th\perp s}} \frac{\mathrm{J}_1(a_s)}{a_s} \right]^2 \mathrm{e}^{-v_\perp^2/v^2_{th\perp s}} = 2\Gamma_1(\alpha_s). \tag{3.14c}$$

In addition, we can express the integrals over the parallel velocity space in the $\Gamma^\parallel_{\ell m}(\xi_s, \alpha_s)$ coefficients in terms of the (Maxwellian) plasma dispersion function $Z_M(\xi)$:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\mathrm{d}v_\parallel}{v_{th\parallel s}} \frac{v_\parallel}{v_\parallel - \omega/k_\parallel} \mathrm{e}^{-v_\parallel^2/v^2_{th\parallel s}} = 1 + \xi_s Z_M(\xi_s), \tag{3.15}$$

where $\xi_s \doteq \omega/k_\parallel v_{th\parallel s}$ is the dimensionless phase speed (Fried & Conte 1961). Thus,

† While this paper was in preparation, a paper appeared on the arXiv (Verscharen *et al.* 2017) in which the linear gyrokinetic theory for a bi-Maxwellian ion-electron plasma was derived using the nonlinear gyrokinetic theory presented in Paper I and the long-wavelength limit was analyzed. Where there is overlap with the results presented in this section, agreement is found.

combining (3.14) and (3.15), we have

$$\Gamma^{\parallel}_{0\ell}(\alpha_s) = \Gamma_\ell(\alpha_s)\big[1+\xi_s Z_M(\xi_s)\big] \tag{3.16}$$

for integer ℓ. Similarly, $C^{\perp}_{\ell s} = 1$.

With these simplifications, (3.6)–(3.8) may be written succinctly in matrix form:

$$\begin{bmatrix} \mathcal{A} & \mathcal{A}-\mathcal{B} & \mathcal{C} \\ \mathcal{A}-\mathcal{B} & \mathcal{A}-\mathcal{B}-\dfrac{\alpha_*}{\varpi^2} & \mathcal{C}+\mathcal{E} \\ \mathcal{C} & \mathcal{C}+\mathcal{E} & \mathcal{D}-\dfrac{2}{\beta_{\perp i}} \end{bmatrix} \begin{bmatrix} \varphi_{\boldsymbol{k}} \\ -\dfrac{\omega A_{\parallel\boldsymbol{k}}}{k_\parallel c} \\ \dfrac{T_{\perp 0i}}{q_i}\dfrac{\delta B_{\parallel\boldsymbol{k}}}{B_0} \end{bmatrix} = 0, \tag{3.17}$$

where we have employed the shorthand notation (cf. §2.6 of Howes *et al.* 2006)

$$\mathcal{A} \doteq 1+\Gamma_0(\alpha_i)\left[\Delta_i+\xi_i Z_M(\xi_i)\frac{T_{\perp 0i}}{T_{\parallel 0i}}\right] + \frac{\tau_{\perp i}}{Z_i}\left\{1+\Gamma_0(\alpha_e)\left[\Delta_e+\xi_e Z_M(\xi_e)\frac{T_{\perp 0e}}{T_{\parallel 0e}}\right]\right\}, \tag{3.18a}$$

$$\mathcal{B} \doteq 1-\Gamma_0(\alpha_i)+\frac{\tau_{\perp i}}{Z_i}\big[1-\Gamma_0(\alpha_e)\big], \tag{3.18b}$$

$$\mathcal{C} \doteq \Gamma_1(\alpha_i)\left[\Delta_i+\xi_i Z_M(\xi_i)\frac{T_{\perp 0i}}{T_{\parallel 0i}}\right] - \Gamma_1(\alpha_e)\left[\Delta_e+\xi_e Z_M(\xi_e)\frac{T_{\perp 0e}}{T_{\parallel 0e}}\right], \tag{3.18c}$$

$$\mathcal{D} \doteq \Gamma_2(\alpha_i)\left[\Delta_i+\xi_i Z_M(\xi_i)\frac{T_{\perp 0i}}{T_{\parallel 0i}}\right] + \Gamma_2(\alpha_e)\frac{Z_i}{\tau_{\perp i}}\left[\Delta_e+\xi_e Z_M(\xi_e)\frac{T_{\perp 0e}}{T_{\parallel 0e}}\right], \tag{3.18d}$$

$$\mathcal{E} \doteq \Gamma_1(\alpha_i)-\Gamma_1(\alpha_e), \tag{3.18e}$$

$$\alpha_* \doteq \alpha_i + \frac{\beta_{\parallel i}}{2}\Delta_i\big[1-\Gamma_0(\alpha_i)\big] + \frac{\beta_{\parallel e}}{2}\Delta_e\big[1-\Gamma_0(\alpha_e)\big]\frac{m_i}{Z_i m_e}\frac{\tau_{\perp i}}{Z_i}, \tag{3.18f}$$

and $\varpi \doteq \omega/k_\parallel v_A$.

Setting the determinant of the 3×3 matrix in (3.17) equal to zero yields the gyrokinetic dispersion relation, which may be written in the following compact form after multiplying by $\mathcal{A}$ (cf. eq. 41 of Howes *et al.* 2006):

$$\underbrace{\left(\frac{\alpha_*\mathcal{A}}{\varpi^2}-\mathcal{A}\mathcal{B}+\mathcal{B}^2\right)}_{\text{Alfvén}}\underbrace{\left(\frac{2\mathcal{A}}{\beta_{\perp i}}-\mathcal{A}\mathcal{D}+\mathcal{C}^2\right)}_{\text{slow}} = \underbrace{\big(\mathcal{A}\mathcal{E}+\mathcal{B}\mathcal{C}\big)^2}_{\text{FLR coupling}}. \tag{3.19}$$

We have labelled each factor in the dispersion relation (3.19) according to its physical meaning: the first term in parentheses corresponds to the Alfvén-wave branch, the second corresponds to the slow-wave branch, and the right-hand side represents the finite-Larmor-radius (FLR) coupling between the two branches that occurs as $k_\perp\rho_s$ approaches and exceeds unity. For a hydrogenic plasma (i.e. $Z_i = 1$, $m_i/m_e \approx 1836$), the complex eigenvalue solution ϖ to (3.19) depends on five dimensionless parameters: the ratio of the ion Larmor radius to the perpendicular wavelength, $k_\perp\rho_i$; the ion plasma beta $\beta_{\parallel i}$; the ion-electron perpendicular temperature ratio, $\tau_{\perp i}$; the ion pressure anisotropy, Δ_i; and the electron pressure anisotropy, Δ_e.† In what follows, we vary these parameters to obtain asymptotic limits of the dispersion relation (3.19).

† Alternatively, one may specify $\tau_{\perp i}$, $\tau_{\parallel i}$, and Δ_i, which, combined, implies a choice of Δ_e.

3.6.1. *KRMHD limit:* $Z_i m_e/m_i,\ k_\perp \rho_i \to 0$

In the limit where $k_\perp \rho_i$ and the electron-ion mass ratio $Z_i m_e/m_i$ are both asymptotically small, one should obtain the linear theory for bi-Maxwellian KRMHD (cf. §4.4 of Paper I). In this limit, $\mathcal{B} \simeq \alpha_i$, $\mathcal{E} \simeq -(3/2)\alpha_i$, and the dispersion relation (3.19) becomes

$$\mathcal{A}\left(\frac{\alpha_*}{\overline{\omega}^2} - \alpha_i\right)\left(\frac{2\mathcal{A}}{\beta_{\perp i}} - \mathcal{A}\mathcal{D} + \mathcal{C}^2\right) = 0. \tag{3.20}$$

Setting $\mathcal{A} = 0$, we obtain the dispersion relation for Landau-damped ion acoustic waves:

$$\xi_i Z_M(\xi_i) = -\left(1 + \frac{\tau_{\parallel i}}{Z_i}\right). \tag{3.21}$$

Setting the second factor of (3.20) to zero and simplifying $\alpha_* \simeq \alpha_i[1 + (\beta_{\parallel i}/2)\Delta_i + (\beta_{\parallel e}/2)\Delta_e]$ (see (3.18*f*)), we obtain the dispersion relation for undamped Alfvén waves modified by the ion and electron pressure anisotropies:

$$\omega = \pm k_\parallel v_A \sqrt{1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e}. \tag{3.22}$$

Again, when

$$1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e < 0, \tag{3.23}$$

the effective Alfvén speed becomes imaginary and the firehose instability results (cf. (3.13)).

Setting the third factor of (3.20) to zero, and using the leading-order expressions for $\mathcal{C} \simeq (T_{\perp 0i}/T_{\parallel 0i})[1 + \xi_i Z_M(\xi_i) - \tau_{\parallel i}/\tau_{\perp i}]$ and $\mathcal{D} \simeq 2\mathcal{C} + 2(1 + Z_i/\tau_{\perp i})\Delta_e$, we obtain after some straightforward but tedious algebra the dispersion relation for the compressive fluctuations,

$$\left[1 + \xi_i Z_M(\xi_i) - \Lambda^+\right]\left[1 + \xi_i Z_M(\xi_i) - \Lambda^-\right] = 0, \tag{3.24}$$

where

$$\Lambda^\pm = -\frac{\tau_{\parallel i}}{Z_i} + \frac{p_{\parallel 0i}}{p_{\perp 0i}}\frac{\varsigma_i}{\beta_{\perp i}} \pm \sqrt{\left(\frac{\tau_{\parallel i}}{\tau_{\perp i}} + \frac{\tau_{\parallel i}}{Z_i}\right)^2 + \left(\frac{p_{\parallel 0i}}{p_{\perp 0i}}\frac{\varsigma_i}{\beta_{\perp i}}\right)^2}, \tag{3.25}$$

$$\varsigma_i \doteq 1 - \beta_\perp \Delta_e. \tag{3.26}$$

An important limit of (3.24) is obtained for $\beta_s \sim 1/\Delta_s \gg 1$, in which the "+" compressive branch, consisting primarily of magnetic-field-strength fluctuations, is collisionlessly damped at a rate

$$\gamma \doteq -\mathrm{i}\omega = -\frac{|k_\parallel| v_A}{\sqrt{\pi \beta_{\parallel i}}}\frac{p_{\parallel 0i}^2}{p_{\perp 0i}^2}\left(1 - \sum_s \beta_{\perp s}\Delta_s\right). \tag{3.27}$$

This expression captures the effect of pressure anisotropy on the Barnes (1966) damping of slow modes (in the limit $k_\parallel/k_\perp \ll 1$), which is due to Landau-resonant particles interacting with the mirror force associated with the magnetic compressions in the wave. When

$$1 - \beta_{\perp i}\Delta_i + \beta_{\perp e}\Delta_e < 0, \tag{3.28}$$

the proportional increase (for $\Delta_s > 0$) in the number of large-pitch-angle particles in the magnetic troughs ($\delta B_\parallel < 0$) of the slow mode results in more perpendicular pressure than can be stably balanced by the magnetic pressure. The result is the *mirror instability* (e.g. Southwood & Kivelson 1993). We refer the reader to §4.4.2 of Paper I for further analysis and discussion.

Thus, the Alfvén- and slow-wave branches of the gyrokinetic dispersion relation decouple in the long-wavelength limit and we obtain the linear theory of KRMHD.

3.6.2. *Kinetic-Alfvén-wave limit: $k_\perp\rho_e \ll 1 \ll k_\perp\rho_i$*

In the limit $k_\perp\rho_e \ll 1 \ll k_\perp\rho_i$, we have $\Gamma_0(\alpha_i)$, $\Gamma_1(\alpha_i) \to 0$ for the ions and $\Gamma_0(\alpha_e) \simeq \Gamma_1(\alpha_e) \simeq 1$ for the electrons, whence $\mathcal{B} \simeq 1$ and $\mathcal{E} \simeq -1$. We also drop the electron plasma dispersion functions to lowest order in $k_\perp\rho_e$, a simplification that will be justified *a posteriori*. The gyrokinetic dispersion relation (3.19) becomes

$$\left(\frac{\alpha_*\mathcal{A}}{\varpi^2} - \mathcal{A} + 1\right)\left(\frac{2\mathcal{A}}{\beta_{\perp i}} - \mathcal{A}\mathcal{D} + \mathcal{C}^2\right) = \left(-\mathcal{A} + \mathcal{C}\right)^2, \tag{3.29}$$

where $\mathcal{A} \simeq 1 + T_{\perp 0i}/Z_i T_{\| 0e}$, $\mathcal{C} \simeq -\Delta_e$, and $\mathcal{D} \simeq 2(Z_i/\tau_{\perp i})\Delta_e$.† The solutions are

$$\omega = \pm\frac{k_\| v_A k_\perp \rho_i}{\sqrt{\beta_{\perp i} + 2/(1 + Z_i T_{\|0e}/T_{\perp 0i}) - 2\mathcal{H}}}\sqrt{\left(1 - \beta_{\perp e}\Delta_e + \mathcal{H}\right)\left(1 + \frac{\beta_{\| e}}{2}\Delta_e\right)}, \tag{3.30}$$

where

$$2\mathcal{H} \doteq \frac{\beta_{\| e}\Delta_e^2}{1 + Z_i T_{\|0e}/T_{\perp 0i}}. \tag{3.31}$$

Equation (3.30) is a generalisation of the standard kinetic-Alfvén-wave dispersion relation (e.g. Kingsep *et al.* 1990),

$$\omega = \pm\frac{k_\| v_A k_\perp \rho_i}{\sqrt{\beta_i + 2/(1 + Z_i T_{0e}/T_{0i})}},$$

for bi-Maxwellian plasmas. Note that, for this solution, $\xi_e \sim \mathcal{O}(k_\perp\rho_e) \ll 1$, as promised.

The equations governing the corresponding sub-ion-scale fluctuations in the electron density, the parallel flow velocity, and the magnetic-field strength are obtained from the gyrokinetic field equations (I–C88)–(I–C90) with all the $\Gamma_{\ell m}(\alpha_i)$ coefficients set to zero; they are, respectively,

$$\frac{\delta n_e}{n_{0e}} = -\frac{Z_i e\varphi}{T_{\perp 0i}} = -\frac{2}{\sqrt{\beta_{\perp i}}}\frac{\Phi}{\rho_i v_{\rm A}}, \tag{3.32a}$$

$$u_{\| e} = \frac{c}{4\pi e n_{0e}}\left(1 + \frac{\beta_{\| e}}{2}\Delta_e\right)\nabla_\perp^2 A_\| = -\frac{\rho_i}{\sqrt{\beta_{\perp i}}}\left(1 + \frac{\beta_{\| e}}{2}\Delta_e\right)\nabla_\perp^2\Psi, \tag{3.32b}$$

$$u_{\| i} = -\frac{c}{4\pi Z e n_{0i}}\beta_{\| i}\Delta_i\frac{A_\|}{\rho_i^2} = \frac{\rho_i}{\sqrt{\beta_{\perp i}}}\beta_{\| i}\Delta_i\frac{\Psi}{\rho_i^2}, \tag{3.32c}$$

$$\begin{aligned}\frac{\delta B_\|}{B_0} &= \frac{\beta_{\perp i}}{2}\left(1 + \frac{Z_i}{\tau_{\perp i}}\right)\left(1 - \frac{\beta_{\perp e}}{2}\Delta_e\right)^{-1}\frac{Z_i e\varphi}{T_{\perp 0i}} \\ &= \sqrt{\beta_{\perp i}}\left(1 + \frac{Z_i}{\tau_{\perp i}}\right)\left(1 - \frac{\beta_{\perp e}}{2}\Delta_e\right)^{-1}\frac{\Phi}{\rho_i v_A},\end{aligned} \tag{3.32d}$$

where we have introduced the stream and flux functions Φ and Ψ (see (I–C54*a*,*b*)) via

$$\varphi = \frac{B_0}{c}\Phi \qquad \text{and} \qquad A_\| = -\frac{B_0}{v_{\rm A}}\Psi. \tag{3.33}$$

Equations (3.32*a*,*b*,*c*,*d*) are to be compared with equations (221)–(223) of Schekochihin

† Note that the final term in the definition of α_* (equation 3.18*f*) must be retained in this limit, despite its dependence on the higher-order term $1 - \Gamma_0(\alpha_e)$. This is because its leading-order term is proportional to $\alpha_e(m_i/Z_i m_e)(\tau_{\perp i}/Z_i) = \alpha_i \gg 1$. Thus, $\alpha_* \simeq \alpha\,(1 + \beta_{\| e}\Delta_e/2)$.

et al. (2009). They reflect the fact that, for $k_\perp\rho_i \gg 1$, the ion response is effectively Boltzmann (see (2.25)), with the gyrokinetic response h_i contributing nothing to the fields or flows. Note that the parallel ion flow velocity (3.32*c*) is $\sim(k_\perp\rho_i)^{-2} \ll 1$ smaller than the corresponding pressure-anisotropic terms in the parallel electron flow velocity (3.32*b*), and thus contributes almost nothing to the parallel current.

There are several things to note about the kinetic-Alfvén-wave dispersion relation (3.30). First, kinetic Alfvén waves in a bi-Maxwellian plasma are unstable at *both* the mirror (cf. (3.28)) *and* firehose (cf. (3.23)) instability thresholds, the geometric mean of which appears as the final term in (3.30). This makes sense, as Alfvénic and compressive fluctuations are coupled in the kinetic Alfvén wave by finite-Larmor-radius effects. Indeed, the eigenfunctions corresponding to the frequencies (3.30) are (cf. (231) of Schekochihin *et al.* 2009)

$$\begin{aligned}\Theta^\pm_{\boldsymbol{k}} = &\sqrt{\left(1+\frac{Z_iT_{\parallel0e}}{T_{\perp0i}}\right)\left[2+\beta_{\perp i}\left(1+\frac{Z_iT_{\parallel0e}}{T_{\perp0i}}\right)-\beta_{\parallel e}\Delta_e^2\right]}\,\frac{(1-\beta_{\perp e}\Delta_e+\mathcal{H})^{1/2}}{1-\beta_{\perp e}\Delta_e/2}\,\frac{\Phi_{\boldsymbol{k}}}{\rho_i}\\ &\mp\sqrt{1+\frac{\beta_{\parallel e}}{2}\Delta_e}\;k_\perp\Psi_{\boldsymbol{k}}.\end{aligned}\tag{3.34}$$

The factor $1+\beta_{\parallel e}\Delta_e/2$, related to the firehose threshold, is seen to be associated with the Alfvénic fluctuation $\delta B_{\perp\boldsymbol{k}} \propto k_\perp\Psi_{\boldsymbol{k}}$; the factor $1-\beta_{\perp e}\Delta_e$, related to the mirror threshold, is seen to be associated with the compressive fluctuation $\delta B_{\parallel\boldsymbol{k}} \propto \Phi_{\boldsymbol{k}}/\rho_i$. The kinetic-Alfvén-wave eigenfunctions $\Theta^\pm_{\boldsymbol{k}}$ combine both effects.

Secondly, the ion pressure anisotropy does not appear in (3.30). Physically, this is because the ion response is essentially purely Boltzmann (see (2.25)), with an isothermal pressure response,

$$\delta p_{\perp i} = T_{\perp0i}\delta n_i \qquad\text{and}\qquad \delta p_{\parallel i} = T_{\parallel0i}\delta n_i.\tag{3.35}$$

By contrast, the electron response satisfies (see (I–2.45*a*,*b*))

$$\delta p_{\perp e} = T_{\perp0e}\delta n_e - p_{\perp0e}\Delta_e\frac{\delta B_\parallel}{B_0} \qquad\text{and}\qquad \delta p_{\parallel e} = T_{\parallel0e}\delta n_e,\tag{3.36}$$

so that a magnetic-field-strength fluctuation produces a perpendicular temperature fluctuation proportional to the electron pressure anisotropy. The difference is because, at scales satisfying $k_\perp\rho_i \gg 1$, the ions do not "see" the magnetic-field-strength fluctuation, which varies rapidly along the ion gyro-orbit and is thus ring averaged away. In this situation, the ions have no reason to adiabatically adjust their perpendicular pressure according to the changes in field strength. (Note also that $k_\perp\rho_i \gg 1$ implies $|\xi_i| \ll 1$.) The ion pressure anisotropy is also absent from the firehose factor $1+(\beta_{\parallel e}/2)\Delta_e$ in (3.30) and (3.34) for a similar reason: such pressure-anisotropy corrections to the effective tension in the magnetic-field lines stem from the $\hat{\boldsymbol{b}}\delta\hat{\boldsymbol{b}}(p_{\perp0s}-p_{\parallel0s})$ term in the perturbed magnetized pressure tensor, which is only effective if species s can "see" the field fluctuation $\delta\hat{\boldsymbol{b}}$.

Thirdly, the kinetic Alfvén wave in the gyrokinetic limit satisfies perpendicular pressure balance:

$$\frac{B_0\delta B_\parallel}{4\pi}+\delta p_{\perp e}+\delta p_{\perp i} = p_{\perp0i}\left(1-\frac{\beta_{\perp e}}{2}\Delta_e\right)\frac{2}{\beta_{\perp i}}\frac{\delta B_\parallel}{B_0}+p_{\perp0i}\left(1+\frac{Z_i}{\tau_{\perp i}}\right)\frac{\delta n_e}{n_{0e}} = 0,\tag{3.37}$$

which follows from combining (3.32*a*), (3.32*d*), (3.35), and (3.36). This equation states that an increase in number density must be accompanied by a decrease in the magnetic-field strength, the amount of decrease depending upon the factor $1-(\beta_{\perp e}/2)\Delta_e$. If $\Delta_e > 0$, then the magnetic-field lines must inflate further in order to maintain perpendicular pressure balance as large-pitch-angle particles are squeezed into the magnetic troughs. When the concentration of these particles leads to more perpendicular pressure than can be stably balanced by the magnetic pressure, the troughs must grow deeper to compensate. In the long-wavelength limit, the pressure-balanced slow mode then goes unstable to the mirror instability. In the short-wavelength limit, the kinetic Alfvén wave goes unstable for the same reason.

Comparing (3.22) and (3.30), we see that the location of the wavenumber transition from Alfvén waves to kinetic Alfvén waves during a turbulent cascade (at $k_\perp \rho_i \sim 1$ for a Maxwellian plasma) is generally a function of the electron pressure anisotropy. This dependence may be tested by looking for a shift in the ion-Larmor-scale spectral break in measurements of Alfvénic turbulence in the non-Maxwellian solar wind and in simulations of gyrokinetic turbulence in bi-Maxwellian plasmas. Furthermore, in a Maxwellian, high-β plasma, the dispersion relation exhibits a sharp frequency jump at the Alfvén-wave–kinetic-Alfvén-wave transition (see Figure 8c of Schekochihin *et al.* 2009). This jump is accompanied by very strong ion Landau damping. It is easy to imagine from the above discussion that the electron pressure anisotropy, by affecting the rate of collisionless damping, would thus play an important role in determining the fraction of wave energy that is damped on the ions versus cascaded down to electron scales. This is manifest in numerical solutions to (3.19), which we now present.

3.6.3. *Numerical solutions*

In Appendix A, we derive additional approximate solutions to (3.19) in the analytically tractable limits of high and low $\beta_{\parallel i}$. These solutions reveal that Alfvén waves suffer weak collisionless damping at $(k_\perp \rho_i)^2 \sim 1$ in the high-beta limit ... and other stuff. Here we present numerical solutions of (3.19). Not yet finished...

4. Generalised free energy and the kinetic cascade

In Paper I, we showed that the long-wavelength Alfvénic and compressive fluctuations satisfy the following nonlinear conservation law:

$$\frac{\mathrm{d}W_{KRMHD}}{\mathrm{d}t} = -\int \mathrm{d}^3\boldsymbol{r} \sum_s u'_{\parallel 0s} \left(q_s \delta n_s E_\parallel - \delta p_{\perp s} \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} \frac{\delta B_\parallel}{B_0} \right), \tag{4.1}$$

where

$$W_{KRMHD} \doteq \int \mathrm{d}^3\boldsymbol{r} \left\{ \sum_s \int \mathrm{d}^3\boldsymbol{v}\, \frac{T_{\parallel 0s} \delta \widetilde{f}_s^2}{2 f_{0s}^\parallel} + \frac{\rho_0 u_\perp^2}{2} \right. \\ \left. + \left[1 + \sum_s \frac{\beta_{\parallel s}}{2} \left(\Delta_s - \frac{2u'^2_{\parallel 0s}}{v^2_{th\parallel s}} \right) \right] \frac{\delta B_\perp^2}{8\pi} + \left(1 - \sum_s \beta_{\perp s} \Delta_{2s} \right) \frac{\delta B_\parallel^2}{8\pi} \right\} \tag{4.2}$$

is the generalised free energy of KRMHD,

$$\delta \widetilde{f}_s = \delta f_s(v_\parallel, w_\perp) + \frac{v_\perp^2}{v^2_{th\perp s}} \frac{\delta B_\parallel}{B_0} \mathfrak{D} f_{0s} \tag{4.3}$$

is the (long-wavelength) perturbed distribution function in the frame of the Alfvénic fluctuations (see §I–4.2 and (2.47)), and

$$\Delta_{2s} \doteq \left(\frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v}\, \frac{1}{2} \frac{v_\perp^4}{v^4_{th\perp s}} f^\parallel_{0s} \right) \frac{p_{\perp 0s}}{p_{\parallel 0s}} - 1; \tag{4.4}$$

the parallel electric field $E_\parallel$ on the right-hand side of (4.1) is given by (I–2.37) (we have no need of restating it here). In the absence of background interspecies drifts where the right-hand side of (4.1) is zero, W_{KRMHD} is a quadratic invariant representing the turbulent cascade of generalised free energy in a pressure-anisotropic plasma to small scales in phase space. It is comprised of three parts: two Alfvénic invariants (I–3.10) representing forward- and backward-propagating nonlinear Alfvén waves and a compressive invariant (I–4.7), which, in the pressure-isotropic case (eq. (201) of Schekochihin *et al.* 2009), is related to the perturbed entropy of the system. For a bi-Maxwellian distribution, the compressive invariant factors further into three independent collisionless cascades (see §I–4.5). In the presence of background interspecies drifts, the right-hand side of (4.1) corresponds to the change in the free energy due to the work done on the system by the fluctuating parallel electric and magnetic-mirror forces acting on the parallel drifts.

Our goal in this section is to derive the gyrokinetic version of (4.1), valid at both long and short wavelengths. The starting point is the gyrokinetic equation (2.44), written in terms of the gyrokinetic response $\widetilde{h}_s$. Multiplying that equation by $T_{\parallel 0s}\widetilde{h}_s/f^{\parallel}_{0s}$ and integrating over the velocities and gyrocentres, we find that the nonlinear term conserves the variance of $\widetilde{h}_s$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{v}\int \mathrm{d}^3\boldsymbol{R}_s\,\frac{T_{\parallel 0s}\widetilde{h}_s^2}{2f^{\parallel}_{0s}} = \int \mathrm{d}^3\boldsymbol{v}\int \mathrm{d}^3\boldsymbol{R}_s\, q_s\left(\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t} + u'_{\parallel 0s}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial z}\right)\widetilde{h}_s. \tag{4.5}$$

We now sum this equation over all species. The right-hand side becomes

$$\begin{aligned}
&\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\int \mathrm{d}^3\boldsymbol{R}_s\left(\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t} + u'_{\parallel 0s}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial z}\right)\widetilde{h}_s \\
&\quad = \int \mathrm{d}^3\boldsymbol{r}\,\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\left\langle\left(\frac{\partial\chi}{\partial t} + u'_{\parallel 0s}\frac{\partial\chi}{\partial z}\right)\widetilde{h}_s\right\rangle_{\boldsymbol{r}} \\
&\quad = \int \mathrm{d}^3\boldsymbol{r}\left[\frac{\partial\varphi}{\partial t}\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\,\langle\widetilde{h}_s\rangle_{\boldsymbol{r}} - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\boldsymbol{\cdot}\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\,\langle\boldsymbol{v}\widetilde{h}_s\rangle_{\boldsymbol{r}}\right. \\
&\qquad\quad \left. + \sum_s q_s u'_{\parallel 0s}\int \mathrm{d}^3\boldsymbol{v}\left\langle\frac{\partial\chi}{\partial z}\,\widetilde{h}_s\right\rangle_{\boldsymbol{r}}\right].
\end{aligned} \tag{4.6}$$

The first term on the right-hand side of (4.6) can be written in terms of the potentials $(\varphi, A_\parallel, \delta B_\parallel)$ by using (2.43) to expand $\widetilde{h}_s$, employing the quasi-neutrality constraint (2.49), and performing the resulting integrals using the notation defined in Appendix B. The second term on the right-hand side is most easily dealt with by using Faraday's law (2.2) and Ampère's law (2.4) to write

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,\frac{|\delta\boldsymbol{B}|^2}{8\pi} &= \int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{j} = \int \mathrm{d}^3\boldsymbol{r}\left(-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\boldsymbol{\cdot}\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\delta f_s\right) \\
&= \int \mathrm{d}^3\boldsymbol{r}\left[-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\boldsymbol{\cdot}\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\left\langle\boldsymbol{v}\left(\delta f_{1s,Boltz} - \frac{q_s\langle\chi\rangle_{\boldsymbol{R}_s}}{T_{\perp 0s}}\,\mathfrak{D}f_{0s} + \widetilde{h}_s\right)\right\rangle_{\boldsymbol{r}}\right].
\end{aligned} \tag{4.7}$$

Then, substituting (2.25) for $\delta f_{1s,Boltz}$ in the final equality above and performing the resulting integrals (again, with the aid of Appendix B), we may use (4.7) to write the second term on the right-hand side of (4.6) in terms of the potentials $(\varphi, A_\parallel, \delta B_\parallel)$ and the rate-of-change of the magnetic energy. The third and final term on the right-hand side of (4.6) is markedly simplified by using (3.1) to move from $\widetilde{h}_s$ to g_s:

$$\int \mathrm{d}^3\boldsymbol{r} \sum_s q_s u'_{\parallel 0s} \int \mathrm{d}^3\boldsymbol{v} \left\langle \frac{\partial\chi}{\partial z}\,\widetilde{h}_s \right\rangle_{\boldsymbol{r}} = \int \mathrm{d}^3\boldsymbol{r} \sum_s q_s u'_{\parallel 0s} \int \mathrm{d}^3\boldsymbol{v} \left\langle \frac{\partial\chi}{\partial z}\, g_s \right\rangle_{\boldsymbol{r}}$$
$$+ \int \mathrm{d}^3\boldsymbol{r} \sum_s q_s u'_{\parallel 0s} \int \mathrm{d}^3\boldsymbol{v} \frac{q_s f^{\parallel}_{0s}}{T_{\parallel 0s}} \left\langle \frac{\partial\chi}{\partial z} \left\langle \chi + \frac{(v_\parallel - u'_{\parallel 0s})A_\parallel}{c} \right\rangle_{\boldsymbol{R}_s} \right\rangle_{\boldsymbol{r}}, \tag{4.8}$$

from which we may remove the entire second line after integrating by parts with respect to z (to eliminate the first term) and $v_\parallel$ (to eliminate the second).

Assembling (4.6)–(4.8) and expending much algebraic effort, we find that (4.5), summed over species, is equivalent to the following conservation law:

$$\frac{\mathrm{d}W_{GK}}{\mathrm{d}t} = \int \mathrm{d}^3\boldsymbol{r} \sum_s u'_{\parallel 0s} \int \mathrm{d}^3\boldsymbol{v}\, q_s g_s \left[\langle\hat{\boldsymbol{b}}\rangle_{\boldsymbol{R}_s} \boldsymbol{\cdot} \boldsymbol{\nabla} \left\langle \varphi - \frac{\boldsymbol{v}_\perp \boldsymbol{\cdot} \boldsymbol{A}_\perp}{c} \right\rangle_{\boldsymbol{R}_s} + \frac{1}{c}\frac{\partial \langle A_\parallel\rangle_{\boldsymbol{R}_s}}{\partial t} \right], \tag{4.9}$$

where

$$W_{GK} \doteq \int \mathrm{d}^3\boldsymbol{r} \Bigg\{ \sum_s \int \mathrm{d}^3\boldsymbol{v}\, \frac{T_{\parallel 0s}\delta\widetilde{f}^2_s}{2 f^{\parallel}_{0s}} - \sum_s \frac{q_s^2 n_{0s}}{T_{\perp 0s}} \left(\widehat{\Gamma}^{\perp}_{00s} - C^{\perp}_{0s}\right) \frac{\varphi^2}{2} + \frac{\delta B^2_\perp}{8\pi}$$
$$+ \sum_s \frac{q_s^2 n_{0s}}{2T_{\perp 0s}} \left[\frac{p_{\perp 0s}}{p_{\parallel 0s}} \left(\widehat{\Gamma}_{00s} - 1\right) - \left(\widehat{\Gamma}^{\perp}_{02s} - C^{\perp}_{2s}\right)\left(1 + \frac{2u'^2_{\parallel 0s}}{v^2_{th\parallel s}}\right)\right] \frac{v^2_{th\parallel s} A^2_\parallel}{2c^2}$$
$$+ \left[1 - \sum_s \beta_{\perp s} \left(\frac{p_{\perp 0s}}{p_{\parallel 0s}} \frac{\widehat{\Gamma}^{\parallel}_{20s}}{2} - \frac{\widehat{\Gamma}^{\perp}_{20s}}{2} \right)\right] \frac{\delta B^2_\parallel}{8\pi} - \sum_s q_s n_{0s} u'_{\parallel 0s} \widehat{\Gamma}^{\perp}_{11s} \frac{A_\parallel \delta B_\parallel}{cB_0} \Bigg\} \tag{4.10}$$

is the appropriately generalised gyrokinetic free energy. Here, $\delta\widetilde{f}_s$ is given by (2.47). The differential operators in (4.10), conspicuously ornamented with the symbol $\widehat{\ }$, are defined in Appendix B by (B 7). To leading order in $k^2_\perp \rho^2_s \ll 1$, they satisfy $\widehat{\Gamma}^{\perp}_{20s} \approx 2$, $\widehat{\Gamma}^{\perp}_{11s} \approx 1$, $\widehat{\Gamma}^{\parallel}_{20s} \approx (1/n_{0s}) \int \mathrm{d}^3\boldsymbol{v}\, (v^4_\perp / v^4_{th\perp s}) f^{\parallel}_{0s}$, and

$$\left(\widehat{\Gamma}^{\perp}_{00s} - C^{\perp}_{0s}\right)\Psi^2 \approx \left(\widehat{\Gamma}_{00s} - 1\right)\Psi^2 \approx \left(\widehat{\Gamma}^{\perp}_{02s} - C^{\perp}_{2s}\right)\Psi^2 \approx -\sum_{\boldsymbol{k}} \frac{1}{2} k^2_\perp \rho^2_s |\Psi_{\boldsymbol{k}}|^2$$

for any function $\Psi(\boldsymbol{r})$. Substituting these long-wavelength expressions into (4.10), eliminating its final term by using $\sum_s q_s n_{0s} u'_{\parallel 0s} = 0$, and manipulating (3.11) to write $\sum_{\boldsymbol{k}} k^2_\perp |\varphi_{\boldsymbol{k}}|^2$ and $\sum_{\boldsymbol{k}} k^2_\perp |A_{\parallel\boldsymbol{k}}|^2$ in terms of $u^2_\perp$ and $\delta B^2_\perp$, respectively, we find that the gyrokinetic invariant reduces to its KRMHD counterpart (4.2), as it should.

Each of the terms in (4.9) deserves some discussion. The first term ($\propto \delta\widetilde{f}^2_s / f^{\parallel}_{0s}$) is due to the non-Alfvénic piece of the distribution function that represents changes in the kinetic energy of the particles due to interactions with the compressive fluctuations. In it are contributions from Landau-resonant particles, whose energy is changed by the parallel electric and magnetic-mirror forces in such a way as to facilitate Landau and Barnes damping of ion-acoustic waves and slow modes. In the pressure-isotropic case, this term is simply the perturbed entropy of the system in the frame of the Alfvénic fluctuations. The second term ($\propto \varphi^2$) represents the energy associated with the $\boldsymbol{E} \times \boldsymbol{B}$ motion. At long wavelengths, it is equal to $\rho_0 u^2_\perp / 2$ (see (3.11*a*)). The next two terms represent the energetic cost of bending the magnetic-field lines, with an increase or decrease in this cost dependent upon the pressure anisotropy of the mean distribution function and the presence of interspecies drifts. The first term on the third line of (4.10) signals a change in the energetic cost of compressing/rarefying the magnetic-field lines due pressure anisotropy. The final term has no long-wavelength limit. It is related to conservation of helicity of the perturbed magnetic field, $\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}\boldsymbol{\cdot}\delta\boldsymbol{B}$, which is broken by parallel electric fields.

The right-hand side of (4.9) is the fluctuating parallel force on gyrocentres multiplied by the number density of gyrocentres and the parallel interspecies drifts. In the long-wavelength limit, it is precisely the right-hand side of (4.10) – the work done on the system by the fluctuating parallel electric and magnetic-mirror forces acting on the background parallel drifts. The only difference is that, in the gyrokinetic limit, these (ring-averaged) parallel forces act on the guiding centres instead of the particles.

In Paper I, the long-wavelength invariant W_{KRMHD} was used to elucidate how the generalized free energy is partitioned as it cascades to small scales in phase space across the inertial range of KRMHD turbulence. It was shown that W_{KRMHD} can be split into three independent cascades of the generalised Alfvénic and compressive-fluctuation energies: W^+_{AW}, W^-_{AW}, and W_{compr}. In the case of a single-ion-species bi-Maxwellian plasma, W_{compr} can be further decomposed into three independently cascading parts: W^+_{compr}, W^-_{compr}, and $W_{\widetilde{g}_i}$, the latter of which represents a purely kinetic cascade. All three cascade channels lead to small perpendicular spatial scales via passive mixing by the Alfvénic turbulence and to small scales in $v_\parallel$ via the linear parallel phase mixing. The rates of mixing are generally functions of the velocity-space anisotropy of the equilibrium function.

With a general invariant W_{GK} (4.10) in hand, valid across all scales (that satisfy the gyrokinetic ordering), we now ask how the phase-space cascade begun in the inertial range proceeds through the sub-Larmor dissipation range.

5. Sub-Larmor cascades

Not yet finished! Free energy conservation is from (4.10). Here we show that it has two pieces: a kinetic-Alfvén-wave cascade and an ion-entropy cascade. Split occurs at $k_\perp\rho_i \sim 1$; involves redistribution of power arriving from the inertial range (see Paper I).

5.1. *Kinetic-Alfvén-wave cascade*

In appendix C.8 of Paper I, we derived nonlinear equations describing the electron kinetics for $k_\perp\rho_e \sim k_\perp\rho_i(m_e/m_i)^{1/2} \ll 1$, obtained via a mass-ratio expansion. For the purposes of this paper, the two most important electron equations are that specifying the parallel electric field (see (I–C72)),

$$E_\parallel \doteq -\frac{1}{c}\frac{\partial A_\parallel}{\partial t} - \hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi = -\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{T_{\parallel 0e}}{e}\left(\frac{\delta n_e}{n_{0e}} + \Delta_e\frac{\delta B_\parallel}{B_0}\right), \tag{5.1}$$

and what amounts to a reduced electron continuity equation (see (I–C78) and accompanying discussion in §I–C.8.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta n_e}{n_{0e}} - \frac{\delta B_\parallel}{B_0}\right) + \hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}u_{\parallel e} + \frac{cT_{\perp 0e}}{eB_0}\left\{\frac{\delta n_e}{n_{0e}}, \frac{\delta B_\parallel}{B_0}\right\} = 0, \tag{5.2}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}_E\boldsymbol{\cdot}\boldsymbol{\nabla} = \frac{\partial}{\partial t} + \frac{c}{B_0}\{\varphi, \dots\} \tag{5.3}$$

is the Lagrangian time derivative measured in a frame transported at the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity, $\boldsymbol{u}_E = -c\boldsymbol{\nabla}_\perp\varphi\times\hat{\boldsymbol{z}}/B_0$. Using (3.32*a*), (3.32*b*), and (3.32*d*) for the electron density fluctuation, the parallel electron flow velocity, and the magnetic-field-strength fluctuation in the sub-ion-Larmor range, and introducing Φ and Ψ via (3.33), equations (5.1) and (5.2) become, respectively,

$$\frac{\partial \Psi}{\partial t} = v_A \left\{ 1 + \frac{Z_i T_{\parallel 0e}}{T_{\perp 0i}} \left[1 - \frac{\beta_{\perp e}}{2} \Delta_e \left(1 + \frac{\tau_{\perp i}}{Z_i} \right) \left(1 - \frac{\beta_{\perp e}}{2} \Delta_e \right)^{-1} \right] \right\} \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} \Phi, \tag{5.4}$$

$$\frac{\partial \Phi}{\partial t} \left[2 + \beta_{\perp i} \left(1 + \frac{Z_i}{\tau_{\perp i}} \right) \left(1 - \frac{\beta_{\perp e}}{2} \Delta_e \right)^{-1} \right] = -v_A \left(1 + \frac{\beta_{\parallel e}}{2} \Delta_e \right) \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} (\rho_i^2 \nabla_\perp^2 \Psi). \tag{5.5}$$

These equations generalize the linear theory of kinetic Alfvén waves (§3.6.2) to the nonlinear regime. Introducing the perturbed magnetic-field vector

$$\frac{\delta \boldsymbol{B}}{B_0} = \frac{1}{v_A} \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Psi + \hat{\boldsymbol{z}} \frac{\delta B_\parallel}{B_0} \tag{5.6}$$

with $\delta B_\parallel$ given by (3.32d), equations (5.4) and (5.5) can be recast as two coupled evolution equations for the perpendicular and parallel components of the perturbed magnetic field, respectively.

It is straightforward to show that (5.4) and (5.5) conserve

$$\begin{aligned} W_{KAW} &\doteq \int \mathrm{d}^3\boldsymbol{r}\, \frac{m_i n_{0i}}{2} \left\{ \left(1 + \frac{\beta_{\parallel e}}{2} \Delta_e \right) |\boldsymbol{\nabla}_\perp \Psi|^2 \right. \\ &\quad \left. + \frac{1 - \beta_{\perp e} \Delta_e + \mathcal{H}}{(1 - \beta_{\perp e} \Delta_e / 2)^2} \left(1 + \frac{Z_i T_{\parallel 0e}}{T_{\perp 0i}} \right) \left[2 + \beta_{\perp i} \left(1 + \frac{Z_i T_{\parallel 0e}}{T_{\perp 0i}} \right) - \beta_{\parallel e} \Delta_e^2 \right] \frac{\Phi^2}{\rho_i^2} \right\} \qquad (5.7a) \\ &= \int \mathrm{d}^3\boldsymbol{r}\, \frac{m_i n_{0i}}{4} \left(|\Theta^+|^2 + |\Theta^-|^2 \right), \qquad (5.7b) \end{aligned}$$

which is the sum of the energies of the "+" and "−" linear kinetic-Alfvén-wave eigenmodes (see (3.34)). At the firehose threshold, Ψ is energetically free; at the mirror threshold, Φ is energetically free; gives link between linear stability and nonlinear stability.

W_{KAW} is sum of the energies of the "+" and "−" linear kinetic-Alfvén-wave eigenmodes, which are also exact nonlinear solutions. However, the two do not cascade independently and can exchange free energy.

5.2. *Entropy cascade*

Not yet finished! Gyrokinetic equation in sub-ion-Larmor range is:

$$\frac{\partial \widetilde{h}_i}{\partial t} + v_\parallel \frac{\partial \widetilde{h}_i}{\partial z} + \{ \langle \chi \rangle_{\boldsymbol{R}_i}, \widetilde{h}_i \} = \frac{2}{\sqrt{\beta_{\perp i}} \rho_i v_A} \frac{T_{\perp 0i}}{T_{\parallel 0i}} \frac{\partial \langle \Phi \rangle_{\boldsymbol{R}_i}}{\partial t} f_{0i}^{\parallel} \tag{5.8}$$

From this, can derive conservation law:

$$\frac{1}{T_{\parallel 0i}} \frac{\mathrm{d} W_{\widetilde{h}_i}}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{v} \int \mathrm{d}^3\boldsymbol{R}_i\, \frac{\widetilde{h}_i^2}{2 f_{0i}^{\parallel}} = \frac{2}{\sqrt{\beta_{\perp i}} \rho_i v_A} \frac{T_{\perp 0i}}{T_{\parallel 0i}} \int \mathrm{d}^3\boldsymbol{v} \int \mathrm{d}^3\boldsymbol{R}_i\, \frac{\partial \langle \Phi \rangle_{\boldsymbol{R}_i}}{\partial t} \widetilde{h}_i, \tag{5.9}$$

where

$$\widetilde{h}_i = h_i + \frac{Z_i e}{T_{\perp 0i}} \Delta_i \langle \chi \rangle_{\boldsymbol{R}_i}.$$

Source of entropy cascade (right-hand side of (5.9)) can be large or small depending upon temperature anisotropy; at fixed ρ_i, more perpendicular energy gives more source, which make sense: imagine if the perpendicular distribution function were very spiked, then the different particles wouldn't have very different $v_\perp/\Omega_i$, so that the nonlinear phase mixing would be reduced. If the perpendicular distribution were very broad, then different particles would have very different $v_\perp/\Omega_i$, so that the nonlinear phase mixing would be enhanced. The perpendicular phase mixing occurs from the $\{\langle\chi\rangle, \widetilde{h}_i\}$ term in the gyrokinetic equation.

See discussion in Section 7.9 of Schekochihin *et al.* (2009) for more on entropy cascade.

6. Discussion

Support for M.W.K. was provided by US DOE Contract DE-AC02-09-CH11466. Support for I.G.A. was provided by the Princeton Center for Theoretical Science and .... A.A.S. was supported in part by grants from UK STFC and EPSRC. The authors thank J. Burby, J. TenBarge, and especially S. Cowley for useful conversations. The completion of this work was facilitated by the generous hospitality and material support provided by the Wolfgang Pauli Institute in Vienna.

Appendix A. Limiting cases of the bi-Maxwellian linear gyrokinetic dispersion relation

In this Appendix, the transition between the long-wavelength solutions of Section 3.6.1 and the short-wavelength solutions of Section 3.6.2 is treated in the analytically tractable limits of high and low $\beta_{\parallel i}$. This follows the procedure in the appendix of Howes *et al.* (2006) of identifying the analytically solvable cases.

A.1. *High-$\beta_{\parallel i}$ limit: $\beta_{\parallel i} \gg 1$, $k_\perp\rho_i \sim 1$*

For $\beta_{\parallel i} \gg 1$, we have $\xi_i = \overline{\omega}/\beta_{\parallel i}^{1/2} \ll 1$ and $\xi_e = (m_e/m_i)^{1/2}(T_{\parallel 0i}/T_{\parallel 0e})^{1/2}\xi_i \ll 1$, and we can use the small-argument expansion of the plasma dispersion function, $Z_M(\xi_s) \simeq \mathrm{i}\sqrt{\pi}$. [This requires $\tau_{\parallel i} \ll (m_i/m_e)\beta_{\parallel i}$, which is not particularly restrictive.] We also take $\alpha_e \ll 1$ because $m_e/m_i \ll 1$, as well as order the pressure anisotropy $\Delta_s \sim 1/\beta_{\parallel s}$. Retaining $k_\perp\rho_i \sim 1$, the coefficients of the gyrokinetic dispersion relation (3.18) become

$$\mathcal{A} \simeq 1 + \Gamma_0(\alpha_i)\Delta_i + \frac{\tau_{\perp i}}{Z_i}\frac{T_{\perp 0e}}{T_{\parallel 0e}} + \mathrm{i}\sqrt{\pi}\xi_i\frac{T_{\perp 0i}}{T_{\parallel 0i}}\left[\Gamma_0(\alpha_i) + \left(\frac{\tau_{\parallel i}}{Z_i}\right)^{3/2}\left(\frac{Z_i m_e}{m_i}\right)^{1/2}\right], \tag{A 1a}$$

$$\mathcal{B} \simeq 1 - \Gamma_0(\alpha_i), \tag{A 1b}$$

$$\mathcal{C} \simeq \Gamma_1(\alpha_i)\Delta_i - \Delta_e + \mathrm{i}\sqrt{\pi}\xi_i\frac{T_{\perp 0i}}{T_{\parallel 0i}}\left[\Gamma_1(\alpha_i) - \frac{\tau_{\parallel i}}{\tau_{\perp i}}\left(\frac{\tau_{\parallel i}}{Z_i}\frac{Z_i m_e}{m_i}\right)^{1/2}\right], \tag{A 1c}$$

$$\begin{aligned}\mathcal{D} &\simeq 2\left[\Gamma_1(\alpha_i)\Delta_i + \frac{Z_i}{\tau_{\perp i}}\Delta_e\right] + 2\mathrm{i}\sqrt{\pi}\xi_i\frac{T_{\perp 0i}}{T_{\parallel 0i}}\left[\Gamma_1(\alpha_i) + \frac{\tau_{\parallel i}^2}{\tau_{\perp i}^2}\left(\frac{Z_i}{\tau_{\parallel i}}\frac{Z_i m_e}{m_i}\right)^{1/2}\right] \\ &\doteq \frac{2}{\beta_{\perp i}}\left[1 - \mathcal{F}(\alpha_i)\right] + 2\mathrm{i}\sqrt{\pi}\xi_i\mathcal{G}(\alpha_i),\end{aligned} \tag{A 1d}$$

$$\mathcal{E} \simeq \Gamma_1(\alpha_i) - 1, \tag{A 1e}$$

where we have dropped all terms of order 1 and higher in $Z_i m_e/m_i$. As in Section 3.6.2, we must be careful to retain the final term in the definition of α_* (3.18f), despite its dependence on the higher-order $1 - \Gamma_0(\alpha_e)$ factor. The auxiliary functions $\mathcal{F}(\alpha_i)$ and $\mathcal{G}(\alpha_i)$, which are defined implicitly by (A 1d), will become useful below.

We proceed by taking two instructive limits:

(i) The limit $k_\perp \rho_i \sim \mathcal{O}(\beta_{\parallel i}^{-1/4})$, $\varpi \sim \mathcal{O}(1)$.

In this ordering, we have $\alpha_i \sim \xi_i \sim \mathcal{O}(\beta_{\parallel i}^{-1/2})$, and so we may expand $\Gamma_0(\alpha_i) \simeq 1 - \alpha_i$ and $\Gamma_1(\alpha_i) \simeq 1 - (3/2)\alpha_i$. We find from (A 1) that $\mathcal{A} \sim \mathcal{O}(1)$ and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E} \sim \mathcal{O}(\beta_{\parallel i}^{-1/2})$. Then, the dispersion relation (3.19) becomes

$$-\left(\frac{\alpha_*}{\varpi^2} - \mathcal{B}\right)\mathcal{D} = \mathcal{E}^2, \tag{A 2}$$

where, to leading order, we have $\mathcal{B} \simeq \alpha_i$, $\mathcal{E} \simeq -(3/2)\alpha_i$, $\mathcal{D} \simeq 2\mathrm{i}\varpi(\pi/\beta_{\parallel i})^{1/2}$, and $\alpha_* \simeq \alpha_i[1 + (\beta_{\parallel i}/2)\Delta_i + (\beta_{\parallel e}/2)\Delta_e]$. This is a quadratic equation for ϖ, whose solutions are

$$\varpi = -\mathrm{i}\frac{9}{16}\sqrt{\frac{\beta_{\parallel i}}{\pi}}\alpha_i \pm \sqrt{1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e - \left(\frac{9}{16}\sqrt{\frac{\beta_{\parallel i}}{\pi}}\alpha_i\right)^2}. \tag{A 3}$$

In the subsidiary limit, $k_\perp \rho_i \ll 1/\beta_{\parallel i}^{1/4}$, we recover, as expected, the Alfvén wave, now with weak collisionless damping (cf. equation 3.22):

$$\varpi = \pm\sqrt{1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e} - \mathrm{i}\frac{9}{16}\frac{k_\perp^2\rho_i^2}{2}\sqrt{\frac{\beta_{\parallel i}}{\pi}}. \tag{A 4}$$

In the intermediate asymptotic limit $\beta_{\parallel i}^{-1/4} \ll k_\perp \rho_i \ll 1$, we have

$$\varpi = -\mathrm{i}\frac{8}{9}\left(1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e\right)\left(\frac{k_\perp^2\rho_i^2}{2}\right)^{-1}\sqrt{\frac{\pi}{\beta_{\parallel i}}} \quad \text{(weakly damped);} \tag{A 5a}$$

$$\varpi = -\mathrm{i}\frac{9}{8}\frac{k_\perp^2\rho_i^2}{2}\sqrt{\frac{\beta_{\parallel i}}{\pi}} \quad \text{(strongly damped).} \tag{A 5b}$$

(ii) The limit $k_\perp \rho_i \sim \mathcal{O}(1)$, $\varpi \sim \mathcal{O}(\beta_{\parallel i}^{-1/2})$.

In this ordering, $\alpha_i \sim \mathcal{O}(1)$ and $\xi_i \sim \mathcal{O}(\beta_{\parallel i}^{-1})$. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E} \sim \mathcal{O}(1)$, and $\mathcal{C}$, $\mathcal{D} \sim \mathcal{O}(\beta_{\parallel i}^{-1})$. The dispersion relation (3.19) becomes

$$\frac{\alpha_*}{\varpi^2}\left(\frac{2}{\beta_{\perp i}} - \mathcal{D}\right) = \mathcal{E}^2. \tag{A 6}$$

Since $\mathcal{D} \simeq (2/\beta_{\perp i})[1 - \mathcal{F}(\alpha_i)] + 2\mathrm{i}\sqrt{\pi}\xi_i\mathcal{G}(\alpha_i)$, this is again a quadratic equation for ϖ, with solutions given by

$$\varpi = -\mathrm{i}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{\alpha_*\,\mathcal{G}(\alpha_i)}{[\Gamma_1(\alpha_i) - 1]^2} \pm \sqrt{\frac{2}{\beta_{\perp i}}\frac{\alpha_*\,\mathcal{F}(\alpha_i)}{[\Gamma_1(\alpha_i) - 1]^2} - \left\{\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{\alpha_*\,\mathcal{G}(\alpha_i)}{[\Gamma_1(\alpha_i) - 1]^2}\right\}^2}. \tag{A 7}$$

In the long-wavelength limit, $k_\perp \rho_i \ll 1$, these become

$$\varpi = -\frac{\mathrm{i}}{\sqrt{\pi\beta_{\parallel i}}}\frac{p_{\parallel 0i}^2}{p_{\perp 0i}^2}\left(1 - \beta_{\perp i}\Delta_i - \beta_{\perp e}\Delta_e\right) \tag{A 8a}$$

$$\varpi = -\mathrm{i}\frac{8}{9}\left(1 + \frac{\beta_{\parallel i}}{2}\Delta_i + \frac{\beta_{\parallel e}}{2}\Delta_e\right)\left(\frac{k_\perp^2\rho_i^2}{2}\right)^{-1}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{p_{\perp 0i}}{p_{\parallel 0i}}. \tag{A 8b}$$

The first solution is the Barnes-damped (or mirror-unstable) slow wave (cf. I–4.33);

the second solution matches the weakly damped Alfvén wave in the intermediate limit (equation A 5*a*).

In the short-wavelength limit, $k_\perp \rho_i \gg 1$, we have $\Gamma_1(\alpha_i) \to 0$, $\mathcal{F}(\alpha_i) \to 1-\beta_{\perp e}\Delta_e$, and $G(\alpha_i) \to (T_{\perp 0e}/T_{\parallel 0e})(\tau_{\parallel i}/\tau_{\perp i})(Z_i/\tau_{\parallel i})^{1/2}(Z_i m_e/m_i)^{1/2}$. Equation (A 7) then reproduces the $\beta_{\parallel i} \gg 1$ limit of the kinetic-Alfvén-wave dispersion relation (cf. 3.30):

$$\begin{aligned}\overline{\omega} = &\pm \frac{k_\perp \rho_i}{\sqrt{\beta_{\perp i}}}\left(1 + \frac{\beta_{\parallel e}}{2}\Delta_e\right)^{1/2}\left(1 - \beta_{\perp e}\Delta_e\right)^{1/2} \\ &- \mathrm{i}\frac{k_\perp^2 \rho_i^2}{2}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\left(1 + \frac{\beta_{\parallel e}}{2}\Delta_e\right)\frac{T_{\perp 0e}}{T_{\parallel 0e}}\frac{\tau_{\parallel i}}{\tau_{\perp i}}\left(\frac{Z_i}{\tau_{\parallel i}}\frac{Z_i m_e}{m_i}\right)^{1/2}.\end{aligned}$$

A.2. *Low-$\beta_{\parallel i}$ limit: $\beta_{\parallel i} \ll 1$, $k_\perp \rho_i \sim 1$*

For $\beta_{\parallel i} \ll 1$ and $\Delta_s \sim 1$, the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E} \sim \mathcal{O}(1)$. Then the gyrokinetic dispersion relation (3.19) reduces to

$$\left(\frac{\alpha_* \mathcal{A}}{\overline{\omega}^2} - \mathcal{A}\mathcal{B} + \mathcal{B}^2\right)\frac{2\mathcal{A}}{\beta_{\perp i}} = 0. \tag{A 9}$$

The long-wavelength limit of the second factor ($\mathcal{A} = 0$) gives the Landau-damped ion acoustic wave (equation 3.21). For the first factor, we order $\overline{\omega} \sim \mathcal{O}(1)$ and consider two interesting limits:

(*i*) *The limit $(Z_i m_e/m_i)(\tau_{\perp i}/Z_i) \ll \beta_{\parallel i} \ll 1$.*

In this limit, $\xi_i = \overline{\omega}/\sqrt{\beta_{\parallel i}} \gg 1$ and $\xi_e = (Z_i m_e/m_i)^{1/2}(\tau_{\perp i}/Z_i)^{1/2} \ll 1$ (slow ions, fast electrons). Expanding the ion and electron plasma dispersion functions in large and small arguments, respectively, we get

$$\begin{aligned}\mathcal{A} \simeq\ & 1 - \Gamma_0(\alpha_i) + \frac{\tau_{\perp i}}{Z_i}\left[1 + \Gamma_0(\alpha_e)\Delta_e\right] \\ &+ \mathrm{i}\overline{\omega}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{T_{\perp 0i}}{T_{\parallel 0i}}\left[\Gamma_0(\alpha_i)\exp\left(-\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right) + \left(\frac{\tau_{\parallel i}}{Z_i}\right)^{3/2}\left(\frac{Z_i m_e}{m_i}\right)^{1/2}\Gamma_0(\alpha_e)\right].\end{aligned} \tag{A 10}$$

The dispersion relation (A 9) then becomes

$$\frac{\tau_{\perp i}}{Z_i}\frac{T_{\perp 0e}}{T_{\parallel 0e}}\mathcal{B}\Gamma_0(\alpha_e)\overline{\omega}^2 - \alpha_*\left\{1 - \Gamma_0(\alpha_i) + \frac{\tau_{\perp i}}{Z_i}\left[1 + \Delta_e\Gamma_0(\alpha_e)\right]\right\} = -\mathrm{i}\big(\mathcal{B}\overline{\omega}^2 - \alpha_*\big)\mathrm{Im}(\mathcal{A}). \tag{A 11}$$

This equation may be iteratively solved to find

$$\mathrm{Re}(\overline{\omega}) = \pm\sqrt{\frac{\alpha_*\left\{1 - \Gamma_0(\alpha_i) + (\tau_{\perp i}/Z_i)\left[1 + \Delta_e\Gamma_0(\alpha_e)\right]\right\}}{(\tau_{\perp i}/Z_i)(T_{\perp 0e}/T_{\parallel 0e})\mathcal{B}\Gamma_0(\alpha_e)}}, \tag{A 12a}$$

$$\begin{aligned}\overline{\gamma} = &-\frac{\alpha_*}{2\left[(\tau_{\parallel i}/Z_i)\Gamma_0(\alpha_e)\right]^2}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{T_{\parallel 0i}}{T_{\perp 0i}} \\ &\times\left[\Gamma_0(\alpha_i)\exp\left(-\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right) + \left(\frac{\tau_{\parallel i}}{Z_i}\right)^{3/2}\left(\frac{Z_i m_e}{m_i}\right)^{1/2}\Gamma_0(\alpha_e)\right],\end{aligned} \tag{A 12b}$$

where $\overline{\gamma} \doteq \mathrm{Im}(\omega)/k_\parallel v_A$. In the limit $\alpha_i \ll 1$, (A 12*a*) reduces to the Alfvén wave solution (3.22).

(*ii*) *The limit $\beta_{\parallel i} \sim Z_i m_e/m_i \ll 1$, $\tau_{\perp i}/Z_i \gg 1$.*

In this limit, $\xi_i \sim (m_i/m_e)^{1/2} \gg 1$, and so $\xi_e \sim (\tau_{\perp i}/Z_i)^{1/2} \gg 1$ (cold ions and electrons). Expanding all plasma dispersion functions in their large arguments, the coefficient

$$\mathcal{A} \simeq \mathcal{B} - \frac{\Gamma_0(\alpha_e)}{2\overline{\omega}^2}\frac{m_i}{Z_i m_e}\beta_{\perp i} + \mathrm{i}\overline{\omega}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{T_{\perp 0i}}{T_{\parallel 0i}} \times \left[\Gamma_0(\alpha_i)\exp\left(-\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right) + \left(\frac{\tau_{\parallel i}}{Z_i}\right)^{3/2}\left(\frac{Z_i m_e}{m_i}\right)^{1/2}\Gamma_0(\alpha_e)\exp\left(-\frac{T_{\parallel 0i}}{T_{\parallel 0e}}\frac{m_e}{m_i}\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right)\right]. \tag{A 13}$$

The dispersion relation (A 9) then becomes

$$\frac{\alpha_*\Gamma_0(\alpha_e)}{2\overline{\omega}^2}\frac{m_i}{Z_i m_e}\beta_{\perp i} - \mathcal{B}\left[\alpha_* + \frac{\Gamma_0(\alpha_e)}{2}\frac{m_i}{Z_i m_e}\beta_{\perp i}\right] = -\mathrm{i}\big(\mathcal{B}\overline{\omega}^2 - \alpha_*\big)\mathrm{Im}(\mathcal{A}), \tag{A 14}$$

which may be iteratively solved to find†

$$\overline{\omega} = \pm\sqrt{\frac{\alpha_*\Gamma_0(\alpha_e)(m_i/Z_i m_e)\beta_{\perp i}}{\big[2\alpha_* + \Gamma_0(\alpha_e)(m_i/Z_i m_e)\beta_{\perp i}\big]\mathcal{B}}}, \tag{A 15a}$$

$$\overline{\gamma} = -\frac{2\alpha_*^3\Gamma_0(\alpha_e)(m_i/Z_i m_e)\beta_{\perp i}}{\big[2\alpha_* + \Gamma_0(\alpha_e)(m_i/Z_i m_e)\beta_{\perp i}\big]^3\mathcal{B}^2}\sqrt{\frac{\pi}{\beta_{\parallel i}}}\frac{T_{\perp 0i}}{T_{\parallel 0i}} \times \left[\Gamma_0(\alpha_i)\exp\left(-\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right) + \left(\frac{\tau_{\parallel i}}{Z_i}\right)^{3/2}\left(\frac{Z_i m_e}{m_i}\right)^{1/2}\Gamma_0(\alpha_e)\exp\left(-\frac{T_{\parallel 0i}}{T_{\parallel 0e}}\frac{m_e}{m_i}\frac{\overline{\omega}^2}{\beta_{\parallel i}}\right)\right]. \tag{A 15b}$$

Appendix B. Definitions of $C^{\perp}_{\ell s}$, $C^{\parallel}_{\ell s}$, $\Gamma^{\perp}_{\ell m}$, and $\Gamma^{\parallel}_{\ell m}$ coefficients

This paper is replete with velocity-space integrals, which we have allowed to masquerade as deceptively benign coefficients. The first set of these are the $C^{\perp}_{\ell s}$ coefficients:

$$C^{\perp}_{0s} \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\, f^{\perp}_{0s}, \tag{B 1a}$$

$$C^{\perp}_{1s} \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{v_\parallel}{v_{th\parallel s}} f^{\perp}_{0s} \times \left(\frac{u'_{\parallel 0s}}{v_{th\parallel s}}\right)^{-1}, \tag{B 1b}$$

$$C^{\perp}_{2s} \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{v_\parallel^2}{v^2_{th\parallel s}} f^{\perp}_{0s} \times \left(\frac{1}{2} + \frac{u'^2_{\parallel 0s}}{v^2_{th\parallel s}}\right)^{-1}, \tag{B 1c}$$

normalized so that $C^{\perp}_{\ell s} = 1$ for a parallel-drifting bi-Maxwellian distribution function. The next set of coefficients were borne out of the linear theory:

$$C^{\parallel}_{\ell s}(\xi_s) = \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{\ell!}\left(\frac{v_\perp}{v_{th\perp s}}\right)^{2\ell}\frac{v_\parallel - u'_{\parallel 0s}}{v_\parallel - \omega/k_\parallel} f^{\parallel}_{0s} \tag{B 2}$$

for integer ℓ, where $\xi_s \doteq (\omega - k_\parallel u'_{\parallel 0s})/k_\parallel v_{th\parallel s}$ is the dimensionless Doppler-shifted phase velocity of the (linear) fluctuations. The functions defined by (B 2) engender suitable

† There is a type-setting error in equation (D25) of Howes *et al.* (2006).

generalisations of the plasma dispersion function for non-Maxwellian distributions: e.g.

$$C^{\parallel}_{\ell s}(\xi_s) = 1 + \xi_s Z_M(\xi_s) \text{ for a bi-Maxwellian, where } Z_M(\xi) \doteq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-x^2}}{x - \xi} \tag{B 3}$$

is the plasma dispersion function (Fried & Conte 1961).

Accounting for finite-Larmor-radius effects is most easily achieved in the Fourier domain, and the above $C_{\ell s}$ coefficients can be profitably generalised by including various combinations of the nth-order Bessel function $J_n(a_s)$, where $a_s \doteq k_\perp v_\parallel / \Omega_s$. Suitably normalised, they are

$$\Gamma_{00}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \left[\mathrm{J}_0(a_s)\right]^2 f_{0s} = 1 - \alpha_s + \dots \tag{B 4a}$$

$$\Gamma^{\perp}_{00}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \left[\mathrm{J}_0(a_s)\right]^2 f^{\perp}_{0s} = C^{\perp}_{0s} - \alpha_s + \dots \tag{B 4b}$$

$$\Gamma^{\perp}_{01}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \left[\mathrm{J}_0(a_s)\right]^2 \left(\frac{v_\parallel}{v_{th\parallel s}}\right) f^{\perp}_{0s} \times \left(\frac{u'_{\parallel 0s}}{v_{th\parallel s}}\right)^{-1} = C^{\perp}_{1s} - \alpha_s + \dots \tag{B 4c}$$

$$\Gamma^{\perp}_{02}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \left[\mathrm{J}_0(a_s)\right]^2 \left(\frac{v_\parallel}{v_{th\parallel s}}\right)^2 f^{\perp}_{0s} \times \left(\frac{1}{2} + \frac{u'^2_{\parallel 0s}}{v^2_{th\parallel s}}\right)^{-1} = C^{\perp}_{2s} - \alpha_s + \dots \tag{B 4d}$$

$$\Gamma^{\perp}_{10}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \, \frac{v^2_\perp}{v^2_{th\perp s}} \frac{2\mathrm{J}_0(a_s)\mathrm{J}_1(a_s)}{a_s} f^{\perp}_{0s} = 1 - \frac{3}{2}\alpha_s + \dots \tag{B 4e}$$

$$\Gamma^{\perp}_{11}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \, \frac{v^2_\perp}{v^2_{th\perp s}} \frac{2\mathrm{J}_1(a_s)\mathrm{J}_1(a_s)}{a_s} \left(\frac{v_\parallel}{v_{th\parallel s}}\right) f^{\perp}_{0s} \times \left(\frac{u'_{\parallel 0s}}{v_{th\parallel s}}\right)^{-1} = 1 - \frac{3}{2}\alpha_s C_{11s} + \dots \tag{B 4f}$$

$$\Gamma^{\perp}_{20}(\alpha_s) \doteq \frac{1}{n_{0s}} \int \mathrm{d}^3\boldsymbol{v} \left[\frac{2v^2_\perp}{v^2_{th\perp s}} \frac{\mathrm{J}_1(a_s)}{a_s}\right]^2 f^{\perp}_{0s} = 2\left(1 - \frac{3}{2}\alpha_s C_{20s} + \dots\right), \tag{B 4g}$$

$$\Gamma^{\|}_{00}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\left[\mathrm{J}_0(a_s)\right]^2 \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} = C^{\|}_{0s} - \alpha_s C^{\|}_{1s} + \dots, \tag{B 5a}$$

$$\Gamma^{\|}_{01}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\left[\mathrm{J}_0(a_s)\right]^2 \left(\frac{v_\|}{v_{th\|s}}\right) \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} \times \left(\frac{u'_{\|0s}}{v_{th\|s}}\right)^{-1}$$
$$= \frac{\omega}{k_\| u'_{\|0s}} \Gamma^{\|}_{00}(\xi_s,\alpha_s), \tag{B 5b}$$

$$\Gamma^{\|}_{02}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\left[\mathrm{J}_0(a_s)\right]^2 \left(\frac{v_\|}{v_{th\|s}}\right)^2 \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} \times \left(\frac{1}{2} + \frac{u'^2_{\|0s}}{v^2_{th\|s}}\right)^{-1}$$
$$= \left[\Gamma_{00}(\xi_s,\alpha_s) + \frac{2u'^2_{\|0s}}{v^2_{th\|s}} \frac{\omega^2}{k^2_\| u'^2_{\|0s}} \Gamma^{\|}_{00}(\xi_s,\alpha_s)\right] \left(1 + \frac{2u'^2_{\|0s}}{v^2_{th\|s}}\right)^{-1}, \tag{B 5c}$$

$$\Gamma^{\|}_{10}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{v^2_\perp}{v^2_{th\perp s}} \frac{2\mathrm{J}_0(a_s)\mathrm{J}_1(a_s)}{a_s} \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} = C^{\|}_{1s} - \frac{3}{2}\alpha_s C^{\|}_{2s} + \dots, \tag{B 5d}$$

$$\Gamma^{\|}_{11}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{v^2_\perp}{v^2_{th\perp s}} \frac{2\mathrm{J}_0(a_s)\mathrm{J}_1(a_s)}{a_s} \left(\frac{v_\|}{v_{th\|s}}\right) \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} \times \left(\frac{u'_{\|0s}}{v_{th\|s}}\right)^{-1}$$
$$= \frac{\omega}{k_\| u'_{\|0s}} \Gamma^{\|}_{10}(\xi_s,\alpha_s), \tag{B 5e}$$

$$\Gamma^{\|}_{20}(\xi_s,\alpha_s) \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\left[\frac{2v^2_\perp}{v^2_{th\perp s}} \frac{\mathrm{J}_1(\alpha_s)}{a_s}\right]^2 \frac{v_\| - u'_{\|0s}}{v_\| - \omega/k_\|} f^{\|}_{0s} = 2\left(C^{\|}_{2s} - \frac{3}{2}\alpha_s C^{\|}_{3s} + \dots\right), \tag{B 5f}$$

where the argument $\alpha_s \doteq (k_\perp \rho_s)^2/2$, and

$$C_{11s} \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{v^2_\perp}{v^2_{th\perp s}} \frac{v_\|}{u'_{\|0s}} f_{0s} \quad\text{and}\quad C_{20s} \doteq \frac{1}{n_{0s}}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{2}\frac{v^4_\perp}{v^4_{th\perp s}} f_{0s}, \tag{B 6}$$

both of which equate to unity for a drifting bi-Maxwellian distribution (2.9). To facilitate comparison with the long-wavelength results of Paper I, the final equalities in (B 4) and (B 5) provide their leading-order expansions in $\alpha_s \ll 1$. It is helpful to note the numbering scheme used for the $\Gamma_{\ell m}$ subscripts, which reflects the number of powers ℓ of $v^2_\perp$ and m of $v_\|$ in the integrand.

In Section 4, we promoted several of these Fourier-space $\Gamma_{\ell m}(\alpha_s)$ integrals to real-space operators by dressing them with hats. Their action on an arbitrary squared function $\Psi^2(\boldsymbol{r})$ is best expressed in Fourier space, where $\Psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$:

$$\int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}_{00}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma_{00}(\alpha_s)|\Psi_{\boldsymbol{k}}|^2, \quad \int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}^{\perp}_{00}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma^{\perp}_{00}(\alpha_s)|\Psi_{\boldsymbol{k}}|^2, \tag{B 7a,b}$$

$$\int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}^{\perp}_{02}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma^{\perp}_{02}(\alpha_s)|\Psi_{\boldsymbol{k}}|^2, \quad \int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}^{\perp}_{11}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma^{\perp}_{11}(\alpha_s)|\Psi_{\boldsymbol{k}}|^2, \tag{B 7c,d}$$

$$\int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}^{\|}_{20}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma^{\|}_{20}(0,\alpha_s)|\Psi_{\boldsymbol{k}}|^2, \quad \int \mathrm{d}^3\boldsymbol{r}\,\widehat{\Gamma}^{\perp}_{20}\Psi^2(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \Gamma^{\perp}_{20}(\alpha_s)|\Psi_{\boldsymbol{k}}|^2. \tag{B 7e,f}$$

This completes our catalog of integrals.

B.1. *Gyrokinetic dispersion relation for an electron-ion bi-kappa plasma*

A bi-kappa distribution function is often used to describe the non-thermal electron population in the solar wind and, in particular, its suprathermal ($T_e \sim 60$ eV) halo (e.g. Vasyliunas 1968; Maksimovic *et al.* 1997*a*,*b*, 2005). In this section, we specialize the linear gyrokinetic theory of Section 3.3 for a mean distribution function equal to

$$f_{bi\text{-}\kappa,s}(v_\|, v_\perp) \doteq \frac{n_{0s}}{\sqrt{\pi\kappa}\theta_{\|s}} \frac{1}{\pi\kappa\theta_{\perp s}^2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left[1 + \frac{(v_\| - u'_{\|0s})^2}{\kappa\theta_{\|s}^2} + \frac{v_\perp^2}{\kappa\theta_{\perp s}^2}\right]^{-(\kappa+1)}, \quad \text{(B 8)}$$

where Γ is the Gamma function, $\kappa > 3/2$ is the spectral index, and

$$\theta_{\|s} \doteq v_{th\|s}\sqrt{1-\frac{3}{2\kappa}} \quad \text{and} \quad \theta_{\perp s} \doteq v_{th\perp s}\sqrt{1-\frac{3}{2\kappa}} \qquad \text{(B 9)}$$

are the effective parallel and perpendicular thermal speeds, respectively. At low and thermal energies, the bi-kappa distribution approaches a Maxwellian distribution, whereas at high energies it exhibits a non-thermal tail that can be described as a decreasing power law.

Coefficients evaluated for $f_{bi\text{-}\kappa,s}$ in Appendix D of Paper I.

KAW limit:

$$\omega = \pm \frac{k_\| v_A k_\perp \rho_i}{\sqrt{\beta_{\perp i} + 2/(1+Z_i T_{\|0e}/T_{\perp 0i}) - 2\mathcal{H}_\kappa}} \left(1 + \frac{\beta_{\|e}}{2}\Delta_e\right)^{1/2} \left(1 - \beta_{\perp e}\Delta_e + \frac{\mathcal{H}}{C_\kappa}\right)^{1/2}, \qquad \text{(B 10)}$$

where

$$\mathcal{H}_\kappa \doteq \mathcal{H}\,\frac{(2C_\kappa - 1)T_{\perp 0e}/T_{\|0e} - 1}{T_{\perp 0e}/T_{\|0e} - 1} \quad \text{and} \quad C_\kappa \doteq \left(1 - \frac{1}{2\kappa}\right)\left(1 - \frac{3}{2\kappa}\right)^{-1}. \qquad \text{(B 11)}$$

B.2. *GK section including drifts*

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